SUBALGEBRAS OF C^*-ALGEBRAS

BY

WILLIAM B. ARVESON

University of California, Berkeley, Calif. U.S.A. (1)

Contents

(1) Research supported by NSF grant GP-5585 and the U.S. Army Research Office, Durham.

Introduction

In a broad sense, the objective of this paper is to call attention to certain relations that exist between non self-adjoint operator algebras on Hilbert space and the C^*-algebras they generate. These relations make it possible to predict, from knowledge of the subalgebra alone, certain features of its generated C^*-algebra. As a typical application, one is able to conclude that certain isometric linear maps between certain non self-adjoint operator algebras are implemented by $*$-isomorphisms of their generated C^*-algebras (2.2.5). In turn, the latter makes it possible to obtain a classification (to unitary equivalence) of certain Hilbert space operators which are neither normal nor compact (3.6.12 and 3.2.11). The invariants of this classification involve an infinite-dimensional analogue of the minimum polynomial of a matrix.

A principal concept underlying these results is that of boundary representations. Let B be an (abstract) C^*-algebra and let A be a linear subspace of B. An irreducible representation π of B on a Hilbert space $\mathfrak{H}$ is called a boundary representation for A if the only completely positive linear map of B into $L(\mathfrak{H})$ which agrees with π on A is π itself. Thus, boundary representations have *unique* completely positive linear extensions from their restrictions to A. It is crucial for the applications that this definition make sense in general, requiring no *a priori* relationship between A and B (for example, $A+A^*$ need not be dense in B). The properties of boundary representations are developed in Chapter 2.

Chapter 3 contains a variety of examples of boundary representations, along with applications to operators on Hilbert space. We regard this as the main chapter, at least in terms of immediate applications, and refer the reader to the introductory paragraphs of chapter 3 for a summary of its contents.

The first chapter contains a discussion of completely positive linear maps of C^*-algebas. The most basic result here is an extension theorem, of Hahn-Banach type, for operator-valued linear maps of subspaces of C^*-algebras (1.2.3). Most of the results of this paper depend, ultimately, upon this extension theorem. In section 1.3 we identify the commutant of the image of a C^*-algebra under a completely positive linear map, and in the last section 1.4 we give solutions to a number of extremal problems in the partially ordered cone of completely positive maps of a C^*-algebra.

Our original plan was to include two additional chapters dealing with a generalized dilation theory for the Hilbert space representations of arbitrary Banach algebras. These chapters have been omitted, due to the length of Chapter 3, and we will take up dilation theory in a subsequent paper.

For the most part, our terminology follows [4], with the following exceptions. The term C^*-algebra means a complex involutive Banach algebra B satisfying $\|x^*x\| = \|x\|^2$ $(x \in B)$

and which contains a multiplicative identity. $L(\mathfrak{H})$ (resp. $LC(\mathfrak{H})$) denotes the algebra of all bounded (resp. compact) operators on a Hilbert space $\mathfrak{H}$. For $T \in L(\mathfrak{H})$ we shall write $P(T)$ for the norm-closed algebra generated by all polynomials in T, and $C^*(T)$ for the C^*-algebra generated by T and the identity. More generally, $C^*(S)$ for S a subset of a C^*-algebra B means the C^*-subalgebra of B generated by S and the identity.

We will say two operators $T_i \in L(\mathfrak{H}_i)$ $(i=1, 2)$ are algebraically equivalent if there is a $*$-isomorphism α of $C^*(T_1)$ on $C^*(T_2)$ such that $\alpha(T_1)=T_2$. It is easy to see that two normal operators are algebraically equivalent iff they have the same spectrum (the spectrum of T will be written sp (T)). Thus, one may regard algebraically equivalent nonnormal operators as having the same "spectrum" in a generalized sense. T_1 and T_2 are said to be quasi-equivalent if the above map α can be extended to a $*$-isomorphism between the respective von Neumann algebras generated by T_1 and T_2. Again, for normal operators T_i on separable spaces, one can show that quasi-equivalence is the same as requiring that sp $(T_1)=$ sp (T_2) and the spectral measures of T_1 and T_2 be mutually absolutely continuous. Finally, and in a more familiar sense, T_1 and T_2 are unitarily equivalent if there is a unitary operator U from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ such that $UT_1=T_2U$. Each of these equivalence relations clearly implies the preceding one.

Sets of (bounded, linear) operators are written with script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$, etc., and $\mathcal{R}'$ denotes the commutant of $\mathcal{R}$. German letters stand for Hilbert spaces and their subsets, Greek letters stand for vectors, and the usual brackets are employed for closed linear spans (e.g., $[\mathcal{A}\mathfrak{H}]$ denotes the closed linear span of all vectors $T\xi$, $T \in \mathcal{A}$, $\xi \in \mathfrak{H}$). The spectrum of an operator $T \in L(\mathfrak{H})$ is written sp (T). A reducing subspace for a subset $\mathcal{A} \subseteq L(\mathfrak{H})$ is a closed subspace of $\mathfrak{H}$ which is invariant under both $\mathcal{A}$ and $\mathcal{A}^*$; $\mathcal{A}$ is irreducible if only the trivial subspaces, 0 and $\mathfrak{H}$, reduce $\mathcal{A}$. Remaining notations are (we hope) defined in context.

We remark, finally, that some of the results of this paper were announced in [1].

Chapter 1. Completely positive maps

1.1. *Preliminaries.* This section begins with a discussion of a theorem of Stinespring characterizing completely positive operator-valued linear maps of C^*-algebras, and some associated material, much of which is known. We then describe, for later use in section 1.2, some topological properties of certain spaces of operator-valued linear maps.

Let B and B' be C^*-algebras, and let φ be linear map of B into B'. φ is *positive* if $\varphi(x) \geqslant 0$ for every positive x in B. For every integer $n \geqslant 1$, let M_n be the C^*-algebra of all complex $n \times n$ matrices. There is a natural way to make the algebra $B \otimes M_n$ of all $n \times n$ matrices over B into a $*$-algebra (for example, the involution is $(x_{ij})^* = (x_{ji}^*)$), and moreover, there is a *unique* C^*-norm on this $*$-algebra (existence follows by tensoring faithful representations of B and M_n, and uniqueness follows from p. 18 of [4]). Thus, it is not ambiguous to speak of $B \otimes M_n$ as a C^*-algebra. Note that unique means identical, not merely equivalent, so that the preceding statement would be false for a general Banach algebra in place of B. Now given the linear map φ, one can define a linear map φ_n: $B \otimes M_n \to B' \otimes M_n$ by applying φ element by element to each matrix over B. φ is called *completely positive* if each φ_n is positive, $n \geq 1$. The term is due to W. F. Stinespring [23], as are some of the results we will presently describe.

A $*$-homomorphism is easily seen to be completely positive. It is shown in [23] that every positive map of a commutative C^*-algebra into $L(\mathfrak{H})$ is completely positive, as is every scalar-valued positive linear map of a general C^*-algebra. It follows easily from the latter that a positive map into a commutative C^*-algebra is completely positive (see, for example, the proof of 1.2.2). It follows that a positive linear map of B into B' is completely positive if either B, or B', is commutative.

In even the simplest non-commutative cases, however, there exist positive maps which are not completely positive. While an example is given in [23], we shall describe here a somewhat simpler one. Let $n \geq 2$ be an integer and let $B = B' = M_n$. Let φ be the positive linear map of M_n into itself which takes every matrix to its transpose (note that φ is an anti-automorphism of M_n). We will show that φ_n is not completely positive. Let $\{E_{ij}: 1 \leq i, j \leq n\}$ be the canonical system of matrix units for M_n, and define $E \in M_n \otimes M_n$ to be the $n \times n$ matrix (E_{ij}). Note that $(1/n)E$ is a self-adjoint projection, and so is positive. But $\varphi_n(E)$ is the matrix $(\varphi(E_{ij})) = (E_{ji})$, which is self-adjoint, nonscalar, and satisfies $\varphi_n(E)^2 = I$ (I denoting the identity in $M_n \otimes M_n$); i.e., $\varphi_n(E)$ is a nonscalar self-adjoint unitary element. Such an operator must have the form $2P - I$, where P is a self-adjoint projection different from 0 and I, and obviously no such operator is positive. Thus, φ is not completely positive.

Let $\mathfrak{H}$ be a Hilbert space, and let B be a C^*-algebra. If V is a bounded linear operator from $\mathfrak{H}$ into some other Hilbert space $\mathfrak{K}$, and π is a representation of B on $\mathfrak{K}$, then $\varphi(x) = V^*\pi(x)V$ defines a linear map of B into $L(\mathfrak{H})$. It is easy to see that φ is completely positive; for if (x_{ij}) is a positive $n \times n$ matrix over B, and $\xi_1, \ldots, \xi_n \in \mathfrak{H}$, then choose $z_{ij} \in B$ such that $(x_{ij}) = (z_{ij})^*(z_{ij})$ and observe that

$$\sum_{i,j} (\varphi(x_{ij})\,\xi_j, \xi_i) = \sum_{i,j} (\pi(x_{ij})V\xi_j, V\xi_i) = \sum_{i,j,k} (\pi(z_{kj})V\xi_j, \pi(z_{ki})V\xi_i) = \sum_k \Big\| \sum_j \pi(z_{kj})V\xi_j \Big\|^2 \geq 0.$$

This implies that the operator matrix $(\varphi(x_{ij}))$ is positive, as an operator on $\mathfrak{H} \otimes \mathbf{C}^n = \mathfrak{H} \oplus \ldots \oplus \mathfrak{H}$, and hence φ_n is positive. Because n was arbitrary, the complete positivity of

φ is established. W. F. Stinespring proved in [23] that this in fact characterizes completely positive maps. For the reader's convenience, we will state this formally and outline a (slightly simplified) proof.

THEOREM 1.1.1 (Stinespring). *Let B be a C^*-algebra with identity and let $\mathfrak{H}$ be a Hilbert space. Then every completely positive linear map of B into $L(\mathfrak{H})$ has the form $\varphi(x) = V^*\pi(x)V$, where π is a representation of B on some Hilbert space $\mathfrak{K}$ and V is a bounded operator from $\mathfrak{H}$ to $\mathfrak{K}$*

Proof. Consider the vector space tensor product $B\otimes\mathfrak{H}$, and define a bilinear form $\langle\cdot,\cdot\rangle$ on $B\otimes\mathfrak{H}$ as follows; if $u = x_1\otimes\xi_1 + \dots + x_m\otimes\xi_m$ and $v = y_1\otimes\eta_1 + \dots + y_n\otimes\eta_n$, put

$$\langle u, v\rangle = \sum_{i,j} (\varphi(y_i^* x_j)\,\xi_j, \eta_i),$$

φ being the given map of B into $L(\mathfrak{H})$. The fact that φ is completely positive guarantees that $\langle\cdot,\cdot\rangle$ is positive semi-definite. For each $x\in B$, define a linear transformation $\pi_0(x)$ on $B\otimes\mathfrak{H}$ by $\pi_0(x)$: $\sum x_j\otimes\xi_j \to \sum xx_j\otimes\xi_j$. π_0 is an algebra homomorphism for which $\langle u, \pi_0(x)v\rangle = \langle\pi_0(x^*)u, v\rangle$, for all $u, v\in B\otimes\mathfrak{H}$. It follows that, for fixed u, $\varrho(x) = \langle\pi_0(x)u, u\rangle$ defines a positive linear functional on B, i.e., $\varrho(x^*x)\geqslant 0$, hence

$$\langle\pi_0(x)u, \pi_0(x)u\rangle = \langle\pi_0(x^*)\pi_0(x)u, u\rangle = \langle\pi_0(x^*x)u, u\rangle = \varrho(x^*x) \leqslant \|x^*x\|\varrho(e) = \|x\|^2\langle u, u\rangle,$$

where e is the identity of B.

Now let $\mathfrak{N} = \{u\in B\otimes\mathfrak{H}: \langle u, u\rangle = 0\}$. $\mathfrak{N}$ is a linear subspace of $B\otimes\mathfrak{H}$, invariant under $\pi_0(x)$ for every $x\in B$ (by the preceding sentence), and $\langle\cdot,\cdot\rangle$ determines a positive definite inner product on the quotient $B\otimes\mathfrak{H}/\mathfrak{N}$ in the usual way: $\langle u+\mathfrak{N}, v+\mathfrak{N}\rangle = \langle u, v\rangle$. Letting $\mathfrak{K}$ be the Hilbert space completion of the quotient, the preceding paragraph implies that there is a unique representation π of B on $\mathfrak{K}$ such that

$$\pi(x)(u+\mathfrak{N}) = \pi_0(x)u + \mathfrak{N}, \quad x\in B,\ u\in B\otimes\mathfrak{H}.$$

Finally, define a linear map V of $\mathfrak{H}$ into $\mathfrak{K}$ by $V\xi = e\otimes\xi + \mathfrak{N}$. It follows that $\|V\xi\|^2 = (\varphi(e)\xi, \xi) \leqslant \|\varphi(e)\|\,\|\xi\|^2$, so that V is bounded, and the required formula $\varphi(x) = V^*\pi(x)V$ follows from the definition of V by a routine computation.

Remarks. Let $\varphi(x) = V^*\pi(x)V$ be as in the theorem. Letting $\mathfrak{K}_0 = [\pi(B)V\mathfrak{H}]$, then the restriction π_0 of π to $\mathfrak{K}_0$ also satisfies $\varphi(x) = V^*\pi_0(x)V$, and so there is no essential loss if we require that $[\pi(B)V\mathfrak{H}] = \mathfrak{K}$. Such a pair (π, V) will be called *minimal*. Observe that a minimal pair is uniquely determined by φ in the following sense. Let π_1 and π_2 be representations of B on Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, and let $V_i \in L(\mathfrak{H}, \mathfrak{K}_i)$ be such that $[\pi_i(B) V_i \mathfrak{H}] = \mathfrak{K}_i$ and $V_1^* \pi_1(x) V_1 = V_2^* \pi_2(x) V_2$ for every $x \in B$; then there is a unitary map U of $\mathfrak{K}_1$ on $\mathfrak{K}_2$ such that $UV_1 = V_2$ and $U\pi_1(x) = \pi_2(x) U$ for all $x \in B$ (for the proof, simply check that the mapping

$$\sum_{i=1}^{n} \pi_1(x_i) V_1 \xi_i \rightarrow \sum_{i=1}^{n} \pi_2(x) V_2 \xi_i, \quad x_i \in B, \ \xi_i \in \mathfrak{H}$$

is a densely defined isometry of $\mathfrak{K}_1$ on a dense subspace of $\mathfrak{K}_2$, whose unitary extension U has the stated properties).

Let $\varphi(x) = V^* \pi(x) V$ be as in the theorem. Note that if $\varphi(e) = I$ then $V^* V = I$, that is, V is an *isometric* embedding of $\mathfrak{H}$ in $\mathfrak{K}$. Using V, then, we can identify $\mathfrak{H}$ with a subspace of $\mathfrak{K}$, and the original equation takes the form $\varphi(x) = P\pi(x)|_{\mathfrak{H}}$, P being the projection of $\mathfrak{K}$ on $\mathfrak{H}$ (the new V is the inclusion map of $\mathfrak{H}$ into $\mathfrak{K}$, whose adjoint is P).

It should also be pointed out that a theorem very similar to Stinespring's was found independently by Sz.-Nagy [25]. We have given Stinespring's version for two reasons. First, it is formulated in terms of C^*-algebras, with which we are concerned in this paper. More importantly, however, it makes explicit the role of complete positivity, in terms of the "matrix" algebras $B \otimes M_n$, $n = 1, 2, \ldots$. Indeed, the results of this paper have strongly indicated that to effectively study general (non self-adjoint) operator algebras on Hilbert space, one should study not only the algebra $\mathcal{A}$ but also the sequence of algebras $\mathcal{A} \otimes M_n$ (each regarded as a subalgebra of the corresponding C^*-algebra $C^*(\mathcal{A}) \otimes M_n$). Accordingly, given a nonnormal operator T, we shall consider "matrix-valued" (as well as scalar-valued) polynomials in T (cf., 3.6 and 3.7).

We now describe certain topological properties of the space of all operator-valued linear maps of a subspace of a C^*-algebra, for use later on in section 1.2. Let S be a linear subspace of a C^*-algebra B, and let $\mathfrak{H}$ be a Hilbert space, fixed throughout the remainder of this section. $\mathcal{B}(S, \mathfrak{H})$ will denote the vector space of all bounded linear maps of S into $L(\mathfrak{H})$. Note that $\mathcal{B}(S, \mathfrak{H})$ is a Banach space in the obvious norm. We shall endow $\mathcal{B}(S, \mathfrak{H})$ with a certain weak topology, relative to which it becomes the dual of another Banach space.

For $r > 0$, let $\mathcal{B}_r(S, \mathfrak{H})$ denote the closed ball of radius r: $\mathcal{B}_r(S, \mathfrak{H}) = \{\varphi \in \mathcal{B}(S, \mathfrak{H})$: $\|\varphi(a)\| \leqslant r\|a\|$ for all $a \in S\}$. First, topologize $\mathcal{B}_r$ as follows: by definition, a net $\varphi_\nu \in \mathcal{B}_r(S, \mathfrak{H})$ converges to $\varphi \in \mathcal{B}_r(S, \mathfrak{H})$ if $\varphi_\nu(a) \rightarrow \varphi(a)$ in the weak operator topology, for every $a \in S$. A convex subset $\mathcal{U}$ of $\mathcal{B}(S, \mathfrak{H})$ is *open* if $\mathcal{U} \cap \mathcal{B}_r(S\ \mathfrak{H})$ is an open subset of $\mathcal{B}_r(S, \mathfrak{H})$, for every $r > 0$. The convex open sets form a base for a locally convex Hausdorff topology on $\mathcal{B}(S, \mathfrak{H})$, which we shall call the BW-topology (this topology is Hausdorff because the convex sets of the form $\mathcal{U}_{\xi,\eta,a,t} = \{\varphi \in \mathcal{B}(S, \mathfrak{H})$: $\mathrm{Re}\,(\varphi(a)\xi, \eta) < t\}$, $\xi, \eta \in \mathfrak{H}$, $a \in S$, $t \in \mathbf{R}$, are BW-open and separate elements of $\mathcal{B}(S, \mathfrak{H})$). Equivalently, the BW-topology is the strongest locally convex topology on $\mathcal{B}(S, \mathfrak{H})$ which relativizes to the prescribed topology on each ball $\mathcal{B}_r(S, \mathfrak{H})$, $r>0$.

It is clear that a linear functional f on $\mathcal{B}(S, \mathfrak{H})$ is BW-continuous iff the restriction of f to every $\mathcal{B}_r(S, \mathfrak{H})$ is continuous. By linearity, we conclude that *f is* BW-*continuous iff the restriction of f to $\mathcal{B}_1(S, \mathfrak{H})$ is continuous.*

There are other ways the BW-topology could have been defined (for example, see 1.1.4), but the description above is easiest to apply for our immediate purposes. In fact, we shall require only one or two properties of this topology.

Remark 1.1.2. For every $r>0$, $\mathcal{B}_r(S, \mathfrak{H})$ is compact in the relative BW-topology. Indeed, this is an immediate consequence of a general theorem of R. V. Kadison [14].

Remark 1.1.3. The restriction map $\varphi \to \varphi|_S$ of $\mathcal{B}(B, \mathfrak{H})$ into $\mathcal{B}(S, \mathfrak{H})$ is BW-continuous. For since restriction is linear, it suffices to show that $\varphi \to f(\varphi|_S)$ is a BW-continuous linear functional on $\mathcal{B}(B, \mathfrak{H})$, for every BW-continuous linear functional f on $\mathcal{B}(S, \mathfrak{H})$; and by the above remarks, this will follow from the BW-continuity on $\mathcal{B}_1(B, \mathfrak{H})$. But if φ_ν is a net in $\mathcal{B}(B, \mathfrak{H})$, $\|\varphi_\nu\| \leq 1$, and $\varphi_\nu \to \varphi$ (BW), then in particular $\varphi_\nu(a) \to \varphi(a)$ in the weak operator topology, for every $a \in S$, and thus $\varphi_\nu|_S \to \varphi|_S$ in the relative BW-topology of $\mathcal{B}_1(S, \mathfrak{H})$. Thus $\varphi_\nu|_S \to \varphi|_S$ (BW), by definition of the topology, and $f(\varphi_\nu|_S) \to f(\varphi|_S)$ follows.

This topology has a number of pleasant properties, which we do not need, some of which we now describe (without proof) for the benefit of the reader. The proofs are not difficult and, by and large, the methods are adapted from those on pp. 427–429 and p. 512 of [6]. Let $\mathcal{B}(S, \mathfrak{H})_*$ denote the vector space of all BW-continuous linear functionals on $\mathcal{B}(S, \mathfrak{H})$. Because such functionals are necessarily bounded relative to the norm topology on $\mathcal{B}(S, \mathfrak{H})$, $\mathcal{B}(S, \mathfrak{H})_*$ becomes a normed linear space with the norm $\|f\| = \sup \{|f(\varphi)|: \varphi \in \mathcal{B}_1(S, \mathfrak{H})\}$. Then we have:

1.1.4.
- (i) *$\mathcal{B}(S, \mathfrak{H})_*$ is a Banach space.*
- (ii) *The duality $\langle \varphi, f \rangle = f(\varphi)$, $\varphi \in \mathcal{B}(S, \mathfrak{H})$, $f \in \mathcal{B}(S, \mathfrak{H})_*$ defines an isometric isomorphism of $\mathcal{B}(S, \mathfrak{H})$ onto the dual of $\mathcal{B}(S, \mathfrak{H})_*$ which identifies the* BW-*topology with the weak*-topology.*
- (iii) *The elements of $\mathcal{B}(S, \mathfrak{H})_*$ are precisely those linear functionals that admit a representation of the form $f(\varphi) = \sum_{n=1}^{\infty} \varrho_n(\varphi(a_n))$, where $\{a_n\}$ is a bounded sequence in S and $\{\varrho_n\}$ is a sequence of ultraweakly continuous linear functionals on $L(\mathfrak{H})$ such that $\sum \|\varrho_n\| < \infty$.*

The preceding discussion fits nicely into a more general format. It is not hard to see that, if one replaces $\mathcal{B}(S, \mathfrak{H})$ with the Banach space $\mathcal{B}(X, Y^*)$ of all bounded linear maps of a Banach space X into the dual of a Banach space Y, and if one imitates the definition of the BW-topology in this setting, then all the preceding statements remain true (note, incidentally, that $L(\mathfrak{H})$ is the dual of the Banach space $L(\mathfrak{H})_*$ of all ultraweakly continuous linear functionals on $L(\mathfrak{H})$ [5], so that $\mathcal{B}(S, \mathfrak{H})$ does have the form $\mathcal{B}(X, Y^*)$). The representation (iii), for example, becomes $f(\varphi)=\sum_{n=1}^{\infty}\langle\varphi(x_n), y_n\rangle$, where $\{x_n\}$ is a bounded sequence in X, $y_n \in Y$ is such that $\sum\|y_n\| < \infty$, and $\langle\cdot,\cdot\rangle$ is the canonical pairing of Y^* and Y.

1.2. *An extension theorem.* Let S be a self-adjoint linear subspace of a C^*-algebra B, such that the identity e of B belongs to S. A familiar theorem of M. Krein ([17], p. 227) implies that every positive linear functional on S has a positive linear extension to B (ϱ: $S \to C$ is *positive* if $\varrho(a) \geqslant 0$ for every positive element a in S). The fact that $e \in S$ insures that there are plenty of positive elements in S, indeed $\|a\|e-a$ is positive for every self-adjoint a; and from this it follows easily that a positive linear functional on S is necessarily self-adjoint (cf. the proof of 1.2.3). We shall require a generalization of Krein's theorem to operator-valued maps, under the additional requirement that S be norm-closed. A linear map φ of S into another C^*-algebra B' is called *positive* if $\varphi(a) \geqslant 0$ for every positive element a of S. Significantly, the obvious generalization of Krein's theorem is false: an operator-valued positive linear map φ: $S \to L(\mathfrak{H})$ ($\mathfrak{H}$ denoting a Hilbert space) need not have a positive extension to B, even when B is commutative and $\mathfrak{H}$ is finite-dimensional (an example is given in appendix A.2).

The proper generalization involves the notion of complete positivity. For $S \subseteq B$ as above and n a positive integer, the linear space $S \otimes M_n$ of all $n \times n$ matrices over S is a subspace of the C^*-algebra $B \otimes M_n$, and a linear map φ of S into another C^*-algebra B' induces a linear map φ_n: $S \otimes M_n \to B' \otimes M_n$ by applying φ element by element to each matrix over S.

Definition 1.2.1. φ is called *completely positive, completely contractive*, or *completely isometric* according as each φ_n is positive, contractive (i.e., $\|\varphi_n\| \leqslant 1$), or isometric.

Theorem 1.2.3 below asserts that a completely positive linear map of S into $L(\mathfrak{H})$ has a completely positive extension: the following result implies that a scalar-valued positive linear map is already completely positive. Thus, 1.2.3 generalizes Krein's theorem.

PROPOSITION 1.2.2. *Let S be a self-adjoint subspace of a C^*-algebra B, and let B' be a commutative C^*-algebra. Then every positive linear map of S into B' is completely positive.*

Proof. We can assume that $B'=C(X)$, for X a compact Hausdorff space. Let φ be a positive map of S into $C(X)$, let n be a positive integer, and let (a_{ij}) be a positive element of $B \otimes M_n$ such that $a_{ij} \in S$ for all i, j.

Let $f_{ij}=\varphi(a_{ij})\in C(X)$; we must show that the matrix (f_{ij}) is a positive element of $C(X)\otimes M_n$. This will follow if we show that $(f_{ij}(x))$ is a positive matrix for every $x\in X$ (one way to check this known result is to use the fact that every pure state of $C(X)\otimes M_n$ has the form $\delta_x\otimes\varrho$ where ϱ is a pure state of M_n and δ_x is the evaluation functional for some $x\in X$, see [27]; thus $\sigma((f_{ij}))\geqslant 0$ for every pure state σ of $C(X)\otimes M_n$ and it is apparent from this that $(f_{ij})\geqslant 0$). But if $\lambda_1, ..., \lambda_n\in \mathbf{C}$ then for each $x\in X$ we have

$$\sum_{i,j} f_{ij}(x)\,\lambda_j\bar{\lambda}_i=(\sum f_{ij}\lambda_j\bar{\lambda}_i)\,(x)=(\sum\varphi(a_{ij})\,\lambda_j\bar{\lambda}_i)\,(x)=\varphi(\sum a_{ij}\lambda_j\bar{\lambda}_i)\,(x)\geqslant 0,$$

because $\sum a_{ij}\lambda_j\bar{\lambda}_i$ is a positive element of S and φ is a positive linear map. That completes the proof.

We can now state the main extension theorem.

THEOREM 1.2.3. *Let S be a norm-closed self-adjoint linear subspace of a C^*-algebra B, which contains the identity of B, and let $\mathfrak{H}$ be a Hilbert space. Then for every completely positive linear map φ: $S\to L(\mathfrak{H})$, there is a completely positive linear map φ_1: $B\to L(\mathfrak{H})$ such that $\varphi_1|_S=\varphi$.*

The proof will occupy a number of steps, some of which we state as lemmas. First let $CP\ (S, \mathfrak{H})$ (resp. $CP\ (B, \mathfrak{H})$) denote the set of all completely positive linear maps of S (resp. B) into $L(\mathfrak{H})$. Each is a subset of $\mathcal{B}(S, \mathfrak{H})$ and $\mathcal{B}(B, \mathfrak{H})$, respectively, and thus inherits a BW-topology from the larger space (cf. section 1.1). In addition, it is apparent that both $CP\ (S, \mathfrak{H})$ and $CP\ (B, \mathfrak{H})$ are convex cones, and the set $CP\ (B, S)|_S$ of all restrictions of maps in $CP\ (B, \mathfrak{H})$ to S is a subcone of $CP\ (S, \mathfrak{H})$. We must prove, of course, that $CP\ (B, \mathfrak{H})|_S=CP\ (S, \mathfrak{H})$.

LEMMA 1.2.4. *$CP\ (B, \mathfrak{H})|_S$ is a closed cone in $\mathcal{B}(S, \mathfrak{H})$, relative to the* BW*-topology.*

Proof. We claim first that $\|\varphi\|=\|\varphi|_S\|$, for every $\varphi\in CP\ (B, \mathfrak{H})$. Choose π and V, as in Theorem 1.1.1., such that $\varphi(x)=V^*\pi(x)\,V$, $x\in B$. Then $\|\varphi\|\leqslant\|V^*\|\cdot\|V\|=\|V^*V\|=\|\varphi(e)\|$; since $e\in S$ it follows that $\|\varphi\|\leqslant\|\varphi|_S\|$. The opposite inequality is trivial.

Next, observe that $CP(B, \mathfrak{H})$ is a BW-closed subset of $\mathcal{B}(B, \mathfrak{H})$; indeed, since $CP(B,\mathfrak{H})$ is convex, then by definition it is closed iff $CP\ (B,\mathfrak{H})\cap\mathcal{B}_r(B,\mathfrak{H})$ is (relatively) closed, for every $r>0$. But if φ_ν is a bounded net in $CP\ (B, \mathfrak{H})$ such that $\varphi_\nu\to\varphi\in\mathcal{B}(B, \mathfrak{H})$ (BW), then $\varphi_\nu(x)\to\varphi(x)$ in the weak operator topology, for every $x\in B$, and this makes it plain that φ must also be completely positive.

By remark 1.1.2, it follows that for every $r>0$, $CP\ (B,\mathfrak{H})\cap\mathcal{B}_r(B, \mathfrak{H})$ is BW-compact. The first paragraph of the proof shows that the restriction map $\varphi\to\varphi|_S$ carries $CP\ (B,\mathfrak{H})\cap$

$\mathcal{B}_r(B, \mathfrak{H})$ onto $CP\ (B, \mathfrak{H})|_S \cap \mathcal{B}_r(S, \mathfrak{H})$, and by remark 1.1.3, restriction is BW-continuous; we conclude that $CP\ (B, \mathfrak{H})|_S \cap \mathcal{B}_r(S, \mathfrak{H})$ is compact, and therefore closed. Since $CP\,(B, \mathfrak{H})|_S$ is convex, it follows from the definition of the BW-topology that this set is closed, and the proof of the lemma is complete.

Now let f be an arbitrary BW-continuous linear functional such that

$$\operatorname{Re} f(CP\ (B, \mathfrak{H})|_S) \geq 0;$$

we will show that $\operatorname{Re} f(\varphi) \geq 0$ for every $\varphi \in CP\ (S, \mathfrak{H})$. This, along with 1.2.4 and a standard separation theorem, leads to the desired conclusion $CP\ (S, \mathfrak{H}) \subseteq CP\ (B, \mathfrak{H})|_S$.

The first step is to find a complex-linear functional g on $\mathcal{B}(S, \mathfrak{H})$ which agrees with $\operatorname{Re} f$ on $CP\ (S, \mathfrak{H})$, as follows. Introduce an involution $\varphi \to \varphi^\sim$ in $\mathcal{B}(S, \mathfrak{H})$ by $\varphi^\sim(x) = \varphi(x^*)^*$ (here we use the fact that $S = S^*$). Note that every $\varphi \in \mathrm{CP}\ (S, \mathfrak{H})$ is selfadjoint in the sense that $\varphi = \varphi^\sim$, or what is the same, $\varphi(a) = \varphi(a)^*$ for every self-adjoint a in S. Indeed, both $\|a\|e$ and $\|a\|e - a$ are positive elements of S, thus $\varphi(a) = \varphi(\|a\|e) - \varphi(\|a\|e - a)$ is a difference of positive operators in $L(\mathfrak{H})$, so $\varphi(a)$ is self-adjoint. Now define g on $\mathcal{B}(S, \mathfrak{H})$ by the equation $g(\psi) = \frac{1}{2}(f(\psi) + \overline{f(\psi^\sim)})$. It is clear that $\psi \to \psi^\sim$ is BW-continuous on bounded subsets of $\mathcal{B}(S, \mathfrak{H})$ (because $X \to X^*$ is a weakly continuous map of $L(\mathfrak{H})$), and so by definition of the BW-topology $\psi \to \psi^\sim$ is continuous. g is therefore a complex-linear BW-continuous functional, and the preceding remarks show that $g = \operatorname{Re} f$ on $CP\ (S, \mathfrak{H})$. What we must prove, therefore, is that $g(CP\ (B, \mathfrak{H})|_S) \geq 0$ implies $g(CP\ (S, \mathfrak{H})) \geq 0$.

Assume, from here on in the proof, that $g(CP\ (B, \mathfrak{H})|_S) \geq 0$.

Now let $\mathcal{J}$ be the net of all finite-dimensional projections in $L(\mathfrak{H})$, directed in the increasing sense by the usual partial order $P \leq Q$. We will define a net g_P of linear functionals as follows. First, define $P\varphi P$ for $\varphi \in \mathcal{B}(S, \mathfrak{H})$ by $P\varphi P(a) = P\varphi(a)P$, $a \in S$. It is clear that for fixed P, $\varphi \to P\varphi P$ is linear and BW-continuous (again, it suffices to check continuity on bounded sets, but that is obvious), and carries $CP\ (S, \mathfrak{H})$ into itself. Now let $g_P(\varphi) = g(P\varphi P)$.

Lemma 1.2.5. *$\lim_P g_P(\varphi) = g(\varphi)$, for every $\varphi \in \mathcal{B}(S, \mathfrak{H})$.*

Proof. Since g is BW-continuous, it suffices to show that $\lim_P P\varphi P = \varphi$ in the BW-topology, for every $\varphi \in \mathcal{B}(S, \mathfrak{H})$. Now the net $\{P\}$ converges to the identity operator in the strong operator topology, and since multiplication is strongly continuous on the unit ball of $L(\mathfrak{H})$, it follows that $PXP \to X$ strongly, for every $X \in L(\mathfrak{H})$. In particular, $P\varphi(a)P \to \varphi(a)$ in the weak operator topology, for every $a \in S$; and since $\{P\varphi P\}$ is a bounded net, it follows from the definition of the BW-topology that $\lim_P P\varphi P = \varphi$, completing the proof.

Using 1.2.5, then, $g(CP(S, \mathfrak{H})) \geq 0$ will follow if we prove that $g_P(CP(S, \mathfrak{H})) \geq 0$, for every finite-dimensional projection $P \in L(\mathfrak{H})$. Now fix such a P, and let n be the dimension of P. The next step is the decisive one.

LEMMA 1.2.6. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be an orthonormal base for $P\mathfrak{H}$. Then there is an $n \times n$ array a_{ij} of elements of S such that*

$$g_P(\varphi) = \sum_{i,j} (\varphi(a_{ij})\, \xi_j, \xi_i),$$

for every $\varphi \in \mathcal{B}(S, \mathfrak{H})$.

Proof. Let $\{E_{ij}\}$ be a family of partial isometries in $L(P\mathfrak{H})$ such that $E_{ij}\xi_k = \delta_{jk}\xi_i$, $i, j, k \leq n$. For the moment, fix i and j, $1 \leq i, j \leq n$. Every bounded linear functional F on S defines an element $F \otimes E_{ij}$ of $\mathcal{B}(S, \mathfrak{H})$ in the following way: $F \otimes E_{ij}(a) = F(a)\, E_{ij}$, $a \in S$. Now if g is as above define an element α_{ij} in the bidual of S by $\alpha_{ij}(F) = g(F \otimes E_{ij})$. We claim: α_{ij} is a weak*-continuous linear functional on the dual of S. By the Krein-Smulyan theorem ([6], p. 429), it suffices to show that α_{ij} is weak*-continuous on the unit ball. But if F_ν is a net of functionals on S such that $\|F_\nu\| \leq 1$ and $F_\nu \to F$ (weak*), then $F_\nu(a)\, E_{ij}$ tends boundedly to $F(a)\, E_{ij}$ in the weak operator topology of $L(\mathfrak{H})$, for every $a \in S$. Thus, $F_\nu \otimes E_{ij} \to F \otimes E_{ij}$ in the BW-topology of $B(S, \mathfrak{H})$, and since g is BW-continuous, we see that

$$\alpha_{ij}(F_\nu) = g(F_\nu \otimes E_{ij}) \to g(F \otimes E_{ij}) = \alpha_{ij}(F),$$

as asserted.

Because S is norm-closed, there is an array $a_{ij} \in S$ such that $g(F \otimes E_{ij}) = \alpha_{ij}(F) = F(a_{ij})$, for every bounded functional F on S. Now fix $\varphi \in \mathcal{B}(S, \mathfrak{H})$, and define functionals F_{ij} on S by $F_{ij}(a) = (\varphi(a)\xi_j, \xi_i)$. Letting P_i be the projection on $[\xi_i]$ we have, for every $a \in S$, $P_i\varphi(a)P_j = (\varphi(a)\xi_j, \xi_i)\, E_{ij} = F_{ij} \otimes E_{ij}(a)$, and therefore

$$g_P(\varphi) = g(P\varphi P) = \sum_{i,j} g(P_i \varphi P_j) = \sum_{i,j} g(F_{ij} \otimes E_{ij}) = \sum \alpha_{ij}(F_{ij}) = \sum F_{ij}(a_{ij}) = \sum (\varphi(a_{ij})\, \xi_j, \xi_i).$$

The proof of the lemma is complete.

Now, in the notation of the preceding lemma, we claim that the $n \times n$ matrix (a_{ij}) is a positive element of $B \otimes M_n$. Choose a *faithful* representation π of B on some Hilbert space $\mathfrak{K}$. Then the canonical representation π_n: $B \otimes M_n \to L(\mathfrak{K} \otimes \mathbf{C}^n)$ defined by $\pi_n(x_{ij}) = (\pi(x_{ij}))$ (the latter regarded as an $n \times n$ operator matrix, acting on $\mathfrak{K} \oplus \ldots \oplus \mathfrak{K}$) is also faithful, and thus it suffices to show that the operator matrix $(\pi(a_{ij})) = \pi_n(a_{ij})$ is positive. Choose an arbitrary set of n vectors $\zeta_1, \ldots, \zeta_n$ from $\mathfrak{K}$. Since $\xi_1, \ldots, \xi_n$ are linearly independent vectors in $P\mathfrak{H}$, there is a unique bounded linear transformation $V \in L(\mathfrak{H}, \mathfrak{K})$ defined by $V\xi_i = \zeta_i$, $1 \leq i \leq n$, and $V = 0$ on $P\mathfrak{H}^\perp$. We can now write

$$\sum_{i,j} (\pi(a_{ij})\,\zeta_j, \zeta_i) = \sum_{i,j} (V^*\pi(a_{ij})V\xi_j, \xi_i) = g_P(V^*\pi V|_S) = g(V^*\pi V|_S) \geq 0,$$

because $PV^*\pi VP = V^*\pi V$, $V^*\pi V$ belongs to $CP\,(B, \mathfrak{H})$, and $g(CP\,(B,\mathfrak{H})|_S) \geq 0$. This shows that $(\pi(a_{ij})) \geq 0$, proving the assertion.

We can now prove that $g_P(CP\,(S, \mathfrak{H})) \geq 0$. Indeed, if $\varphi \in CP\,(B, \mathfrak{H})$ then we have, by 1.2.6, $g_P(\varphi) = \sum(\varphi(a_{ij})\xi_j, \xi_i)$; but $(\varphi(a_{ij}))$ is a positive operator matrix, by the preceding paragraph and the fact that φ is completely positive. Thus $g_P(\varphi) \geq 0$, and the proof of the theorem is complete.

No doubt, one could weaken the requirement that $e \in S$ by assuming merely that S contains a bounded approximate identity for B. For our purposes, however, 1.2.3 will be enough.

We shall now indicate how 1.2.3 can be adapted to cover the case where S is not necessarily self-adjoint. Recall that the *numerical radius* $w(T)$ of an operator $T \in L(\mathfrak{H})$ is defined by

$$w(T) = \sup\,\{|(T\xi, \xi)| : \xi \in \mathfrak{H},\ \|\xi\| = 1\}$$

LEMMA 1.2.7. *Let A be a linear subspace of a C^*-algebra B, such that $e \in A$, and let φ be a linear map of A into $L(\mathfrak{H})$, for some Hilbert space $\mathfrak{H}$, such that $\varphi(e) = I$ and $\|\varphi\| = 1$. Then $w(\varphi(a) + \varphi(b)^*) \leq \|a + b^*\|$, for every $a, b \in A$.*

Proof. Fix $\xi \in \mathfrak{H}$, $\|\xi\| = 1$. Then the linear functional $a \in A \to (\varphi(a)\xi, \xi)$ has norm at most 1, and takes the value 1 at e. By the Hahn-Banach theorem it has a norm-preserving extension ϱ to B. Clearly $\|\varrho\| = \varrho(e) = 1$, so that ϱ is a state, and in particular $\overline{\varrho(a)} = \varrho(a^*)$ for every $a \in A$. Thus,

$$|((\varphi(a) + \varphi(b)^*)\,\xi, \xi)| = |(\varphi(a)\,\xi, \xi) + \overline{(\varphi(b)\,\xi, \xi)}| = |\varrho(a) + \overline{\varrho(b)}| = |\varrho(a + b^*)| \leq \|a + b^*\|.$$

The required conclusion follows by taking the supremum over $\{\|\xi\| = 1\}$.

PROPOSITION 1.2.8. *Let A be a linear subspace of a C^*-algebra B, such that $e \in A$, and let S be the norm-closure of $A + A^*$. Then every contractive linear map φ of A in $L(\mathfrak{H})$, for which $\varphi(e) = I$, has a unique bounded self-adjoint linear extension φ_1 to S. φ_1 is positive, and it is completely positive if φ is completely contractive.*

Proof. It is plain that, if a bounded self-adjoint extension to S exists at all, it must be unique. By 1.2.7 we have, for $a, b \in A$, $\|\varphi(a) + \varphi(b)^*\| \leq 2w(\varphi(a) + \varphi(b)^*) \leq 2\|a + b^*\|$, and thus there is a bounded linear map φ_1 of S such that $\varphi_1(a + b^*) = \varphi(a) + \varphi(b)^*$, $a, b \in A$. φ_1 is clearly a self-adjoint extension of φ to S.

To see that φ_1 is positive, choose a unit vector $\xi \in \mathfrak{H}$. As in the proof of 1.2.7, there is a state ϱ of B such that $\varrho(a) = (\varphi(a)\xi, \xi)$, $a \in A$. Because ϱ and φ_1 are both self-adjoint, we have $\varrho(z) = (\varphi_1(z)\xi, \xi)$ for all $z \in S$. So if z is a positive element in S we see that $(\varphi_1(z)\xi, \xi) = \varrho(z) \geq 0$; it follows that φ_1 is positive.

Now assume φ is completely contractive. For each $n \geq 1$, note that $A \otimes M_n + (A \otimes M_n)^*$ is dense in $S \otimes M_n$, so that the argument of the preceding paragraph shows that $\varphi_{1,n} \geq 0$. That completes the proof.

We can now state an analogoue of theorem 1.2.3 for linear subspaces which are not necessarily self-adjoint.

THEOREM 1.2.9. *Let A be a linear subspace of a C^*-algebra B, such that $e \in A$, and let $\mathfrak{H}$ be a Hilbert space. Let φ be a completely contractive linear map of A into $L(\mathfrak{H})$ such that $\varphi(e) = I$. Then φ has a completely positive linear extension to B.*

Proof. By 1.2.8, φ has a unique completely positive extension to the closure of $A + A^*$, and now 1.2.3 applies to complete the proof.

Remarks. One can regard the preceding theorem as providing operator-valued "representing measures" for certain linear maps of subspaces of C^*-algebras.

Combining 1.2.9 with 1.1.1, we see that there is a representation π of B on a Hilbert space $\mathfrak{K}$ and a linear map $V \in L(\mathfrak{H}, \mathfrak{K})$ such that $\varphi(a) = V^*\pi(a)V$, $a \in A$. The condition $\varphi(e) = I$ implies that V is an isometry. If A is a subalgebra of B and φ is a (completely contractive) homomorphism of A, then it follows from the multiplicativity of φ that $V\mathfrak{H}$ is a semi-invariant subspace of $\mathfrak{K}$ for the algebra $\pi(A)$ (see A.1). Thus, the pair (π, V) gives a generalized "dilation" of φ, completely analogous to the unitary (power) dilation of a contraction. We will take up dilation theory in a subsequent paper.

We shall make repeated use of the following two observations.

PROPOSITION 1.2.10. *Let S be a closed self-adjoint linear subspace of a C^*-algebra B, such that $e \in S$, and let φ be a completely positive linear map of S into a C^*-algebra B_1. Then for every $n \geq 1$, φ_n has norm $\|\varphi(e)\|$.*

Proof. There is no loss if we assume B_1 is a sub-C^*-algebra of $L(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$. Note first that $\|\varphi\| = \|\varphi(e)\|$; for by 1.2.3 and Stinespring's theorem (1.1.1), there is a representation π of B on a Hilbert space $\mathfrak{K}$ and an operator $V \in L(\mathfrak{H}, \mathfrak{K})$ such that $\varphi(a) = V^*\pi(a)V$, $a \in S$. Thus, $\|\varphi(a)\| \leq \|V^*\| \cdot \|\pi(a)\| \cdot \|V\| \leq \|a\| \cdot \|V^*V\| = \|a\| \cdot \|\varphi(e)\|$, and the opposite inequality is trivial.

If $n \geq 1$, then φ_n is a completely positive map of $S \otimes M_n$, so that $\|\varphi_n\| = \|\varphi(e)\|$ follows from the preceding and the fact that if e_n is the identity of $B\otimes M_n$ then $\|\varphi_n(e_n)\| = \|\varphi(e)\|$. That completes the proof.

We remark that 1.2.10 is false for positive linear maps of S (see A.2). Note also that 1.2.10 and 1.2.9 together imply that if φ is a linear map of S into $L(\mathfrak{H})$ such that $\varphi(e)=I$, then φ is completely positive if, and only if, it is completely contractive.

PROPOSITION 1.2.11. *Let A be a linear subspace of a C^*-algebra B, such that $e\in A$. Then every contractive linear map of A into a commutative C^*-algebra, which preserves the identity, is completely contractive.*

Proof. Call the map φ, and let S be the norm-closure of $A+A^*$. According to 1.2.8 φ has a unique positive linear extension φ_1 to S. By 1.2.2, φ_1 is completely positive, thus the conclusion follows from the preceding proposition.

1.3. *Lifting Commutants.* Let B be a C^*-algebra with identity, let $\mathfrak{H}$ be a Hilbert space, and let φ be a completely positive linear map of B into $L(\mathfrak{H})$. According to Stinespring's theorem (1.1.1) there is a representation π of B on a Hilbert space $\mathfrak{K}$, and a bounded linear map V: $\mathfrak{H}\to\mathfrak{K}$ such that $\varphi(x)=V^*\pi(x)V$ and $[\pi(B)V\mathfrak{H}]=\mathfrak{K}$. In the sequel, we shall require information about operators commuting with the self-adjoint linear space of operators $\varphi(B)$. Because of the arbitrariness in the relation of V and the subspace $[V\mathfrak{H}]$ to $\pi(B)$ (for example, $[V\mathfrak{H}]$ need not be affiliated with either $\pi(B)'$ or $\pi(B)''$), it may be somewhat unexpected that there is an intimate relation between $\varphi(B)'$ and $\pi(B)'$. This is based on the following.

THEOREM 1.3.1. *Let $\mathfrak{H}$, $\mathfrak{K}$ be Hilbert spaces, let V be a bounded linear operator from $\mathfrak{H}$ into $\mathfrak{K}$, and let $\mathcal{B}$ be a self-adjoint subalgebra of $L(\mathfrak{K})$ such that $[\mathcal{B}V\mathfrak{H}]=\mathfrak{K}$. Then for every $T\in L(\mathfrak{H})$ which commutes with $V^*\mathcal{B}V$, there is a unique operator $T_1\in L(\mathfrak{K})$ having the properties*

(i) $T_1\in\mathcal{B}'$
(ii) $T_1V=VT$.

*The map $T\to T_1$ is an ultraweakly continuous surjective *-homomorphism of $V^*\mathcal{B}V'$ on $\mathcal{B}'\cap\{VV^*\}'$, for which $T_1=0$ iff $VT=VT^*=0$. In particular, when V has trivial nullspace, $T\to T_1$ is a *-isomorphism.*

Proof. Fix $T\in V^*\mathcal{B}V'$. T_1 is constructed as follows. Let $\xi_1, \dots, \xi_n\in\mathfrak{H}$, $A_1, \dots, A_n\in\mathcal{B}$; we claim that $\|\sum A_kVT\xi_k\|\leqslant\|T\|\cdot\|\sum A_kV\xi_k\|$. Assume first that $n=1$. Then $\|AVT\xi\|^2=(V^*A^*AVT\xi, T\xi)$. Now $V^*A^*AV\in V^*\mathcal{B}V$ is a positive operator which commutes with T, and so must its positive square root K. Thus,

$$\|AVT\xi\|^2 = (K^2T\xi, T\xi) = \|KT\xi\|^2 = \|TK\xi^2\| \leqslant \|T\|^2\|K\xi\|^2 = \|T\|^2\|AV\xi\|^2,$$

proving the claim for $n=1$. The case of a general integer n is reduced to the preceding by the following device. Let $\mathfrak{H}' = \mathbf{C}^n \otimes \mathfrak{H}$ (resp. $\mathfrak{K}' = \mathbf{C}^n \otimes \mathfrak{K}$), let $T' = I_n \otimes T \in L(\mathfrak{H}')$ (I_n denoting the identity on $\mathbf{C}^n$), $V' = I_n \otimes V \in L(\mathfrak{H}', \mathfrak{K}')$, and let A' be the operator on $\mathfrak{K}'$ given by the matrix

$$A' = \begin{pmatrix} A_1 & A_2 & \dots & A_n \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix}.$$

Then $V'^*A'^*A'V' \in L(\mathfrak{H}')$ has the matrix $(V^*A_i^*A_jV)$, which commutes with T' because $V^*A_i^*A_jV \in V^*\mathcal{B}V$ for all i, j. So if we put $\xi' = \xi_1 \oplus \dots \oplus \xi_n \in \mathfrak{H}'$ then $\|\sum A_jVT\xi_j\|^2 = \sum_{i,j}(V^*A_i^*A_jVT\xi_j, T\xi_i) = (V'^*A'^*A'V'T'\xi', T'\xi')$ which is not greater than $\|T'\|^2\|A'V'\xi'\|^2 = \|T\|^2\|\sum A_jV\xi_j\|^2$ by the argument already given. That proves the claim. Therefore the operator

$$T_1 : \sum A_jV\xi_j \mapsto \sum A_jVT\xi_j$$

is well-defined, and extends uniquely to an operator on $[\mathcal{B}V\mathfrak{H}] = \mathfrak{K}$ of norm at most $\|T\|$, denoted by the same letter T_1. Now $[\mathcal{B}\mathfrak{K}]$ contains $[\mathcal{B}V\mathfrak{H}] = \mathfrak{K}$, so that $\mathcal{B} = \mathcal{B}^*$ has trivial nullspace; the double commutant theorem now shows that the strong closure of $\mathcal{B}$ contains the identity, and from the relation $T_1AV\xi = AVT\xi$ ($\xi \in \mathfrak{H}$, $A \in \mathcal{B}$) we may conclude that $T_1V = VT$ by allowing A to approach I. That T_1 commutes with $\mathcal{B}$ is evident from its definition.

The remainder of the proof is routine, and we merely sketch the details. The uniqueness of the operator T_1 satisfying (i) and (ii) is an immediate consequence of $[\mathcal{B}V\mathfrak{H}] = \mathfrak{K}$. It follows that products and linear combinations behave right under the map $T \to T_1$. $(T_1)^* = (T^*)_1$ means $T_1^*V = VT^*$, or equivalently $V^*T_1 = TV^*$; to see this, let $\xi \in \mathfrak{H}$, $A \in \mathcal{B}$, and write $V^*T_1AV\xi = V^*AT_1V\xi = V^*AVT\xi = TV^*AV\xi$, using the fact that T commutes with V^*AV. The conclusion follows since $[\mathcal{B}V\mathfrak{H}] = \mathfrak{K}$. This argument also shows that T_1 commutes with VV^*, for $T_1VV^* = VTV^* = V(VT^*)^* = V(T_1^*V)^* = VV^*T_1$. Thus, $T \to T_1$ is a $*$-homomorphism of $V^*\mathcal{B}V'$ into $\mathcal{B}' \cap \{VV^*\}'$. The kernel is easily identified; indeed $T_1 = 0$ implies $T_1^* = 0$, so $VT = T_1V = 0$ and $VT^* = T_1^*V = 0$. The converse implication is clear from the relation (ii).

It remains to show that $T \to T_1$ is surjective and continuous. Let $T_1 \in \mathcal{A}'$, $T_1VV^* = VV^*T_1$. Let $V = HW$ be the polar decomposition of V, where H is the positive square root of VV^*, W is a partial isometry in $L(\mathfrak{H}, \mathfrak{K})$, and $WW^*H = HWW^* = H$. Define $T = W^*T_1W \in L(\mathfrak{H})$. Then $VT = HWW^*T_1W = HT_1W = T_1HW = T_1V$, since T_1 commutes with $H =$

$(VV^*)^{\frac{1}{2}}$. A similar calculation shows that T commutes with $V^*\mathcal{B}V$. Thus, $T \to T_1$ is a surjective $*$-homomorphism whose kernel is ultraweakly closed. It is well known that such a homomorphism is ultraweakly continuous. That completes the proof.

Now let B be a C^*-algebra, and let φ, π, and V be as in the discussion preceding 1.3.1; $\varphi(x) = V^*\pi(x)V$, $x \in B$. The following characterization of $\varphi(B)'$ is an immediate consequence of the preceding theorem.

Corollary 1.3.2. *Assume V has trivial nullspace. Then there is a canonical $*$-isomorphism between the von Neumann algebras $\varphi(B)'$ and $\pi(B)' \cap \{VV^*\}'$.*

This corollary allows one to make certain gross statements about the "size" of $\varphi(B)'$ in terms of π, when V has trivial nullspace. For example, if $\pi(B)'$ is a finite von Neumann algebra then so is $\varphi(B)'$; if $\pi(B)$ is multiplicity-free (i.e., $\pi(B)'$ is abelian) then so is $\varphi(B)$; and if π is an irreducible representation of B then $\varphi(B)$ is an irreducible family of operators.

If T is a contraction on a Hilbert space $\mathfrak{H}$ such that the powers of T^* tend strongly to 0, then the minimal unitary dilation of T is the shift of multiplicity dim $\mathfrak{H}$ [9], and one may associate with T a characteristic inner function U ([10], p. 103). It is natural to ask how one may characterize certain properties of T, such as irreducibility, in terms of U. We shall indicate how theorem 1.3.1 can be used to give quite a concrete answer to one of these questions. We begin with a general lemma. Recall [20] that a subspace $\mathfrak{H}$ of a Hilbert space $\mathfrak{K}$ is called *semi-invariant* under a subalgebra $\mathcal{A}$ of $L(\mathfrak{K})$ if the map $A \in \mathcal{A} \to PA|_{\mathfrak{H}}$ (P denoting the projection of $\mathfrak{K}$ on $\mathfrak{H}$) is multiplicative. If $\mathfrak{M}_2 = [\mathcal{A}\mathfrak{H}]$ and $\mathfrak{M}_1 = \mathfrak{M}_2 \ominus \mathfrak{H}$, then $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$, each $\mathfrak{M}_2$ is $\mathcal{A}$-invariant (cf. [20], Lemma 0), and $\mathfrak{H} = \mathfrak{M}_2 \ominus \mathfrak{M}_1$. Let $\mathcal{R}$ be the von Neumann algebra generated by $\mathcal{A}$, and suppose $[\mathcal{R}\mathfrak{H}] = \mathfrak{K}$.

Lemma 1.3.3. *Let $\mathcal{A}$, $\mathcal{R}$, and $\mathfrak{H} = \mathfrak{M}_2 \ominus \mathfrak{M}_1$ be as above. Assume the linear space of operators $\mathcal{A} + \mathcal{A}^*$ is weakly dense in $\mathcal{R}$. Then for every $T \in \mathcal{R}'$, one has $T\mathfrak{H} \subseteq \mathfrak{H}$ if, and only if, $T\mathfrak{M}_2 \subseteq \mathfrak{M}_2$ and $T^*\mathfrak{M}_1 \subseteq \mathfrak{M}_1$.*

Proof. We note first that $[\mathcal{A}^*\mathfrak{H}] \oplus \mathfrak{M}_1$ is a direct sum decomposition of $\mathfrak{K}$. Indeed, if $\xi \in \mathfrak{H}$, $\zeta \in \mathfrak{M}_1$ and $A \in \mathcal{A}$, then $(A^*\xi, \zeta) = (\xi, A\zeta) = 0$, since $A\zeta \in \mathfrak{M}_1$ and $\mathfrak{M}_1 \perp \mathfrak{H}$. The sum is therefore direct; it clearly contains $[\mathcal{A}^*\mathfrak{H}]$ and $[\mathcal{A}\mathfrak{H}] = \mathfrak{H} \oplus \mathfrak{M}_1$, so that it contains $[(\mathcal{A} + \mathcal{A}^*)\mathfrak{H}] = [\mathcal{R}\mathfrak{H}] = \mathfrak{K}$.

If $T\mathfrak{H} \subseteq \mathfrak{H}$ and $T \in \mathcal{R}'$, then $T\mathfrak{M}_2 = T[\mathcal{A}\mathfrak{H}] \subseteq [\mathcal{A}T\mathfrak{H}] \subseteq [\mathcal{A}\mathfrak{H}] = \mathfrak{M}_2$, and similarly $T[\mathcal{A}^*\mathfrak{H}] \subseteq [\mathcal{A}^*\mathfrak{H}]$. But by the above note, $[\mathcal{A}^*\mathfrak{H}] = \mathfrak{M}_1^\perp$, and so $T^*\mathfrak{M}_1 \subseteq \mathfrak{M}_1$ follows from $T\mathfrak{M}_1^\perp \subseteq \mathfrak{M}_1^\perp$. Conversely, if $T \in L(\mathfrak{H})$ is such that $T\mathfrak{M}_2 \subseteq \mathfrak{M}_2$ and $T^*\mathfrak{M}_1 \subseteq \mathfrak{M}_1$, then $T\mathfrak{M}_1^\perp \subseteq \mathfrak{M}_1^\perp$ so that $\mathfrak{H} = \mathfrak{M}_2 \ominus \mathfrak{M}_1 = \mathfrak{M}_2 \cap \mathfrak{M}_1^\perp$ is an intersection of T-invariant subspaces, therefore invariant itself. That completes the proof.

The following is an immediate consequence.

COROLLARY 1.3.4. *Under the above hypotheses, an operator* $T \in \mathcal{R}'$ *is reduced by* $\mathfrak{H}$ *if, and only if, it is reduced by both* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$.

Now let $\mathfrak{C}$ be a separable Hilbert space, let $\mathbf{T}$ be the unit circle with normalized Lebesgue measure $d\sigma$, and consider the Hilbert space $L^2(\mathbf{T}, \sigma; \mathfrak{C})$ of all square-integrable measurable $\mathfrak{C}$-valued functions on $\mathbf{T}$. We want to consider a certain semi-invariant subspace for the unitary operator L_z (multiplication by $e^{i\theta}$), as follows. Let $H^2_{\mathfrak{C}}$ be the closed linear span of all functions of the form $\xi_0 + e^{i\theta}\xi_1 + \ldots + e^{in\theta}\xi_n$, $\xi_j \in \mathfrak{C}$, $n \geqslant 0$, and let $U(e^{i\theta})$ be a (weakly) measurable function on $\mathbf{T}$ taking values in the unitary group of $L(\mathfrak{C})$. We assume U is an *inner* function in the sense that all the negative Fourier coefficients of each function $(U(e^{i\theta})\xi, \eta)$ $(\xi, \eta \in \mathfrak{C})$ vanish. U gives rise to a unitary operator L_U on $L^2(\mathbf{T}, \sigma; \mathfrak{C})$ (L_U is "pointwise" multiplication by U in the usual sense) and the analyticity requirement cited above insures that $L_U H^2_{\mathfrak{C}} \subseteq H^2_{\mathfrak{C}}$. Define $\mathfrak{H} = H^2_{\mathfrak{C}} \ominus L_U H^2_{\mathfrak{C}}$, and let S_U be the projection of L_z onto $\mathfrak{H}$:

$$S_U = P_{\mathfrak{H}} L_z|_{\mathfrak{H}}.$$

Let $\mathcal{R}$ be the von Neumann algebra generated by L_z, and let $\mathcal{A}$ be the algebra of all polynomials in L_z. It is known that $\mathcal{R}$ is the algebra of all multiplications by scalar $L^\infty(\mathbf{T}, \sigma)$ functions, $\mathcal{R}'$ is the algebra of all multiplications by $L(\mathfrak{C})$-valued bounded measurable functions, and that $\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{R}$ (equivalently, trigonometric polynomials are weak*-dense in $L^\infty(\mathbf{T}, d\sigma)$; for the details see [10]).

The inner function U has a canonical analytic extension to the interior $D = \{|z| < 1\}$ of the unit disc, and we shall write $U(D)$ for the set of operators $\{U(z): z \in D\}$. Now it follows, from the known convergence $U(e^{i\theta}) = \text{weak} \lim_{r\to 1} U(re^{i\theta})$ almost everywhere on $\mathbf{T}$, that almost every unitary operator $U(e^{i\theta})$ belongs to the weak closure of $U(D)$. Moreover, the subspace $L_U H^2_{\mathfrak{C}}$ is unaffected if we replace U by the function UW, where W is any (constant) unitary operator in $L(\mathfrak{C})$. Therefore we shall assume U is so normalized that the identity operator belongs to the weak closure of $U(D)$ (e.g., replace U with $U \cdot U(e^{i\theta})^*$ for an appropriate choice of θ). Finally, we shall assume that U is *completely nonconstant* (i.e., the only vector $\xi \in \mathfrak{C}$ for which $z \to U(z)\xi$ is constant is $\xi = 0$; cf. appendix A.1.). The following result implies that S_U is irreducible precisely when U takes on enough values so that $U(D)$ is an irreducible subset of $L(\mathfrak{C})$.

THEOREM 1.3.5. *Let T be an operator in $L(\mathfrak{H})$ which commutes with S_U and S_U^*. Then there is a (constant) operator $A \in L(\mathfrak{C})$ such that A commutes with $U(D) \cup U(D)^*$ and $T = L_A|_{\mathfrak{H}}$. The correspondence $T \leftrightarrow A$ is a $*$-isomorphism between the von Neumann algebras $\{S_U, S_U^*\}'$ and $(U(D) \cup U(D)^*)'$. In particular, S_U is irreducible if, and only if, $U(D)$ is an irreducible set of operators in $L(\mathfrak{C})$.*

Proof. Let T have the stated property. Taking for V the inclusion map: $\mathfrak{H}\to\mathfrak{K}$, we see that V^* is the projection of $\mathfrak{K}$ on $\mathfrak{H}$, and $S_U=V^*L_zV$. Clearly T commutes with $S_U^n=V^*L_{z^n}V$ and $S_U^{*n}=V^*L_{\bar{z}^n}V$ for all $n\geqslant 0$. So if $\mathcal{A}$ is the algebra of all polynomials in L_z, we have $T\in V^*(\mathcal{A}+\mathcal{A}^*)V'$; and since $\mathcal{A}+\mathcal{A}^*$ is weakly dense in the von Neumann algebra $\mathcal{R}$ generated by L_z we conclude that $T\in V^*\mathcal{R}V'$. Now $[\mathcal{R}\mathfrak{H}]=L^2(\mathbf{T},\sigma;\mathfrak{C})$, because U is completely nonconstant (see A.1.3 and A.1.1), and so we may apply Theorem 1.3.1. Thus, there exists $T_1\in\mathcal{R}'$ such that T_1 commutes with $VV^*=P_{\mathfrak{H}}$ and $T_1|_{\mathfrak{H}}=T$. The preceding remarks indicate that there is a measurable bounded $L(\mathfrak{C})$-valued function $A(e^{i\theta})$ on $\mathbf{T}$ such that $T_1=L_A$. Now, using A.1.3 once again, we see that $[\mathcal{A}\mathfrak{H}]=H^2_{\mathfrak{C}}$ and $[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}=L_UH^2_{\mathfrak{C}}$; so an application of 1.3.4 gives $L_AH^2_{\mathfrak{C}}\subseteq H^2_{\mathfrak{C}}$, $L_{A^*}H^2_{\mathfrak{C}}\subseteq H^2_{\mathfrak{C}}$, $L_AL_UH^2_{\mathfrak{C}}\subseteq L_UH^2_{\mathfrak{C}}$, and $L_{A^*}L_UH^2_{\mathfrak{C}}\subseteq L_UH^2_{\mathfrak{C}}$. Now the first two inclusions imply that both A and A^* are in $H^\infty_{\mathfrak{C}}$... i.e., A is constant a.e. σ (we identify A with its constant value). The second two imply that both $U(e^{i\theta})^*AU(e^{i\theta})$ and $(U(e^{i\theta})^*AU(e^{i\theta}))^*=U(e^{i\theta})^*A^*U(e^{i\theta})$ are in $H^\infty_{\mathfrak{C}}$, so there exists a constant operator C such that $U(e^{i\theta})^*AU(e^{i\theta})=C$, or $AU(e^{i\theta})=U(e^{i\theta})C$ almost everywhere on $\mathbf{T}$. This formula extends to the interior of the disc to give $AU(z)=U(z)C$, $|z|<1$. Since $I\in U(D)^-$, we may take an appropriate weak limit to conclude that $C=A$; thus $A\in U(D)'$. Now replace A with A^* to obtain $A\in U(D)^{*\prime}$. Note that L_A, and therefore A, is uniquely determined by T (1.3.1); and since V has trivial nullspace, the mapping $T\to A$ is $1-1$. A routine calculation shows that the algebraic operations are preserved (including the $*$-operations), and finally the above steps can, in an obvious way, be reversed to show that every $A\in U(D)'$ is the image of some $T\in\{S_U, S_U^*\}'$. That completes the proof of the theorem.

It seems worth pointing out the fact, proved implicitly above, that a necessary and sufficient condition for a (constant) operator $A\in L(\mathfrak{C})$ to have the property $L_A(H^2_{\mathfrak{C}}\ominus UH^2_{\mathfrak{C}})\subseteq H^2_{\mathfrak{C}}\ominus UH^2_{\mathfrak{C}}$ is that A commute with $U(D)$ (provided, of course, that U is normalized so that the identity belongs to the weak closure of $U(D)$).

1.4. *The order structure of CP $(B,\mathfrak{H})$.* Let B be a C^*-algebra and $\mathfrak{H}$ a Hilbert space. We wish to analyze the set CP $(B,\mathfrak{H})$ of all completely positive linear maps of B into $L(\mathfrak{H})$, one goal of which is to give complete solutions to three extremal problems associated with completely positive maps. While there is a considerable literature dealing with similar problems in the set of positive maps (cf. [24] and [13] for two notable examples), the known results are not always definitive, and it is somewhat surprising that the much more tractable family of completely positive maps has not received very much attention.

The results in the later portions of this section go somewhat beyond our immediate needs in this paper; we feel, however, that these results may be interesting, and that they will prove useful in future developments. We also remark that, while a few of the results of this section resemble results in [24], a close reading shows that they are somewhat different.

There is a natural partial ordering on $CP(B, \mathfrak{H})$, defined by $\psi \leqslant \varphi$ if $\varphi - \psi$ is completely positive. We begin by describing this ordering in terms of the representations π of B associated with elements $\varphi \in CP(B, \mathfrak{H})$ through the relation $\varphi = V^*\pi V$.

LEMMA 1.4.1. *Let φ_1 and φ_2 belong to $CP(B, \mathfrak{H})$, and suppose $\varphi_1 \leqslant \varphi_2$. Let $\varphi_i(x) = V_i^*\pi_i(x)V_i$ be the canonical expression of φ_i, where π_i is a representation of B on $\mathfrak{K}_i$ such that $[\pi_i(B)V_i\mathfrak{H}] = \mathfrak{K}_i$, $i = 1, 2$. Then there exists a contraction $T \in L(\mathfrak{K}_2, \mathfrak{K}_1)$ such that*

(i) $TV_2 = V_1$, *and*
(ii) $T\pi_2(x) = \pi_1(x)T$, $x \in B$.

Proof. Let $\xi_1, \dots, \xi_n \in \mathfrak{H}$, $x_1, \dots, x_n \in B$. Then

$$\|\sum_j \pi_1(x_j)V_1\xi_j\|^2 = \sum_{i,j}(V_1^*\pi_1(x_i^*x_j)V_1\xi_j, \xi_i)$$
$$= \sum_{i,j}(\varphi_1(x_i^*x_j)\xi_j, \xi_i) \leqslant \sum_{i,j}(\varphi_2(x_i^*x_j)\xi_j, \xi_i) = \|\sum_j \pi_2(x_j)V_2\xi_j\|^2,$$

since $\varphi_1 \leqslant \varphi_2$ and the matrix $(x_i^*x_j)$ is a positive element of $B \otimes M_n$. Therefore, there is a unique contraction T defined on $[\pi_2(B)V_2\mathfrak{H}] = \mathfrak{K}_2$ which satisfies $T\pi_2(x)V_2\xi = \pi_1(x)V_1\xi$, for all $x \in B$, $\xi \in \mathfrak{H}$. Taking $x = e$, we have $TV_2 = V_1$, and $T\pi_2(x) = \pi_1(x)T$ follows from the definition via $T\pi_2(x)\pi_2(y)V_2\xi = T\pi_2(xy)V_2\xi = \pi_1(xy)V_1\xi = \pi_1(x)\pi_1(y)V_1\xi = \pi_1(x)T\pi_2(y)V_2\xi$, using once more the fact that $[\pi_2(B)V_2\mathfrak{H}] = \mathfrak{K}_2$. That proves the lemma.

The next result can be thought of as a Radon-Nikodym theorem, and gives quite a useful description of the order relation in the set of completely positive maps. Some notation will be of help: for $\varphi \in CP(B, \mathfrak{H})$, let $[0, \varphi] = \{\psi \in \mathrm{CP}(B, \mathfrak{H}): \psi \leqslant \varphi\}$. $[0, \varphi]$ is a convex set, which is at the same time an order ideal in $CP(B, \mathfrak{H})$. Let $\varphi(x) = V^*\pi(x)V$ be the canonical expression of φ, where π is a representation of B in $L(\mathfrak{K})$ and $V \in L(\mathfrak{H}, \mathfrak{K})$ is such that $[\pi(B)V\mathfrak{H}] = \mathfrak{K}$. For each operator $T \in \pi(B)'$, define a linear map φ_T: $B \to L(\mathfrak{H})$ by $\varphi_T(x) = V^*T\pi(x)V$. Clearly the correspondence $T \to \varphi_T$ is linear, and it is injective because if $\varphi_T = 0$, then for all $x, y \in B$ and $\xi, \eta \in \mathfrak{H}$, one has $(T\pi(x)V\xi, \pi(y)V\eta) = (V^*T\pi(y^*x)V\xi, \eta) = (\varphi_T(y^*x)\xi, \eta) = 0$ and from $[\pi(B)V\mathfrak{H}] = \mathfrak{K}$ it follows that $T = 0$.

THEOREM 1.4.2. *$T \to \varphi_T$ is an affine order isomorphism of the partially ordered convex set of operators $\{T \in \pi(B)': 0 \leqslant T \leqslant I\}$ onto $[0, \varphi]$.*

Proof. The preceding remarks show that the correspondence is affine and $1-1$. Let

$T \in \pi(B)'$, $0 \leqslant T \leqslant I$. We claim $\varphi_T \in [0, \varphi]$. Indeed, if $\xi_1, \ldots, \xi_n \in \mathfrak{H}$ and (x_{ij}) is a positive matrix in $B \otimes M_n$, then letting $K \in \pi(B)'$ be the positive square root of T, we have

$$\sum_{i,j} (\varphi_T(x_{ij})\, \xi_j, \xi_i) = \sum (K^2 \pi(x_{ij}) V\xi_j, V\xi_i) = \sum (\pi(x_{ij})\, KV\xi_j, KV\xi_i) \geqslant 0,$$

since the matrix $(\pi(x_{ij}))$ is a positive element of $L(\mathfrak{K}) \otimes M_n$. This shows that φ_T is completely positive. Replacing T with $I - T$, we conclude also that $\varphi_T \leqslant \varphi$. Thus, $\varphi_T \in [0, \varphi]$. In the same way, we see that if $T_i \in \pi(B)'$ and $0 \leqslant T_1 \leqslant T_2 \leqslant I$ then $0 \leqslant \varphi_{T_1} \leqslant \varphi_{T_2} \leqslant \varphi$ (Consider $T_2 - T_1$).

We claim next that if $T \in \pi(B)'$ and φ_T is completely positive, then $T \geqslant 0$. Indeed, if $\zeta \in \mathfrak{K}$ has the form $\zeta = \pi(x_1) V\xi_1 + \ldots + \pi(x_n) V\xi_n$ ($\xi_j \in \mathfrak{H}$, $x_j \in B$), then

$$(T\zeta, \zeta) = \sum_{i,j} (V^* \pi(x_i)^* T\pi(x_j) V\xi_j, \xi_i) = \sum (V^* T\pi(x_i^* x_j) V\xi_j, \xi_i) = \sum (\varphi_T(x_i^* x_j)\, \xi_j, \xi_i) \geqslant 0,$$

since $(x_i^* x_j)$ is a positive element of $B \otimes M_n$. $T \geqslant 0$ follows because such ζ's are dense in $\mathfrak{K}$. By considering differences as in the preceding paragraph, we conclude from the above that if $T_1, T_2 \in \pi(B)'$ and $0 \leqslant \varphi_{T_1} \leqslant \varphi_{T_2} \leqslant \varphi$, then $0 \leqslant T_1 \leqslant T_2 \leqslant I$.

It remains to show that every $\psi \in [0, \varphi]$ is of the form φ_T, for some $T \in \pi(B)'$, $0 \leqslant T \leqslant I$. Since ψ is completely positive, there is a representation σ of B on $\mathfrak{H}_1$ and a linear map W of $\mathfrak{H}$ into $\mathfrak{K}_1$ such that $[\sigma(B) W\mathfrak{H}] = \mathfrak{K}_1$ and $\psi(x) = W^* \sigma(x) W$. By lemma 1.4.1, there is a contraction X: $\mathfrak{K} \to \mathfrak{K}_1$ such that $XV = W$ and $X\pi(x) = \sigma(x) X$, $x \in B$. Put $T = X^*X$. Then clearly $0 \leqslant T \leqslant I$, and $T\pi(x) = X^* \sigma(x) X = \pi(x) T$, so that $T \in \pi(B)'$. Finally, we have, for $\xi, \eta \in \mathfrak{H}$,

$$\begin{aligned}(\varphi_T(x)\xi, \eta) &= (X^*X\pi(x)\, V\xi,\ V\eta) = (X\pi(x)\, V\eta,\ XV\eta) = (\sigma(x) XV\xi,\ XV\eta) \\ &= (\sigma(x)\, W\xi,\ W\eta) = (\psi(x)\xi, \eta).\end{aligned}$$

That completes the proof.

There are a number of extremal problems associated with completely positive maps, of which we shall consider three. The problems are to identity the following sets:

(i) *the extreme rays of the cone* $CP\,(B, \mathfrak{H})$

(ii) *the extreme points of* $[0, \varphi]$ *(for a fixed* φ *in* $CP\,(B, \mathfrak{H})$*)*

(iii) *the extreme points of the set* $CP\,(B, \mathfrak{H}; K) = \{\varphi \in CP\,(B, \mathfrak{H}) : \varphi(e) = K\}$, *where* K *is a fixed positive operator in* $L(\mathfrak{H})$.

The descriptions of (i) and (ii) are almost immediate consequences of the preceding theorem. First, let us call a completely positive map $\varphi \in CP\,(B, \mathfrak{H})$ *pure* if, for every $\psi \in CP(B, \mathfrak{H})$, $\psi \leqslant \varphi$ implies ψ is a scalar multiple of φ; equivalently, φ is pure if the only possible decompositions of φ of the form $\varphi(x)=\varphi_1(x)+\varphi_2(x)$ ($\varphi_i \in CP\ (B, \mathfrak{H})$) are when each φ_i is a scalar multiple of φ. The extreme rays ([15], p. 133, and [16], p. 87, 123) of $CP\ (B, \mathfrak{H})$ can be characterized as the half-lines $\{t\varphi: t \geqslant 0\}$, where φ is a pure element of $CP\ (B, \mathfrak{H})$. Thus, the solution of (i) is given by:

COROLLARY 1.4.3. *The nonzero pure elements of $CP\ (B, \mathfrak{H})$ are precisely those of the form $\varphi(x) = V^*\pi(x)V$ where π is an irreducible representation of B on some Hilbert space $\mathfrak{K}$ and $V \in L(\mathfrak{H}, \mathfrak{K})$, $V \neq 0$.*

Proof. Let φ be a nonzero pure element, and let $\varphi(x) = V^*\pi(x)V$ be its canonical representation. 1.4.2 shows that $\{T \in \pi(B)': 0 \leqslant T \leqslant I\}$ consists of scalar multiples of the identity, which implies that π is irreducible. Conversely, if π is any irreducible representation of B on $\mathfrak{K}$ and V is any nonzero element of $L(\mathfrak{H}, \mathfrak{K})$, then $[V\mathfrak{H}] \neq 0$ is necessarily cyclic for $\pi(B)$, and another application of 1.4.2, along with the fact that $\pi(B)' =$scalars, shows that $\varphi(x) = V^*\pi(x)V$ is pure. The proof is complete.

Note that 1.4.3 generalizes a familiar theorem of Gelfand and Segal about positive linear functionals on C^*-algebras. The commutative case $B = C(X)$, X compact Hausdorff, is also noteworthy. The nonzero pure elements of $CP\ (C(X), \mathfrak{H})$ are those of the form $\varphi(f) = f(p)H$, $f \in C(X)$, where p is a point of X and H is a positive operator of rank 1 (here, $\mathfrak{K}$ is one-dimensional, $\pi(f) = f(p)I_{\mathfrak{K}}$, and V has rank 1 ... so V^*V has rank 1 and the representation follows by taking $H = V^*V$).

The description of (ii) is an equally direct consequence.

COROLLARY 1.4.4. *Let $\varphi(x) = V^*\pi(x)V$ be nonzero and completely positive. Then the extreme points of $[0, \varphi]$ are those maps of the form φ_P, where P is a projection in $\pi(B)'$.*

Proof. This follows from 1.4.2, and the well-known fact that for any von Neumann algebra $\mathcal{R}$, the extreme points of $\{T \in \mathcal{R}: 0 \leqslant T \leqslant I\}$ are the projections in $\mathcal{R}$.

We turn now to the extremal problem (iii). Recall that a closed subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{H}$ is said to be a *separating* subspace for a von Neumann algebra $\mathcal{R} \subseteq L(\mathfrak{H})$ if for every $X \in \mathcal{R}$, $X\mathfrak{M} = \{0\}$ implies $X = 0$; equivalently, the linear map $\varphi(X) = P_{\mathfrak{M}}X|_{\mathfrak{M}}$ of $L(\mathfrak{H})$ into $L(\mathfrak{M})$ satisfies the condition: $\varphi(X^*X) = 0$ implies $X = 0$, for every $X \in \mathcal{R}$. The following property, which is somewhat stronger, plays an essential role in the discussion to follow.

Definition 1.4.5. A closed subspace $\mathfrak{M}$ of $\mathfrak{H}$ is said to be *faithful* for a von Neumann algebra $\mathcal{R}$ if, for every $X \in \mathcal{R}$, $PX|_{\mathfrak{M}} = 0$ implies $X = 0$, P denoting the projection of $\mathfrak{H}$ on $\mathfrak{M}$.

Before proceeding with the extremal problem, we give a few examples of faithful subspaces. Note first that a faithful subspace $\mathfrak{M}$ (for $\mathcal{R}$) must also be separating, which is the same as being cyclic for $\mathcal{R}'$: $[\mathcal{R}'\mathfrak{M}]=\mathfrak{H}$ ([5], p. 6). On the other hand, as some of the following examples show, a cyclic subspace for $\mathcal{R}'$ need not be faithful for $\mathcal{R}$. In the special case where $\mathfrak{M}$ reduces $\mathcal{R}$, however, then $PX|_{\mathfrak{M}}=0$ if $X|_{\mathfrak{M}}=0$, for every $X\in\mathcal{R}$, so that $\mathfrak{M}$ is faithful for $\mathcal{R}$ if and only if $[\mathcal{R}'\mathfrak{M}]=\mathfrak{H}$.

There are interesting examples of faithful subspaces which are affiliated with neither $\mathcal{R}$ nor $\mathcal{R}'$. As one example, let m be Haar measure on the unit circle $\mathbf{T}$, and let $\mathfrak{H}=L^2(\mathbf{T}, m)$. Let $\mathcal{R}$ be the von Neumann algebra of all multiplications L_f by bounded measurable functions f and, as usual, let H^2 be the closed linear span of the functions $e_n(e^{i\theta})=e^{ni\theta}$, $n\geqslant 0$. It is a familiar fact that $\mathcal{R}=\mathcal{R}'$. Note also that H^2 is a faithful subspace for $\mathcal{R}$; indeed if $f\in L^\infty(\mathbf{T}, m)$, then for every $m, n\geqslant 0$ one has

$$(P_{H^2}L_f e_n, e_m)=(f e_n, e_m)=\int f(e^{i\theta})\, e^{i(n-m)\theta}\, dm,$$

and from the condition $P_{H^2}L_f|_{H^2}=0$ it follows that every Fourier coefficient of f vanishes, hence $L_f=0$ (for rather different purposes, this fact has already been pointed out in [2]). Note also that it follows from the above argument that if S is *any* set of integers such that $S-S=\mathbf{Z}$ (e.g., $S=\{0, 1, 3, 5, 7, \ldots\}$), then $\mathfrak{M}=[e_n\colon n\in S]$ is a faithful subspace for the multiplication algebra $\mathcal{R}$.

If U is any unitary operator which normalizes $\mathcal{R}$ ($U\mathcal{R}U^{-1}=\mathcal{R}$) and $\mathfrak{M}$ is a faithful subspace for $\mathcal{R}$, then so is $U\mathfrak{M}$, as a very simple argument shows. So if $\mathfrak{M}$ and $\mathcal{R}$ are as in the preceding paragraph and $\psi\in L^\infty(\mathbf{T}, m)$ is such that $|\psi|=1$ almost everywhere, then $\psi\cdot\mathfrak{M}=L_\psi\mathfrak{M}$ is faithful for $\mathcal{R}$. A different class of examples arises as the subspaces of the form $U\mathfrak{M}$, where U is the (normalizing) unitary operator induced by an invertible measure-preserving transformation of $\mathbf{T}$.

The following examples of subspaces of $L^2(\mathbf{T}, dm)$ which are *not* faithful for the multiplication algebra will be of interest in the sequel. Let ψ be an inner function in $L^\infty(\mathbf{T}, m)$ (i.e., $|\psi|=1$ a.e. and $(\psi, e_n)=0$ for all $n<0$), and let $\mathfrak{M}=H^2\ominus\psi H^2$. It is shown in A.1 that $\mathfrak{M}$ is cyclic for the multiplication algebra $\mathcal{R}$; but $\mathfrak{M}$ is not faithful because $\psi\cdot\mathfrak{M}\subseteq\psi\cdot H^2\subseteq\mathfrak{M}^\perp$, hence $P_{\mathfrak{M}}L_\psi|_{\mathfrak{M}}=0$, while of course $L_\psi\neq 0$.

We can now state the solution of the extremal problem (iii). Let B be a C^*-algebra with identity, let $\mathfrak{H}$ be a Hilbert space, and let K be a positive operator in $L(\mathfrak{H})$. Let φ be a completely positive map of B into $L(\mathfrak{H})$, and let $\varphi(x)=V^*\pi(x)V$ be the canonical expression for φ, with π a representation of B on $\mathfrak{K}$ and $V\in L(\mathfrak{H}, \mathfrak{K})$. Then of course, $\varphi\in CP(B, \mathfrak{H}; K)$ if, and only if, $V^*V=K$.

THEOREM 1.4.6. *Let $\varphi=V^*\pi V$ be as above, with $V^*V=K$. Then φ is an extreme point of $CP(B, \mathfrak{H}; K)$ if, and only if, $[V\mathfrak{H}]$ is a faithful subspace for the commutant $\pi(B)'$ of $\pi(B)$.*

Proof. Suppose first that $\mathfrak{M}=[V\mathfrak{H}]$ is faithful for $\pi(B)'$, and let $\varphi=t\psi_1+(1-t)\psi_2$ with $\psi_i\in CP\,(B,\mathfrak{H};K)$, $0<t<1$. Then $\varphi\geqslant t\psi_1$, so by 1.4.2 there is an operator $T\in\pi(B)'$, $0\leqslant T\leqslant I$, such that $t\psi_1(x)=V^*T\pi(x)V$, $x\in B$. Setting $x=e$ we obtain $V^*TV=tK=tV^*V$, and so $(P_{\mathfrak{M}}TV\xi,\,V\eta)=(V^*TV\xi,\eta)=t(V\xi,\,V\eta)$ for all $\xi,\eta\in\mathfrak{H}$. It follows that $P_{\mathfrak{M}}T|_{\mathfrak{M}}=tI_{\mathfrak{M}}$ and hence $T=tI_{\mathfrak{R}}$ because $\mathfrak{M}$ is faithful. Thus $t\psi_1=tV^*\pi V=t\varphi$ and we conclude $\psi_1=\psi_2=\varphi$. Thus, φ is extreme.

Conversely, suppose φ is extreme. Define the positive linear map μ: $\pi(B)'\rightarrow L(\mathfrak{M})$ by $\mu(X)=P_{\mathfrak{M}}X|_{\mathfrak{M}}$. We must show that μ is injective. Take $X\in\mathcal{R}'$ such that $\mu(X)=0$; we claim $X=0$. Since μ preserves adjoints, it follows that $\mu(\mathrm{Re}\,X)=\mu(\mathrm{Im}\,X)=0$, and so we can assume that X is self-adjoint. Choose positive scalars s,t such that $\frac{1}{4}I_{\mathfrak{R}}\leqslant sX+tI_{\mathfrak{R}}\leqslant\frac{3}{4}I_{\mathfrak{R}}$ and put $A=sX+tI_{\mathfrak{R}}$. Then $\frac{1}{4}I_{\mathfrak{M}}\leqslant\mu(A)=tI_{\mathfrak{M}}\leqslant\frac{3}{4}I_{\mathfrak{M}}$, so that $0<t<1$. Define $\psi_1(x)=V^*A\pi(x)V$ and $\psi_2(x)=V^*(I-A)\pi(x)V$. Then ψ_1 and ψ_2 are completely positive (1.4.2) because $0\leqslant A\leqslant I_{\mathfrak{R}}$, one has $\psi_1(e)=V^*AV=V^*\mu(A)V=tV^*V=tK$, and similarly $\psi_2(e)=V^*\mu(I-A)V=(1-t)K$. Of course, $\psi_1+\psi_2=\varphi$. Since $t^{-1}\psi_1$ and $(1-t)^{-1}\psi_2$ are in $CP\,(B,\mathfrak{H};K)$, it follows from the extremality of φ that $t^{-1}\psi_1=(1-t)^{-1}\psi_2=\varphi$. In particular, $V^*A\pi(x)V=\psi_1(x)=tV^*\pi(x)V$ for every $x\in B$. From the uniqueness statement of 1.4.2 we conclude that $A=t\cdot I_{\mathfrak{R}}$; so finally sX, and therefore X itself, must be 0. That completes the proof.

The following two corollaries describe the multiplicative properties of certain extremal positive maps.

COROLLARY 1.4.7. *Let φ be an extreme point of $CP\,(B,\mathfrak{H};I)$ and let Z be the center of B. Assume $\varphi(Z)\subseteq\varphi(B)'$. Then $\varphi(xz)=\varphi(x)\varphi(z)$, for every $x\in B$, $z\in Z$.*

Proof. Let $\varphi=V^*\pi V$ be the canonical expression for φ, with π: $B\rightarrow L(\mathfrak{R})$ and $V\in L(\mathfrak{H},\mathfrak{R})$. Then $V^*V=\varphi(e)=I$, so V is an isometry and hence VV^* is a projection in $L(\mathfrak{R})$. Let $z\in Z$. Then $\varphi(z)\in\varphi(B)'$ so by 1.3.1 there is an operator $T\in\pi(B)'$, $TVV^*=VV^*T$, such that $TV=V\varphi(z)$, hence $V^*TV=\varphi(z)$. On the other hand, $\pi(z)\in\pi(B)'$ and $V^*\pi(z)V=\varphi(z)=V^*TV$. But the map $T\in\pi(B)'\rightarrow V^*TV$ is injective, by extremality of φ and 1.4.6, so that $\pi(z)=T$; in particular, $\pi(z)$ commutes with VV^*. Thus, $\varphi(xz)=V^*\pi(x)\pi(z)V=V^*\pi(x)\pi(z)VV^*V=V^*\pi(x)VV^*\pi(z)V=\varphi(x)\varphi(z)$, completing the proof.

Note in particular that the preceding gives a new proof of the known result that an extreme point of $CP\,(B,\mathfrak{H};I)$ is multiplicative, when both B and $\varphi(B)$ are commutative. Also, the reader will have no trouble modifying the preceding proof to obtain the following result, which should be compared with Theorem 3.1 of [24]. Let $\varphi=V^*\pi V$ be an extreme point of $CP\,(B,\mathfrak{H};I)$ (B is a general C^*-algebra with identity) and let $T\in\pi(B)'$ be such that $T_0=V^*TV$ commutes with $\varphi(B)$; then $V^*\pi(x)TV=\varphi(x)T_0$, for every $x\in B$.

It is known that for *commutative* C^*-algebras B_1 and B_2, the extreme points of the identity preserving positive maps of B_1 into B_2 are precisely the $*$-homomorphisms [24], [12]. It is known (and follows directly from 1.4.6, for example, by taking $\mathfrak{H}=\mathfrak{K}$, $\varphi=\pi$, and $V=$identity) that a *representation* of B is extremal in CP $(B, \mathfrak{H}; I)$. On the other hand, an extreme point of CP $(B, \mathfrak{H}; I)$ need not be multiplicative, even when B is commutative (an example relevant to this work follows 1.4.8). Recall, incidentally, that when B is commutative, positive maps of B are identical with completely positive maps [23]. One natural question, for a commutative B, is how "close" do the extreme points of CP $(B, \mathfrak{H}; I)$ come to being multiplicative? The answer is that the kernel, at least, of an extreme point is the same as the kernel of a representation.

COROLLARY 1.4.8. *Let B be a commutative C^*-algebra with identity, and let K be a positive operator on a Hilbert space $\mathfrak{H}$. Then for every extreme point φ of* CP $(B, \mathfrak{H}; K)$, $\{x \in B: \varphi(x)=0\}$ *is a closed ideal in B.*

Proof. Let $\varphi(x)=V^*\pi(x)V$ be the representation of φ as in 1.4.6. We show that $\ker\varphi=\ker\pi$. It is apparent that $\pi(x)=0$ implies $\varphi(x)=0$. Conversely if $\varphi(x)=0=V^*\pi(x)V$, then if follows that $P_{[V\mathfrak{H}]}\pi(x)|_{[V\mathfrak{H}]}=0$. $\pi(B)$ is commutative and thus $\pi(x)\in\pi(B)'$; since $[V\mathfrak{H}]$ is faithful for $\pi(B)'$ we conclude that $\pi(x)=0$, and the proof is complete.

We digress, momentarily, to present a simple example. Consider the commutative C^*-algebra $C(\mathbf{T})$ of continuous functions on the unit circle $\mathbf{T}$, let m be Haar measure on $\mathbf{T}$, and let H^2 be the usual subspace of $L^2(\mathbf{T}, m)$. Define a positive linear map φ of $C(\mathbf{T})$ into $L(H^2)$ by

$$\varphi(f)=P_{H^2}L_f|_{H^2}, \quad f\in C(\mathbf{T}),$$

L_f denoting "multiplication by f". Letting π be the representation of $C(\mathbf{T})$ on $L^2(\mathbf{T}, m)$ defined by $\pi(f)=L_f$, and V the inclusion map of $H^2\subseteq L^2$ into L^2, then $\varphi=V^*\pi V$ is the canonical representation of φ. Now $\pi(C(\mathbf{T}))'$ is the algebra of all multiplications by $L^\infty(\mathbf{T}, m)$ functions, and the discussion following 1.4.5 shows that H^2 is faithful for $\pi(C(\mathbf{T}))'$. It follows from 1.4.6 that φ is an extreme point of CP $(C(\mathbf{T}), H^2; I)$. Clearly φ is not multiplicative on $C(\mathbf{T})$, while in this case $\ker\varphi$ is the trivial ideal 0.

We conclude this section with some information about the question: when is a sum of extremal maps extremal? These results allow us to give a complete description of the extremal positive maps of $C(X)$ into a matrix algebra, thus solving a problem taken up in [24].

Now, it is not hard to see that if φ is an extreme point of CP $(B, \mathfrak{H}; K)$ and $\psi\in[0,\varphi]$, then ψ is an extreme point of CP $(B, \mathfrak{H}; \psi(e))$ (the proof makes use of 1.4.2 to obtain the canonical expression for ψ in terms of that for φ, so that 1.4.6 can be applied in a straightforward manner; we omit the details since this result does not bear directly on the sequel).

It follows that if $\varphi_i \in CP(B, \mathfrak{H}; K_i)$ and $\varphi_1 + \ldots + \varphi_n$ is an extreme point of $CP(B, \mathfrak{H}; K_1 + \ldots + K_n)$, then each φ_i is extreme in $CP(B, \mathfrak{H}; K_i)$. The following lemma gives a partial converse.

We shall say an operator $T \in L(\mathfrak{H})$ *lives in* a closed subspace $\mathfrak{M}$ of $\mathfrak{H}$ if both T and T^* vanish on $\mathfrak{M}^{\perp}$: equivalently, $\mathfrak{M}$ contains the range of both T and T^*. A finite collection $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_n\}$ of subspaces of $\mathfrak{H}$ is called *weakly independent* if, whenever T_i lives in $\mathfrak{M}_i$ and $T_1 + \ldots + T_n = 0$, then $T_1 = \ldots = T_n = 0$. A linearly independent family of subspaces (i.e., $\xi_i \in \mathfrak{M}_i$ and $\xi_1 + \ldots + \xi_n = 0$ implies $\xi_1 = \ldots = \xi_n = 0$) is necessarily weakly independent, but the converse is false. A simple example is obtained by taking $\mathfrak{M}_1 = [\xi]$, $\mathfrak{M}_2 = [\eta]$ and $\mathfrak{M}_3 = [\xi + \eta]$, where ξ and η are linearly independent vectors; $\{\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3\}$ is not linearly independent, but some simple calculations (which we leave to the reader) will show that it is a weakly independent family. We also remark that when $\mathfrak{H}$ is finite dimensional, weak independence of $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_n\}$ is equivalent to the *linear* independence of the family $\{\mathfrak{N}_i\}$ of subspaces of $\mathfrak{H} \otimes \mathfrak{H}$ defined by $\mathfrak{N}_i = [\xi \otimes \eta: \xi, \eta \in \mathfrak{M}_i]$. This rests on the fact that the vector space $L(\mathfrak{H})$ is isomorphic with $\mathfrak{H} \otimes \mathfrak{H}$, in such a way that the subspace of operators that live on $\mathfrak{M}_i$ corresponds to $\mathfrak{N}_i$ (the details of which we again leave to the reader).

We also recall Mackey's notion of disjointness: Two representations π_1 and π_2 of a C^*-algebra B are *disjoint* ($\pi_1 \, \delta \, \pi_2$) if no nonzero subrepresentation of π_1 is equivalent to any subrepresentation of π_2. It follows that if π_1 and π_2 are respectively multiples of two inequivalent irreducible representations, then $\pi_1 \, \delta \, \pi_2$.

LEMMA 1.4.9. *Let B be a C^*-algebra, $\mathfrak{H}$ a Hilbert space, let $K_1, \ldots, K_n$ be positive operators on $\mathfrak{H}$, and let $\varphi_i = V_i^* \pi_i V_i$ be an extreme point of $CP(B, \mathfrak{H}; K_i)$ $1 \leq i \leq n$. Assume that*

(i) *$\pi_i \, \delta \, \pi_j$ if $i \neq j$, and*

(ii) *$\{[K_1 \mathfrak{H}], \ldots, [K_n \mathfrak{H}]\}$ is a weakly independent family of subspaces of $\mathfrak{H}$.*

Then $\varphi_1 + \ldots + \varphi_n$ is an extreme point of CP $(B, \mathfrak{H}; K_1 + \ldots + K_n)$.

Proof. First, we shall exhibit the canonical expression for $\varphi = \varphi_1 + \ldots + \varphi_n$. Let $\mathfrak{K}_i$ be the Hilbert space on which π_i acts; we can assume, of course, that $[\pi_i(B) V_i \mathfrak{H}] = \mathfrak{K}_i$. Define the operator $V \in L(\mathfrak{H}, \mathfrak{K}_1 \oplus \ldots \oplus \mathfrak{K}_n)$ by $V\xi = V_1 \xi \oplus \ldots \oplus V_n \xi$, and put $\pi(x) = \pi_1(x) \oplus \ldots \oplus \pi_n(x) \in L(\mathfrak{K}_1 \oplus \ldots \oplus \mathfrak{K}_n)$, $x \in B$. Then $V^* \pi V = \sum V_i^* \pi_i V_i = \sum \varphi_i$, so that $V^* \pi V$ will be the canonical expression for $\varphi_1 + \ldots + \varphi_n$ provided that $[\pi(B) V \mathfrak{H}] = \oplus_{i=1}^{n} \mathfrak{K}_i$. But since the π_j are pairwise disjoint it follows that $\pi_j \, \delta \, \oplus_{i \neq j} \pi_i$ ([4], Prop. 5.2.3), and so the projection E_j of $\mathfrak{K}_1 \oplus \ldots \oplus \mathfrak{K}_n$ on its jth coordinate space belongs to the center of $\pi(B)''$ ([4], Prop. 5.2.4). Thus, $[\pi(B) V \mathfrak{H}]$ contains $[\pi(B) E_j V \mathfrak{H}] = 0 \oplus \ldots \oplus [\pi_j(B) V_j \mathfrak{H}] \oplus \ldots \oplus 0 = 0 \oplus \ldots \oplus \mathfrak{K}_j \oplus \ldots \oplus 0$ (the nonzero summand occurs in the jth place) for $1 \leq j \leq n$, and hence $[\pi(B) V \mathfrak{H}]$ contains $\mathfrak{K}_1 \oplus \ldots \oplus \mathfrak{K}_n$, as required.

To show that $\varphi_1 + \ldots + \varphi_n$ is extremal, we make use of 1.4.6. Choose $T \in \pi(B)'$ such that $V^*TV = 0$. For each j, we have $TE_j \in \pi(B)E_j'$, so that T has the "diagonal" form $T = T_1 \oplus \ldots \oplus T_n$, where $T_j \in L(\mathfrak{K}_j)$ and commutes with $\pi_j(B)$. Now $0 = V^*TV = V_1^* T_1 V_1 + \ldots + V_n^* T_n V_n$, and of course $V_j^* T_j V_j$ lives in $[V_j^* V_j \mathfrak{K}_j] = [K_j \mathfrak{H}]$. By (ii), then, we have $V_j^* T_j V_j = 0$, $1 \leqslant j \leqslant n$. Since each φ_j is extremal and $T_j \in \pi_j(B)'$, 1.4.6 implies that $T_1 = \ldots = T_n = 0$, hence $T = T_1 \oplus \ldots \oplus T_n = 0$, and the proof is completed by another application of 1.4.6.

We can now describe the extremal positive maps from a commutative C^*-algebra into a matrix algebra.

THEOREM 1.4.10. *Let X be a compact Hausdorff space, let $\mathfrak{H}$ be a finite dimensional Hilbert space, and let K be a positive operator on $\mathfrak{H}$. Then the extreme points of $CP\ (C(X), \mathfrak{H}; K)$ are the positive maps of the form*

$$\varphi(f) = f(x_1) K_1 + \ldots + f(x_n) K_n, \quad f \in C(X),$$

where $n \geqslant 1$, $x_1, \ldots, x_n$ are distinct points of X, and $K_1, \ldots, K_n$ are positive operators satisfying

(i) *$K_1 + \ldots + K_n = K$, and*
(ii) *$\{[K_1 \mathfrak{H}], \ldots, [K_n \mathfrak{H}]\}$ is a weakly independent family of subspaces.*

Proof. Suppose φ has the given form $\varphi(f) = \sum_j f(x_j) K_j$, and set $\varphi_j(f) = f(x_j) K_j$. Let $\mathfrak{K}_j = [K_j \mathfrak{H}]$, $V_j = K_j^{\frac{1}{2}}$, and let $\pi(f)_j$ be the scalar operator $f(x_j) I_{\mathfrak{K}_j}$. Then $\varphi_j(f) = V_j^* \pi_j(f) V_j$, and clearly $[V_j \mathfrak{H}] = [K_j^{\frac{1}{2}} \mathfrak{H}] = [K_j \mathfrak{H}] = \mathfrak{K}_j$, so that this is the canonical form for φ_j. If $T \in L(\mathfrak{K}_j)$ and $V_j^* T V_j = 0 = K_j^{\frac{1}{2}} T K_j^{\frac{1}{2}}$, then clearly $T = 0$ because $K_j^{\frac{1}{2}}$ is an isomorphism of $[K_j \mathfrak{H}]$ onto itself. 1.4.6 now shows that φ_j is extreme. Since $x_i \neq x_j$ we have $\pi_i \mathbin{\delta} \pi_j$ by the remark preceding 1.4.9, and so by 1.4.9 it follows that $\varphi_1 + \ldots + \varphi_n$ is extreme in $CP\ (C(X), \mathfrak{H}; K_1 + \ldots + K_n)$.

Conversely, let φ be an extreme point of $CP\ (B, \mathfrak{H}; K)$, and let $\varphi = V^* \pi V$ be its canonical representation, with π a representation of $C(X)$ on $\mathfrak{K}$, $V \in L(\mathfrak{H}, \mathfrak{K})$, and $[\pi(C(X)) V \mathfrak{H}] = \mathfrak{K}$. By 1.4.6, the linear map $T \in \pi(C(X))' \to V^*TV \in L(\mathfrak{H})$ is injective, so that $\pi(C(X))'$ is a finite dimensional von Neumann algebra. In particular, the reducing subspaces for the C^*-algebra $\pi(C(X))$ satisfy the chain conditions, and a familiar argument shows that π is a finite direct sum of irreducible representations. Each irreducible representation is of the form $f \to f(x) I$ (I being a one-dimensional identity operator and x being a point of X), so that we can arrange π in the form

$$\pi(f) = f(x_1) E_1 + \ldots + f(x_n) E_n, \quad f \in C(X),$$

where $x_i \neq x_j$ if $i \neq j$, and $E_1, \ldots, E_n$ is a set of mutually orthogonal projections in $L(\mathfrak{K})$ with sum I. Since the subrepresentations $f \to f(x_i) E_i$ are mutually disjoint, it follows that

$E_i \in \pi(C(X))''$ ([4], Prop. 5.2.4), $1 \leqslant i \leqslant n$. Now put $K_j = V^* E_j V$, and note that $\varphi(f) = \sum_j V^* f(x_j) E_j V = \sum_j f(x_j) K_j$. Clearly $\sum K_j = \varphi(1) = K$; we shall complete the proof by showing $\{[K_1 \mathfrak{H}], ..., [K_n \mathfrak{H}]\}$ is a weakly independent family of subspaces of $\mathfrak{H}$.

Choose $T_j \in L(\mathfrak{H})$ such that T_j lives in $[K_j \mathfrak{H}]$ and $T_1 + ... + T_n = 0$. Put $V_j = E_j V \in L(\mathfrak{H}, E_j \mathfrak{K})$. Then $V_j^* V_j = K_j$, so that the polar decomposition of V_j takes the form $V_j = U_j K_j^{\frac{1}{2}}$, where U_j is a partial isometry with initial space $[K_j \mathfrak{H}]$ and range $[E_j V \mathfrak{H}]$. Now the restriction of $K_j^{\frac{1}{2}}$ to $[K_j \mathfrak{H}]$ is injective, so it has a positive inverse $K_j^{-\frac{1}{2}} \in L([K_j \mathfrak{H}])$. Define the operator $A_j \in L(\mathfrak{K})$ by $A_j = U_j K_j^{-\frac{1}{2}} T_j K_j^{-\frac{1}{2}} U_j^*$; then A_j lives in $[U_j \mathfrak{H}] = [E_j V \mathfrak{H}]$, and we have $V_j^* A_j V_j = T_j$, $1 \leqslant j \leqslant n$. So if we put $A = A_1 + ... + A_n \in L(\mathfrak{K})$ (note that the sum is direct), then each A_i commutes with $\{E_j\}$, hence $A \in \pi(C(X))'$. Moreover, $V^* A V = \sum_j V_j^* A_j V_j = \sum_j T_j = 0$. Since φ is extreme, 1.4.6 implies that $A = 0$, hence $A_j = E_j A = 0$, hence $T_j = V_j^* A_j V_j = 0$, $1 \leqslant j \leqslant n$. That completes the proof.

Chapter 2. Boundary representations and Silov boundaries

We now take up the general problem mentioned in the introduction: to what extent does an algebra of operators on a Hilbert space determine the structure of the C^*-algebra it generates? More precisely, let $\mathfrak{H}$ be a Hilbert space and let $\mathcal{A}$ be a subalgebra of $L(\mathfrak{H})$ which contains the identity operator. The meaning of this question can be illustrated in terms of invariants. Let us say a property of $C^*(\mathcal{A})$ is invariant (relative to $\mathcal{A}$) if, for every operator algebra $\mathcal{A}_1$ which is completely isometrically isomorphic to $\mathcal{A}$, $C^*(\mathcal{A}_1)$ has the property when, and only when, $C^*(\mathcal{A})$ has it. Accordingly, if there are enough invariant properties to determine $C^*(\mathcal{A})$ to within $*$-isomorphism, then in an obvious sense $\mathcal{A}$ determines the structure of its generated C^*-algebra.

It is not obvious that invariant properties should exist at all. We will show, however, that certain irreducible representations of $C^*(\mathcal{A})$ (the boundary representations) are $\mathcal{A}$-invariant in the above sense. This leads to a body of general results, relating to analogues of Silov boundaries and the problem of implementing certain linear maps of operator algebras with $*$-isomorphisms (Section 2.2).

In sections 2.3 and 2.4, we obtain a characterization of boundary representations which is more useful for specific applications, a number of which are taken up later in Chapter 3.

2.1. *Boundary Representations.* Let A be a linear subspace of a C^*-algebra B, such that $B = C^*(A)$. We assume, throughout this chapter, that such an A always contains the identity of B. If ω is any representation of B, then $\omega|A$ has just one *multiplicative* completely positive extension to B (namely ω); in general, however, there may be other linear completely positive extensions of $\omega|A$.

Definition 2.1.1. An irreducible representation ω of B is called a *boundary representation* for A if $\omega|A$ has a unique completely positive linear extension to B.

When $B=C(X)$ (for X a compact Hausdorff space), the irreducible representations correspond to point evaluations; and if A is a separating linear subspace of $C(X)$ then the boundary representations correspond to points of X which have unique representing measures (relative to A). This is one of the characteristic properties of points in the Choquet boundary of X relative to A [16]. When dealing with one-dimensional boundary representations of a general C^*-algebra B, the analogy with Choquet boundary points carries over quite well (cf. section 3.1). General boundary representations, on the other hand, can possess properties for which there is no commutative counterpart (cf. section 3.5), and one should probably not try to push the analogy too far.

The very useful feature of boundary representations of B is their invariance relative to A, as described in the following theorem.

THEOREM 2.1.2. *Let B and B_1 be C^*-algebras and let A and A_1 be linear subspaces of B and B_1, respectively. Assume $B=C^*(A)$ and $B_1=C^*(A_1)$. Let φ be a completely isometric linear map of A on A_1 such that $\varphi(e)=e$. Then for every boundary representation ω of B (relative to A) there exists a boundary representation ω_1 of B_1 (relative to A_1) such that $\omega_1\circ\varphi(a)=\omega(a)$, $a\in A$.*

Proof. We may assume that B_1 acts on a Hilbert space $\mathfrak{H}$. By 1.2.9, φ may be extended to a completely positive linear map of B into $L(\mathfrak{H})$, which we denote by the same letter φ.

Now the map $\varphi(a)\to\omega(a)$, $a\in A$, is a completely contractive linear map of A_1 into $\omega(A)\subseteq L(\mathfrak{H}_\omega)$ which takes the identity to the identity, $\mathfrak{H}_\omega$ being the Hilbert space on which $\omega(B)$ acts, so by 1.2.9, there exists a completely positive linear map ϱ: $B_1\to L(\mathfrak{H}_\omega)$ such that $\varrho\circ\varphi(a)=\omega(a)$, $a\in A$. We will show that such a ϱ must be a representation of B_1. This will complete the proof, for two *representations* which agree on $\varphi(A)=A_1$ must agree on $C^*(A_1)=B_1$, and such a ϱ must be irreducible because $\varrho(B_1)=C^*(\varrho\circ\varphi(A))=C^*(\omega(A))=\omega(B)$ and ω is irreducible; thus ϱ is the required boundary representation of B_1, relative to A_1.

Now clearly $C^*(\varphi(B))\supseteq C^*(\varphi(A))=B_1$, so there is, by 1.2.3, a completely positive map $\tilde{\varrho}$: $C^*(\varphi(B))\to L(\mathfrak{H}_\omega)$ such that $\tilde{\varrho}=\varrho$ on B_1. By 1.1.1, there is a representation π of $C^*(\varphi(B))$ on $\mathfrak{K}$ and an operator $V\in L(\mathfrak{H}_\omega, \mathfrak{K})$ such that $\tilde{\varrho}(x)=V^*\pi(x)V$, $x\in C^*(\varphi(B))$, and $V\mathfrak{H}_\omega$ is cyclic for $\pi(C^*(\varphi(B))$. For $a\in A$, $\omega(a)=\tilde{\varrho}\circ\varphi(a)=V^*\pi\circ\varphi(a)V$, so that $x\in B\to V^*\pi\circ\varphi(x)V$ is a completely positive extension of $\omega|A$ to B. Since ω is a boundary representation, we conclude that $\omega(x) = V^*\pi\circ\varphi(x)V$, for all $x \in B$.

We claim now that V is unitary. Indeed, $V^*V = V^*\pi\circ\varphi(e)V = \omega(e) = I$, so V is isometric, and it suffices to show that $[V\mathfrak{H}_\omega] = \mathfrak{K}$. But $[V\mathfrak{H}_\omega]$ is cyclic for $\pi(C^*(\varphi(B))) = C^*(\pi\circ\varphi(B))$, and so $[V\mathfrak{H}_\omega] = \mathfrak{K}$ follows if we prove that the self-adjoint family of operators $\pi\circ\varphi(B)$ leaves $[V\mathfrak{H}_\omega]$ invariant. Choose a unitary element u in B. Then for $\xi \in \mathfrak{H}_\omega$, we have

$$\|\pi\circ\varphi(u)V\xi - V\omega(u)\xi\|^2 = \|\pi\circ\varphi(u)V\xi\|^2 - 2\,\mathrm{Re}\,(V^*\pi\circ\varphi(u)V\xi,\ \omega(u)\xi) + \|\omega(u)\xi\|^2$$

$$= \|\pi\circ\varphi(u)V\xi\|^2 - \|\omega(u)\xi\|^2 = \|\pi\circ\varphi(u)V\xi\|^2 - \|\xi\|^2 \leqslant \|\xi\|^2 - \|\xi\|^2 = 0,$$

since $V^*\pi\circ\varphi(u)V = \omega(u)$, $\omega(u)$ is unitary, and $\|\pi\circ\varphi(u)\| \leqslant 1$. Thus, $\pi\circ\varphi(u)V\xi = V\omega(u)\xi \in [V\mathfrak{H}_\omega]$, for every $\xi \in \mathfrak{H}_\omega$, and hence $\pi\circ\varphi(u)$ leaves $[V\mathfrak{H}_\omega]$ invariant. Since B is the norm-closed linear span of its unitary elements, we see that $\pi\circ\varphi(B)$ leaves $[V\mathfrak{H}_\omega]$ invariant; by the above comments, V is unitary.

Thus $\tilde{\varrho} = V^{-1}\pi V$ is a representation of $C^*(\varphi(B))$, and hence $\varrho = \tilde{\varrho}\,|\,B_1$ is a representation of B_1. That completes the proof.

Note that the proof shows somewhat more than we have claimed, when B_1 acts on a Hilbert space $\mathfrak{H}$; namely, for every completely positive extension $\tilde{\varphi}$ of φ to B, there exists a unique completely positive map ϱ: $C^*(\tilde{\varphi}(B)) \to L(\mathfrak{H})$, which is necessarily a $*$-representation, such that $\varrho\circ\tilde{\varphi}(x) = \omega(x)$, $x \in B$.

We shall give a number of applications of this theorem, a basic result of this paper, in the following two sections and in Chapter 3. Chapter 3 also contains a number of examples of boundary representations.

Let X be a compact Hausdorff space and let A be a linear subspace of $C(X)$ which contains the constants and separate points of X. Then there is a smallest closed subset K of X such that every function in A achieves its maximum modulus on K, called the *Silov boundary* of X relative to A [16]. We now introduce a non-commutative generalization of the Silov boundary.

Definition 2.1.3. Let A be a linear subspace of a C^*-algebra B such that A contains the identity and generates B as a C^*-algebra. A closed (two-sided) ideal J in B is called a *boundary ideal* for A if the canonical quotient map q: $B \to B/J$ is completely isometric on A. A boundary ideal is called the *Silov boundary* for A if it contains every other boundary ideal.

If $B = C(X)$ and K is a closed subset of X, then $J = \{f \in C(X): f(K) = 0\}$ is a closed ideal in B, and the quotient norm in B/J is given by $\|f\,|\,K\| = \sup_{x\in K} |f(x)|$, for $f \in C(X)$. Thus, J is a boundary ideal for A iff K is a boundary for A in the sense of the discussion preceding

2.1.3 (here we use the fact that a constant-preserving isometric linear map between subspaces of abelian C^*-algebras is completely isometric, by 1.2.11). The known correspondence between closed subsets of X and ideals in $C(X)$ now shows that 2.1.3 reduces to the usual definition of the Silov boundary for subspaces of commutative C^*-algebras.

Note that the Silov boundary of $A \subseteq B$ is unique, whenever it exists. Whether or not it always exists under the general conditions of 2.1.3 is, however, still an open question. In the commutative case $B = C(X)$, it is known that the closure of the set of all Choquet boundary points is the Silov boundary. We will show in the next section that a similar fact is true for "admissible" subspaces of arbitrary C^*-algebras, but that is the best general result we now know. For reasons brought out clearly in the next section, this question has significance in the development of an abstract theory of (non self-adjoint) operator algebras.

2.2. *Admissible subspaces of C^*-algebras.* In this section, we show that the Silov boundary exists for "admissible" subspaces, and we obtain some consequences; toward the end of the section we discuss a sufficient condition for admissibility.

Let A be a linear subspace of a C^*-algebra B such that $B = C^*(A)$. We remind the reader that A is always assumed to contain the identity.

Definition 2.2.1. A is called an *admissible* subspace of B if the intersection of the kernels of the boundary representations (for A) is a boundary ideal for A.

Let bd A denote the class of all boundary representations for A (to avoid set-theoretic difficulties, one should regard bd A as a set of representatives, one taken from each unitary equivalence class of boundary representations: we shall be deliberately casual about this kind of distinction). The reader can easily see that A is admissible if, and only if, it satisfies the condition: *for every integer $N \geqslant 1$ and every $N \times N$ matrix (a_{ij}) over A, one has*

$$\|(a_{ij})\| = \sup_{\omega \in \operatorname{bd} A} \|(\omega(a_{ij}))\|,$$

where the norm of (a_{ij}) is inherited from $B \otimes M_N$. Note also that, since a representation of B is always completely contractive, one need only check the inequality $\leqslant$. It is significant that admissibility is an invariant for completely isometric linear maps:

THEOREM 2.2.2. *Let A (resp. A_1) be a linear subspace of a C^*-algebra B (resp. B_1) such that $B = C^*(A)$ (resp. $B_1 = C^*(A_1)$) and suppose there is a completely isometric linear map φ of A on A_1 such that $\varphi(e) = e$. If A is admissible then so is A_1.*

Proof. We will show that A_1 satisfies the condition of the preceding paragraph. Let N be a positive integer and let (b_{ij}) be an $N \times N$ matrix over A_1. Then there exists elements

$a_{ij} \in A$ such that $\varphi(a_{ij}) = b_{ij}$. For each $\omega \in \mathrm{bd}\, A$, let ω_1 be the element of $\mathrm{bd}\, A_1$ satisfying $\omega_1 \circ \varphi = \omega$ (by 2.1.2). Since A is admissible and φ is completely isometric, we have

$$\|(b_{ij})\| = \|(a_{ij})\| = \sup_{\omega \in \mathrm{bd}\, A} \|(\omega(a_{ij}))\| = \sup_{\omega \in \mathrm{bd}\, A} \|(\omega_1(b_{ij}))\| \leqslant \sup_{\omega_1 \in \mathrm{bd}\, A_1} \|(\omega_1(b_{ij}))\|,$$

and the proof is complete.

PROPOSITION 2.2.3. *Let A be a subspace of B such that $C^*(A) = B$, let J be a boundary ideal for A, and let $\omega \in \mathrm{bd}\, A$. Then $J \subseteq \ker \omega$.*

Proof. Let q be the quotient map of B into B/J. Then $q|A$ is completely isometric, $q(e) = e$, and $C^*(q(A)) = q(C^*(A)) = q(B) = B/J$. By 2.1.2, there exists a representation ω_1 of B/J such that $\omega_1 \circ q = \omega$ on A. Since $\omega_1 \circ q$ and ω are $*$-homomorphisms and A generates B, we have $\omega_1 \circ q = \omega$ on B. Thus, if $x \in J = \operatorname{Ker} q$, then $\omega(x) = \omega_1 \circ q(x) = 0$; thus $J \subseteq \ker \omega$, completing the proof.

THEOREM 2.2.3. *Let A be an admissible subspace of B, and let K be the intersection of all kernels of boundary representations. Then K is the Silov boundary ideal for A.*

Proof. By hypothesis, K is a boundary ideal. If J is any other boundary ideal and if $\omega \in \mathrm{bd}\, A$, then by 2.2.2 we have $J \subseteq \operatorname{Ker} \omega$; hence, $J \subseteq K$, and we are done.

Now let A be an admissible subspace of B, let $K = \bigcap_{\omega \in bdA} \operatorname{Ker} \omega$ be the Silov boundary for A, and let q be the quotient map of B onto B/K. The process of passing from A to $q(A) \subseteq B/K$ is analogous to passing from a subspace A of $C(X)$ to the space of restrictions $A|\partial X \subseteq C(\partial X)$, and in dealing with "abstract" admissible subspaces, it is convenient to do this. Note, for example, that we have the following.

PROPOSITION 2.2.4. *Let A be an admissible subspace of a C^*-algebra B such that $B = C^*(A)$, let K be the Silov boundary ideal for A, and let q be the quotient map of B in B/K. Then $q(A)$ is an admissible subspace of B/K which has $\{0\}$ as its Silov boundary.*

Proof. The admissibility of $q(A)$ is evident from 2.2.2. By 2.1.2, the relation $\omega_1 \circ q = \omega$ sets up a bijective correspondence $\omega \leftrightarrow \omega_1$ between the boundary representations for A and those for $q(A)$ (note that the equation $\omega_1 \circ q = \omega$ on A entails its validity on $B = C^*(A)$, since ω, ω_1, and q are all $*$-homomorphisms). Thus, for $x \in B$, $q(x) \in \bigcap \ker \omega_1$ implies $\omega(x) = \omega_1 \circ q(x) = 0$ for all $\omega \in \mathrm{bd}\, A$, hence $x \in \bigcap_{\omega \in bdA} \operatorname{Ker} \omega = K$, and so $q(x) = 0$. 2.2.3 now shows that $\{0\}$ is the Silov boundary for $q(A)$.

It is natural to ask the extent to which a subalgebra or subspace, A, of a C^*-algebra B determines the structure of B. Even when B is commutative and is generated by A, there can be quite a variation of structure. For example, let D be the closed unit disc, and let A be the closed subalgebra of $C(D)$ consisting of all sup-norm limits of polynomials. A separates points of D and thus $C^*(A)=C(D)$. On the other hand, the unit circle $\mathbf{T}$ is a subset of D, and the restriction map $f\in C(D)\to f|_{\mathbf{T}}\in C(\mathbf{T})$ is a $*$-homomorphism of $C(D)$ on $C(\mathbf{T})$ which is completely isometric on A by the maximum modulus principle (cf. 1.2.11). Thus, $A_1=A|_{\mathbf{T}}$ is the "same" as A, whereas $C^*(A_1)=C(\mathbf{T})$ is quite different from $C(D)$.

The following result implies, among other things, that an admissible subspace completely determines the structure of its generated C^*-algebra, once one has factored by the Silov boundary ideal.

THEOREM 2.2.5. *Let A (resp. A_1) be an admissible subspace of a C^*-algebra B (resp. B_1) such that $B=C^*(A)$ (resp. $B_1=C^*(A_1)$). Assume that both A and A_1 have trivial Silov boundary ideals. Then every completely isometric linear map of A on A_1, which takes e to e, is implemented by a $*$-isomorphism of B on B_1.*

Proof. Let S be the set of all equivalence classes of boundary representations of B (for A). For each $n\in S$ choose a representative ω_n for n. Let φ be a completely isometric linear map of A on A_1 such that $\varphi(e)=e$. By 2.1.2, there exists, for each $n\in S$, a boundary representation ω_n' of B_1 (for A_1) such that $\omega_n'\circ\varphi=\omega_n$ on A. As n runs over S, ω_n' runs over all (classes) of boundary representations of B_1. So by hypothesis, and 2.2.3, we have $\bigcap_n \operatorname{Ker}\omega_n=\bigcap_n \operatorname{Ker}\omega_n'=\{0\}$. Thus the representations $\pi=\oplus_n\omega_n$ and $\pi'=\oplus_n\omega_n'$ are, respectively, faithful representations of B and B_1. Moreover, $\pi'\circ\varphi=\pi$ on A, by construction. It follows that $\pi'(B_1)=C^*(\pi'\circ\varphi(A))=C^*(\pi(A))=\pi(B)$, and since both π and π' are injective, the mapping $\sigma=(\pi')^{-1}\circ\pi$ is in fact a $*$-isomorphism of B on B_1. The relation $\pi'\circ\varphi(a)=\pi(a)$, $a\in A$, implies $\sigma|A=\varphi$, and the proof is complete.

COROLLARY 2.2.6. *Let A be an admissible subalgebra of a C^*-algebra B, such that $B=C^*(A)$. Then every completely isometric linear mapping of A onto itself which leaves the identity fixed is an algebra automorphism.*

Proof. Let K be the Silov boundary ideal for A and let q be the quotient map of B on B/K. Let φ be a completely isometric map of A on A such that $\varphi(e)=e$, and put $\varphi_1=q\circ\varphi\circ q^{-1}$: $q(A)\to q(A)$. By 2.2.5, φ_1 is implemented by a $*$-automorphism of B/K, and in particular φ_1 is multiplicative on $q(A)$. Since q is an algebra isomorphism of A on $q(A)$, it follows that $\varphi=q^{-1}\circ\varphi_1\circ q$ is multiplicative on A. That completes the proof.

We conclude this section with a discussion of one sufficient condition for a subspace to be admissible. This condition is not always satisfied, but it is effective in dealing with a variety of examples. Also, we point out that some related questions will be taken up in section 2.4.

Let A be a linear subspace of a C^*-algebra B and let ω be an irreducible representation of B. Define the set M_ω to be all b in the closure of $A+A^*$ for which $\omega(b)$ has the form $\|b\|U$, where U is a unitary operator. M_ω consitsts of those elements of $(A+A^*)^-$ which, in a sense, take on their "maximum modulus" in ω. Note, too, that $M_\omega^* = M_\omega$.

THEOREM 2.2.7. *Let A be a linear subspace of a C^*-algebra B and let ω be an irreducible representation of B such that M_ω generates B as a C^*-algebra. Then ω is a boundary representation for A.*

Proof. Let ϱ be any completely positive extension of $\omega|A$, say $\varrho = V^*\pi V$, where π is a representation of B on a Hilbert space $\mathfrak{K}$ and V is an operator from the Hilbert space $\mathfrak{H}$ on which ω acts to $\mathfrak{K}$, such that $[\pi(B)V\mathfrak{H}] = \mathfrak{K}$. As in 2.1.2, it suffices to show that ϱ is a representation, or what is the same, that V is unitary. Now $V^*V = \varrho(e) = \omega(e) = I$, so V is isometric, and we need only prove that $[V\mathfrak{H}] = \mathfrak{K}$.

Note that ϱ must equal ω on the closure of $A+A^*$, since both are bounded self-adjoint linear maps. Take $z \in (A+A^*)^-$. Then for every $\xi \in \mathfrak{H}$, we have

$$\begin{aligned}\|\pi(z)V\xi - V\omega(z)\xi\|^2 &= \|\pi(z)V\xi\|^2 - 2\,\mathrm{Re}\,(V^*\pi(z)V\xi, \omega(z)\xi) + \|V\omega(z)\xi\|^2 \\ &= \|\pi(z)V\xi\|^2 - \|\omega(z)\xi\|^2,\end{aligned}$$

since $V^*\pi(z)V = \omega(z)$ and V is isometric. So if $z \in M_\omega$, then $\|\omega(z)\xi\|^2 = \|z\|^2\|\xi\|^2$, hence $\|\pi(z)V\xi\|^2 - \|\omega(z)\xi\|^2 \leq 0$, and it follows that $\pi(z)V\xi = V\omega(z)\xi \in [V\mathfrak{H}]$. Thus, $[V\mathfrak{H}]$ is invariant under the self-adjoint family of operators $\pi(M_\omega)$, which generates $\pi(B)$ as a C^*-algebra. We conclude that $\mathfrak{K} = [\pi(B)V\mathfrak{H}] \subseteq [V\mathfrak{H}]$, as required.

Some examples are noteworthy. Suppose A is a linear subspace of B such that the unitary elements of $(A+A^*)^-$ generate B as a C^*-algebra; for example, $A+A^*$ could be dense in B, or A could be the algebra generated by a semigroup of unitary operators which contains e and generates B as a C^*-algebra. Then by 2.2.7, *every* irreducible representation of B is a boundary representation for A; hence A is admissible and, in fact, the Silov boundary of A is the trivial ideal $\{0\}$. Thus, making use also of 2.2.5, we can state the following.

COROLLARY 2.2.8. *Let A be a subspace of a C^*-algebra B such that the unitary elements in $(A+A^*)^-$ generate B as a C^*-algebra. Then every irreducible representation of B is a boundary representation for A, the Silov boundary ideal for A is trivial, and every completely isometric linear map of A onto itself which leaves the identity fixed is implemented by a $*$-automorphism of B.*

We remark that the same conclusion can be drawn from weaker, though less easily verified, hypotheses. For convenience, let us call an irreducible representation ω of B

peaking for A if $M_\omega (\subseteq (A+A^*)^-)$ generates B as a C^*-algebra. Then the conclusion of 2.2.8 is valid provided only that the intersection of the kernels of all peaking representations is trivial.

2.3. *Finite representations of operator algebras.* This section serves two purposes; it contains material which is preparatory for the characterization of boundary representations in 2.4, and we introduce here certain notions and terminology which will be used in the entire sequel.

In this section and the next, we shall be interested in subalgebras (rather than subspaces) of C^*-algebras. Recall first the definition of semi-invariant subspaces. A closed subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{H}$ is said to be semi-invariant under a subalgebra $\mathcal{A}$ of $L(\mathfrak{H})$ ($\mathcal{A}$ is assumed to contain the identity) if the map $\varphi(T) = P_{\mathfrak{M}} T|_{\mathfrak{M}}$ is multiplicative on $\mathcal{A}$. The definition is due to Sarason [20], who pointed out the following characterization. If $\mathfrak{M}$ is semi-invariant for $\mathcal{A}$, then $\mathfrak{M}_0 = [\mathcal{A}\mathfrak{M}] \ominus \mathfrak{M}$ is $\mathcal{A}$-invariant, so that $\mathfrak{M} = [\mathcal{A}\mathfrak{M}] \ominus \mathfrak{M}_0$ is a nested difference of $\mathcal{A}$-invariant subspaces; conversely if $\mathfrak{N} = \mathfrak{M}_1 \ominus \mathfrak{M}_0$ where $\mathfrak{M}_0 \subseteq \mathfrak{M}_1$ are $\mathcal{A}$-invariant, then $\mathfrak{N}$ is semi-invariant for $\mathcal{A}$. Thus when $\mathcal{A}$ is a self-adjoint algebra the semi-invariant subspaces are reducing subspaces. In general, of course, a semi-invariant subspace need not even be invariant.

Let A be a subalgebra of a C^*-algebra B; A is always assumed to contain the identity of B. A *representation* of A is a homomorphism φ of A into the algebra of operators on some Hilbert space, such that

(i) $\varphi(e) = I$, and

(ii) $\|\varphi(a)\| \leq \|a\|$, for all $a \in A$.

Our reason for imposing the condition (ii) is to make this definition coincide with the usual usage of the term for C^*-algebras. Indeed, if φ is a representation of $A = B$, then (i) and (ii) together imply that φ is positive (1.2.8), and thus $\varphi(x^*) = \varphi(x)^*$ for all $x \in B$. In general, if φ is a representation of A on $\mathfrak{H}$ and $\mathfrak{M}$ is a semi-invariant subspace for $\varphi(A)$, then we may define a new representation φ_0 of A on $\mathfrak{M}$ by

$$\varphi_0(a) = P_{\mathfrak{M}} \varphi(a)|_{\mathfrak{M}}, \quad a \in A.$$

Such a φ_0 is called a *subrepresentation* of φ. Note that, by the preceding comment about semi-invariant subspaces for $*$-algebras, this definition too reduces to the usual one in case A is self-adjoint.

Two representations φ_1 and φ_2 of A are said to be *equivalent* if there is a unitary operator U between their respective Hilbert spaces such that $U\varphi_1(a) = \varphi_2(a)\, U$ for all $a \in A$.

Definition 2.3.1. Let A be a subalgebra of a C^*-algebra and let φ be a representation of A. φ is called infinite if it is equivalent to a proper subrepresentation $\varphi_0 \neq \varphi$; φ is called finite if it is not infinite.

Some examples are given after 2.3.4. The following description of finite representations will be useful.

PROPOSITION 2.3.2. *Let φ be a representation of A on $\mathfrak{H}$. Then φ is finite if, and only if, for every isometry $V \in L(\mathfrak{H})$ the condition $V^*\varphi(a)V = \varphi(a)$, for all $a \in A$, implies V is unitary.*

Proof. Assume, first, that φ is finite, and let V be an isometry in $L(\mathfrak{H})$ such that $V^*\varphi V = \varphi$. Let $P = VV^*$ be the projection on the range of V. We will prove that V is unitary by showing $P = I$. Now $V^*\varphi V = \varphi$ implies $P\varphi P = V\varphi V^*$; hence for $a, b \in A$ we have $P\varphi(a)P\varphi(b)P = V\varphi(a)V^*V\varphi(b)V^* = V\varphi(a)\varphi(b)V^*$, because $V^*V = I$. Thus $P\mathfrak{H}$ is semi-invariant under $\varphi(A)$. Define the subrepresentation $\varphi_0(a) = P\varphi(a)|_{P\mathfrak{H}}$, of A on $P\mathfrak{H} = V\mathfrak{H}$. Then V is a unitary map of $\mathfrak{H}$ on $V\mathfrak{H}$, and $\varphi_0 V = P\varphi V = VV^*\varphi V = V\varphi$. As φ is finite, we have $\varphi_0(a) = \varphi(a)$, $a \in A$, and so $P = I$ follows by taking a to be the identity.

Conversely, suppose that φ is infinite. Then there is a semi-invariant subspace $\mathfrak{M}$ for $\varphi(A)$, $\mathfrak{M} \neq \mathfrak{H}$, and an isometric operator V of $\mathfrak{H}$ onto $\mathfrak{M}$ such that $P_{\mathfrak{M}}\varphi(a)V\xi = V\varphi(a)\xi$ for all $\xi \in \mathfrak{H}$. If we regard V as an element of $L(\mathfrak{H})$, then $P_{\mathfrak{M}} = VV^*$, so the above equation is $VV^*\varphi V = V\varphi$; multiply on the left by V^* to get $V^*\varphi V = \varphi$. V cannot be unitary because $V\mathfrak{H} = \mathfrak{M} \neq \mathfrak{H}$, and that completes the proof.

If $A = B$ is *self-adjoint* and φ is a representation of A on $\mathfrak{H}$, then as we have already noted, the subrepresentations of φ correspond to projections in the commutant $\varphi(A)'$ of $\varphi(A)$; and a proper subrepresentation of φ is equivalent to φ iff there is an isometry in $\varphi(B)'$ with range the indicated semi-invariant subspace. Thus we have the known result that a representation φ of a C^*-algebra is finite iff the von Neumann algebra $\varphi(B)'$ is finite (a von Neumann algebra is finite if it contains no non-unitary isometries).

We know of no analogous characterization of finite representations of general (non self-adjoint) algebras. Consider the following example. Let A be a subalgebra of a C^*-algebra B and let π be a representation of B on a Hilbert space $\mathfrak{K}$. Then $\varphi = \pi|A$ is a representation of A, and so is every subrepresentation of φ. Note that the subrepresentations of φ may differ greatly from φ; for instance, if B is abelian then the operators in $\varphi(A)$ are normal, while no such thing is true for the images of A under subrepresentations. Nevertheless, in the special case where $A + A^*$ is dense in B, there is an effective method for determining when subrepresentations of φ are finite (2.3.4).

LEMMA 2.3.3. *Let $\mathcal{A}$ be an algebra of operators on a Hilbert space $\mathfrak{K}$ such that $\mathcal{A}+\mathcal{A}^*$ is weakly dense in a von Neumann algebra $\mathcal{R}$. Let $\mathfrak{H}$ be a subspace of $\mathfrak{K}$ and define $\varphi(X)=P_{\mathfrak{H}}X|_{\mathfrak{H}}\in L(\mathfrak{H})$, $X\in L(\mathfrak{K})$. Then for every isometry $V\in L(\mathfrak{H})$ such that $V^*\varphi(X)V=\varphi(X)$, $X\in\mathcal{A}$ there exists an isometry $V_1\in\mathcal{R}'$ such that $V_1|_{\mathfrak{H}}=V$.*

Proof. Let us write the projection on $\mathfrak{H}$ simply as P. Note that we have $V^*\varphi(X)V=\varphi(X)$ for all $X\in\mathcal{R}$, because both sides of the equation are weakly continuous $*$-preserving linear functions of X, equality was assumed to hold on $\mathcal{A}$, and $\mathcal{A}+\mathcal{A}^*$ is weakly dense in $\mathcal{R}$.

Let $\xi,\eta\in\mathfrak{H}$ and $X, Y\in\mathcal{R}$. Then since $PV=V$ we have $(XV\xi, YV\eta)=(V^*Y^*XV\xi,\eta)=(V^*PY^*XV\xi,\eta)=(V^*\varphi(Y^*X)V\xi,\eta)=(\varphi(Y^*X)V\xi,\eta)=(X\xi, Y\eta)$. Thus we can define an isometric linear map V_0: $[\mathcal{R}\mathfrak{H}]\to[\mathcal{R}V\mathfrak{H}]\subseteq[\mathcal{R}\mathfrak{H}]$ by the formula

$$V_0: \sum_{j=1}^{n} X_j\xi_j \to \sum_{j=1}^{n} X_jV\xi_j,$$

where $\xi_1,\ldots,\xi_n\in\mathfrak{H}$, $X_1,\ldots,X_n\in\mathcal{R}$.

A trivial calculation shows that $V_0X=XV_0$ on $[\mathcal{R}\mathfrak{H}]$, so if we define $V_1\in L(\mathfrak{K})$ by $V_1\xi=V_0\xi$ for $\xi\in[\mathcal{R}\mathfrak{H}]$ and $V_1\xi=\xi$ for $\xi\in[\mathcal{R}\mathfrak{H}]^{\perp}$, then V_1 is an isometry in $\mathcal{R}'$, and clearly $V_1\xi=V\xi$ for $\xi\in\mathfrak{H}$. That completes the proof.

THEOREM 2.3.4. *Let A be a subalgebra of a C^*-algebra B such that $A+A^*$ is dense in B. Let π be a finite representation of B in $L(\mathfrak{K})$, let $\mathfrak{H}$ be a semi-invariant subspace for $\pi(A)$, and let $\varphi(a)=P_{\mathfrak{H}}\pi(a)|_{\mathfrak{H}}$ for $a\in A$. Then φ is a finite representation of A if, and only if, every unitary operator in $\pi(B)'$ which leaves $\mathfrak{H}$ invariant is reduced by $\mathfrak{H}$.*

Proof. Assume first that φ is finite, and let U be a unitary operator in $\pi(B)'$ such that $U\mathfrak{H}\subseteq\mathfrak{H}$. If $\mathfrak{H}$ does not reduce U, then $V=U|_{\mathfrak{H}}$ is a non-unitary isometry in $L(\mathfrak{H})$. Letting P stand for the projection on $\mathfrak{H}$, we have $V^*=PU^*|_{\mathfrak{H}}$, so that if $a\in A$ and $\xi\in\mathfrak{H}$ then $V^*\varphi(a)V\xi=PU^*P\pi(a)U\xi=PU^*\pi(a)U\xi=P\pi(a)\xi=\varphi(a)\xi$, using $PU^*P=PU^*$. By 2.3.2, φ must be infinite, a contradiction.

Conversely, assume φ is infinite. We will show that there is a unitary operator $U\in\pi(B)'$ such that $U\mathfrak{H}$ is properly contained in $\mathfrak{H}$ (thus $\mathfrak{H}$ does not reduce U). By 2.3.2 there is a non-unitary isometry $V\in L(\mathfrak{H})$ such that $V^*\varphi(a)V=\varphi(a)$, $a\in A$. Now the norm closure of $\pi(A)+\pi(A)^*=\pi(A+A^*)$ is $\pi(B)$, so that the weak operator closure of $\pi(A)+\pi(A)^*$ is the von Neumann algebra $\pi(B)''$. By 2.3.3, there is an isometry $U\in\pi(B)'$ such that $U|_{\mathfrak{H}}=V$. Since $\pi(B)'$ is a finite von Neumann algebra (by the remarks following 2.3.2), U must in fact be unitary. We have $U\mathfrak{H}=V\mathfrak{H}$ so $U\mathfrak{H}$ is a proper subspace of $\mathfrak{H}$, and thus the condition of the theorem is violated.

We can now give some pertinent examples. It is well-known (and follows from the discussion after 2.3.2) that the subrepresentations of a finite representation of a C^*-*algebra* are always finite. Such is not the case for representations of non-self-adjoint algebras, as the following examples show.

Let $\mathbf{T}$ be the unit circle, let $B = C(\mathbf{T})$, and let A be the closure in $C(\mathbf{T})$ of the polynomials. Then A is a subalgebra of $C(\mathbf{T})$ for which $A + A^*$ is dense in $C(\mathbf{T})$ (i.e., A is a Dirichlet algebra). Let m be Haar measure on $\mathbf{T}$ and let $\pi(f) =$ "multiplication by f", $f \in C(\mathbf{T})$, be the usual representation of $C(\mathbf{T})$ on $L^2(\mathbf{T}, m)$. Since $\pi(C(\mathbf{T}))'$ is the finite von Neumann algebra of all multiplications by $L^\infty(\mathbf{T}, m)$ functions, we see that π is a finite representation of $C(\mathbf{T})$.

Example 1: Let $\varphi(f) = \pi(f)$, for $f \in A$. Then φ is a representation of A on $\mathfrak{H} = L^2(\mathbf{T}, m)$, and trivially, every unitary operator in $\pi(C(\mathbf{T}))'$ which leaves $L^2(\mathbf{T}, m)$ invariant is reduced by $L^2(\mathbf{T}, m)$. By 2.3.3, φ is finite.

Example 2: Let π be as above, and take $\mathfrak{H} = H^2$ (H^2 is the closed linear span of $\{e^{in\theta};\ n \geq 0\}$). Then H^2 is invariant, hence semi-invariant, for $\pi(A)$ so that $\varphi(f) = P_{H^2}\pi(f)|_{H^2}$ is a subrepresentation of $\pi|_A$ on H^2. Now the unitary operator "multiplication by $e^{i\theta}$", commutes with $\pi(C(\mathbf{T}))$ and has H^2 as a non-reducing invariant subspace. By 2.3.4 we conclude that φ is an infinite representation of A. Since φ is a subrepresentation of the one in example 1, we have here an infinite subrepresentation of a finite representation.

Example 3: Let ψ be a non-constant inner function in H^∞ (see the discussion preceding 1.4.6 for definitions), and let $\mathfrak{H} = H^2 \ominus \psi H^2$. If π is as before, then $\mathfrak{H}$ is semi-invariant under $\pi(A)$, so that $\varphi(f) = P_{\mathfrak{H}}\pi(f)|_{\mathfrak{H}}$ gives a representation of A. We claim that φ is finite.

Indeed, by A.1.3 we have that $[\pi(A)\mathfrak{H}] = H^2$; since H^2 is obviously a cyclic subspace for $\pi(C(\mathbf{T}))$ we have that $[\pi(C(\mathbf{T}))\mathfrak{H}] = L^2(\mathbf{T}, m)$. Thus by 2.3.4, it suffices to show that if U is a unitary operator in $\pi(C(\mathbf{T}))'$ such that $U\mathfrak{H} \subseteq \mathfrak{H}$, then $\mathfrak{H}$ reduces U. By 1.3.3 we see that $UH^2 \subseteq H^2$ and $U^*\psi H^2 \subseteq \psi H^2$. Now U is multiplication by a function $u \in L^\infty(\mathbf{T}, m)$ such that $|u| = 1$ almost everywhere. It follows, then, that $uH^2 \subseteq H^2$ and $\bar{u}H^2 \subseteq H^2$; in particular $u \in H^2$ and $\bar{u} \in H^2$, so that u must be a constant. Thus U is a scalar, which of course is reduced by $\mathfrak{H}$. By 2.3.4, φ is finite.

There is an analogous result on finiteness for subrepresentations of *finite* direct sums of π (i.e. projections of shifts of finite multiplicity). It is also easy to see that the projection of a shift of infinite multiplicity onto one of its semi-invariant (but not invariant) subspaces gives rise to a representation which need not be finite.

Example 4: As a final example, let A be a subalgebra of a C^*-algebra B, and let ω

be a boundary representation for A. Then $\omega|_A$ is a finite representation of A. This is proved in the following section (2.4.1).

2.4. *A characterization of boundary representations.* Let A be a subalgebra of a C^*-algebra B. It is not at all clear from the definition, in most examples, which irreducible representations of B give rise to boundary representations for A. In particular, if A is an irreducible algebra of operators on some Hilbert space, when is the identity representation a boundary representation? What is required is an intrinsic description of boundary representations in terms of their restrictions to A. We shall obtain three necessary conditions which, together, are also sufficient (2.4.5); two are intrinsic, while the third is easily checked in specific examples. These results are essential for many of the applications in Chapter 3.

LEMMA 2.4.1. *Let A be a subalgebra of a C^*-algebra B and let ω be an irredubible representation of B. If ω is a boundary representation for A, then $\omega|_A$ is finite.*

Proof. Suppose ω acts on the Hilbert space $\mathfrak{H}$ and let V be an isometry in $L(\mathfrak{H})$ such that $V^*\omega(a)V=\omega(a)$ for all $a\in A$. Then $x\in B\rightarrow V^*\omega(x)V$ is a completely positive extension of $\omega|_A$, and hence $V^*\omega V=\omega$ on B, because ω is a boundary representation. Thus $V^*\omega V$ is a representation of the C^*-algebra B; it follows that $V\mathfrak{H}(\neq 0)$ is a semi-invariant, hence reducing, subspace of $\mathfrak{H}$ for $\omega(B)$ (cf. the discussion preceding 2.3.1). Since $\omega(B)$ is irreducible we must have $V\mathfrak{H}=\mathfrak{H}$, i.e., V is unitary. By 2.3.2 we conclude that $\omega|_A$ is finite, as required.

We now introduce an extension, to linear subspaces of C^*-algebras, of the notion of pure completely positive map.

Definition 2.4.2. Let S be a self-adjoint linear subspace (containing the identity) of a C^*-algebra B, and let φ be a completely positive linear map of S into $L(\mathfrak{H})$, for some Hilbert space $\mathfrak{H}$. φ is called *pure* if the only linear maps ψ: $S\rightarrow L(\mathfrak{H})$, for which both ψ and $\varphi-\psi$ are completely positive, are scalar multiples of φ.

LEMMA 2.4.3. *Let A, B and ω be as in 2.4.1, and suppose ω is a boundary representation for A. Then the restriction of ω to the closure of $A+A^*$ is pure.*

Proof. Let $S=(A+A^*)^-$, and let $\mathfrak{H}$ be the space on which ω acts. Let $\varphi_1, \varphi_2\in CP(S,\mathfrak{H})$ be such that $\omega|_S=\varphi_1+\varphi_2$. By 1.2.3 there are completely positive linear maps ψ_i: $B\rightarrow L(\mathfrak{H})$ such that $\psi_i|_S=\varphi_i$, $i=1,2$. In particular, the linear map $x\in B\rightarrow\psi_1(x)+\psi_2(x)$ is a completely positive extension of $\omega|_A$. Since ω is a boundary representation for A we must have $\psi_1(x)+\psi_2(x)=\omega(x)$ for all $x\in B$. Now ω is an irreducible representation of B, so by 1.4.3 it is a pure element of $CP\,(B,\mathfrak{H})$. Thus, there are scalars $t_i\geqslant 0$ such that $\psi_i=t_i\omega$ on B. Restricting to S we see $\varphi_i=t_i\omega|_S$, and thus $\omega|_S$ is pure.

Definition 2.4.4. Let A be a subalgebra of a C^*-algebra B, and let ω be an irreducible representation of B. Say that A *separates* ω if, whenever π is an irreducible representation of B such that $\omega|_A$ is equivalent to a subrepresentation of $\pi|_A$, then π and ω are equivalent representations of B. A is called a *separating subalgebra* if it separates every irreducible representation of B.

Note that if $B = C(X)$ with X a compact Hausdorff space, then a subalgebra is separating iff it separates points of X; by the Stone-Weierstrass theorem, this is equivalent to the assertion $B = C^*(A)$. One noncommutative analogue of this assertion is: A is a separating subalgebra of B iff $B = C^*(A)$. Neither implication is known to be true in general (see, however, [8], and [4] p. 223).

Let A be a subalgebra of B, and let ω be an irreducible representation of B on a Hilbert space $\mathfrak{H}$. It is easily seen that A separates ω iff the following condition is satisfied: *for every irreducible representation π of B on $\mathfrak{H}$ and every isometry $V \in L(\mathfrak{H}, \mathfrak{K})$, $V^*\pi(a)V = \omega(a)$ for all $a \in A$ implies π and ω are equivalent representations of B.* In this form, this definition makes sense if A is merely a linear subspace of B. We shall have no need for such a definition here, however.

Some non-commutative examples of separating subalgebras are given in Chapter 3.

Remarks. Let ω be an irreducible representation of B which is a boundary representation for a subalgebra A of B. We claim: A separates ω. Making use of the preceding observation, it suffices to show that if π is an irreducible representation of B and V is an isometry such that $V^*\pi(a)V = \omega(a)$, $a \in A$, then π and ω are equivalent. But $V^*\pi V$ is a completely positive linear extension of $\omega|_A$, and since ω is a boundary representation the preceding formula implies $V^*\pi(x)V = \omega(x)$ for all $x \in B$. Now argue as in the proof of 2.4.1 to conclude that V is an isometry whose range reduces the (irreducible) C^*-algebra $\pi(B)$; therefore V is unitary, and the formula $V^{-1}\pi V = \omega$ now shows that π and ω are equivalent representations of B.

We now have the promised characterization of boundary representations.

THEOREM 2.4.5. *Let A be a subalgebra of a C^*-algebra B such that $B = C^*(A)$, and let ω be an irreducible representation of B. Then ω is a boundary representation for A if, and only if, the following three conditions are satisfied:*

(i) *$\omega|_A$ is a finite representation of A*

(ii) *the restriction of ω to $(A + A^*)^-$ is pure*

(iii) *A separates ω.*

Proof. The necessity of the conditions is established in 2.4.1, 2.4.3 and the above remark. So assume (i), (ii), and (iii) are satisfied. Let $K = \{\varphi \in CP\,(B, \mathfrak{H}) : \varphi|_A = \omega|_A\}$. We must show that K is a singleton $\{\omega\}$. Now K is a convex subset of $CP\,(B, \mathfrak{H})$ which is closed in the BW-topology of $CP(B, \mathfrak{H})$; K is not empty ($\omega \in K$), and for every $\varphi \in K$ we have $\|\varphi\| = \|\varphi(e)\| = \|\omega(e)\| = 1$. Hence by remark 1.1.2, K is BW-compact. The Krein-Milman theorem implies that K is the closed convex hull of its extreme points, and thus we need only prove that an extreme point of K must necessarily be ω.

Let φ be an extreme point of K. We claim first that φ is a pure element of $CP(B, \mathfrak{H})$. To see this, choose nonzero elements $\varphi_1, \varphi_2 \in CP(B, \mathfrak{H})$ such that $\varphi_1(x) + \varphi_2(x) = \varphi(x)$, $x \in B$. Now both φ and ω are self-adjoint bounded linear maps of B, and they agree on A; therefore they must agree on $(A + A^*)^-$. Hence, $\varphi_1(b) + \varphi_2(b) = \omega(b)$ for all $b \in (A + A^*)^-$. By (ii), there are scalars $t_i \geq 0$ such that $\varphi_i(b) = t_i \omega(b)$, $b \in (A + A^*)^-$. If $t_1 = 0$, then since $e \in A$ we have $\varphi_1(e) = 0$, hence $\varphi_1 = 0$, contrary to the choice of φ_1. Thus $t_1 > 0$, and similarly $t_2 > 0$. By taking $b = e$ in the preceding equations we also see that $t_1 + t_2 = 1$. Now put $\psi_i = t_i^{-1}\varphi_i$. Then $\psi_i \in K$ and $t_1\psi_1 + t_2\psi_2 = \varphi$. By extremality of φ we conclude that $\psi_1 = \psi_2 = \varphi$, thus $\varphi_1 = t_i\varphi$. This proves that φ is pure.

By 1.1.1 and 1.4.3, there is an irreducible representation π of B on a Hilbert space $\mathfrak{K}$ and an operator $V \in L(\mathfrak{H}, \mathfrak{K})$ such that $\varphi = V^*\pi V$. Restricting to A, we have $\omega(a) = V^*\pi(a) V$ for $a \in A$. Taking $a = e$ we see that $I = V^*V$, so V is an isometry. Because A separates ω it follows that π is equivalent to ω, that is, there is a unitary operator $U \in L(\mathfrak{K}, \mathfrak{H})$ such that $\pi = U^{-1}\omega U$. The above formula now becomes $\omega(a) = (UV)^*\omega(a)\, UV$ for all $a \in A$. Now UV is an isometry in $L(\mathfrak{H})$ and according to (i) $\omega|_A$ is finite; so by 2.3.2, the preceding implies UV is unitary. Hence $V = U^{-1}UV$ is a unitary operator in $L(\mathfrak{H}, \mathfrak{K})$. The original equation $\omega|_A = V^*\pi V|_A$ now becomes $\omega(a) = V^{-1}\pi(a)\, V$, $a \in A$. $V^{-1}\pi V$ is a representation of B which agrees with ω on A, hence $V^{-1}\pi V$ agrees with ω on $C^*(A) = B$. Therefore, $\varphi = \omega$ on B, and the proof is complete.

Combining the preceding with 1.2.3., one may prove the following result, which seems noteworthy: *Let S be a closed self-adjoint linear subspace of a C^*-algebra B, such that $e \in S$. Then every pure element of $CP(S, \mathfrak{H})$ extends to a pure element of $CP(B, \mathfrak{H})$.* Thus, using 1.1.1 and 1.4.3, every pure element of $CP(S, \mathfrak{H})$ has a representation $\varphi(a) = V^*\pi(a)\, V$ ($a \in S$) where π is an *irreducible* representation of B on a Hilbert space $\mathfrak{H}$ and $V \in L(\mathfrak{H}, \mathfrak{K})$.

Note also that essentially no use was made of the fact that A is an algebra, in proving 2.4.5, and in fact the theorem is true for linear subspaces as well. One need only give definitions which make sense in context.

Chapter 3. Applications to nonnormal operators

This chapter contains a number of applications of the preceding theory to certain operators on Hilbert space. The main results are in sections 3.1, 3.2, 3.5, 3.6, and 3.7.

In 3.1 we show that for an arbitrary Hilbert space operator T, spectral points of T which are on the boundary of the numerical range of T correspond to one-dimensional boundary representations of $C^*(T)$ for $P(T)$. In 3.2 we classify certain operators which satisfy a polynomial equation $p(T)=0$. Sections 3.3 and 3.4 contain results of a preliminary nature for the discussion in 3.6. In 3.5 we determine the boundary representations of $C^*(T)$ for $P(T)$ where T is the projection of the bilateral shift (of multiplicity 1) onto certain of its semi-invariant subspaces. 3.6 contains a classification theorem, and associated results, for certain operators on Hilbert space; these results are probably the most significant applications that we have at the present time. In 3.7, we show how the Volterra operator $Vf(x)=\int_0^x f(t)\,dt$ ($f\in L^2(0, 1)$) can be characterized by the norms of certain polynomials in V.

3.1. *Characters of* $C^*(T)$ *and* $\operatorname{sp}(T)\cap\partial W(T)$. Let T be an operator on a Hilbert space $\mathfrak{H}$. We prove, in this section, the useful and perhaps surprising fact that points in the spectrum of T which lie on the boundary of the numerical range of T correspond to characters (i.e., complex homomorphisms) of $C^*(T)$; moreover, these characters give rise to one-dimensional boundary representations for $P(T)$.

Recall that the *numerical range* of T is the set of complex numbers $W(T)=\{(T\xi, \xi): \xi\in\mathfrak{H}, \|\xi\|=1\}$. Note that $\operatorname{Re} T\geqslant 0$ if, and only if, $W(T)$ is contained in the right half-plane $\{\operatorname{Re} z\geqslant 0\}$; in this event $I+T$ is invertible (because the spectrum $\operatorname{sp}(T)$ is contained in the closure of $W(T)$ [9]), and in fact $(I-T)(I+T)^{-1}$ has norm at most 1 ([18], pp. 442–443). The following lemma provides a bit more information.

LEMMA 3.1.1. *If* $\operatorname{Re} T\geqslant 0$, *then* $(I+T)^{-1}$ *can be norm-approximated by polynomials in* T.

Proof. Note that $P(T)$ is a commutative Banach algebra with identity, and we have to show that $I+T$ is invertible in $P(T)$. Suppose $\lambda\in\mathbf{C}$ is such that $T-\lambda I$ is not invertible n $P(T)$; then we claim that $\operatorname{Re}\lambda\geqslant 0$ (this yields the desired conclusion). Since $T-\lambda I$ lies in a proper maximal ideal, there is a nontrivial complex homomorphism ω of $P(T)$ such that $\omega(T)=\lambda$. We have $\|\omega\|=1=\omega(I)$, and so there is a linear functional ϱ on $C^*(T)$ such that $\|\varrho\|=1$ and $\varrho=\omega$ on $P(T)$ (by the Hahn-Banach theorem). The conditions $\|\varrho\|=1=\varrho(I)$ imply that ϱ is positive ([4], p. 25), hence $\operatorname{Re}\lambda=\operatorname{Re}\varrho(T)=\varrho(\operatorname{Re} T)\geqslant 0$, completing the proof.

THEOREM 3.1.2. *Let* T *be an operator on* $\mathfrak{H}$ *and let* $\lambda\in\operatorname{sp}(T)\cap\partial W(T)$. *Then there exists a character* χ *of* $C^*(T)$ *such that* $\chi(T)=\lambda$; χ *is a one-dimensional boundary representation for* $P(T)$.

Proof. We first make a reduction. Since $W(T)$ is a convex set ([9], p. 110) which contains λ on its boundary, there is a supporting tangent line at λ, i.e. a complex number

$\alpha \neq 0$ such that Re $\alpha\lambda \leqslant$ Re $\alpha(T\xi, \xi)$, for all $\xi = \mathfrak{H}$, $\|\xi\| = 1$. By replacing T with $\alpha(T - \lambda I)$, we may assume $\lambda = 0$ and $W(T)$ is contained in the right half plane $\{\text{Re}\, z \geqslant 0\}$.

Now define a linear functional ω on $P(T)$ as follows. For every polynomial p, we have (by the spectral mapping theorem)

$$|p(0)| \leqslant \sup \{|p(z)| : z \in \text{sp}\,(T)\} = \sup |\text{sp}(p(T))| \leqslant \|p(T)\|,$$

so there exists a unique bounded linear functional ω on $P(T)$ such that $\omega(p(T)) = p(0)$ for every polynomial p. Clearly ω is multiplicative, $\omega(T) = \lambda = 0$, and $\|\omega\| = \omega(I) = 1$. We must show, first, that there is a character on $C^*(T)$ which extends ω, and second, that this character is the only positive extension of ω to $C^*(T)$. Since $P(T)$ generates $C^*(T)$ as a C^*-algebra, two characters which agree on $P(T)$ must agree everywhere; thus it suffices to show, first, that ω *has* a positive extension to $C^*(T)$ and second, that every positive extension is a character.

The first conclusion is immediate from the Hahn-Banach theorem: choose a linear functional ϱ on $C^*(T)$ such that $\varrho = \omega$ on $P(T)$ and $\|\varrho\| = \|\omega\| = 1$. Thus $\|\varrho\| = \varrho(I)$ and it follows ([4], p. 25) that ϱ is positive.

We claim now that any such positive extension ϱ is a character. For this, define the operator $S = (I - T)(I + T)^{-1}$. Then $C^*(S) = C^*(T)$, $\|S\| \leqslant 1$ by the preceding remarks, and Lemma 3.1.1 shows that $S \in P(T)$. Since ϱ is multiplicative on $P(T)$ we have $\varrho(S) = (1 - \varrho(T))(1 + \varrho(T))^{-1} = 1$. A familiar theorem of Gelfand and Segal ([4], p. 32–33) provides a representation π of $C^*(S)$ on a Hilbert space $\mathfrak{K}$ and a unit vector $\zeta \in \mathfrak{H}$ such that $\varrho(X) = (\pi(X)\zeta, \zeta)$, for all $X \in C^*(S)$. We will show that the one-dimensional subspace $[\zeta]$ is invariant under $\pi(C^*(S))$; the theorem will follow, because then $\pi(X)\zeta = \varrho(X)\zeta$ for every $X \in C^*(S)$, and hence ϱ is multiplicative everywhere. Now we can write

$$\|\pi(S)\zeta - \zeta\|^2 = \|\pi(S)\zeta\|^2 - 2\text{Re}\,(\pi(S)\zeta, \zeta) + 1 = \|\pi(S)\zeta\|^2 - 2\text{Re}\,\varrho(S) + 1 = \|\pi(S)\zeta\|^2 - 1 \leqslant 0,$$

since $\|\pi(S)\| \leqslant \|S\| \leqslant 1$. Therefore, $\pi(S)\zeta = \zeta$. Since $\varrho(S^*) = \overline{\varrho(S)} = 1$, the same argument shows $\pi(S^*)\zeta = \zeta$. Thus, $[\zeta]$ is invariant under the self-adjoint family of operators $\{\pi(S), \pi(S^*), I\}$, and since the norm-closed algebra generated by the latter is $\pi(C^*(S))$, the proof is complete.

As one noteworthy application, let $T \in L(\mathfrak{H})$, and suppose λ is a point in the spectrum of T such that $|\lambda| = \|T\|$. Since sp $(T) \subseteq W(T)^- \subseteq \{|z| \leqslant \|T\|\}$, λ must be a boundary point of $W(T)$. Thus, there is a unique character χ of $C^*(T)$ such that $\chi(T) = \lambda$. To restate the argument, suppose T is such that $C^*(T)$ has no maximal ideals of codimension 1. Then for every spectral value λ, we must have $|\lambda| < \|T\|$; i.e., $r(T) < \|T\|$ ($r(T)$ denoting the spectral radius of T). That proves:

COROLLARY 3.1.3. *Let T be a Hilbert space operator such that $C^*(T)$ has no characters. Then the spectral radius of T is less than $\|T\|$.*

Note that an operator T is normal iff there are enough characters of $C^*(T)$ to separate points. In this case, of course, we have $r(f(T)) = \|f(T)\|$ for every bounded Borel function f on sp (T). 3.1.3 shows how thoroughly the latter fails for operators at the opposite extreme from normal operators.

3.2. *Simple algebraic operators.* In this section we consider simple algebraic operators, that is, operators T for which $C^*(T)$ is simple and which satisfy a polynomial equation $p(T)=0$. A natural question is, to what extent is such an operator determined by its minimum polynomial p? The most obvious examples of simple algebraic operators are irreducible operators on finite-dimensional spaces; but even here there is apparently little relation between the minimum polynomial of T and, say, $C^*(T)$. In infinite dimensions, the situation is more complicated by the fact that algebraic operators have no particular tendency to generate type I C^*-algebras. Consider, for example, an operator $T_0 \in L(\mathfrak{H})$ such that $C^*(T_0)$ is an infinite-dimensional UHF algebra [7] (such operators exist, by [28]), and define $T \in L(\mathbf{C}^3 \otimes \mathfrak{H})$ by the operator matrix

$$T = \begin{pmatrix} 0 & I & T_0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}.$$

A laborious but routine calculation shows that $C^*(T)$ is the algebra $M_3 \otimes C^*(T_0)$ of all 3×3 matrices over $C^*(T_0)$, which is again a (simple) UHF algebra. Clearly $T_3 = 0$, and thus we have a simple algebraic operator for which $C^*(T)$ is antiliminal (the preceding is a modification of an example due to C. Pearcy). Indeed, this observation shows that for every simple operator T_0, $M_3 \otimes C^*(T_0)$ has the form $C^*(T)$ for some simple *algebraic* operator T. Since a great variety of separable C^*-algebras are singly-generated as C^*-algebras, the situation for general simple algebraic operators is about as complicated as it can get.

It may be somewhat surprising, therefore, that in the presence of one additional condition on norms (maximality), it is possible not only to predict the structure of $C^*(T)$ from the minimum polynomial of T, but also to classify such operators to unitary equivalence (3.2.11–3.2.13).

Let T be a simple algebraic operator having minimum polynomial $p(z) = (z-a_1)^{n_1}$ $(z-a_2)^{n_2} \dots (z-a_k)^{n_k}$. To avoid trivialities, we will always assume that T is not a scalar; and there is no essential loss if we also require $\|T\| = 1$. Since each a_i belongs to the spectrum of T, we conclude from 3.1.3 that $|a_i| < 1$. Let ψ be the Blaschke product having p as its numerator:

$$\psi(z) = \left(\frac{z-a_1}{1-\bar{a}_1 z}\right)^{n_1} \cdots \left(\frac{z-a_k}{1-\bar{a}_k z}\right)^{n_k}, \quad |z| \leqslant 1.$$

Note that $\psi(T)=0$ and $\lambda(T)\neq 0$ for every proper divisor λ of ψ (proper means non-proportional). A divisor ψ_0 of ψ is called *large* if its degree is one less than the degree of ψ; these are of the form $\psi_i(z) = \left(\frac{1-\bar{a}_i z}{z-a_i}\right)\psi(z)$, for $1 \leqslant i \leqslant k$. Now if ψ_0 is any proper divisor of ψ then it follows that $\|\psi_0(T)\| \leqslant 1$ (because the closed unit disc is a spectral set for T); we shall call T *maximal* if there is a large Blaschke divisor ψ_0 of ψ for which $\|\psi_0(T)\|=1$. Note that this entails $\|\lambda(T)\|=1$ for every Blaschke divisior λ of ψ_0 (indeed, $\|\lambda(T)\|\leqslant 1$ is automatic, and if $\psi_0=\lambda_1\lambda$ where λ_1 is a Blaschke product then we have $1=\|\psi_0(T)\| \leqslant \|\lambda_1(T)\|\cdot\|\lambda(T)\| \leqslant \|\lambda(T)\|$). So for example, if the minimum polynomial of T is $p(z)=z^n$, $n>1$, then T is maximal iff $\|T\|=\|T^2\|=\ldots=\|T^{n-1}\|=1$. One exemple of such a T is given by the operator on $\mathbf{C}^n\otimes\mathfrak{H}$ whose matrix is

$$\begin{pmatrix} 0 & T_1 & 0 \ldots & & 0 \\ 0 & 0 & T_2 \ldots & & 0 \\ & & & & \vdots \\ & & & 0 & T_{n-1} \\ 0 & & & 0 & 0 \end{pmatrix}$$

where $T_i \in L(\mathfrak{H})$ and $\|T_1\|=\ldots=\|T_{n-1}\|=\|T_1T_2\ldots T_{n-1}\|=1$.

Another way maximality could be defined is to require $\|\lambda(T)\|=1$ for every proper divisor λ of ψ, or what is the same, $\|\psi_0(T)\|=1$ for *every* large divisor ψ_0. While this appears to be stronger than the above definition, the results below imply that the two are in fact equivalent.

The first few results provide some facts about certain special maximal operators. H^2, as usual, denotes all functions in L^2 (of the unit circle) whose negative Fourier coefficients vanish, and for an inner function ψ, S_ψ denotes the projection of the unilateral shift S_+ (i.e., multiplication by $e^{i\theta}$, qua an operator on H^2) onto $H^2\ominus\psi H^2$. It is a familiar fact that, for every $\alpha\in\mathbf{C}$, $|\alpha|<1$, the function $e_\alpha(e^{i\theta})=(1-|\alpha|^2)^{\frac{1}{2}}(1-\bar{\alpha}e^{i\theta})^{-1}$ is a unit eigenvector for S_+^* having eigenvalue $\bar{\alpha}$.

LEMMA 3.2.1. *Let ψ be an inner function and let α be a zero of ψ in the interior of the unit disc. Then $\lambda e_\alpha\in H^2\ominus\psi H^2$ for every divisor λ of $\left(\frac{1-\bar{\alpha}z}{z-\alpha}\right)\psi(z)$.*

Proof. Let ψ_0 be the inner function $\left(\frac{1-\bar{\alpha}z}{z-\alpha}\right)\psi(z)$, and let λ be a divisor of ψ_0. Then $\psi_0=\lambda\mu$ for some inner function μ, and hence $\psi(z)=\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)\lambda\mu$. Clearly $\lambda e_\alpha\in H^2$; and for every $g\in H^2$ we have

$$(\lambda e_\alpha, \psi g) = (\lambda e_\alpha, (S_+ - \alpha I)(I - \bar{\alpha} S_+)^{-1}\lambda \mu g) = (e_\alpha, (S_+ - \alpha I)(I - \bar{\alpha} S_+)^{-1}\mu g)$$
$$= ((S_+^* - \bar{\alpha} I) e_\alpha, (I - \bar{\alpha} S_+)^{-1}\mu g) = 0,$$

because "multiplication by ψ" is an isometry which commutes with S_+, and because $S_+^* e_\alpha = \bar{\alpha} e_\alpha$. Thus, $\psi e_\alpha \in H^2 \ominus \psi_1 H^2$, as asserted.

Note, in particular, that $e_\alpha \in H^2 \ominus \psi H^2$ for every zero α of ψ in the interior of the unit disc.

COROLLARY 3.2.2. *Let ψ be a finite Blaschke product of degree $\geqslant 2$. Then $\|S_\psi\| = 1$, and S_ψ is a maximal contraction whose minimum polynomial is the numerator of ψ.*

Proof. It is clear that $\psi(S_\psi) = 0$ (for if $f \in H^2 \ominus \psi H^2$, then $\psi(S_+) f = \psi \cdot f \in \psi H^2$, so that $\psi(S_\psi) f = P\psi(S_+) f = 0$, P denoting the projection of H^2 on $H^2 \ominus \psi H^2$); so if p is the numerator of ψ then we have $p(S_\psi) = 0$.

Since ψ is not constant, it must have at least one zero α in the interior of the unit disc. Let $\psi_0(z) = \left(\dfrac{1 - \bar{\alpha} z}{z - \alpha}\right) \psi(z)$. We claim that $\|\psi_0(S_\psi)\| = 1$. Indeed, by 3.2.1 we have $\|\psi_0(S_\psi)\| \geqslant \|\psi_0(S_\psi) e_\alpha\| = \|P\psi_0 \cdot e_\alpha\| = \|\psi_0 \cdot e_\alpha\| = 1$, because $|\psi_0| = 1$ identically on the unit circle and e_α is a unit vector in L^2.

Note that the preceding paragraph actually shows that $\|\psi_0(S_\psi)\| = 1$ for *every* large divisor ψ_0 of ψ. It follows that $q(S_\psi) \neq 0$ for every polynomial q properly dividing p, so that p is the minimum polynomial of S_ψ.

All that remains is to show that $\|S_\psi\| = 1$. Note that one inequality is immediate from $\|S_\psi\| = \|PS_+|_{H^2 \ominus \psi H^2}\| \leqslant \|S_+\| = 1$. Now if ψ_0 is constant, then ψ has degree 1, and this contradicts the hypothesis. So ψ_0 necessarily has at least one zero β inside the unit disc. Thus ψ_0 can be factored

$$\psi_0(z) = \left(\frac{z - \beta}{1 - \bar{\beta} z}\right) \psi_1(z)$$

where ψ_1 is an inner (in fact, Blaschke) function. Now we have $\|\psi_1(S_\psi)\| \leqslant 1$ (because the unit disc is a spectral set for S_ψ, see [18], p. 442) and hence $1 = \|\psi_0(S_\psi)\| \leqslant \|(S_\psi - \beta I)(I - \bar{\beta} S_\psi)^{-1}\|$. On the other hand, if $\|S_\psi\| = r < 1$, then the (closed) disc of radius r is a spectral set for S_ψ so that

$$\|(S_\psi - \beta I)(I - \bar{\beta} S_\psi)^{-1}\| \leqslant \sup_{|z| \leqslant r} \left|\frac{z - \beta}{1 - \bar{\beta} z}\right| < 1,$$

contradicting the above inequality. We conclude $\|S_\psi\| \geqslant 1$, and the proof is complete.

LEMMA 3.2.3. *Let ψ be a nonconstant inner function. Then $1 - \overline{\psi(0)}\psi$ is a cyclic vector for S_ψ in $H^2 \ominus \psi H^2$.*

Proof. Note first that $|\psi(0)|<1$, by the maximum modulus principle, so that $(1-\overline{\psi(0)}\psi)^{-1}$ is bounded; in particular, $1-\overline{\psi(0)}\psi$ is an outer function.

Next, observe that $1-\overline{\psi(0)}\psi\in H^2\ominus\psi H^2$. Indeed, if $g\in H^2$ then $(1-\overline{\psi(0)}\psi,\ \psi g)=(1,\ (\psi-\psi(0))g)=0$ since $(\psi-\psi(0))g$ vanishes at the origin. Thus, $1-\overline{\psi(0)}\psi$ is orthogonal to ψH^2, and it clearly is in H^2.

Now let P be the projection of H^2 on $H^2\ominus\psi H^2$. If $g\in H^2$ is such that $g\perp S_\psi^n(1-\overline{\psi(0)}\psi)$ for every $n\geqslant 0$, then $Pg\perp S_+^n(1-\overline{\psi(0)}\psi,\ n\geqslant 0$. Now $1-\overline{\psi(0)}\psi$ is cyclic for S_+ (since it is an outer function), so that $Pg\perp H^2$. Therefore, $Pg=0$, or $g\in\psi H^2$, and this proves that $H^2\ominus\psi H^2\subseteq[S_\psi^n(1-\overline{\psi(0)}\psi)\colon n\geqslant 0]$. The conclusion follows.

COROLLARY 3.2.4. *If ψ is a finite Blaschke product and α is a zero of ψ, then e_α is a cyclic vector for S_ψ. Moreover, if $\psi_0(z)=\left(\dfrac{1-\bar{\alpha}z}{z-\alpha}\right)\psi(z)$, then $H^2\ominus\psi H^2$ is linearly spanned by $\{\varphi e_\alpha\colon \varphi$ is a Blaschke divisor of $\psi_0\}$.*

Proof. We prove the second statement first. Let $\alpha_1, \ldots, \alpha_n$ be the zeroes of ψ, repeated according to multiplicities. Then clearly there is a polynomial p of degree n such that $p(S_\psi)=0$ (the numerator of ψ is one such), and by 3.2.3 S_ψ has a cyclic vector f; it follows that $H^2\ominus\psi H^2$ is spanned by $f, S_\psi f, \ldots, S_\psi^{n-1}f$ so that the dimension of $H^2\ominus\psi H^2$ is at most n. Now by 3.2.1 we have $\lambda e_\alpha\in H^2\ominus\psi H^2$ for every divisor λ of ψ_0, so it suffices to show that there are at least n linearly independent elements of the form λe_α, λ dividing ψ_0. This we can do as follows. Suppose $\alpha=\alpha_n$. Then put $\lambda_1=1$ and

$$\lambda_j(z)=\left(\frac{z-\alpha_1}{1-\bar{\alpha}_1 z}\right)\cdots\left(\frac{z-\alpha_{j-1}}{1-\bar{\alpha}_{j-1}z}\right),$$

for $1<j\leqslant n$. Clearly $\{\lambda_1 e_\alpha, \ldots, \lambda_n e_\alpha\}$ is a linearly independent set of functions, and $\lambda_j|\psi_0$ by construction, so the second assertion follows.

The fact that e_α is cyclic is an immediate consequence, for if P denotes the projection of H^2 on $H^2\ominus\psi H^2$, then for $1\leqslant j\leqslant n$ we have, using 3.2.1 again, $\lambda_j(S_\psi)e_\alpha=P\lambda_j(S_+)e_\alpha=P\lambda_j e_\alpha=\lambda_j e_\alpha$, and so $[\lambda_j(S_\psi)e_\alpha\colon 1\leqslant j\leqslant n]=H^2\ominus\psi H^2$. That completes the proof.

COROLLARY 3.2.5. *Let ψ be a nonconstant finite Blaschke product and let ψ_0 be a large divisor of ψ. Then $P(S_\psi)$ is linearly spanned by operators of the form $\lambda(S_\psi)$, where λ is a Blaschke divisor of ψ_0.*

Proof. Let $X\in P(S_\psi)$ and let α be the zero of ψ/ψ_0, inside the unit disc. By the second statement of 3.2.4, there are divisions $\lambda_1, \ldots, \lambda_m$ of ψ_0 and scalars $c_1, \ldots, c_m$ such that $Xe_\alpha=$

$\sum c_k \lambda_k e_\alpha$. Putting $X_0 = \sum c_k \lambda_k (S_\psi)$, then we have $Xe_\alpha = X_0 e_\alpha$, and $X = X_0$ now follows from the first statement of 3.2.4, completing the proof.

Now let T be an algebraic operator on a Hilbert space such that $\|T\| = 1$, and the spectral radius of T is less than 1. Let ψ be the finite Blaschke product which has the minimum polynomial of T as its numerator. In the next two results we prove that if T is maximal, then the map $p(T) \mapsto p(S_\psi)$ (p running over all polynomials) is a completely contractive map of $P(T)$ on $P(S_\psi)$.

LEMMA 3.2.6. *Let T be an algebraic contraction on a Hilbert space $\mathfrak{H}$, such that the spectral radius of T is less than* 1, *and let ψ be the Blaschke product associated with the minimum polynomial of T. Assume there is a unit cyclic vector ξ for T such that $\|\psi_0(T)\xi\| = 1$ for some large Blaschke divisor ψ_0 of ψ. Then T is unitarily equivalent to S_ψ.*

Proof. Let U be the minimal unitary dilation of T; we can assume U acts on $\mathfrak{K} \supseteq \mathfrak{H}$, and $T^n = PU^n|_{\mathfrak{H}}$, $n \geq 0$, where P is the projection of $\mathfrak{K}$ on $\mathfrak{H}$. First, we claim $\lambda(U)\xi = \lambda(T)\xi$ for every Blaschke divisor λ of ψ_0. Indeed, since $|\lambda| = 1$ on $\operatorname{sp}(U) \subseteq \{|z| = 1\}$, it follows from the operational calculus for normal operators that $\lambda(U)$ is unitary. Note also that $\|\lambda(T)\xi\| = 1$. For $\|\lambda(T)\xi\| \leq \|\lambda(T)\| \leq 1$ because the closed unit disc is a spectral set for T, and if μ is the Blaschke product satisfying $\psi_0 = \mu\lambda$ then we have $1 = \|\psi_0(T)\xi\| = \|\mu(T)\lambda(T)\xi\| \leq \|\lambda(T)\xi\|$, because $\|\mu(T)\| \leq 1$ (as above). Now we can write

$$\|\lambda(U)\xi - \lambda(T)\xi\|^2 = 1 - 2 \operatorname{Re} (\lambda(U)\xi, \lambda(T)\xi) + 1 = 0,$$

because $(\lambda(U)\xi, \lambda(T)\xi) = (P\lambda(U)\xi, \lambda(T)\xi) = (\lambda(T)\xi, \lambda(T)\xi) = 1$, proving the assertion.

Now $\psi(z)/\psi_0(z)$ has the form $(z-\alpha)/(1-\bar{\alpha}z)$ for some $\alpha \in \mathbf{C}$, $|\alpha| < 1$. Define the unitary operator V on $\mathfrak{K}$ by $V = (U - \alpha I)(I - \bar{\alpha}U)^{-1}$. We will define a unitary mapping of $[V^n\xi: n = 0, \pm 1, \pm 2, \ldots]$ on $L^2(\mathbf{T})$ ($\mathbf{T}$ denoting the unit circle) as follows. Note first that $V^n\xi \perp V^m\xi$ if $m \neq n$. Indeed, if $n \geq 1$ then

$$PV^n\psi_0(U)\xi = PV^{n-1}\psi(U)\xi = PV^{n-1}P\psi(U)\xi = PV^{n-1}\psi(T)\xi = 0,$$

because $(z-\alpha)/(1-\bar{\alpha}z)\,\psi_0(z) = \psi(z)$ and the map $X \mapsto PX|_{\mathfrak{H}}$ is multiplicative on $P(U)$. It follows, because $\psi_0(U)$ is a unitary operator commuting with V, that

$$(V^n\xi, \xi) = (V^n\psi_0(U)\xi, \psi_0(U)\xi) = (V^n\psi_0(U)\xi, \psi_0(T)\xi) = (PV^n\psi_0(U)\xi, \psi_0(T)\xi) = 0,$$

(note that we used the fact that $\psi_0(U)\xi = \psi_0(T)\xi$). The conclusion $V^n\xi \perp V^m\xi$ ($n \neq m$) is now an immediate consequence of the above. On the other hand, if $e_\alpha(e^{i\theta}) = (1 - |\alpha|^2)^{\frac{1}{2}}$

$(1-\bar{\alpha}e^{i\theta})$ and $u(e^{i\theta})=(e^{i\theta}-\alpha)(1-\bar{\alpha}e^{i\theta})^{-1}$, then a routine calculation shows that $\{u^n e_\alpha: n=0, \pm 1, \pm 2, ...\}$ is a complete orthonormal set in $L^2(\mathbf{T})$. Therefore,

$$L: \sum_{-\infty}^{+\infty} c_n V^n \xi \mapsto \sum_{-\infty}^{+\infty} c_n u^n e_\alpha$$

defines a unitary map of $[V^n\xi: n=0, \pm 1, ...]$ on $L^2(\mathbf{T})$. If S denotes the bilateral shift on $L^2(\mathbf{T})$, then the definition of L implies $L(U-\alpha I)(I-\bar{\alpha}U)^{-1}=(S-\alpha I)(I-\bar{\alpha}S)^{-1}L$, and another calculation (i.e., solving the equation $w=(z-\alpha)(1-\bar{\alpha}z)^{-1}$ for z) shows that $LU=SL$.

We claim:

(i) $[V^n\xi: n\geqslant 0]=[U^n\mathfrak{H}: n\geqslant 0]$,
(ii) $L[V^n\xi: n\geqslant 0]=H^2$, and
(iii) $L([V^n\xi: n\geqslant 0]\ominus\mathfrak{H})=\psi H^2$.

Note that (i)–(iii), together with $LU=SL$, imply the conclusion of the lemma. To see that it does, note that $\mathfrak{M}=[V^n\xi: n\geqslant 0]\ominus\mathfrak{H}=[U^n\mathfrak{H}: n\geqslant 0]\ominus\mathfrak{H}$ is a U-invariant subspace of $\mathfrak{K}$ (since $\mathfrak{H}$ is semi-invariant for U) and $[U^n\mathfrak{H}: n\geqslant 0]\ominus\mathfrak{M}=\mathfrak{H}$, and thus L maps $\mathfrak{H}$ onto $H^2\ominus\psi H^2$; and from this and the equation $LU=SL$, it follows in a routine manner that the restriction L_0 of L to $\mathfrak{H}$ is a unitary map of $\mathfrak{H}$ on $H^2\ominus\psi H^2$ which intertwines the projection of U on $\mathfrak{H}$ (i.e., T) and the projection of S on $H^2\ominus\psi H^2$ (i.e. S_ψ). That is what the lemma requires.

For (i), note that $V=(U-\alpha I)(I-\bar{\alpha}U)^{-1}$ implies $U=(V+\alpha I)(I+\bar{\alpha}V)^{-1}$, so that $[V^n\xi: n\geqslant 0]=[U^n\xi: n\geqslant 0]$. Clearly this is contained in $[U^n\mathfrak{H}: n\geqslant 0]$ because $\xi\in\mathfrak{H}$; on the other hand, since vectors of the form $\lambda(T)\xi=\lambda(U)\xi$ span $\mathfrak{H}$ (for λ a divisor of ψ_0, by the first paragraph of the proof), we have $\mathfrak{H}\subseteq[U^n\xi: n\geqslant 0]$, proving (i).

For (ii), we have by definition of L that $L[\xi, V\xi, V^2\xi, ...]=[e_\alpha, ue_\alpha, u^2e_\alpha, ...]$. Now S and the operator "multiplication by u" are related in the same way as U and V; hence by the preceding paragraph we have $[u^n e_\alpha: n\geqslant 0]=[S^n e_\alpha: n\geqslant 0]$. Since e_α is an outer function in H^2, it is a cyclic vector for S_+ and thus $[S^n e_\alpha: n\geqslant 0]=[S_+^n e_\alpha: n\geqslant 0]=H^2$.

Next, we claim $L\mathfrak{H}=H^2\ominus\psi H^2$; (iii) follows from this, (i) and (ii) by taking orthogonal complements in $[V^n\xi: n\geqslant 0]$ and H^2, respectively. Let $\mathfrak{D}$ be the set of all Blaschke divisors of ψ_0. Now if $\lambda\in\mathfrak{D}$ then by the first paragraph of the proof we have $\lambda(T)\xi=\lambda(U)\xi$, so that $L\lambda(T)\xi=L\lambda(U)\xi=\lambda(S)L\xi=\lambda(S)e_\alpha=\lambda\cdot e_\alpha$; 3.2.4 shows that $[\lambda\cdot e_\alpha: \lambda\in\mathfrak{D}]=H^2\ominus\psi H^2$, so that L maps $[\lambda(T)\xi: \lambda\in\mathfrak{D}]$ onto $H^2\ominus\psi H^2$. On the other hand, T and S_ψ have the same minimum polynomial (by the definition of ψ), and so $p(S_\psi)\mapsto p(T)$ (p ranging over polynomials) is an algebra isomorphism of $P(S_\psi)$ on $P(T)$. Now by 3.2.5 $P(S_\psi)$ is spanned by $\{\lambda(S_\psi): \lambda\in\mathfrak{D}\}$. Since $[P(T)\xi]=\mathfrak{H}$ by hypothesis, we conclude that

$$L\mathfrak{H} = L[\lambda(T)\xi \colon \lambda \in \mathcal{D}] = [\lambda \cdot e_\alpha \colon \lambda \in \mathcal{D}] = H^2 \ominus \psi H^2,$$

as required.

The proof is now complete.

The next theorem supplies a key step in the proof of 3.2.11, and seems to be of some interest in itself.

THEOREM 3.2.7. *Let T be an algebraic contraction on a Hilbert space, such that the spectral radius of T is less than 1, and let ψ be the finite Blaschke product associated with the minimum polynomial of T. Assume $\|\psi_0(T)\| = 1$ for some large Blaschke divisor ψ_0 of ψ. Then the map $p(T) \mapsto p(S_\psi)$ (p ranging over all polynomials) extends to a completely contractive homomorphism of $P(T)$ on $P(S_\psi)$.*

Proof. We will construct a representation π of $C^*(T)$ on a Hilbert space $\mathfrak{H}$, and a unit vector $\xi \in \mathfrak{H}$ for which $\|\psi_0(\pi(T))\xi\| = 1$. Letting $\mathfrak{H}_0 = [\xi, \pi(T)\xi, \pi(T)^2\xi, \ldots]$, then clearly the map $X \in P(T) \mapsto \pi(X)|_{\mathfrak{H}_0}$ is a completely contractive homomorphism of $P(T)$, and by 3.2.6 this map is unitarily equivalent to the given homomorphism $p(T) \mapsto p(S_\psi)$. Thus the theorem will follow.

π is obtained as follows. $\psi_0(T)^*\psi_0(T)$ is a positive operator of norm 1 in $C^*(T)$, so there is a state ϱ of $C^*(T)$ such that $\varrho(\psi_0(T)^*\psi_0(T)) = 1$. Simply let π be the canonical representation of $C^*(T)$ associated with ϱ and let ξ be the unit vector for which $(\pi(X)\xi, \xi) = \varrho(X)$, $X \in C^*(T)$. Clearly $\psi_0(\pi(T)) = \pi(\psi_0(T))$, and we have $\|\psi_0(\pi(T))\xi\|^2 = \varrho(\psi_0(T)^*\psi_0(T)) = 1$. That completes the proof.

The following result will not be used in this section, but is of some interest for the questions taken up in section 3.6.

COROLLARY 3.2.8. *Let ψ be a nonconstant finite Blaschke product. Then every isometric representation of $P(S_\psi)$ is completely isometric. (Representations are defined in 2.3.)*

Proof. Let φ be an isometric representation of $P(S_\psi)$. By 3.6.8, φ is completely contractive. Let ψ_0 be any large divisor of ψ. 3.2.2. and the subsequent remark show that $\|\psi_0(S_\psi)\| = 1$. Letting $T = \varphi(S_\psi)$, it follows that $\|\psi_0(T)\| = 1$, because φ is isometric, and we conclude from 3.2.7 that φ^{-1} is completely contractive. The proof is complete.

Our next step is to show that for operators T as in 3.2.7, the map $p(T) \mapsto p(S_\psi)$ is implemented by a representation of $C^*(T)$ (3.2.10). At this point, because we have the preceding corollary, it would be possible to prove 3.2.10 using the general results of sections 3.3 through 3.6. In this special case, however, it is possible to give a more direct proof which, we feel, may be of some interest in its own right.

Let $\mathfrak{H}$ be a Hilbert space. Recall that a *conjugation* of $\mathfrak{H}$ is a conjugate-linear isometry

γ of $\mathfrak{H}$ onto itself such that $\gamma^2 = I$ (i.e., $\gamma^{-1} = \gamma$). Let $\mathcal{A}$ be a subalgebra of $L(\mathfrak{H})$. The existence of a conjugation γ of $\mathfrak{H}$ for which $\gamma T \gamma = T^*$, for all $T \in \mathcal{A}$, is an indication of symmetry between $\mathcal{A}$ and $\mathcal{A}^*$; note, for example, that this implies that if $\mathcal{A}$ has a cyclic vector then so does $\mathcal{A}^*$. Note also that the condition $\gamma T \gamma = T^*$, $T \in \mathcal{A}$, implies $\mathcal{A}$ is abelian (for S, $T \in \mathcal{A}$ we have $T^* S^* = (ST)^* = \gamma ST \gamma = \gamma S \gamma \gamma T \gamma = S^* T^*$). If one requires such a symmetry condition for noncommutative algebras, it is necessary to incorporate an anti-automorphism $T \mapsto T^t$ of $\mathcal{A}$ as follows: $\gamma T^t \gamma = T^*$.

Let φ be a representation of a subalgebra A of a C^*-algebra B, on a Hilbert space $\mathfrak{H}$. A unit vector $\xi \in \mathfrak{H}$ is called a *special vector* for φ if $\{a \in A: \|\varphi(a)\xi\| = \|a\|\}$ has all of A as its closed linear span, and in addition ξ is cyclic for $\varphi(A)$. The next result is more general than we shall actually require.

THEOREM 3.2.9. *Let A be a commutative closed subalgebra of a (perhaps non-commutative) C^*-algebra B, such that $e \in A$ and $B = C^*(A)$. Let φ be a completely contractive representation of A on a Hilbert space $\mathfrak{H}$, satisfying:*

(i) *φ has a special vector, and*

(ii) *there exists a conjugation γ of $\mathfrak{H}$ such that $\gamma\varphi(a)\gamma = \varphi(a)^*$, $a \in A$.*

Then φ is implemented by a representation π of B. Moreover, π is the only completely positive linear extension of φ to B.

Proof. Note that by 1.2.8, there is a (unique) completely positive extension of φ to the closure of $A + A^*$, which we denote by the same symbol φ. By 1.2.3, there is at least one completely positive extension of φ to B. Note, then, that the theorem will follow if we prove that *every* completely positive extension of φ is a representation; for two representations of B which agree on A must agree on $B = C^*(A)$.

Choose any completely positive extension $\varphi_1: B \to L(\mathfrak{H})$ of φ. By Stinespring's theorem (1.1.1), there is a representation ω of B on a Hilbert space $\mathfrak{K}$ and an isometry $V \in L(\mathfrak{H}, \mathfrak{K})$ such that $V^*\omega(x)V = \varphi_1(x)$, and $[\omega(B)V\mathfrak{H}] = \mathfrak{K}$.

First, we claim $\omega(A)V\mathfrak{H} \subseteq V\mathfrak{H}$. Let ξ be a special vector for φ and put $S = \{a \in A: \|\varphi(a)\xi\| = \|a\|\}$. If $a \in S$, then

$$\begin{aligned}\|\omega(a)V\xi - V\varphi(a)\xi\|^2 &= \|\omega(a)V\xi\|^2 - 2\mathrm{Re}\,(V^*\omega(a)V\xi, \varphi(a)\xi) + \|V\varphi(a)\xi\|^2 \\ &= \|\omega(a)V\xi\|^2 - \|\varphi(a)\xi\|^2 = \|\omega(a)V\xi\|^2 - \|a\|^2 \leqslant 0.\end{aligned}$$

Thus, $\omega(a)V\xi = V\varphi(a)\xi$ holds for all $a \in S$. Since A is the closed linear span of S, this identity in fact holds for all $a \in A$. It follows that $\omega(a)V\zeta = V\varphi(a)\zeta$, $a \in A$, $\zeta \in \mathfrak{H}$; indeed, if $a, b \in A$ then $\omega(a)V\varphi(b)\xi = \omega(a)\omega(b)V\xi = \omega(ab)V\xi = V\varphi(ab)\xi = V\varphi(a)\varphi(b)\xi$, and the assertion follows from the fact that $[\varphi(A)\xi] = \mathfrak{H}$. The desired property, $\omega(A)V\mathfrak{H} \subseteq V\mathfrak{H}$, is now immediate.

Next, we claim that $\omega(A^*)V\mathfrak{H}\subseteq V\mathfrak{H}$. Indeed, $\varphi|_{A^*}$ is a (completely contractive) representation of A^*, and if we can show that $\varphi|_{A^*}$ has a special vector, then the assertion follows from the same argument as in the preceding paragraph. Let γ be the conjugation described in (ii). We claim: $\gamma\xi$ is a special vector for $\varphi|_{A^*}$. Indeed, $[\varphi(A^*)\gamma\xi]=[\varphi(A)^*\gamma\xi]=[\gamma\varphi(A)\gamma^2\xi]=\gamma\mathfrak{H}=\mathfrak{H}$, because ξ is cyclic for $\varphi(A)$. Moreover, if $b\in S$, then $\|\varphi(b^*)\gamma\xi\|=\|\gamma\varphi(b)\xi\|=\|\varphi(b)\xi\|=\|b\|=\|b^*\|$, so that $\{c\in A^*\colon \|\varphi(c)\gamma\xi\|=\|c\|\}$ contains S^*; since S^* spans A^* (because S spans A), we see that $\gamma\xi$ is a special vector for $\varphi|_{A^*}$. As we pointed out already, this implies $\omega(A^*)V\mathfrak{H}\subseteq V\mathfrak{H}$.

Thus, $V\mathfrak{H}$ is invariant under $\omega(A)\cup\omega(A)^*$, and hence $\omega(B)V\mathfrak{H}\subseteq V\mathfrak{H}$. Since $[\omega(B)V\mathfrak{H}]=\mathfrak{K}$, it follows that V is unitary, and hence $\varphi_1=V^{-1}\pi V$ is a representation. That completes the proof.

We remark that if $\varphi(A)$ is an irreducible family of operators, then π is an irreducible representation of B, and hence 3.2.9 implies π is a boundary representation for A.

The decisive step in the proof of 3.2.11 can now be taken.

COROLLARY 3.2.10. *Let T and ψ be as in 3.2.7. Then there is a representation π of $C^*(T)$ such that $\pi(T)=S_\psi$.*

Proof. Define $\varphi\colon P(T)\to P(S_\psi)$ by $\varphi(p(T))=p(S_\psi)$, where p is an arbitrary polynomial. By 3.2.7, φ is a completely contractive homomorphism, and we want to show that φ is implemented by a representation of $C^*(T)$. By 3.2.9, it suffices to show that φ has a special vector, and that there is a conjugation γ of $H^2\ominus\psi H^2$ such that $\gamma S_\psi\gamma=S_\psi{}^*$.

For the special vector, let α be a zero of ψ inside the unit disc, and let e_α be as in 3.2.4. Let $\mathcal{D}$ be the set of all Blaschke divisors of $\psi_0(z)=\dfrac{1-\bar{\alpha}z}{z-\alpha}\,\psi(z)$. 3.2.4 shows that $e_\alpha\in H^2\ominus\psi H^2$ and $\lambda\cdot e_\alpha\in H^2\ominus\psi H^2$ for every $\lambda\in\mathcal{D}$. Thus, if P denotes the projection of H^2 on $H^2\ominus\psi H^2$, we have

$$\|\lambda(S_\psi)e_\alpha\|=\|P\lambda(S_+)e_\alpha\|=\|P\lambda\cdot e_\alpha\|=\|\lambda\cdot e_\alpha\|=1.$$

Since $\|\lambda(T)\|\leqslant 1$ (because the unit disc is a spectral set for T), we see that $\{X\in P(T)\colon \|\varphi(X)e_\alpha\|=\|X\|\}$ contains $\{\lambda(T)\colon \lambda\in\mathcal{D}\}$, and the latter spans $P(T)$ by 3.2.5 and the fact that φ is a vector space isomorphism. e_α is cyclic for $P(S_\psi)=\varphi(P(T))$ by the first sentence of 3.2.4. Thus, e_α is a special vector for φ.

We now define the conjugation γ. First, define $\gamma_1\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$ ($\mathbf{T}$ denoting the unit circle) by $\gamma_1 f(e^{i\theta})=e^{-i\theta}\psi(e^{i\theta})\bar{f}(e^{i\theta})$, $f\in L^2(T)$. Clearly γ_1 is a conjugate-linear isometry for which $\gamma_1^2=I$. Moreover, if S denotes the bilateral shift and $f\in L^2(\mathbf{T})$, then $\gamma_1 Sf(e^{i\theta})=e^{-2i\theta}\psi(e^{i\theta})\bar{f}(e^{i\theta})=(S^*\gamma_1 f)(e^{i\theta})$. Thus, $\gamma_1 S\gamma_1=S^*$. A routine calculation now shows that $\{\bar{f}\colon f\in H^2\ominus\psi H^2\}=z\bar{\psi}(H^2\ominus\psi H^2)$ (where $z\in L^2(T)$ is the function $z(e^{i\theta})=e^{i\theta}$), which is equivalent to the assertion $\gamma_1(H^2\ominus\psi H^2)=H^2\ominus\psi H^2$. Thus, $\gamma=\gamma_1|_{H^2\ominus\psi H^2}$ is a conjugation of $H^2\ominus\psi H^2$.

We claim that $\gamma_1 P=P\gamma_1$, where P is the projection of $L^2(\mathbf{T})$ on $H^2\ominus\psi H^2$. Note first that the usual polarization argument shows that $(\gamma_1 f, \gamma_1 g)=(g, f)$ for all $f, g\in L^2(\mathbf{T})$, and since $\gamma_1^2=I$ it follows that $(\gamma_1 f, g)=(\gamma_1 g, f)$. Thus, noting that $P\gamma_1 P=\gamma_1 P$ (by the preceding paragraph), we can write

$$(\gamma_1 Pf, g) = (P\gamma_1 Pf, g) = (\gamma_1 Pf, Pg) = (\gamma_1 Pg, Pf) = (P\gamma_1 Pg, f)$$
$$= (\gamma_1 Pg, f) = (\gamma_1 f, Pg) = (P\gamma_1 f, g),$$

for all $f, g\in L^2(\mathbf{T})$, proving that $P\gamma_1=\gamma_1 P$.

$\gamma S_\psi\gamma=S_\psi{}^*$ now follows, for if $\zeta\in H^2\ominus\psi H^2$ then we have $\gamma S_\psi\gamma\zeta=\gamma_1 PS\gamma_1\zeta=P\gamma_1 S\gamma_1\zeta=PS^*\zeta=S_\psi{}^*\zeta$, as required. The proof is complete.

We can now state the principal result of this section.

THEOREM 3.2.11. *Let T be a simple algebraic operator of norm* 1, *and let ψ be the finite Blaschke product associated with the minimum polynomial of T. If T is maximal, then it is unitarily equivalent to an operator of the form $I\otimes S_\psi$, where I is the identity operator on some Hilbert space. $C^*(T)$ is $*$-isomorphic with M_n, n being the degree of the minimum polynomial of T.*

Proof. By 3.2.10, there is a representation π of $C^*(T)$ such that $\pi(T)=S_\psi$. $\ker\pi$ is an ideal in $C^*(T)$ which is not all of $C^*(T)$; therefore $\ker\pi=0$ by simplicity. It follows that $\sigma=\pi^{-1}$ is a representation of $C^*(S_\psi)$, and of course $\sigma(S_\psi)=T$. Now S_ψ is an irreducible operator on $H^2\ominus\psi H^2$ and the latter has dimension n (cf. the proof of 3.2.4); therefore $C^*(S_\psi)=L(H^2\ominus\psi H^2)$, which is ($*$-isomorphic with) M_n. Now a familiar variation of a classical theorem of Burnside asserts that every representation of the C^*-algebra $L(\mathfrak{H})$ (for $\mathfrak{H}$ finite dimensional) is equivalent to a multiple of the identity representation. Thus, σ is equivalent to a representation $X\in C^*(S_\psi)\to I\otimes X$, where I is the identity operator on some Hilbert space. In particular, $T=\sigma(S_\psi)$ is equivalent to $I\otimes S_\psi$.

We have already observed that $C^*(S_\psi)$ is isomorphic with M_n, and so the last sentence of the theorem follows because σ is a faithful representation. That completes the proof.

We remark that a converse of this theorem is obvious, namely, $I\otimes S_\psi$ is a maximal simple algebraic operator of norm 1 (3.2.2). Moreover, $I\otimes S_\psi$ determines uniquely the dimension of I (if $\dim I=m$, then the commutant of $C^*(I\otimes S_\psi)=I\otimes C^*(S_\psi)$ is $L(\mathfrak{K})\otimes I$, $\mathfrak{K}$ being the space on which I acts, which is a factor of type I_m). This gives a complete classification, to unitary equivalence, of all maximal simple algebraic operators of norm 1 which have the same minimum polynomial as T.

In particular, we have:

COROLLARY 3.2.12. *Two irreducible maximal simple algebraic operators of norm* 1 *are unitarily equivalent if, and only if, they have the same minimum polynomial.*

Noting that irreducible operators on finite dimensional spaces are always simple and algebraic, we have the following application to matrices which, so far as we can tell, is also new.

COROLLARY 3.2.13. *Let S and T be irreducible operators of norm* 1, *acting on finite-dimensional spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Suppose S and T are maximal, and have the same minimum polynomial. Then* $\dim \mathfrak{H} = \dim \mathfrak{K}$, *and S and T are unitarily equivalent.*

3.3. *Almost simple operators and the commutator ideal in $C^*(T)$.* In this section we define a class of operators and C^*-algebras and collect some general results for use later on. This material provides a general setting for the problems taken up in the remainder of chapter 3.

Because we shall be considering ideals in C^*-algebras as separate entities and because ideals rarely contain an identity, we shall deviate from our usual assumption about the presence of an identity; in this section (and in this section only), C^*-algebras may or may not contain an identity. Our terminology for representations, etc., follows [4].

Let B be a C^*-algebra, which need not contain an identity. The term *ideal* always means closed two-sided ideal; thus, ideals are necessarily self-adjoint. The *commutator ideal* in B is defined as the ideal generated by all elements of the form $xy-yx$, $x, y \in B$. We write this ideal as Comm (B); if $B = C^*(T)$ for some operator T on a Hilbert space, then the commutator ideal is written simply as Comm (T).

It is clear that the quotient $B/\mathrm{Comm}\,(B)$ is commutative. Conversely, if K is any ideal in B such that B/K is commutative, then K must contain Comm (B); hence, Comm (B) is the smallest ideal in B having a commutative quotient. Comm (B) can be 0 (for B commutative) or it can be all of B (for example, when B is simple). There is also a close relation between Comm (B) and characters (i.e., nontrivial complex homomorphisms) of B:

PROPOSITION 3.3.1. Comm (B) *is the intersection of the kernels of all characters of B (if B has no characters the intersection is taken as B itself).*

Proof. Suppose first that B has no characters. We claim Comm $(B) = B$. For if Comm $(B) \neq B$, then $B/\mathrm{Comm}\,(B)$ is a (nonzero) commutative C^*-algebra, which therefore has at least one character ω; and thus the composition of ω with the quotient map of B onto $B/\mathrm{Comm}\,(B)$ gives a character of B. Contradiction.

Assume, then, that B has characters. Every character must vanish on Comm (B), so that one inclusion is obvious. Conversely, suppose $x \in B$ is such that $\chi(x) = 0$ for every character χ. Then we claim $x \in \text{Comm}(B)$. If $\text{Comm}(B)=B$, there is nothing to prove, so assume $\text{Comm}(B) \neq B$. Consider the coset representative $\dot{x}$ for x in $B/\text{Comm}(B)$. Since the quotient is a (nonzero) commutative C^*-algebra, there is a character ω of $B/\text{Comm}(B)$ such that $\|\dot{x}\| = |\omega(\dot{x})|$. But $\chi(z)=\omega(\dot{z})$ is a character of B, so by hypothesis, $\omega(\dot{x})=\chi(x)=0$. Thus, $\dot{x}=0$ and so $x \in \text{Comm}(B)$, as asserted.

By analogy with the commutator subgroup of a group, one might at first expect to obtain an entire sequence of ideals by taking Comm (B), then the commutator of this sub C^*-algebra, and so on. In fact, the process stops after the first step.

COROLLARY 3.3.2. Comm (Comm (B)) = Comm (B).

Proof. By 3.3.1, it suffices to show that Comm (B) has no characters. Suppose it does, and let ω be one. Then ω extends uniquely to a character ω_1 of B ([4], p. 52). But then we have $\text{Comm}(B) \subseteq \ker \omega_1$, and hence the restriction of ω_1 to Comm (B) is 0, a contradiction.

We shall require another simple algebraic fact.

PROPOSITION 3.3.3. *Let B be a C^*-algebra with identity, and let S be a self-adjoint subset of B such that B is generated, as a C^*-algebra, by S and the identity. Then* Comm (B) *is the (closed) ideal generated by* $\{xy-yx\colon x, y \in S\}$.

Proof. For $x, y \in B$, write $[x, y]$ for the commutator $xy-yx$. Let K be the ideal generated by $\{[x, y]\colon x, y \in S\}$. Evidently $K \subseteq \text{Comm}(B)$, and we need only prove the reverse inclusion.

Define $B_0=\{z \in B\colon [z, S] \subseteq K\}$. From the identities $[x^*, y]=-[y^*, x]^*$ and $[xy, z]=x[y, z]+[x, z]y$, and the fact that $S=S^*$, it follows that B_0 is a self-adjoint subalgebra of B. Clearly B_0 is norm-closed, and contains S as well as the identity e of B ($[e, z]=0$ for all $z \in B$). Thus, $B_0=B$, and we have $[B, S] \subseteq K$. Taking the adjoint of this condition we see that $-[S^*, B^*] \subseteq K^*$, or $[S, B] \subseteq K$. The argument can now be repeated with B in place of S to obtain $[B, B] \subseteq K$. Since Comm (B) is the smallest ideal containing $[B, B]$, it follows that $\text{Comm}(B) \subseteq K$, completing the proof.

COROLLARY 3.3.4. *Let T be an operator on a Hilbert space. Then* Comm (T) *is the ideal (in $C^*(T)$) generated by T^*T-TT^*.*

Proof. Simply take $S=\{T, T^*\}$ and note that $[S, S]$ generates the same ideal as $\{T^*T-TT^*\}$.

The "building blocks" for C^*-algebras are the primitive ones (i.e., C^*-algebras having a faithful irreducible representation); the reason is that every C^*-algebra is semi-simple, and is therefore a field of primitive C^*-algebras over a compact (usually non-Hausdorff) space [4]. For the moment, let us call an operator T *primitive* if $C^*(T)$ is a primitive C^*-algebra. Thus, every irreducible operator is primitive, and it seems too much to expect a general classification of these operators in the near future, even with respect to algebraic equivalence.

There is a subclass of primitive operators which we feel is more tractable, and the bulk of this chapter is devoted to a study of some of these operators. Recall that a C^*-algebra B is *simple* if it has no nontrivial ideals. A simple C^*-algebra is automatically primitive (indeed, *every* irreducible representation if faithful, because its kernel is an ideal $\neq B$, which must then be 0 by simplicity). If the intersection of all maximal ideals in B is 0 (i.e., B is strongly semi-simple), then B is a field of simple C^*-algebras in a way analogous to the structure theory alluded to above [17], but not all C^*-algebras have this property ($L(\mathfrak{H})$ does not). The algebras that are of interest here can be described as extensions of simple C^*-algebras by commutative C^*-algebras, where the former need not contain an identity. More precisely:

Definition 3.3.5. A C^*-algebra B is called *almost simple* if Comm (B) is contained in every nonzero ideal of B. An operator T is *almost simple* if $C^*(T)$ has that property.

In this definition, the commutator ideal is allowed to be 0. Thus, a normal operator is almost simple; at the other extreme, a simple operator is almost simple. Here, we shall be concerned almost exclusively with nonnormal almost simple operators; a class of examples is described in 3.3.7.

Note that if B is an almost simple C^*-algebra then Comm (B) is a simple C^*-algebra (the converse is false: consider the direct sum of a simple C^*-algebra and a commutative C^*-algebra). In fact, it is easy to see that the following two conditions are equivalent to almost simplicity, for a *noncommutative* C^*-algebra B:

(i) Comm (B) *is a simple* C^**-algebra.*

(ii) *for every ideal* K *in* B, $K \cap \text{Comm}(B) = 0$ *implies* $K = 0$.

For many examples, however, (ii) is rather more difficult to verify than (i). The following result provides a more tractable replacement for (ii).

PROPOSITION 3.3.6. *Let B be a noncommutative C^*-algebra such that* Comm (B) *is simple. Then B is almost simple if, and only if, it has a faithful irreducible representation.*

Proof. Suppose first that B is almost simple. Then Comm (B) is a C^*-algebra (nonzero, because B is not commutative), and so there is an irreducible (nonzero) representation π of Comm (B) on a Hilbert space $\mathfrak{H}$ ([4], p. 41). π extends uniquely to a representation

π_1 of B ([4], p. 52) on $\mathfrak{H}$. Now $\ker \pi_1$ is an ideal in B, and we want to conclude that $\ker\pi_1 = 0$. But if $\ker \pi_1 \neq 0$, then Comm $(B) \subseteq \ker \pi_1$, since B is almost simple, and this contradicts the original choice of π. Hence, π_1 is a faithful irreducible representation of B.

Conversely, let π be a faithful irreducible representation of B on $\mathfrak{H}$. Let K be a nonzero ideal in B; we must show $K \cap \text{Comm}\,(B) = \text{Comm}\,(B)$. Let L be the ideal generated by $\{xy\colon x \in K,\ y \in \text{Comm}\ B\}$. Then L is contained in both K and Comm (B), and it suffices to prove that $L \neq 0$ (for then $L = \text{Comm}\,(B)$, by simplicity of Comm (B)).

Since B is noncommutative and π faithful, we have $\pi(\text{Comm}(B)) \neq 0$. Thus, $[\pi(\text{Comm}\,(B))\mathfrak{H}]$ is a nonzero $\pi(B)$-invariant subspace, so that $[\pi(\text{Comm}\,(B))\mathfrak{H}] = \mathfrak{H}$, by irreducibility of π. Hence, $[\pi(L)\mathfrak{H}]$ contains $\pi(K)\mathfrak{H}$, and the latter is not 0 because $K \neq 0$ and π is faithful. Thus, $\pi(L)$, and therefore L, is not 0. That completes the proof.

In particular, note that *a noncommutative almost simple C^*-algebra is primitive.* We can now describe a variety of almost simple operators.

COROLLARY 3.3.7. *Let T be an irreducible operator on a (necessarily separable) Hilbert space $\mathfrak{H}$, of dimension greater than* 1, *such that $T^*T - TT^*$ is compact. Then T is almost simple. Moreover,* Comm (T) *is the algebra of all compact operators on $\mathfrak{H}$.*

Proof. Because T is irreducible and $\dim \mathfrak{H} \geqslant 2$, T cannot be normal. Moreover, the identity representation of $C^*(T)$ is irreducible.

Let $LC\,(\mathfrak{H})$ be the C^*-algebra of all compact operators on $\mathfrak{H}$. Then $LC\,(\mathfrak{H})$ is an ideal in $L\,(\mathfrak{H})$, and so 3.3.4 shows that Comm (T) is contained in $LC\,(\mathfrak{H})$. On the other hand, Comm (T) is an ideal in the irreducible C^*-algebra $C^*(T)$ (nonzero because T is nonnormal), and so ([4], p. 53) Comm (T) is itself irreducible. But $LC\,(\mathfrak{H})$ contains no proper irreducible C^*-subalgebras ([4], p. 88), hence, Comm $(T) = LC\,(\mathfrak{H})$.

Since $LC\,(\mathfrak{H})$ is a simple C^*-algebra, we now conclude from 3.3.6 that $C^*(T)$ is almost simple.

Thus, every operator on a finite dimensional Hilbert space is a finite direct sum of irreducible almost simple (in fact, simple) operators. For a general operator $T = T_1 + iT_2$, $T_i = T_i^*$, we have $T^*T - TT^* = 2i(T_1T_2 - T_2T_1)$; thus an irreducible operator having compact real or imaginary part is almost simple. The same is true of irreducible operators which are "almost unitary" in the sense that both $I - T^*T$ and $I - TT^*$ are compact: we consider some of the latter in the following sections.

We conclude this section with a final note on Comm (T). One might wonder about the structure of Comm (T) for general almost simple operators. 3.3.7 shows that Comm (T) can be $LC\,(\mathfrak{H})$ for a number of examples. The following result shows that, in fact, Comm (T)

is LC ($\mathfrak{H}$) whenever $C^*(T)$ is of type I (for our purposes, a separable C^*-algebra B is type I if whenever two irreducible representations of B have the same kernel then they are equivalent; this definition is equivalent to a number of other natural properties [4]). We shall make use of a theorem of A. Rosenberg [29], to the effect that if a separable C^*-algebra B has, to within equivalence, only one irreducible representation, then B is $*$-isomorphic with the algebra of all compact operators on a separable Hilbert space.

PROPOSITION 3.3.8. *Let T be a nonnormal almost simple operator on a Hilbert space, such that $C^*(T)$ is of type I. Then* Comm (T) *is $*$-isomorphic with the algebra of all compact operators on a separable Hilbert space.*

Proof. Comm (T) is nonzero, and it is separable because it is contained in the separable C^*-algebra $C^*(T)$. As it has already been pointed out, Comm (T) is simple; hence any two irreducible representations of Comm (T) have the same kernel, namely 0. Also, Comm (T) is type I, because it is a nontrivial ideal in a type I C^*-algebra (this follows, for example, from the fact that a separable type I C^*-algebra is postliminal [4], p. 168, p. 88). Thus, all irreducible representations of Comm (T) are equivalent. The Rosenberg theorem cited above now yields desired conclusion.

Note that the essential ingredient in the preceding is a proof of the (probably known) fact that a separable, simple, type I C^*-algebra is LC ($\mathfrak{H}$), where $\mathfrak{H}$ is separable.

Although most of the examples of almost simple operators in the following sections do generate type I C^*-algebras, we shall not make that an *a priori* assumption in stating results.

Finally, we point out that the representations of certain almost simple C^*-algebras are determined in the following section.

3.4. *The structure of* $C^*(S_\psi)$. In sections 3.5–3.7 we shall require certain information about the C^*-algebra generated by the projection of the bilateral shift onto one of its semi-invariant subspaces. We need to know that such a C^*-algebra is almost simple and type I, the purpose of which is to give a description of its representation theory. The present section is devoted to this discussion.

It is relevant, perhaps, to point out that a description is given in [3] of the C^*-algebra generated by an isometry. While the problems of this section (as well as our methods) are different from those of [3], it is of interest to note certain similarities in some of the results; e.g., compare 3.4.2 with [3].

Our terminology here will follow [11]. Let σ denote normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let $z(e^{i\theta}) = e^{i\theta}$, and let L_z be the operator multiplication by z in $L^2(\mathbf{T}, d\sigma)$.

Let H^2 denote all functions in L^2 whose negative Fourier coefficients vanish, and let ψ be an inner function (i.e., $\psi \in H^2$ and $|\psi| = 1$ almost everywhere).

Let $\mathfrak{H} = H^2 \ominus \psi H^2$ and let S_ψ be the projection of L_z on $\mathfrak{H}$. Since ψ takes on (scalar) values of modulus 1 almost everywhere, it follows from 1.3.5 that S_ψ is irreducible. We shall show that $I - S_\psi S_\psi^*$ and $I - S_\psi^* S_\psi$ are compact; it will follow then from 3.3.7 and the subsequent remarks that S_ψ is almost simple and Comm (S_ψ) is the algebra $LC\,(\mathfrak{H})$ of all compact operators on $\mathfrak{H}$ (to avoid trivial exceptions, we assume ψ is such that $H^2 \ominus \psi H^2$ has dimension greater than 1). We begin with a routine formula.

LEMMA 3.4.1. *Let $T \in L(\mathfrak{H})$, let $T = UH$ be the polar decomposition of T (where $H = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with initial and final spaces respectively $[T^*\mathfrak{H}]$ and $[T\mathfrak{H}]$), and let P be the projection on the nullspace of T. Then*

$$I - T^*T = P + U^*(I - TT^*)\,U.$$

Proof. Simply note that $I - P$ is the projection on $[T^*\mathfrak{H}]$. Thus $I - P = U^*U$, and we have

$$I - P - U^*(I - TT^*)\,U = U^*TT^*U = U^*UH^2U^*U = U^*UT^*TU^*U = T^*T,$$

from which the formula is evident.

Note that the ranges of the operators P and $U^*(I - TT^*)\,U$ are orthogonal; hence $I - T^*T$ is compact (resp. has finite rank) if, and only if, $I - TT^*$ is compact (resp. has finite rank) and the nullspace of T is finite dimensional.

THEOREM 3.4.2. *$I - S_\psi S_\psi^*$ and $I - S_\psi^* S_\psi$ are both of finite rank; S_ψ is almost simple, and* Comm $(S_\psi) = LC\,(\mathfrak{H})$.

Proof. We only need to point out that $I - S_\psi S_\psi^*$ has finite rank and S_ψ has finite dimensional nullspace; the remaining assertions follow from the preceding remarks and 3.3.7.

Note first that $I - S_\psi S_\psi^*$ has rank 1. Indeed, the rank is at least 1 because S_ψ^* is not an isometry (for example, the powers of S_ψ^* tend strongly to 0). Let P be the projection of L^2 on H^2, and let $S = P_{H^2} L_z|_{H^2}$. Then $I_{H^2} - SS^*$ is the projection on the one-dimensional space of constant functions, $\mathfrak{H}$ is invariant under S^*, and of course $S_\psi = P_\mathfrak{H} S|_\mathfrak{H}$. Thus, using $P_\mathfrak{H} S P_\mathfrak{H} = P_\mathfrak{H} S$ we can write

$$S_\psi S_\psi^* = P_\mathfrak{H} S P_\mathfrak{H} S^*|_\mathfrak{H} = P_\mathfrak{H} SS^*|_\mathfrak{H} = P_\mathfrak{H}(I_{H^2} - (I_{H^2} - SS^*))|_\mathfrak{H} = I_\mathfrak{H} - P_\mathfrak{H}(I_{H^2} - SS^*)|_\mathfrak{H}$$

and thus $I_\mathfrak{H} - S_\psi S_\psi^* = P_\mathfrak{H}(I_{H^2} - SS^*)|_\mathfrak{H}$ has rank at most 1.

Now let f belong to the nullspace of S_ψ. Then in particular $f \in H^2$; and $P_\mathfrak{H} zf = 0$, or

$zf \in \psi H^2$, and hence $z\bar{\psi}f \in H^2$. On the other hand, $f \perp \psi H^2$ implies $\bar{\psi}f \in H^{2\perp} = \overline{zH^2}$, so that $z\bar{\psi}f \in H^2 \cap \overline{H^2}$. It follows that f is a constant multiple of $\bar{z}\psi$, and we conclude that the null-space of S_ψ is of dimension at most 1. That completes the proof.

We shall now find the characters of $C^*(S_\psi)$. First, extend ψ analytically to the interior of the unit disc $D = \{|z| < 1\}$, so that the radial limits of the extension agree almost everywhere on the unit circle $\mathbf{T} = \{|z| = 1\}$ with the original function. Let Z_ψ be the "zero set" of ψ; i.e., Z_ψ consists of the zeroes of ψ inside D, along with all points λ on $\mathbf{T}$ for which ψ cannot be continued analytically from D to λ. It is known that Z_ψ is the spectrum of S_ψ ([10], p. 74; note that it is the spectrum of S_ψ^*, rather than S_ψ, that is described there).

Let $T \mapsto \dot{T}$ be the canonical quotient map of $C^*(S_\psi)$ onto $C^*(S_\psi)/\mathrm{Comm}\,(S_\psi)$. Since the quotient is the abelian C^*-algebra generated by $\dot{S}_\psi$ and the identity, there is an obvious homeomorphism between the maximal ideal space of $C^*(S_\psi)/\mathrm{Comm}\,(S_\psi)$ and $\mathrm{sp}\,(\dot{S}_\psi)$. At the same time, there is a natural bijective correspondence between characters of $C^*(S_\psi)$ and characters of $C^*(S_\psi)/\mathrm{Comm}\,(S_\psi)$ (cf. 3.3.1); thus the characters of $C^*(S_\psi)$ correspond one-to-one with points in the spectrum of $\dot{S}_\psi$. Of course, we have yet to find the characters (or equivalently, $\mathrm{sp}\,(\dot{S}_\psi)$).

Note first that if ψ is a finite Blaschke product then $\mathfrak{H} = H^2 \ominus \psi H^2$ is clearly finite-dimensional; if ψ is not a finite Blaschke product then $\mathfrak{H}$ is infinite-dimensional. The latter is easily seen by making use of the structure theory for inner functions; for example, ψ has an infinite linearly ordered set of nonproportional divisors, and these correspond to an infinite chain of distinct subspaces of $\mathfrak{H}$. We omit the details.

THEOREM 3.4.3. (i) *If ψ is a finite Blaschke product, then $\mathfrak{H}$ is finite-dimensional, $C^*(S_\psi)$ has no characters, and in fact* $\mathrm{Comm}\,(S_\psi) = C^*(S_\psi) = L(\mathfrak{H})$.

(ii) *If ψ is not a finite Blaschke product, then for every point λ in $Z_\psi \cap \mathbf{T}$ there is a unique character χ_λ of $C^*(S_\psi)$ for which $\chi_\lambda(S_\psi) = \lambda$. $\lambda \leftrightarrow \chi_\lambda$ is a bijective correspondence between $Z_\psi \cap \mathbf{T}$ and characters of $C^*(S_\psi)$. $C^*(S_\psi)/\mathrm{Comm}\,(S_\psi)$ is canonically $*$-isomorphic with the continuous functions on $Z_\psi \cap \mathbf{T}$.*

Proof. (i) follows from the fact that S_ψ is an irreducible operator on a finite dimensional space; for then $C^*(S_\psi) = L(\mathfrak{H})$ and the latter is known to have no characters, and now 3.3.1 shows that $\mathrm{Comm}\,(S_\psi)$ must also equal $L(\mathfrak{H})$.

Suppose now that ψ is not a finite Blaschke product. It follows that Z_ψ must have at least one point in common with $\mathbf{T}$. Let λ be such a point. Then $|\lambda| = 1 = \|S_\psi\|$, and by the preceding remarks λ also belongs to the spectrum of S_ψ. 3.1.2 shows that there is a unique character χ_λ of $C^*(S_\psi)$ such that $\chi_\lambda(S_\psi) = \lambda$. Now let χ be any character of $C^*(S_\psi)$; we must show that $\lambda = \chi(S_\psi)$ belongs to $Z_\psi \cap \mathbf{T}$. Clearly $\lambda \in \mathrm{sp}\,(S_\psi) = Z_\psi$ (for if $(S_\psi - \lambda)^{-1} = T$ were to exist, then T would have to belong to $C^*(S_\psi)$ and hence $1 = \chi(T(S_\psi - \lambda)) = \chi(T)(\chi(S_\psi) - \lambda) = 0$, an absurdity), and it suffices to show that $|\lambda| = 1$. But χ vanishes on Comm $(S_\psi) = LC\,(\mathfrak{H})$, and so by 3.4.2 we have $0 = \chi(I - S_\psi^* S_\psi) = 1 - |\chi(S_\psi)|^2$, as required.

The last sentence follows from the remarks preceding the theorem.

We turn now to more general representations of $C^*(S_\psi)$. The following decomposition procedure will be useful. Let B be a C^*-algebra with identity, and let K be an ideal in B. Let μ and σ be representations of K and B/K, respectively, on $\mathfrak{H}$ and $\mathfrak{K}$ (note that K need not possess an identity; a *representation* of K is defined as a $*$-homomorphism of K into $L(\mathfrak{H})$ for which $[\pi(K)\mathfrak{H}] = \mathfrak{H}$). We can manufacture a representation of B out of μ and σ in the following way. Let $z \mapsto \dot{z}$ be the quotient map of B on B/K, and let $\bar{\mu}$ be the unique extension of μ to a representation of B on $\mathfrak{H}$ ([4], p. 52). Define a representation π of B on $\mathfrak{H} \oplus \mathfrak{K}$ as follows:

$$\pi(x) = \bar{\mu}(x) \oplus \sigma(\dot{x}), \quad x \in B.$$

The following lemma shows that this process allows one to reduce the representation theory of B to that of K and B/K. Note first that if π is a representation of K such that $\pi(K) = 0$, then there is already a representation σ of B/K such that $\pi(x) = \sigma(\dot{x})$, $x \in B$. So we assume, in the following, that $\pi(K) \neq 0$.

LEMMA 3.4.4. *For every representation π of B such that $\pi(K) \neq 0$, there are representations μ and σ of K and B/K, respectively, such that $\pi(x) = \bar{\mu}(x) \oplus \sigma(\dot{x})$, $x \in B$.*

Proof. Define $\mathfrak{H}_0 = [\pi(K)\mathfrak{H}]$. Then $\mathfrak{H}_0$ is a nonzero reducing subspace for $\pi(B)$. Define a representation μ of K on $\mathfrak{H}_0$ by $\mu(z) = \pi(z)|_{\mathfrak{H}_0}$, $z \in K$.

Now $\pi(K)$ vanishes on $\mathfrak{H}_0^\perp$, so that the subrepresentation $\pi_1(x) = \pi(x)|_{\mathfrak{H}_0^\perp}$ annihilates K. Thus, there is a representation σ of B/K such that $\pi_1(x) = \sigma(\dot{x})$, for $x \in B$. The required decomposition $\pi(x) = \bar{\mu}(x) \oplus \sigma(\dot{x})$ now follows easily from ([4], p. 52).

If B is of the form $C^*(T)$ for some operator T, then a representation π of $C^*(T)$ is completely determined by the operator $\pi(T)$. It is convenient to specify representations of $C^*(T)$ in this way, by simply giving their values at T.

Now let T be an irreducible operator on a separable Hilbert space of dimension greater than 1, such that $T^*T - TT^*$ is compact. We can describe the representations of $C^*(T)$ as follows. Suppose first that $C^*(T)$ has no characters. Then by 3.3.1, $C^*(T) = \mathrm{Comm}\,(T)$, and by 3.3.7 we see that $C^*(T)$ must be the algebra of all compact operators. In particular, the identity is compact and so the underlying space is finite-dimensional; hence $C^*(T)$ is $*$-isomorphic with a (full) matrix algebra. A familiar theorem of Burnside asserts that every representation of $C^*(T)$ is a multiple of the identity representation. Hence every representation is equivalent to one defined by $\pi(T) = I \otimes T$, where I is the identity operator on some Hilbert space.

Assume, now, that $C^*(T)$ has characters. Note first that there are still representations of the form $\pi(T) = I \otimes T$, where I is as above. As a second example, let $\dot{T}$ be the image of T in the quotient $C^*(T)/\mathrm{Comm}\,(T)$, and let N be a normal operator on some Hilbert space such that $\mathrm{sp}\,(N) \subseteq \mathrm{sp}\,(\dot{T})$. Since both N and $\dot{T}$ are normal, we have by spectral theory that $\|p(N)\| \leqslant \|p(\dot{T})\|$ for every polynomial p in z and $\bar{z}$. Thus, there is a representation σ of $C^*(T)/\mathrm{Comm}\,(T)$ such that $\sigma(\dot{T}) = N$; and we obtain a representation π of $C^*(T)$ by taking $\pi(x) = \sigma(\dot{x})$, $x \in C^*(T)$.

We claim, now, that every representation of $C^*(T)$ is equivalent to one or the other of these two types, or a direct sum of both. By 3.4.4, it suffices to show that every representation of Comm (T) is equivalent to a multiple of the identity representation, and for every representation σ of $C^*(T)/\mathrm{Comm}\,(T)$ the normal operator $N = \sigma(\dot{T})$ has its spectrum contained in $\mathrm{sp}\,(\dot{T})$. But the second follows from the familiar fact that representations shrink spectra, and the first follows from 3.3.7 and the fact that every representation of $LC\,(\mathfrak{H})$ (for $\mathfrak{H}$ separable) is equivalent to a multiple of the identity representation [29].

Applying this to S_ψ, we have:

THEOREM 3.4.5. *If ψ is a finite Blaschke product then every representation of $C^*(S_\psi)$ is equivalent to one defined by $\pi(S_\psi) = I \otimes S_\psi$, with I the identity operator on some Hilbert space.*

If ψ is not a finite Blaschke product then for every unitary operator N such that $\mathrm{sp}\,(N) \subseteq$ $Z_\psi \cap \mathbf{T}$, *there is a representation π of $C^*(S_\psi)$ such that $\pi(S_\psi) = N$. Every representation of $C^*(S_\psi)$ is either of this form, or is equivalent to a multiple of the identity representation ($\pi(S_\psi) = I \otimes S_\psi$), or is a direct sum of these two.*

Proof. We need only note that $Z_\omega \cap \mathbf{T}$ is the spectrum of the image $\dot{S}_\psi$ of S_ψ in $C^*(S_\psi)/$ Comm (S_ψ), by 3.4.3 and the discussion preceding it. The rest follows from 3.4.3 and the preceding discussion.

COROLLARY 3.4.6. *If ψ is a finite Blaschke product then every irreducible representation of $C^*(S_\psi)$ is equivalent to the identity representation.*

If ψ is not a finite Blaschke product, then the irreducible representations of $C^*(S_\psi)$ are, to within unitary equivalence:

(i) *the characters corresponding to points in $Z_\psi \cap \mathbf{T}$ (distinct points giving rise to distinct characters)*

(ii) *the identity representation.*

3.5. *Boundary representations for $P(S_\psi)$.* Let $S_\psi \in L(H^2 \ominus \psi H^2)$ be as in the preceding section. The question taken up in this section is, which irreducible representations of $C^*(S_\psi)$ are boundary representations for the algebra $P(S_\psi)$ of all norm limits of polynomials in S_ψ?

According to 3.4.5, the irreducible representations of $C^*(S_\psi)$ are characters corresponding to points in $Z_\psi \cap \mathbf{T}$ (which are absent when ψ is a finite Blaschke product), and the identity representation (up to equivalence, of course). By 3.1.2, the characters are all one-dimensional boundary representations for $P(S_\psi)$. So the question reduces to this: is the identity representation a boundary representation for $P(S_\psi)$? As we will see, the answer is sometimes yes, and sometimes no.

We begin with a general lemma. Let $\mathcal{A}$ be an algebra of operators on a Hilbert space $\mathfrak{K}$, containing the identity, such that $\mathcal{A}+\mathcal{A}^*$ is norm-dense in $C^*(\mathcal{A})$. Let $\mathfrak{H}$ be a semi-invariant subspace for $\mathcal{A}$ such that $[C^*(\mathcal{A})\mathfrak{H}] = \mathfrak{K}$. Define a (completely positive) linear map φ: $L(\mathfrak{K}) \to L(\mathfrak{H})$ by

$$\varphi(X) = P_{\mathfrak{H}} X|_{\mathfrak{H}}, \quad X \in L(\mathfrak{K}).$$

The restriction of φ to $\mathcal{A}$ is multiplicative, because $\mathfrak{H}$ is semi-invariant, hence $\mathfrak{J} = \{A \in \mathcal{A}: \varphi(A) = 0\}$ is a norm-closed two-sided ideal in $\mathcal{A}$. We write id for the identity map of the norm-closure of $\varphi(\mathcal{A}) + \varphi(\mathcal{A})^*$.

LEMMA 3.5.1. *Let μ be a linear map of the norm closure of $\varphi(\mathcal{A})+\varphi(\mathcal{A})^*$ into $L(\mathfrak{H})$ such that both μ and id$-\mu$ are completely positive. Then there is an operator $T \in L(\mathfrak{K})$ with the properties*

(i) $0 \leq T \leq I$, $T \in C^*(\mathcal{A})'$

(ii) $T[\mathfrak{J}\mathfrak{H}] \subseteq [\mathcal{A}\mathfrak{H}] \ominus \mathfrak{H}$

(iii) $\mu \circ \varphi(X) = \varphi(TX)$, *for all* $X \in \mathcal{A}+\mathcal{A}^*$.

Proof. Since $\mathcal{A}+\mathcal{A}^*$ is norm-dense in $C^*(\mathcal{A})$, $\varphi(C^*(\mathcal{A}))$ is contained in the norm-closure of $\varphi(\mathcal{A})+\varphi(\mathcal{A})^*$. So we can define a linear map μ_1: $C^*(\mathcal{A}) \to L(\mathfrak{H})$ by

$$\mu_1(X) = \mu \circ \varphi(X), \quad X \in C^*(\mathcal{A}).$$

Both μ_1 and $\varphi - \mu_1$ are completely positive maps of the C^*-algebra $C^*(\mathcal{A})$, so by 1.4.2 there is an operator $T \in C^*(\mathcal{A})'$, $0 \leq T \leq I$, such that $\mu_1(X) = \varphi(TX) = P_{\mathfrak{H}} TX|_{\mathfrak{H}}$, $X \in C^*(\mathcal{A})$. (i) and (iii) are immediate, and we need consider only (ii).

Note first that $\varphi(T\mathfrak{J}) = 0$. Indeed, if $A \in \mathfrak{J}$ then by definition of $\mathfrak{J}\varphi(A) = 0$, hence $\mu \circ \varphi(A) = 0$ and thus $\varphi(TA) = 0$, as required. Next, note that $T[\mathfrak{J}\mathfrak{H}]$ is orthogonal to $[\mathcal{A}^*\mathfrak{H}]$. For if $\xi, \eta \in \mathfrak{H}$, $A \in \mathfrak{J}$ and $B \in \mathcal{A}$, then

$$(TA\xi, B^*\eta) = (BTA\xi, \eta) = (TBA\xi, \eta) = (\varphi(TBA)\xi, \eta) = 0,$$

by the preceding, because $BA \in \mathfrak{J}$. But since $\mathcal{A}+\mathcal{A}^*$ is dense in $C^*(\mathcal{A})$ and $\mathfrak{H}$ is cyclic for $C^*(\mathcal{A})$, it follows (as in the proof of 1.3.3) that $[\mathcal{A}^*\mathfrak{H}]^\perp = [\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}$. Therefore (ii) is valid, and that completes the proof.

As we shall see, this lemma is useful only when $\mathfrak{J}$ is a "large" enough subset of $\mathcal{A}$; indeed, if $\mathfrak{J}=0$ then (ii) is vacuous. On the other hand, note that $[\mathfrak{J}\mathfrak{H}]$ is always contained in $[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}$, and if equality happens to hold then (ii) becomes the requirement that T leave the subspace $[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}$ invariant.

The next lemma is known, and is only a minor variation of a construction in [11]. Let A denote the disc algebra, i.e., all continuous functions on the unit circle whose negative Fourier coefficients vanish.

LEMMA 3.5.2. *Let K be a closed set of Lebesgue measure zero in the unit circle. Then there exists a uniformly bounded sequence $f_n \in A$ such that each f_n vanishes precisely on K and $f_n \to 1$ uniformly on compact subsets of $\mathbf{T}\setminus K$.*

Proof. We shall merely indicate how the proof of Fatou's theorem on p. 80 of [11] can be modified to prove this lemma. Let h be the function constructed there, and put $f_n = e^{(1/n)h}$. Then $|f_n| \leqslant e^{-1/n} \leqslant 1$, and it is clear that $f_n \in A$ and tends to 1 uniformly on compact subsets of $\mathbf{T}\setminus K$ (we are indebted to D. E. Sarason for pointing out essentially this proof).

We can now state the main result of this section.

THEOREM 3.5.3. *Let ψ be an inner function such that $Z_\psi \cap \mathbf{T}$ has Lebesgue measure zero. Then the identity representation of $C^*(S_\psi)$ is a boundary representation for $P(S_\psi)$.*

Proof. We shall make use of the criteria of Theorem 2.4.5. It suffices to show that the following three conditions are satisfied:

(i) *$P(S_\psi)$ separates the identity representation of $C^*(S_\psi)$* (cf. 2.4.4),

(ii) *$id|_{P(S_\psi)}$ is a finite representation of $P(S_\psi)$* (cf. 2.3.1), *and*

(iii) *the restriction of id to the* (*norm*) *closure of $P(S_\psi)+P(S_\psi)^*$ is pure* (cf. 2.4.2).

(i) is immediate from 3.4.5. Indeed, if $H^2\ominus\psi H^2$ is one-dimensional, then ψ is a single simple Blaschke factor. By 3.4.5 *id* is the *only* irreducible representation of $C^*(S_\psi)$, and the separation property is trivial. Otherwise, the irreducible representations of $C^*(S_\psi)$ which are not equivalent to *id* are all one-dimensional (3.4.5 again), and since $\dim(H^2\ominus\psi H^2)>1$, *id* cannot be a subrepresentation of one of these.

(ii) follows from example 3 in section 2.3.

We now establish (iii). Let μ be a linear map of the closure of $P(S_\psi)+P(S_\psi)^*$ into $L(H^2\ominus\psi H^2)$ such that both μ and $id-\mu$ are completely positive (on the closure of $P(S_\psi)+P(S_\psi)^*$). We must prove that μ is a scalar multiple of id. In 3.5.1, take $\mathfrak{K}=L^2(\mathbf{T}, m)$ (m is Lebesgue measure), take $\mathcal{A}$ to be the algebra of all multiplications L_f by functions $f\in A$, and let $\mathfrak{H}=H^2\ominus\psi H^2$. Since $A+\bar{A}$ is norm-dense in $C(T)$, it follows that $\mathcal{A}+\mathcal{A}^*$ is dense in $C^*(\mathcal{A})$ (the latter consists of all multiplications by functions in $C(\mathbf{T})$) and that $\mathfrak{H}$ is cyclic for $C^*(\mathcal{A})$ (A.1.1 and A.1.3). Since polynomials in $e^{i\theta}$ are dense in A, polynomials in L_z are dense in $\mathcal{A}$; thus $P_{\mathfrak{H}}\mathcal{A}|_{\mathfrak{H}}$ is a dense subalgebra of $P(S_\psi)$. By 3.5.1, μ has a representation $\mu(P_{\mathfrak{H}}X|_{\mathfrak{H}})=P_{\mathfrak{H}}TX|_{\mathfrak{H}}$ for $X\in\mathcal{A}+\mathcal{A}^*$, where T is a positive operator in the commutant of $\{L_f\colon f\in C(\mathbf{T})\}$ such that $T[\mathcal{J}\mathfrak{H}]\subseteq[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}$, where $\mathcal{J}=\{Z\in\mathcal{A}\colon P_{\mathfrak{H}}Z|_{\mathfrak{H}}=0\}$. Now $\{L_f\colon f\in C(\mathbf{T})\}$ is weakly dense in the multiplication algebra $\{L_f\colon f\in L^\infty(\mathbf{T}, m)\}$ (because $C(\mathbf{T})$ is weak*-dense in L^∞), and the latter is well-known to be a maximal abelian von Neumann algebra. Thus, there is a nonnegative function $h\in L^\infty(\mathbf{T}, m)$ such that $T=L_h$. We will complete the proof by showing that h is constant.

We claim $[\mathcal{J}\mathfrak{H}]=[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}=\psi H^2$. Let A_0 be all functions in A which vanish on $Z_\psi\cap\mathbf{T}$. Then ψA_0 is a closed ideal in A ([11], pp. 68–69, p. 84), and note that $\mathcal{J}$ contains all multiplications by functions in ψA_0. First, we show that $[\mathcal{J}\mathfrak{H}]$ contains ψH^2. For, if f_n is the sequence constructed in 3.5.2 (vanishing on $Z_\psi\cap\mathbf{T}$) and g is in A, then $\psi g f_n$ belongs to ψA_0 and tends boundedly and almost everywhere to ψg on $\mathbf{T}$. An application of the dominated convergence theorem shows that $L_{\psi g f_n}$ tends to $L_{\psi g}$ in the weak operator topology; hence the weak closure of $\mathcal{J}$ contains $\{L_{\psi g}\colon g\in A\}$. It follows that $[\mathcal{J}\mathfrak{H}]$ contains $\psi[\mathcal{A}\mathfrak{H}]$, which is ψH^2 by A.1.3. On the other hand, $[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}=\psi H^2$ (A.1.3), and by the remarks following 3.5.1 we know that $[\mathcal{J}\mathfrak{H}]\subseteq[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}$. Putting all of this together we obtain $\psi H^2\subseteq[\mathcal{J}\mathfrak{H}]\subseteq[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}=\psi H^2$, and the claim is established.

The condition $L_h[\mathcal{J}\mathfrak{H}]\subseteq[\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H}$ now becomes $h\psi H^2\subseteq\psi H^2$, or $hH^2\subseteq H^2$. Applying h to the constant function $1\in H^2$ gives $h\in H^2$, and since h is real-valued, we conclude that h is constant. That completes the proof.

The argument in this proof is not reversible, and one might wonder if in fact the theorem is valid for arbitrary inner functions ψ. The answer is no; we shall indicate how the results of the next section imply that if ψ is an inner function such that Z_ψ contains the entire unit circle, then the identity representation is *not* a boundary representation for $P(S_\psi)$.

Note first that it is easy to come by such inner functions. For example, let $\{\lambda_n\}$ be a countable dense subset of $\mathbf{T}$, and take for ψ the Blaschke product having a simple zero at each point $\zeta_n=(1-n^{-2})\lambda_n$ (cf. [11], p. 64; the condition $\sum(1-|\zeta_n|)<\infty$ guarantees its existence). Clearly every point of $\mathbf{T}$ is a cluster point of $\{\zeta_n\}$, and $Z_\psi\supseteq\mathbf{T}$ follows ([11], p. 68).

To obtain a singular example, let μ be the (singular) measure which assigns mass 2^{-n} to each point λ_n and mass 0 to $\mathbf{T}\setminus\{\lambda_1, \lambda_2, \ldots\}$, and let ψ be the corresponding singular function ([11], pp. 68–69).

Now fix an inner function ψ such that Z_ψ contains $\mathbf{T}$, and let A be the disc algebra: as before, to be regarded as a subalgebra of $C(\mathbf{T})$. By 3.6.3, the mapping $f \in A \mapsto f(S_\psi)$ is a completely isometric representation of A on $P(S_\psi)$, and the same theorem shows that the identity representation of $C^*(S_\psi)$ is not a boundary representation for $P(S_\psi)$. This proves the following:

THEOREM 3.5.4. *If ψ is an inner function such that Z_ψ contains* $\mathbf{T}$, *then the identity representation of $C^*(S_\psi)$ is not a boundary representation for $P(S_\psi)$.*

The question of whether or not *id* is a boundary representation for $P(S_\psi)$ in the intermediate cases ($Z_\psi \cap \mathbf{T}$ of positive measure but different from $\mathbf{T}$) remains unresolved.(1)

3.6. *Representations of the disc algebra.* In the preceding section, it was shown that for certain inner functions ψ, the identity representation of $C^*(S_\psi)$ is a boundary representation for $P(S_\psi)$. We will show in this section how that fact leads to a complete classification of certain almost simple contractions which possess "minimum functions", the latter being an infinite-dimensional analogue of the minimum polynomial of a finite dimensional operator (3.6.12). In a sense, these results generalize those of section 3.2.

It will be convenient, at times, to identify contractions with representations of the disc algebra A (recall that A consists of all continuous functions on the unit circle $\mathbf{T}$ whose negative Fourier coefficients vanish), in the following way. Let φ be a representation of A in $L(\mathfrak{H})$ (i.e., φ is a contractive homomorphism of A into $L(\mathfrak{H})$ for which $\varphi(1)=I$), and let $z \in A$ be the function $z(e^{i\theta})=e^{i\theta}$. Let $T=\varphi(z)$; then $\|T\| \leqslant \|z\|=1$, so that T is a contraction. Conversely, if we start with any contraction T in $L(\mathfrak{H})$, then because the unit disc is a spectral set for T ([18], p. 441), we have for any polynomial p:

$$\|p(T)\| \leqslant \sup_{|z|\leqslant 1} |p(z)| = \sup_{\lambda \in \mathbf{T}} |p(\lambda)|.$$

Since polynomials are norm-dense in A, there is a unique representation φ_T of A which extends the map $p \mapsto p(T)$. We sometimes write $f(T)$ in place of $\varphi_T(f)$, for $f \in A$. Thus, representations of A correspond bijectively with Hilbert space contractions. Clearly, then, one does not expect to classify general representations of A, at least in the forseeable future. We shall concentrate, instead, on those corresponding to almost simple operators T which fit well into the structure of A, in an appropriate sense.

(1) Added in proof Nov. 16, 1969. By using the methods of Theorem 1 of [30], we have settled this question in the affirmative.

We begin with a description of the isometric representations of A. It is desireable to state this material for more general function algebras. Recall that a closed subalgebra A of $C(X)$ (X being a compact Hausdorff space) is called a *Dirichlet algebra* if $1 \in A$ and the linear space $A + \bar{A}$ is norm-dense in $C(X)$ (the latter is equivalent to requiring that the real parts of functions in A be dense in the set of real-valued continuous functions on X).

PROPOSITION 3.6.1. *Let A be a Dirichlet algebra in $C(X)$. Then every representation (resp. isometric representation) of A is completely contractive (resp. completely isometric).*

Proof. Let φ be a representation of A in $L(\mathfrak{H})$. By 1.2.8, there is a unique positive linear extension φ_1 of φ to the closure of $A + \bar{A}$, i.e., $C(X)$. But a positive linear map of $C(X)$ must be completely positive [23], therefore φ_1 is completely contractive (1.2.10), thus φ is completely contractive.

Suppose φ is isometric. By the preceding paragraph, we need only note that φ^{-1} is completely contractive, which follows directly from 1.2.11.

It would be of considerable interest to know if 3.6.1 is true for general function algebras. Appendix A.2 suggests that the answer may be no.

Because the disc algebra is a Dirichlet algebra in $C(\mathbf{T})$, it follows that every contraction T gives rise to a *completely* contractive representation; moreover, if $\|p(T)\| = \sup_{|\zeta| \leq 1} |p(\zeta)|$ for every polynomial p, then the representation is in fact completely isometric (also, see 3.6.3).

In the next result, we assume that the compact Hausdorff space X contains at least two points.

PROPOSITION 3.6.2. *Let A be a closed subalgebra of $C(X)$, which contains* 1 *and separates points. Let φ be a completely isometric representation of A such that $\varphi(A)$ is an irreducible family of operators. Then the identity representation of $C^*(\varphi(A))$ fails to be a boundary representation for $\varphi(A)$.*

Proof. Assume that *id* is a boundary representation for $\varphi(A)$. By 2.1.2, there is a boundary representation ω of $C(X)$, for A, such that $\omega(f) = \varphi(f)$, $f \in A$. Now the irreducible representations of $C(X)$ correspond to points of X, and thus ω has the form $\omega(f) = f(p)I$, where $p \in X$ and I is the identity operator for the space $\mathfrak{H}$ on which $\varphi(A)$ acts (note that $\mathfrak{H}$ is therefore one-dimensional). So for $f \in A$ we have $\varphi(f) = f(p)I$, and thus $\|f\| = |f(p)|$ because φ is isometric. But A separates points of X, and it is therefore evident that $X = \{p\}$ is a singleton, contrary to our initial assumption.

We can apply these results to the disc algebra as follows.

THEOREM 3.6.3. *Let T be a contraction on a Hilbert space. Then T gives rise to a com-*

pletely isometric representation of the disc algebra if, and only if, sp (T) *contains the unit circle. For such a* T, *the identity representation of* $C^*(T)$ *is never a boundary representation for* $P(T)$.

Proof. Let T be a contraction such that sp (T) contains $\mathbf{T}$. By the spectral mapping theorem we have, for every polynomial p,

$$\sup_{|\lambda|=1} |p(\lambda)| \leq \sup_{\zeta \in \mathrm{sp}(T)} |p(\zeta)| = r(p(T)) \leq \|p(T)\|,$$

while of course $\|p(T)\| \leq \sup_{|\lambda|=1} |p(\lambda)|$. Thus, $p \mapsto p(T)$ extends to an isometric representation φ of the disc algebra A. Since A is a Dirichlet algebra in $C(\mathbf{T})$, 3.6.1 shows that φ is completely isometric. Conversely, let T be a contraction such that $\|p(T)\| = \sup_{|\lambda|=1} |p(\lambda)|$ for every polynomial p; then we claim every point $\lambda \in \mathbf{T}$ is in sp (T). Indeed, the equation shows that $|p(\lambda)| \leq \|p(T)\|$ for every polynomial p, so that there is a complex homomorphism ω of $P(T)$ such that $\omega(T) = \lambda$. Thus, λ is in the spectrum of T relative to the Banach algebra $P(T)$. But since $|\lambda| = \|T\| = 1$, λ must be a boundary point of the relative spectrum of T, and is therefore also a spectral value of T relative to the larger Banach algebra $C^*(T)$ ([17], p. 33). This implies $\lambda \in$ sp (T), and the first sentence of the theorem is proved.

The second sentence is immediate from 3.6.2. That completes the proof.

A familiar example of a contraction T whose spectrum includes the unit circle is given by any non-unitary isometry. More interesting is an example of the form $T = S_\psi$, where ψ is an inner function whose "zero set" Z_ψ includes the full unit circle (the discussion following 3.5.3 contains examples). As we have observed already, following 3.4.2, sp $(S_\psi) = Z_\psi$.

In the remainder of this section, we consider contractions T for which the map $f \in A \mapsto f(T)$ has nontrivial kernel; throughout the discussion, A will denote the disc algebra, qua a subalgebra of $C(\mathbf{T})$.

For such a T let $J = \{f \in A: f(T) = 0\}$. Then J is a nonzero closed ideal in A, analogous to the principal ideal generated by the minimum polynomial of a matrix. Indeed, there exists a closed set K of Lebesgue measure zero in $\mathbf{T}$ and an inner function ψ for which $Z_\psi \cap \mathbf{T} \subseteq K$, such that $J = \psi A_K$ where $A_K = \{f \in A: f(K) = 0\}$ ([11], p. 85). This correspondence between ideals and pairs (K, ψ) is bijective provided one identifies proportional inner functions. We shall call J the *order* of T, and ψ will be called the *minimum function* of T.

It is worthwhile to look at the case of a finite-dimensional contraction T in more detail. A known decomposition expresses T as a direct sum $U \oplus T_0$ where U is unitary and T_0 is completely nonunitary (while one of the summands may be absent, we suppose for the sake of illustration that both are present). Let sp $(U) = \{a_1, ..., a_k\}$ and sp $(T_0) =$

$\{b_1, ..., b_l\}$, and suppose $p(z)=(z-b_1)^{n_1} ... (z-b_l)^{n_l}$ is the minimum polynomial of T_0; then of course $(z-a_1) ... (z-a_k)p(z)$ is the minimum polynomial of T. We claim that $|b_j|<1$ for every j. Indeed, if $|b_j|=1$, then by 3.1.2, there exists a character π of $C^*(T_0)$ (the latter regarded as a family of operators on the subspace $\mathfrak{H}_0$ corresponding to the summand T_0) such that $\pi(T_0)=b_j$; a character is a pure state, so it has a pure extension to $L(\mathfrak{H}_0)$, and since $\mathfrak{H}_0$ is finite dimensional it follows that π is a vector state $\pi(x)=(x\xi, \xi)$, $\xi \in \mathfrak{H}_0$. Thus $[\xi]$ reduces T_0, and $T_0\xi=b_j\xi$, contradicting the fact that T_0 is completely nonunitary.

Now let ψ be the finite Blaschke product which has p as its numerator and let $K=\{a_1, a_2, ..., a_k\}$. An easy calculation, which we leave to the reader, shows that the kernel J of the map $f \in A \mapsto f(T)$ is none other than the ideal ψA_K. Note that here the intersection $Z_\psi \cap \mathbf{T}$ is empty. Thus, the minimum function of T corresponds to the "nonunitary" factor of the minimum polynomial of T.

Returning now to the general discussion, we wish to define an analogue of completely isometric representations for representations that have nontrivial kernels. Let T be a contraction on $\mathfrak{H}$ and let $\varphi(f)=f(T)$, $f \in A$. Suppose $\ker \varphi \neq 0$. Then φ induces a canonical homomorphism $\dot{\varphi}$ of the quotient $A/\ker \varphi$ into $L(\mathfrak{H})$ in the usual way: $\dot{\varphi}(\dot{f})=\varphi(f)$, $\dot{f}$ denoting the coset determined by f. Clearly $\|\varphi(f)\|=\|\varphi(f+g)\|$ for every $g \in \ker \varphi$ and it follows that $\dot{\varphi}$ is contractive because φ was. Now for an integer $n>1$, form the Banach algebra $A \otimes M_n$ of all $n \times n$ matrices over A, endowed with the norm inherited from $C(\mathbf{T}) \otimes M_n$. As usual, we obtain a homomorphism φ_n: $A \otimes M_n \to L(\mathfrak{H}) \otimes M_n$, contractive because φ is completely contractive (3.6.1), whose kernel is $\ker \varphi \otimes M_n$. In a similar way, then, each φ_n induces a contractive representation $\dot{\varphi}_n$ of $A \otimes M_n/\ker \varphi \otimes M_n$ into $L(\mathfrak{H}) \otimes M_n$.

Definition 3.6.4. φ (resp. T) is called a *maximal representation* (resp. *maximal operator*) if each $\dot{\varphi}_n$ is isometric.

The term maximal refers to the fact that the norm of $\varphi_n(f)$, for every $f \in A \otimes M_n$ and every $n \geq 1$, is as large as possible. Moreover, using 3.6.7 below and the results of section 3.2, it is easy to see that this term is in harmony with the preceding usage in 3.2.

It is not obvious that maximal operators exist. However, 3.6.6 below shows that if ψ is any inner function for which $Z_\psi \cap \mathbf{T}$ has measure zero, then S_ψ is a maximal operator which has ψ as its minimum function. We first require a lemma which, for $n=1$, is closely related to Lemma 2.1 of [22]. In its proof, we shall make use of a routine fact from the lore of integration theory, which we now state without proof. Let (μ_{ij}) and (σ_{ij}) be $n \times n$ matrices of (complex) Borel measures on $\mathbf{T}$, and let μ and σ be the corresponding linear functionals on $C(\mathbf{T}) \otimes M_n$:

$$\mu(F)=\sum_{i,j}\int f_{ij}d\mu_{ji}, \qquad \sigma(F)=\sum_{i,j}\int f_{ij}d\sigma_{ji},$$

$F=(f_{ij})\in C(\mathbf{T})\otimes M_n$. If each μ_{ij} is singular with respect to each σ_{pq} then $\|\mu+\sigma\|=\|\mu\|+\|\sigma\|$.

We write $\dot{F}$ for the representative of $F\in A\otimes M_n$ in the quotient algebra $A\otimes M_n/\psi A_K\otimes M_n$, and $\mathfrak{H}$ for $H^2\ominus\psi H^2$. We regard $L(\mathfrak{H})\otimes M_n$ as acting on the Hilbert space $\mathfrak{H}\otimes\mathbf{C}^n$, and for $F=(f_{ij})\in A\otimes M_n$, $F(S_\psi)$ will denote the operator matrix $(f_{ij}(S_\psi))\in L(\mathfrak{H})\otimes M_n$.

LEMMA 3.6.5. *Assume $Z_\psi\cap\mathbf{T}=K$ has Lebesgue measure zero. Let ϱ be a linear functional of norm* 1 *on the Banach space $A\otimes M_n/\psi A_K\otimes M_n$. Then there exists a linear functional σ on $C(\mathbf{T})\otimes M_n$, and there are vectors ξ_ν, η_ν* in *$\mathfrak{H}\otimes\mathbf{C}^n$, $1\leqslant\nu\leqslant n$, all of which satisfy:*

(i) $\|\sigma\|\leqslant 1$, *and* $|\sigma(F)|\leqslant\|\sigma\|\sup_{\lambda\in K}\|F(\lambda)\|$ *for all* $F\in C(\mathbf{T})\otimes M_n$ (*i.e.*, σ *lives on* K),

(ii) $\sum_{\nu=1}^{n}\|\xi_\nu\|^2\leqslant 1-\|\sigma\|, \quad \sum_{\nu=1}^{n}\|\eta_\nu\|^2\leqslant 1-\|\sigma\|,$

(iii) $\varrho(\dot{F})=\sum_{\nu=1}^{n}(F(S_\psi)\,\xi_\nu,\eta_\nu)+\sigma(F)$, *for all* $F\in A\otimes M_n$.

Proof. The functional $F\in A\otimes M_n\mapsto\varrho(\dot{F})$ has norm 1, so the Hahn-Banach theorem provides a linear functional ϱ_1 on $C(\mathbf{T})\otimes M_n$, of norm 1, such that $\varrho_1(F)=\varrho(\dot{F})$ for $F\in A\otimes M_n$. Term-by-term application of the Riesz-Markov theorem yields a matrix (τ_{ij}) of measures on $\mathbf{T}$ such that $\varrho_1(F)=\sum_{i,j}\int f_{ij}d\tau_{ji}$, for $F=(f_{ij})$ in $C(\mathbf{T})\otimes M_n$. Now define new measures μ_{ij} and σ_{ij} by $\mu_{ij}(E)=\tau_{ij}(E\setminus K)$, $\sigma_{ij}(E)=\tau_{ij}(E\cap K)$, $E\subseteq\mathbf{T}$. Then $\tau_{ij}=\mu_{ij}+\sigma_{ij}$, all the μ_{ij}'s are singular relative to all the σ_{ij}'s, and if we define the corresponding linear functionals μ and σ, then we have $\varrho_1=\mu+\sigma$. The remarks preceding the Lemma assert that $\|\mu\|+\|\sigma\|=\|\varrho_1\|=1$, thus $\|\sigma\|\leqslant 1$. By definition, σ lives on K, so property (i) is satisfied.

It remains to find the representation (iii) for μ. Now since both σ and ϱ_1 annihilate $\psi A_K\otimes M_n$, so does μ, and it follows that $\int\psi f d\mu_{ij}=0$ for every $f\in A_K$, $1\leqslant i,j\leqslant n$. Choose a bounded sequence $f_n\in A_K$ such that $f_n\to 1$ on $\mathbf{T}\setminus K$ (3.5.2). Then $f_n\to 1$ a.e. $(|\mu_{ij}|)$, and by the bounded convergence theorem we have for every $g\in A$, $\int\psi g d\mu_{ij}=\lim_n\int\psi g f_n d\mu_{ij}=0$. It follows from the F. and M. Riesz theorem that there are functions $h_{ij}\in H_0^1$ (the space of H^1 functions h for which $\int h\,dm=0$, m denoting normalized Lebesgue measure on $\mathbf{T}$) such that $d\mu_{ij}=\bar{\psi}h_{ij}dm$, $1\leqslant i,j\leqslant n$. Thus, for $F=(f_{ij})\in C(\mathbf{T})\otimes M_n$, we have $\mu(F)=\sum_{i,j}\int\bar{\psi}f_{ij}h_{ji}dm=\int\bar{\psi}\operatorname{Tr}(FH)dm$, where $H=(h_{ij})$ and Tr denotes the canonical trace on M_n. Making use of a familiar formula, we conclude $\int\operatorname{Tr}((H^*H)^{\frac{1}{2}})dm=\|\mu\|=1-\|\sigma\|$. By ([22], p. 198), there are $n\times n$ matrices $X=(x_{ij})$ and $Y=(y_{ij})$, both of whose entries are functions in H^2, such that $H=XY$, $Y^*Y=(H^*H)^{\frac{1}{2}}$, and $X^*X=YY^*$. Note that since the entries of $\bar{z}H$ are also in H^1, we can even assume that $\int y_{ij}dm=0$ (just factor $\bar{z}H$ in the above way, then multiply through by z so as to absorb it in the factor Y). The preceding formula for μ now becomes

$$\mu(F)=\int\bar{\psi}\operatorname{Tr}(FXY)\,dm=\sum_{\nu,i,j}(f_{ij}x_{j\nu},\psi\bar{y}_{\nu i}),$$

the inner product being relative to the Hilbert space $L^2(\mathbf{T}, dm)$. Now if P is the projection of L^2 on $\mathfrak{H}$, then it follows exactly as on p. 182 of [22] that

$$(f_{ij}x_{j\nu},\psi\bar{y}_{\nu i})=(f_{ij}Px_{j\nu},P\psi\bar{y}_{\nu i})=(f_{ij}(S_\psi)Px_{j\nu},P\psi\bar{y}_{\nu i}).$$

We define ξ_ν and η_ν as follows. Regarding elements of $\mathfrak{H}\otimes\mathbf{C}^n$ as column vectors (of height n) with coordinates in $\mathfrak{H}$, let ξ_ν be the νth column of the matrix (Px_{ij}) and let η_ν be the transpose of the νth row of $(P\psi\bar{y}_{ij})$. The desired representation $\mu(F)=\sum_\nu(F(S_\psi)\xi_\nu,\eta_\nu)$ follows. As for the norm condition (ii), we have

$$\sum_\nu\|\xi_\nu\|^2=\sum_{i,j}\|Px_{ij}\|^2\leqslant\sum_{i,j}\|x_{ij}\|^2=\int\operatorname{Tr}(X^*X)\,dm=\int\operatorname{Tr}(YY^*)\,dm$$
$$=\int\operatorname{Tr}(Y^*Y)\,dm=\int\operatorname{Tr}((H^*H)^{\frac{1}{2}})\,dm=1-\|\sigma\|.$$

The argument for $\sum_\nu\|\eta_\nu\|^2\leqslant 1-\|\sigma\|$ is similar. That completes the proof.

THEOREM 3.6.6. *Let ψ be an inner function such that $Z_\psi\cap\mathbf{T}=K$ has Lebesgue measure zero. Then φ: $f\in A\mapsto f(S_\psi)$ is a maximal representation of A whose kernel is ψA_K.*

Proof. We first identify the kernel of φ. It is easy to see that ψA_K annihilates S_ψ (indeed, if $f\in A_K$ then $f(L_z)$ maps $H^2\ominus\psi H^2$ into ψH^2, which is orthogonal to $H^2\ominus\psi H^2$, and $f(S_\psi)=P_{H^2\ominus\psi H^2}f(L_z)|_{H^2\ominus\psi H^2}=0$ follows). In particular, $\ker\varphi$ is a closed *nonzero* ideal in A, and thus has the form $\psi_1A_{K_1}$ where K_1 is a closed set in $\mathbf{T}$ of Lebesgue measure zero and ψ_1 is an inner function for which $Z_{\psi_1}\cap\mathbf{T}\subseteq K_1$. The preceding also shows that $\psi A_K\subseteq\psi_1A_{K_1}$, thus ψ_1 divides ψ and $K_1\subseteq K$. Note, next that ψ divides ψ_1 (the conclusion $\ker\varphi=\psi A_K$ follows, because then ψ and ψ_1 are proportional, and so $Z_\psi\cap\mathbf{T}=Z_{\psi_1}\cap\mathbf{T}\subseteq K_1\subseteq K=Z_\psi\cap\mathbf{T}$). To see that, choose a bounded sequence $f_n\in A_{K_1}$ such that $f_n\to 1$ on $\mathbf{T}\setminus K_1$ (by 3.5.2). Then $(\psi_1f_n)(S_\psi)=0$ for all n (because $\psi_1f_n\in\ker\varphi$), and hence $L_{\psi_1f_n}H^2\ominus\psi H^2$ is orthogonal to $H^2\ominus\psi H^2$. Since $\psi_1f_nH^2\subseteq H^2$, it follows that $\psi_1f_n\cdot H^2\ominus\psi H^2\subseteq\psi H^2$ and thus $\psi_1f_nH^2\subseteq\psi H^2$. Hence, $\bar{\psi}\psi_1f_n$ must belong to H^∞, and since $f_n\to 1$ in the weak*-topology of L^∞, we obtain $\bar{\psi}\psi_1\in H^\infty$ and therefore ψ divides ψ_1.

It remains to show that φ is maximal. By 3.6.1, φ_n: $A\otimes M_n\to L(\mathfrak{H})\otimes M_n$ is contractive, for every $n\geqslant 1$, and thus by definition of the quotient norm, $\dot{\varphi}_n$ is also contractive. To see that $\|\dot{\varphi}_n(\dot{F})\| \geqslant \|\dot{F}\|$, choose a function $F \in A \otimes M_n$ for which $\|\dot{F}\| = 1$; we must show that $\|\dot{\varphi}_n(\dot{F})\| = \|\varphi_n(F)\| \geqslant 1$. By the Hahn-Banach theorem, there is a linear functional ϱ on $A \otimes M_n / \psi A_K \otimes M_n$ such that $\|\varrho\| = |\varrho(\dot{F})| = 1$. Using 3.6.5, we can write

$$\varrho(\dot{F}) = \sum_{\nu=1}^{n} (\varphi_n(F)\,\xi_\nu, \eta_\nu) + \sigma(F),$$

where σ, ξ_ν, and η_ν are as described there. Note first, that if $\lambda \in K$, then $\|F(\lambda)\| \leqslant \|\varphi_n(F)\|$. Indeed, by 3.1.2, there is a character χ of $C^*(S_\psi)$ such that $\chi(S_\psi) = \lambda$. Hence the (scalar-valued) map $Z \in C^*(S_\psi) \mapsto \chi(Z)$ is completely contractive, and we have that $\|(\chi(Z_{ij}))\| \leqslant \|(Z_{ij})\|$ for every $n \times n$ matrix $(Z_{ij}) \in C^*(S_\psi) \otimes M_n$. In particular, if $F = (f_{ij}) \in A \otimes M_n$, then $\|F(\lambda)\| = \|(\chi \cdot \varphi(f_{ij}))\| \leqslant \|(\varphi(f_{ij}))\| = \|\varphi_n(F)\|$, as asserted. It then follows that

$$|\sigma(F)| \leqslant \|\sigma\| \sup_{\lambda \in K} \|F(\lambda)\| \leqslant \|\sigma\| \; \|\varphi_n(F)\|.$$

On the other hand, an elementary application of the Schwarz inequality, along with the conditions $\sum \|\xi_\nu\|^2 \leqslant 1 - \|\sigma\|$ and $\sum \|\eta_\nu\|^2 \leqslant 1 - \|\sigma\|$, shows that $|\sum_\nu (\varphi_n(F)\xi_\nu, \eta_\nu)| \leqslant \|\varphi_n(F)\| \cdot (1 - \|\sigma\|)$. Hence, $|\varrho(\dot{F})| \leqslant \|\varphi_n(F)\|(1 - \|\sigma\|) + \|\sigma\| \cdot \|\varphi_n(F)\| = \|\varphi_n(F)\|$, and $1 \leqslant \|\varphi_n(F)\|$ follows because $|\varrho(\dot{F})| = 1$. That completes the proof.

If $C_0, C_1, \ldots, C_m$ are elements of M_n, then $p(z) = \sum_j C_j z^j$ can be regarded as an $n \times n$ matrix-valued polynomial. If $T \in L(\mathfrak{H})$, then we define

$$p(T) = \sum_j C_j \otimes T^j,$$

regarded as an operator on $\mathbf{C}^n \otimes \mathfrak{H}$. When we say a statement holds for all matrix-valued polynomials in T, we mean it to hold for every M_n-valued polynomial, for every $n = 1, 2, \ldots$. The following corollary provides another characterization of maximal operators with non-zero order.

COROLLARY 3.6.7. *Let T be a contraction on a Hilbert space, and suppose $\{f \in A: f(T) = 0\}$ has the form ψA_K, where $K = Z_\psi \cap \mathbf{T}$. Then T is maximal if, and only if, $\|p(T)\| = \|p(S_\psi)\|$ for every matrix-valued polynomial p.*

Proof. Let p be an M_n-valued polynomial, and define the polynomials p_{ij} to be the entries of p: i.e., $p = (p_{ij})$. Then $\|p(T)\| = \|p(S_\psi)\|$ simply means $\|(p_{ij}(T))\| = \|(p_{ij}(S_\psi))\|$, and thus the condition is equivalent to the requirement that $f(S_\psi) \mapsto f(T)$ (for $f \in A$) define a completely isometric representation of $P(S_\psi)$ onto $P(T)$. The corollary now follows from 3.6.6.

Remark. Note that the inequality $\|p(T)\| \leq \|p(S_\psi)\|$ in 3.6.7 is automatic. Indeed, the representation φ: $f \in A \mapsto f(T)$ is completely contractive (3.6.1), and thus for every M_n-valued polynomial p we have

$$\|p(T)\| = \|\varphi_n(p)\| = \inf_{F \in \ker \varphi \otimes M_n} \|\varphi_n(p+F)\| \leq \inf_{F \in \ker \varphi \otimes M_n} \|p+F\| = \|p(S_\psi)\|,$$

the last equality by 3.6.6. Therefore, T is maximal iff

(i) $\|p(T)\| = \|p(S_\psi)\|$ *for every scalar-valued* p, *and*

(ii) $\|p(T)\| \geq \|p(S_\psi)\|$ *for every matrix valued* p.

It is possible that condition (ii) is redundant; put differently, it may be that every isometric representation of $P(S_\psi)$ is completely isometric. Indeed, 3.2.8 shows that this is in fact true when ψ is a finite Blaschke product. We do not know the answer for more general inner functions ψ (for which $Z_\psi \cap \mathbf{T}$ has measure zero), and that is an interesting unsolved problem in this theory. For example, see section 3.7.

We point out, however, that the above argument has proved:

PROPOSITION 3.6.8. *Every contractive representation of* $P(S_\psi)$ *is completely contractive.*

While the above proof only works when $Z_\psi \cap \mathbf{T}$ has measure zero, 3.6.8 is actually true for arbitrary inner functions. We omit the proof since the result is not needed in the sequel. Note, however, that since 3.6.8 is false for the general contraction in place of S_ψ (see A.3.6), it is possible that the answer to the above question is no.

Suppose ψA_K is the order of a contraction T. The next three results take care of the occasionally bothersome case where K is properly larger than $Z_\psi \cap \mathbf{T}$.

A theorem of W. Rudin [19] implies that if K is a closed set of measure zero in $\mathbf{T}$, then A/A_K is canonically isometrically isomorphic with $C(K)$. We shall require the following somewhat more general fact.

LEMMA 3.6.9. *Let* $K_0 \subseteq K$ *be closed sets of Lebesgue measure zero in* $\mathbf{T}$ *and let* ψ *be an inner function such that* $Z_\psi \cap \mathbf{T} \subseteq K_0$ (ψ *may be constant and* K_0 *may be empty*). *Then the homomorphism* φ: $f \in \psi A_{K_0} \mapsto f|_K \in C(K)$ *has kernel* ψA_K, *and canonically induces an isometric isomorphism of* $\psi A_{K_0}/\psi A_K$ *onto the closed ideal* $\{f \in C(K)\colon f(K_0) = 0\}$ *in* $C(K)$.

Proof. φ is clearly a norm-depressing algebra homomorphism, it maps into $I = \{f \in C(K)\colon f(K_0) = 0\}$, and for $f \in \psi A_{K_0}$ we have $f|_K = 0$ iff $f \in \psi A_K$. This identifies $\ker \varphi$, and the induced map is a norm-depressing homomorphism of $\psi A_{K_0}/\psi A_K$ into I. We will complete the proof by showing that, for every $f \in I$, there is an $f_1 \in \psi A_{K_0}$ such that $\|f_1\| = \|f\|$ and $f_1|_K = f$. Choose $f \in I$ and define a function g on K by $g(\lambda) = \bar{\psi}(\lambda) f(\lambda)$ (g is taken as zero on K_0). Since

ψ is continuous and of modulus 1 on $\mathbf{T}\setminus Z_\psi$ and f is continuous and vanishes on $K_0 \supseteq Z_\psi \cap \mathbf{T}$, it follows that $g \in C(K)$ and $|g(\lambda)| = |f(\lambda)|$ for all $\lambda \in K$. Since K is of measure zero, a theorem of Rudin [19] provides a function $g_1 \in A$ such that $\|g_1\| = \|g\| = \|f\|$ and $g_1|_K = g$. Of course, g_1 vanishes on K so that $\psi g_1 \in \psi A_K$. Taking $f_1 = \psi g_1$, we have the desired conclusions $f_1(\lambda) = |\psi(\lambda)|^2 f(\lambda) = f(\lambda)$, for $\lambda \in K$, and $\|f_1\| = \|f\|$.

Note that the theorem of Rudin mentioned above is the case where $\psi = 1$ and K_0 is empty.

LEMMA 3.6.10. *Let K be a closed set of measure zero in $\mathbf{T}$, and let T be a contraction whose order is A_K. Then T is unitary and* sp $(T) = K$.

Proof. Since the map $f \in A \mapsto f(T)$ has kernel A_K, we obtain a contractive homomorphism $\dot{f} \in A/A_K \mapsto f(T)$ ($\dot{f}$ denoting the coset representative of $f \in A$). By 3.6.9 (or, for that matter, by Rudin's theorem above), there is a contractive homomorphism σ of $C(K)$ defined by $\sigma(f|_K) = f(T)$, for every $f \in A$. Note that $\sigma(1)$ is the identity operator. So because $C(K)$ is a C^*-algebra, it follows that σ is a representation (i.e., $*$-preserving), by 1.2.8. Putting $z(\lambda) = \lambda$ for $\lambda \in K$, then $z \in C(K)$ and is of modulus 1; thus $T = \sigma(z)$ is unitary.

Now sp (z) is the range of the function z, namely K, and so sp $(T) \subseteq K$ because representations shrink spectra. On the other hand, if $f \in C(K)$ and f vanishes on sp (T), then $\sigma(f) = f(\sigma(z)) = f(T) = 0$ by the operational calculus. By Rudin's theorem (or 3.6.9), there is an $f_1 \in A$ such that $f_1|_K = f$. Thus, $f_1(T) = \sigma(f_1|_K) = \sigma(f) = 0$, and so $f_1 \in A_K$ by definition of the order of T. Hence $f = f_1|_K = 0$ and we conclude that sp $(T) = K$, completing the proof.

The next result provides the decomposition alluded to in the discussion preceding 3.6.9. In the proof, we shall make use of the following fact about ideals in the disc algebra. Let K_1 and K_2 be closed sets of measure zero in $\mathbf{T}$ and let ψ_1 and ψ_2 be inner functions such that $Z_{\psi_i} \cap \mathbf{T} \subseteq K_i$. Then $\psi_1 A_{K_1} \subseteq \psi_2 A_{K_2}$ if, and only if, ψ_2 divides ψ_1 and $K_2 \subseteq K_1$. It is immediate from this that for two arbitrary ideals $\psi_1 A_{K_1}$ and $\psi_2 A_{K_2}$ we have $\psi_1 A_{K_1} \cap \psi_2 A_{K_2} = \psi A_K$, where $K = K_1 \cup K_2$ and ψ is the least common multiple of ψ_1 and ψ_2 (for the existence of the latter, see chapter 6 of [11]).

PROPOSITION 3.6.11. *Let T be a contraction on a Hilbert space $\mathfrak{H}$ which has order ψA_K, where K is a closed set of Lebesgue measure zero in $\mathbf{T}$ and ψ is an inner function for which $Z_\psi \cap \mathbf{T} \subseteq K$. Let $K_0 = Z_\psi \cap \mathbf{T}$ and $J_0 = \psi A_{K_0}$. Then we have:*

(i) *$J_0(T)$ is a closed self-adjoint subalgebra of $P(T) \cap P(T)^*$, $\mathfrak{H}_0 = [J_0(T)\mathfrak{H}]$ reduces T, and the projection on $\mathfrak{H}_0$ belongs to the weak closure of $J_0(T)$,*

(ii) *$T|_{\mathfrak{H}_0}$ is a unitary operator whose spectrum is the closure of $K \setminus K_0$,*

(iii) *$T|_{\mathfrak{H}_0^\perp}$ has order J_0.*

Proof. Let $I=\{f\in C(K): f(K_0)=0\}$. Then I is a closed ideal in $C(K)$ and is a commutative C^*-algebra (perhaps with no identity). We first define a homomorphism of I into $L(\mathfrak{H})$ in the following way.

The map $f\in J_0\mapsto f(T)$ is a contractive homomorphism of the Banach algebra J_0, having kernel J. Thus it induces canonically a contractive homomorphism of the quotient J_0/J into $L(\mathfrak{H})$. By 3.6.9, the latter may be regarded as a homomorphism of I; more precisely, $\sigma(f|_K)=f(T)$ $(f\in J_0)$ defines a contractive homomorphism of I. Since I is an abelian C^*-algebra, σ is necessarily self-adjoint, hence $\sigma(I)$ is closed ([4], p. 18). Now $I=J_0|_K$, and hence the abelian C^*-algebra $\sigma(I)$ is none other than $J_0(T)$. It follows that $J_0(T)\subseteq P(T)\cap P(T)^*$ and $\mathfrak{H}_0$ reduces T; the projection on $\mathfrak{H}_0$ is in the weak closure of $J_0(T)$ by von Neumann's density theorem ([5], pp. 43–44).

Let $T_1=T|_{\mathfrak{H}_0}$. To prove (ii), it suffices to show that T_1 has order $A_{(K\setminus K_0)^-}$, by 3.6.10. But for every $f\in A$ we have $f(T_1)=0$ iff $f(T)J_0(T)\mathfrak{H}=\{0\}$ iff $fJ_0\subseteq\psi A_K$, because ψA_K is the order of T. Now if $f(K\setminus K_0)=0$ then fJ_0 vanishes on K so that $fJ_0\subseteq\psi A_K$; hence $f(T_1)=0$. Conversely, if $fJ_0\subseteq\psi A_K$ then choose λ in $K\setminus K_0$ and find $g\in J_0$ so that $g(\lambda)\neq 0$ ([11], p. 80). Then fg is in ψA_K, by the last sentence, and in particular $f(\lambda)g(\lambda)=0$. Therefore, $f(\lambda)=0$ and this proves $f=0$ on $K\setminus K_0$.

Consider now $T_2=T|_{\mathfrak{H}_0^\perp}$, and let J_2 be the order of T_2. Since $T=T_1\oplus T_2$ we have $f(T)=f(T_1)\oplus f(T_2)=0$ if, and only if, $f(T_1)=f(T_2)=0$. So the order of T is the intersection of the orders of T_1 and T_2; or, if we put $E=(K\setminus K_0)^-$, then $\psi A_K=J_2\cap A_E$. Consider first the degenerate case $J_2=A$. Then $\psi A_K=A_E$ and in particular ψ is constant. Hence K_0 is empty, $J_0=A$, and we have $\mathfrak{H}_0=[A(T)\mathfrak{H}]=\mathfrak{H}$. This shows that $\mathfrak{H}_0^\perp=0$ so trivially T_2 has order A.

Thus, we can assume $J_2\neq A$; hence J_2 is of the usual form $\psi_2 A_{K_2}$. By the remarks preceding this theorem we conclude from the equation $\psi A_K=J_2\cap A_E$ that ψ_2 and ψ are proportional and $K_2\cup E=K$ (note that the least common multiple of ψ_2 and 1 is ψ_2). This shows in particular that J_2 is contained in $\psi A_{K_0}=J_0$. To see, conversely, that J_0 annihilates T_2, note that, by definition of $\mathfrak{H}_0$, $\mathfrak{H}_0^\perp$ is the intersection of the nullspaces of all operators $f(T)^*$, $f\in J_0$. So if $f\in J_0$ then because $\mathfrak{H}_0^\perp$ reduces T we have $f(T_2)^*=f(T)^*|_{\mathfrak{H}_0^\perp}$ which is zero by the preceding comment. Hence, $f(T_2)=0$ and the proof is now complete.

The next theorem, which is the principal result of this section, allows us to give a complete classification (to unitary equivalence) of all almost simple maximal contractions which have nonzero order. It is significant that, while the conclusion of the theorem implies that such operators generate type I C^*-algebras, no such condition is imposed *a priori*. In the proof, we shall make use of results from sections 2.1, 3.3, 3.4, and 3.5.

THEOREM 3.6.12. *Let T be a nonscalar contraction with nonzero order, say $\{f \in A: f(T)=0\} = \psi A_K$, where K is a closed set of Lebesgue measure zero in $\mathbf{T}$ and ψ is an inner function for which $Z_\psi \cap \mathbf{T} \subseteq K$.*

Then T is maximal and almost simple if, and only if, it is equivalent to one of the following $I \otimes S_\psi$ or $U \oplus (I \otimes S_\psi)$, where I is the identity operator on some Hilbert space and U is a unitary operator for which $\operatorname{sp}(U) \subseteq Z_\psi \cap \mathbf{T}$. Moreover, under these conditions we necessarily have $Z_\psi \cap \mathbf{T} = K$, and ψ cannot be constant.

Remark. Note that the unitary operator U is quite different from the unitary summand of Proposition 3.6.11; indeed, the last sentence shows that the summand of 3.6.11 cannot appear.

Proof. Suppose first that T has the form $U \oplus (I \otimes S_\psi)$ as above. Define a representation μ of $C^*(S_\psi)$ by $\mu(S_\psi) = U \oplus (I \otimes S_\psi)$ (3.4.5). Then μ has the form $\mu = \sigma \oplus (I \otimes id)$, where id is the identity representation and σ is the representation defined by $\sigma(S_\psi) = U$, and hence μ is faithful. Since $\mu(S_\psi) = T$ we have $\mu(C^*(S_\psi)) = C^*(T)$, so that μ is a $*$-isomorphism of $C^*(S_\psi)$ on $C^*(T)$. Since $C^*(S_\psi)$ is almost simple, so is $C^*(T)$, and therefore T. Moreover, a $*$-isomorphism is completely isometric, so by 3.6.7 we conclude that T is maximal.

If T has the form $I \otimes S_\psi$, then $X \in C^*(S_\psi) \to I \otimes X$ defines a $*$-isomorphism between $C^*(S_\psi)$ and $C^*(T)$ which carries S_ψ to T, and we can repeat the above argument to arrive at the same conclusions.

Turning now to the more interesting implication, let T be an almost simple contraction satisfying the stated conditions. First, we claim that $K = Z_\psi \cap \mathbf{T}$. Assume not. We will produce a nonzero ideal $\mathfrak{J}$ in $C^*(T)$ such that $\mathfrak{J} \cap \operatorname{Comm}(T) = 0$, contradicting almost simplicity. Let $\mathfrak{H}_0 \oplus \mathfrak{H}_0^\perp$ be the decomposition of the underlying space described in 3.6.11, and put $T_1 = T|_{\mathfrak{H}_0}$ and $T_2 = T|_{\mathfrak{H}_0^\perp}$. Then $T = T_1 \oplus T_2$ and T_1 is unitary having $(K \setminus Z_\psi)^-$ as its spectrum. In particular, $\mathfrak{H}_0 \neq 0$. Since T_1 is normal we have $\operatorname{Comm}(T) \subseteq 0 \oplus \operatorname{Comm}(T_2)$. Now the map $X \in C^*(T) \mapsto X|_{\mathfrak{H}_0^\perp}$ is a representation, and its kernel $\mathfrak{J}$ is an ideal in $C^*(T)$. Clearly $\mathfrak{J} \subseteq L(\mathfrak{H}_0) \oplus 0$, so the desired conclusion will follow if we show that $\mathfrak{J} \neq 0$. But the projection E on $\mathfrak{H}_0 \oplus 0$ is nonzero, and is a weak limit of operators X in $P(T) \cap P(T)^*$ satisfying $EX = X$, by 3.6.11 (i). In particular, there exists $X \in P(T)$ such that $X = EX \neq 0$. Since E commutes with T we have $X = EX = XE$, hence $X \in \mathfrak{J}$, and the claim is now established.

We may now apply 3.6.7 to conclude that the map $p(T) \mapsto p(S_\psi)$. (p ranging over all scalar polynomials) extends to a completely isometric representation of $P(T)$ on $P(S_\psi)$. 3.5.3 shows that the identity representation of $C^*(S_\psi)$ is a boundary representation for $P(S_\psi)$, and so by 2.1.2 there is a representation ω of $C^*(T)$ such that $\omega(T) = id\,(S_\psi) = S_\psi$.

We claim: ω is faithful. For if not, then $\ker \omega$ is a nonzero ideal in $C^*(T)$ which, by almost simplicity, must contain Comm (T). In particular, $S_\psi^* S_\psi - S_\psi S_\psi^* = \omega(T^*T - TT^*) = 0$, and so S_ψ is normal. Since S_ψ is also irreducible we can only infer that $H^2 \ominus \psi H^2$ is one-dimensional, and therefore ψ is a single simple Blaschke factor $\psi(z) = (z-\alpha)(1-\bar{\alpha}z)^{-1}$ for some α, $|\alpha| < 1$. This implies $K = Z_\psi \cap \mathbf{T}$ is empty, and so the order of T is simply ψA. In particular, $\psi(T) = 0$, which implies $T = \alpha I$, contradicting an original hypothesis.

Hence, $\pi = \omega^{-1}$ is a well-defined representation of $C^*(S_\psi)$, for which $\pi(S_\psi) = T$. By 3.4.5, T must have the form $I \otimes S_\psi$, $U \oplus (I \otimes S_\psi)$, or U, where U and I are as described. The third contingency cannot occur because an almost simple unitary operator must clearly be a scalar, which has been ruled out by hypothesis.

It only remains to show that if T is almost simple (and nonscalar), then ψ is not a constant. But if $\psi = 1$, then the order of T is A_K. If K is empty then $T = 0$ is a scalar; and otherwise, by 3.6.10, T is unitary, and again, almost simplicity implies T is a scalar. That completes the proof.

We remark that this theorem allows one to classify all such T to equivalence. The details are, briefly, as follows. Suppose $T = I \otimes S_\psi$, where I is the identity on an n-dimensional Hilbert space $\mathfrak{K}$ (n being an arbitrary cardinal $\geqslant 1$). Then note that T determines n (the commutant of $C^*(T)$ is $L(\mathfrak{K}) \otimes I$, which is a factor of type I_n). If $T = U \oplus (I \otimes S_\psi)$, then T determines U, or equivalently, T determines the projection $0 \oplus (I \otimes I)$ on the second coordinate space (Comm$(T) = 0 \oplus (I \otimes \text{Comm}(S_\psi))$, hence $0 \oplus (I \otimes I)$ is the projection on the range of Comm (T)). Thus, these operators T are completely classified to equivalence by triples consisting of (a) a cardinal $n \geqslant 1$, (b) an inner function ψ for which $Z_\psi \cap \mathbf{T}$ has Lebesgue measure zero (proportional functions being identified), and (c) a (unitary) equivalence class of unitary operators U for which sp $(U) \subseteq Z_\psi \cap \mathbf{T}$. It is understood that the third component (c) may be absent.

The following consequence of theorem 3.6.12 is noteworthy.

COROLLARY 3.6.13. *Let T_1 and T_2 be irreducible operators, each acting on a Hilbert space of dimension greater than one, such that both commutators $T_i^* T_i - T_i T_i^*$ are compact. Suppose T_1 and T_2 are maximal, and have nonzero orders. Then T_1 and T_2 are equivalent if, and only if, their minimum functions are proportional.*

Proof. First, consider $T_1 \in L(\mathfrak{H}_1)$. The order of T_1 is nonzero, and therefore has the usual form $\psi_1 A_{K_1}$; in particular, T_1 has a minimum function. By 3.3.7, T_1 is almost simple. So theorem 3.6.12, together with irreducibility, shows that T_1 is equivalent to either a one-dimensional unitary operator U or to S_{ψ_1}. The former cannot occur by hypothesis, hence T_1 is equivalent to S_{ψ_1}. Similarly, T_2 is equivalent to S_{ψ_2}, where ψ_2 is the minimum function of T_2. But S_{ψ_1} and S_{ψ_2} are equivalent if, and only if, ψ_1 and ψ_2 are proportional (the less trivial implication follows after comparing their respective orders), and the proof is complete.

We remark that 3.6.13 remains valid when the hypothesis $T_i^* T_i - T_i T_i^*$ compact is replaced with the weaker condition: the commutator ideal in $C^*(T_i)$ is a minimal ideal (cf. 3.3.6).

One might ask if 3.6.13 holds for maximal operators having order 0 (i.e., operators T_i for which $f \in A \mapsto f(T_i)$ is a completely isometric representation of A). The answer is no. For example, choose ψ_1 and ψ_2 to be (nonproportional) inner functions such that Z_{ψ_i} contains $\mathbf{T}$, for $i=1, 2$, and let $T_i = S_{\psi_i}$. Then T_1 and T_2 satisfy the conditions of the first sentence of 3.6.13, both are maximal (3.6.3), yet they are not unitarily equivalent. One way to see this is to extend the mappings $f \in A \to f(T_i)$ to H^∞ in the canonical way (for example, see [22], p. 179). Then note that $\psi(T_i)=0$ for an inner function ψ iff ψ_i divides ψ. Since ψ_1 and ψ_2 are not proportional, we must have either $\psi_1(T_2) \neq 0$ or $\psi_2(T_1) \neq 0$; because $\psi_1(T_1)=\psi_2(T_2)=0$, it then follows that T_1 cannot be equivalent to T_2. The details are left to the reader. The reason the proof of 3.6.13 does not work for these operators is explained by the second statement of 3.6.3.

3.7. *A characterization of the Volterra operator.* We shall indicate how the results of the preceding section can be applied to give a characterization of the Volterra operator V, defined on the Hilbert space $L^2(0, 1)$ by

$$Vf(x) = \int_0^x f(t)\,dt, \quad x \in (0, 1),\ f \in L^2(0, 1).$$

V is known to be an irreducible, compact, quasinilpotent operator for which $\operatorname{Re} V \geqslant 0$ and $\|V\| < 1$ [9]. Recall that if n is a positive integer, $p(z) = C_0 + C_1 z + \dots + C_k z^k$ is an M_n-valued polynomial, and T is a Hilbert space operator, then $p(T)$ is defined as $C_0 \otimes I + C_1 \otimes T + \dots + C_k \otimes T^k$. We shall prove the following:

THEOREM 3.7.1. *Let T be an irreducible operator on a Hilbert space $\mathfrak{H}$ such that $T^*T - TT^*$ is compact and $\|p(T)\| = \|p(V)\|$ for every matrix-valued polynomial p. Then T is unitarily equivalent to V.*

First, we recall some facts about certain transforms of Hilbert space operators. Let X be an operator on a Hilbert space for which $-1 \notin \operatorname{sp}(X)$, and put $Y = (I - X)(I + X)^{-1}$. If $\operatorname{Re} X \geqslant 0$, then $\|Y\| \leqslant 1$ (for example, see [18], p. 442). A calculation shows that $\operatorname{Re} Y = (I + X^*)^{-1}(I - X^*X)(I + X)^{-1}$, so that $\|X\| \leqslant 1$ implies $\operatorname{Re} Y \geqslant 0$. Hence, if $\|X\| \leqslant 1$ and

Re $X \geqslant 0$, then the same is true of Y. Now $f(z) = (1-z)(1+z)^{-1}$ never takes on the value -1 in the (finite) complex plane, and it follows from the spectral mapping theorem that $-1 \notin \mathrm{sp}\,(Y)$; since $(1-f(z))(1+f(z))^{-1} = z$, we see that $(I-Y)(I+Y)^{-1} = X$. According to 3.1.1, if Re $X \geqslant 0$ then $Y \in P(X)$, and by the above remarks, if $\|X\| \leqslant 1$ then $X \in P(Y)$ by the same lemma. We conclude: *if X is a contraction for which* Re $X \geqslant 0$, *then the same is true of Y and moreover*, $P(X) = P(Y)$.

We now take up the proof of 3.7.1. Note first that the space $\mathfrak{H}$ on which T acts is of dimension $\aleph_0$; indeed T satisfies no nontrivial polynomial equation $p(T) = 0$ because V does not, so that dim $\mathfrak{H} \geqslant \aleph_0$, and dim $\mathfrak{H} \leqslant \aleph_0$ follows because $C^*(T)$ is a separable subalgebra of $L(\mathfrak{H})$ which has a cyclic vector.

By hypothesis, the map $p(V) \mapsto p(T)$, defined for all polynomials p, extends uniquely to a completely isometric isomorphism φ of $P(V)$ on $P(T)$. We have $\|T\| = \|V\| < 1$, and Re $T = \mathrm{Re}\,\varphi(V) \geqslant 0$ because Re $V \geqslant 0$, by 1.2.8. Define $V_1 = (I-V)(I+V)^{-1}$ and $T_1 = (I-T)(I+T)^{-1}$. By the preceding remarks, $P(V_1) = P(V)$, $P(T_1) = P(T)$, and of course $\varphi(V_1) = T_1$. Note also that T_1 is irreducible ($C^*(T_1) = C^*(T)$) and $T_1^* T_1 - T_1 T_1^*$ is compact (Comm $(T_1) =$ Comm $(T) = LC(\mathfrak{H})$, by 3.3.7).

Now as it is pointed out in [21], V_1 is unitarily equivalent to S_ψ where ψ is the inner function $\psi(z) = \exp\,(z+1)(z-1)^{-1}$, $|z| < 1$ (note that we have slightly restated the result of [21]). Therefore V_1 and T_1 have the same minimum function ψ, both are maximal (because S_ψ is by 3.6.6), and hence 3.6.13 shows that V_1 and T_1 are equivalent. It follows that V and T are equivalent, and that completes the proof.

It is not known if 3.7.1 is valid when the norm condition $\|p(T)\| = \|p(V)\|$ is assumed to hold only for scalar-valued polynomials p. By the above proof and 3.6.8, the scalar condition implies $\|p(T)\| \leqslant \|p(V)\|$ for all matrix-valued p, but the opposite inequality is in doubt.

Note that 3.7.1 is valid when V is replaced with S_ψ, where ψ is an inner function for which $Z_\psi \cap \mathbf{T}$ has measure zero: that is what the last few lines of the proof showed. But the theorem becomes false with other substitutions. For example, if T_1 and T_2 are irreducible contractions each of whose spectrum contains the unit circle, such that each commutator $T_i^* T_i - T_i T_i^*$ is compact, then $p(T_1) \mapsto p(T_2)$ (p running over the polynomials) is completely isometric (3.6.3), but T_1 and T_2 surely do not have to be equivalent (cf. the discussion following 3.6.13). The second sentence of 3.6.3 points to where the proof of 3.7.1 breaks down: the identity representation of $C^*(T_i)$ is not a boundary representation for $P(T_i)$, $i = 1, 2$.

Appendix

A.1. *Semi-invariant subspaces.* Let $\mathcal{A}$ be an algebra of operators on a Hilbert space $\mathfrak{K}$, which contains the identity, and let $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$ be a nested pair of $\mathcal{A}$-invariant subspaces in $\mathfrak{K}$. Then $\mathfrak{H} = \mathfrak{M}_2 \ominus \mathfrak{M}_1$ is a semi-invariant subspace for $\mathcal{A}$ (cf. the discussion preceding 1.3.3). While $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not uniquely determined by $\mathfrak{H}$ (as obvious examples will show), there is a *canonical* nested pair of $\mathcal{A}$-invariant subspaces $\mathfrak{N}_1 \subseteq \mathfrak{N}_2$ such that $\mathfrak{H} = \mathfrak{N}_2 \ominus \mathfrak{N}_1$, namely $\mathfrak{N}_2 = [\mathcal{A}\mathfrak{H}]$ and $\mathfrak{N}_1 = [\mathcal{A}\mathfrak{H}] \ominus \mathfrak{H}$ (see [20]). In applications, one needs to know when a given pair $\mathfrak{M}_1$, $\mathfrak{M}_2$ is canonical; equivalently, when is $\mathfrak{M}_2$ the smallest $\mathcal{A}$-invariant subspace containing $\mathfrak{H}$? The present section is devoted to a discussion of this point, for a special class of algebras $\mathcal{A}$.

In the following, $\mathcal{A}$ will be a subalgebra of $L(\mathfrak{K})$, which contains the identity of $\mathfrak{K}$, $\mathcal{R}$ will denote the von Neumann algebra generated by $\mathcal{A}$, and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ will be a nested pair of $\mathcal{A}$-invariant subspaces, as above.

THEOREM A.1.1. *Assume that $\mathcal{A} + \mathcal{A}^*$ is weakly dense in $\mathcal{R}$, $[\mathcal{R}\mathfrak{M}_2] = \mathfrak{K}$, and let $\mathfrak{H} = \mathfrak{M}_2 \ominus \mathfrak{M}_1$. Then the following are equivalent:*

(i) $[\mathcal{A}\mathfrak{H}] = \mathfrak{M}_2$,

(ii) $[\mathcal{R}\mathfrak{H}] = \mathfrak{K}$,

(iii) *the only projection $E \in \mathcal{R}'$ for which $E\mathfrak{M}_2 \subseteq \mathfrak{M}_1$ is $E = 0$.*

Proof. That (i) implies (ii) is clear from the hypothesis $[\mathcal{R}\mathfrak{M}_2] = \mathfrak{K}$.

We claim that (ii) implies (iii). Indeed, if E is a projection in $\mathcal{R}'$ such that $E\mathfrak{M}_2 \subseteq \mathfrak{M}_1$, then in particular we have $E\mathfrak{H} \subseteq \mathfrak{M}_1 \subseteq \mathfrak{H}^\perp$, or $E\mathfrak{H} \perp \mathfrak{H}$. Because E is a projection, it follows that $E\mathfrak{H} = 0$, or $\mathfrak{H} \subseteq (E\mathfrak{K})^\perp$. Using (ii) we have $\mathfrak{K} = [\mathcal{R}\mathfrak{H}] \subseteq [\mathcal{R}(E\mathfrak{K})^\perp] \subseteq (E\mathfrak{K})^\perp$, and therefore $E\mathfrak{K} = 0$, as required.

Assume, now, that (iii) is satisfied; we must show that $[\mathcal{A}\mathfrak{H}] = \mathfrak{M}_2$. Note first that $[\mathcal{A}\mathfrak{H}] \subseteq \mathfrak{M}_2$. Consider the restriction $\mathcal{A}|_{\mathfrak{M}_2}$ of $\mathcal{A}$ to the invariant subspace $\mathfrak{M}_2$, and let $E_0 \in L(\mathfrak{M}_2)$ be the projection of $\mathfrak{M}_2$ on $\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}]$. We claim that E_0 commutes with $\mathcal{A}|_{\mathfrak{M}_2}$ or, what is the same, $\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}]$ is invariant under both $\mathcal{A}|_{\mathfrak{M}_2}$ and $(\mathcal{A}|_{\mathfrak{M}_2})^*$. It is clear that $(\mathcal{A}|_{\mathfrak{M}_2})^*$ leaves $\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}]$ invariant, since the latter is the complement of an $\mathcal{A}|_{\mathfrak{M}_2}$-invariant subspace. The other assertion is simply $\mathcal{A}(\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}]) \subseteq \mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}]$, and this will follow if we prove that $\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}] = \mathfrak{M}_1 \cap [\mathcal{R}\mathfrak{H}]^\perp$ (for the right side is an intersection of $\mathcal{A}$-invariant subspaces). Now $\mathfrak{M}_2 = \mathfrak{H} \oplus \mathfrak{M}_1$ and $[\mathcal{A}\mathfrak{H}] = \mathfrak{H} \oplus ([\mathcal{A}\mathfrak{H}] \ominus \mathfrak{H})$, so that we have $\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}] = \mathfrak{M}_1 \ominus ([\mathcal{A}\mathfrak{H}] \ominus \mathfrak{H}) = \mathfrak{M}_1 \cap ([\mathcal{A}\mathfrak{H}] \ominus \mathfrak{H})^\perp$. Because $\mathfrak{M}_1$ is an $\mathcal{A}$-invariant subspace orthogonal to $\mathfrak{H}$, we have $\mathfrak{M}_1 \perp [\mathcal{A}^*\mathfrak{H}]$; it follows that the right side of the above equation is unchanged if we intersect it with $[\mathcal{A}^*\mathfrak{H}]^\perp$, i.e., $\mathfrak{M}_2 \ominus [\mathcal{A}\mathfrak{H}] = \mathfrak{M}_1 \cap ([\mathcal{A}\mathfrak{H}] \ominus \mathfrak{H})^\perp \cap [\mathcal{A}^*\mathfrak{H}]^\perp$.

But since $\mathcal{A}+\mathcal{A}^*$ is dense in $\mathcal{R}$ we have $[\mathcal{A}^*\mathfrak{H}]\oplus([\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H})=[\mathcal{R}\mathfrak{H}]$ (see the proof of 1.3.3), hence $([\mathcal{A}\mathfrak{H}]\ominus\mathfrak{H})^\perp\cap[\mathcal{A}^*\mathfrak{H}]^\perp=[\mathcal{R}\mathfrak{H}]^\perp$, and the required formula $\mathfrak{M}_2\ominus[\mathcal{A}\mathfrak{H}]=\mathfrak{M}_1\cap[\mathcal{R}\mathfrak{H}]^\perp$ follows.

One makes use of the commuting projection E_0 as follows. Letting P be the projection of $\mathfrak{K}$ on $\mathfrak{M}_2$, then because $\mathcal{A}+\mathcal{A}^*$ is dense in $\mathcal{R}$ we see that E_0 commutes with $P\mathcal{R}|_{\mathfrak{M}_2}$. Since $[\mathcal{R}\mathfrak{M}_2]=\mathfrak{K}$, one can apply 1.3.1 (taking the V of 1.3.1 as the inclusion map of $\mathfrak{M}_2$ in $\mathfrak{K}$) to infer the existence of a projection $E\in\mathcal{R}'$ such that $E|_{\mathfrak{M}_2}=E_0$. Thus, $E\mathfrak{M}_2=E_0\mathfrak{M}_2=\mathfrak{M}_2\ominus[\mathcal{A}\mathfrak{H}]=\mathfrak{M}_1\cap[R\mathfrak{H}]^\perp\subseteq\mathfrak{M}_1$. By (iii) we conclude $E=0$, hence $\mathfrak{M}_2\ominus[\mathcal{A}\mathfrak{H}]=0$. This means that $\mathfrak{M}_2=[\mathcal{A}\mathfrak{H}]$, and the proof is complete.

COROLLARY A.1.2. *Let $\mathcal{A}$ be a subalgebra of $L(\mathfrak{K})$, containing the identity, such that $\mathcal{A}+\mathcal{A}^*$ is weakly dense in a von Neumann algebra $\mathcal{R}$, let $\mathfrak{M}$ be an $\mathcal{A}$-invariant subspace such that $[\mathcal{R}\mathfrak{M}]=\mathfrak{K}$, and let $\mathcal{D}$ be the von Neumann algebra $\{T\in\mathcal{R}'\colon T\mathfrak{M}\subseteq\mathfrak{M},\ T^*\mathfrak{M}\subseteq\mathfrak{M}\}$.*

Then for every unitary operator $U\in\mathcal{R}'$, $U\mathfrak{M}$ is an $\mathcal{A}$-invariant subspace. If $U\mathfrak{M}\subseteq\mathfrak{M}$ and $\mathfrak{H}$ is the semi-invariant subspace $\mathfrak{M}\ominus U\mathfrak{M}$, then $[\mathcal{A}\mathfrak{H}]=\mathfrak{M}$ if, and only if, the only projection $E\in\mathcal{D}$ for which $UE\in\mathcal{D}$ is $E=0$.

Proof. It is obvious that $U\mathfrak{M}$ is an $\mathcal{A}$-invariant subspace.

Assume $U\mathfrak{M}\subseteq\mathfrak{M}$. Note that $[\mathcal{A}\mathfrak{H}]=\mathfrak{M}$ iff $[\mathcal{A}\mathfrak{H}_0]=U^{-1}\mathfrak{M}$ where $\mathfrak{H}_0=U^{-1}\mathfrak{H}=U^{-1}\mathfrak{M}\ominus\mathfrak{M}$. Now since $[\mathcal{R}U^{-1}\mathfrak{M}]=U^{-1}[\mathcal{R}\mathfrak{M}]=U^{-1}\mathfrak{K}=\mathfrak{K}$, we can apply A.1.1 to the $\mathcal{A}$-invariant subspaces $\mathfrak{M}_1=\mathfrak{M}$ and $\mathfrak{M}_2=U^{-1}\mathfrak{M}$ to conclude that $[\mathcal{A}\mathfrak{H}]=\mathfrak{M}$ iff the only projection $E\in\mathcal{R}'$ such that $EU^{-1}\mathfrak{M}\subseteq\mathfrak{M}$ is $E=0$. Note, however, that this condition is equivalent to $E\in\mathcal{D}$ and $UE\in\mathcal{D}$. Indeed, $E\mathfrak{M}\subseteq\mathfrak{M}$ follows from $EU^{-1}\mathfrak{M}\subseteq\mathfrak{M}$ because $U^{-1}\mathfrak{M}$ is larger than $\mathfrak{M}$ (thus $E\in\mathcal{D}$), and along with $EU^{-1}\mathfrak{M}\subseteq\mathfrak{M}$ we have $UE\mathfrak{M}\subseteq U\mathfrak{M}\subseteq\mathfrak{M}$ (thus $UE\in\mathcal{D}$). The converse is apparent. That completes the proof.

We are now ready to give an application to shifts of arbitrary multiplicity. Let $\mathfrak{C}$ be a separable Hilbert space and let U be an inner function with values in $L(\mathfrak{C})$ (see the discussion preceding 1.3.4 for definitions and notation). We will say U is *completely nonconstant* if the only $\xi\in\mathfrak{C}$ for which the vector-valued function $z\mapsto U(z)\xi$ is constant (inside the unit disc) is $\xi=0$. Taking $\mathfrak{K}=L^2(\mathbf{T},\sigma;\mathfrak{C})$, $\mathfrak{M}=H^2_{\mathfrak{C}}$, and $\mathcal{A}$ as the algebra of all multiplications by scalar-valued polynomials in $e^{i\theta}$, in the notation of A.1.2, we see that $\mathcal{D}$ is the von Neumann algebra of all multiplications by *constant* $L(\mathfrak{C})$-valued functions on $\mathbf{T}$, thus we conclude from A.1.2:

COROLLARY A.1.3. $[\mathcal{A}(H^2_{\mathfrak{C}}\ominus UH^2_{\mathfrak{C}})]=H^2_{\mathfrak{C}}$ *if, and only if, U is completely nonconstant.*

We remark that A.1.3 seems closely related to a result on p. 43 of [26], and may be a consequence of the latter.

A.2. *A positive linear map with no positive extension.* Let $\mathbf{T}$ be the unit circle, and define $z \in C(\mathbf{T})$ by $z(e^{i\theta}) = e^{i\theta}$, $0 \leqslant \theta < 2\pi$. Let S be the three-dimensional self-adjoint subspace of $C(\mathbf{T})$ spanned by 1, z, and $\bar{z}$. We will give an example of a positive linear map of S into $L(\mathfrak{H})$ which has no positive linear extension to (the commutative C^*-algebra) $C(\mathbf{T})$.

Regard the C^*-algebra M_2 of all complex 2×2 matrices as the ring of all bounded operators on a two-dimensional Hilbert space. Define a linear map φ: $S \to M_2$ by:

$$\varphi(a1 + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix},$$

$a, b, c \in \mathbf{C}$. Clearly φ is well-defined linear, and takes 1 to the identity.

Now the self-adjoint elements of S can be put in the form $t1 + \frac{1}{2}(bz + \bar{b}\bar{z}) = t1 + \operatorname{Re}(bz)$, with t real and b arbitrary, and such an element is positive iff $t \geqslant |b|$. φ takes $t1 + \operatorname{Re}(bz)$ to the matrix

$$\begin{pmatrix} t & b \\ \bar{b} & t \end{pmatrix}$$

which, in turn, is positive iff $t \geqslant |b|$ (a self-adjoint 2×2 matrix is positive iff its trace and determinant are both nonnegative). Thus, φ is an order isomorphism and in particular, it is positive.

Now since a positive linear map of a commutative C^*-algebra is necessarily completely positive (see [23]) and the norm of a completely positive map is achieved at the identity (1.2.10), it follows that if φ were positively extendable then $\|\varphi\| = \|\varphi(1)\| = \|I\| = 1$. But z is of norm 1, while

$$\|\varphi(z)\| = \left\| \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \right\| = 2,$$

and that proves φ has no positive linear extension to $C(\mathbf{T})$.

A.3. *A contractive representation need not be completely contractive.* In this appendix we show by example that a contractive representation of a subalgebra of a C^*-algebra need not be completely contractive, even when the subalgebra is commutative. Indeed, we shall give examples of (finite-dimensional) operators X and Y such that $\|p(X)\| = \|p(Y)\|$ for every (scalar-valued) polynomial p, but $p(X) \mapsto p(Y)$ is not completely contractive.

We remark that it does not seem easy to come by such examples, even with respect to the strongest statement of the first sentence. Our own experience with the examples we tried initially was that when it was possible to make a decision at all, contractive representations turned out to be completely contractive. The problem was finally solved by making use of a general result (Theorem A.3.5) which allowed us to sidestep the more involved calculations. Though A.3.6 is a negative result, it is an important one for the theory, and it would be interesting to have a simpler class of counter-examples. Along these lines, it is not known if a contractive representation of a function algebra must be completely contractive; we conjecture no (the answer is yes, however, for Dirichlet algebras, by 3.6.1).

The example is described as follows. Let $\mathfrak{H}$ be a three-dimensional Hilbert space, and realize $L(\mathfrak{H})$ as the algebra M_3 of all 3×3 matrices over $\mathbf{C}$, relative to a fixed orthonormal base for $\mathfrak{H}$. Let λ and ω be any two complex numbers such that λ is *not* real and $|\omega| = (1+|\lambda|^2)^{\frac{1}{2}}$. Define $S, T \in L(\mathfrak{H})$ by

$$S=\begin{pmatrix}0 & \lambda & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad T=\begin{pmatrix}0 & 0 & \omega\\ 0 & 1 & 0\\ 0 & -\lambda & 0\end{pmatrix},$$

and let $\mathcal{A}$ be the three-dimensional subspace of $L(\mathfrak{H})$ spanned by S, T, and the identity (it follows from A.3.1 that $\mathcal{A}$ is a singly-generated subalgebra of $L(\mathfrak{H})$). Let X^t denote the transpose of a matrix $X\in L(\mathfrak{H})$. Thus, the map φ: $X\in\mathcal{A}\mapsto X^t\in L(\mathfrak{H})$ defines an identity preserving homomorphism of $\mathcal{A}$ into $L(\mathfrak{H})$; since $\|X^t\| = \|X\|$ for every matrix X (a self-adjoint anti-automorphism of M_3 is necessarily isometric), it follows that φ is an isometric representation of $\mathcal{A}$ in $L(\mathfrak{H})$. We will show that φ is not completely contractive.

The first four lemmas represent computations, and we shall merely outline their proof.

LEMMA A.3.1. *$\mathcal{A}$ is a singly-generated subalgebra of $L(\mathfrak{H})$.*

Proof. Note first that $ST = TS = S^2 = 0$ and $T^2 = T - \omega S$. Then if $X = S + T$, for example, it follows that $\mathcal{A} = P(X)$.

LEMMA A.3.2. $C^*(\mathcal{A}) = L(\mathfrak{H})$.

Proof. By the double commutant theorem, one need only prove that the only self-adjoint matrices that commute with both S and T are scalars. This calculation shows, in fact, that $\{S, T\}$ is irreducible provided merely that both ω and λ are nonzero.

LEMMA A.3.3. *$\mathcal{A}+\mathcal{A}^*$ is linearly spanned by its unitary elements.*

Proof. First, show that aS^*+bT is unitary when $|a| = |b| = (1+|\lambda|^2)^{-\frac{1}{2}}$. Thus, the span of the unitaries contains $(1+|\lambda|^2)^{-\frac{1}{2}}(S^*+T)$ and $(1+|\lambda|^2)^{-\frac{1}{2}}(S^*-T)$, and therefore it contains S^* and T. Since the span of the unitaries in $\mathcal{A}+\mathcal{A}^*$ is a self-adjoint subspace which contains the identity, the lemma follows.

LEMMA A.3.4. *There is no unitary matrix U such that $US = S^tU$ and $UT = T^tU$.*

Proof. This is a laborious calculation. Assuming the existence of such a U, one uses the fact that the row vectors of U form an orthonormal set to arrive at the conclusion $|\omega| = |1+\lambda^2|(1+|\lambda|^2)^{-\frac{1}{2}}$. Now by the choice of ω we have $|\omega| = (1+|\lambda|^2)^{\frac{1}{2}}$, hence the above implies $|1+\lambda^2|=1+|\lambda|^2$, from which we conclude $\lambda^2=|\lambda|^2$ (i.e., λ is real), contradicting the choice of λ.

Note that the preceding lemma, together with the proof of A.3.1, shows that $S+T$ is not unitarily equivalent to its transpose.

We now state a general result.

THEOREM A.3.5. *Let S be a self-adjoint linear subspace of a C^*-algebra B, containing the identity of B, such that $B=C^*(S)$. Let $\mathfrak{H}$ be a Hilbert space and let φ be an identity-preserving completely positive linear map of S into $L(\mathfrak{H})$ such that the closed linear span of $\{a\in S$: $\|a\|=1$, $\varphi(a)$ is unitary$\}$ is all of S. Then φ is implemented by a representation of B.*

Proof. Making use of 1.2.3 and 1.1.1, we see that there is a representation π of B on a Hilbert space $\mathfrak{K}$ and a bounded linear map V: $\mathfrak{H}\to\mathfrak{K}$ such that $\varphi(a)=V^*\pi(a)V$, $a\in S$, and $[\pi(B)V\mathfrak{H}]=\mathfrak{K}$. Since $\varphi(e)=I$, it follows that $V^*V=I$, i.e., V is an isometry.

We claim: $V\mathfrak{H}=\mathfrak{K}$ (the theorem follows, for then V is unitary and the representation $V^{-1}\pi V$ implements φ). Since $[\pi(B)V\mathfrak{H}]=\mathfrak{K}$, it suffices to show that $V\mathfrak{H}$ is invariant under $\pi(S)$ (for then $V\mathfrak{H}$ is invariant under the norm-closed algebra generated by $\pi(S)$, namely $\pi(B)$). Let $M=\{a\in S$: $\|a\|=1$, $\varphi(a)$ is unitary$\}$. Then for $a\in M$ and $\xi\in\mathfrak{H}$ we have

$$\begin{aligned}\|\pi(a)V\xi-V\varphi(a)\xi\|^2&=\|\pi(a)V\xi\|^2-2\operatorname{Re}(\pi(a)V\xi,\ V\varphi(a)\xi)+\|V\varphi(a)\xi\|^2\\&=\|\pi(a)V\xi\|^2-\|\varphi(a)\xi\|^2=\|\pi(a)V\xi\|^2-\|\xi\|^2\leqslant\|\xi\|^2-\|\xi\|^2=0,\end{aligned}$$

because $(\pi(a)V\xi,V\varphi(a)\xi)=(V^*\pi(a)V\xi,\ \varphi(a)\xi)=\|\varphi(a)\xi^2\|^2=\|V\varphi(a)\xi\|^2$ and $\varphi(a)$ is unitary. Thus, $\pi(a)V=V\varphi(a)$ for every $a\in M$, and since M spans S this equation persists for $a\in S$. In particular, $\pi(S)V\mathfrak{H}=V\varphi(S)\mathfrak{H}\subseteq V\mathfrak{H}$, and the proof is complete.

We now state the main result, for the isometric representation φ of $\mathcal{A}$ defined above.

THEOREM A.3.6. *φ is not a completely contractive representation of $\mathcal{A}$.*

Proof. Consider the self-adjoint linear map φ_1: $X\in\mathcal{A}+\mathcal{A}^*\mapsto X^t\in M_3$. Note that φ_1 is the unique self-adjoint linear extension of φ to $\mathcal{A}+\mathcal{A}^*$.

Assume, now, that φ is completely contractive. By 1.2.8, φ_1 is a completely positive linear map of $\mathcal{A}+\mathcal{A}^*$. Now by A.3.3, $\mathcal{A}+\mathcal{A}^*$ is linearly spanned by its unitary elements; and since the transpose of a unitary matrix is unitary, it follows that $\{X\in\mathcal{A}+\mathcal{A}^*$: $\|X\|=1$, $\varphi_1(X)$ unitary$\}$ spans $\mathcal{A}+\mathcal{A}^*$. By A.3.2, we have $C^*(\mathcal{A})=M_3$. Hence Theorem A.3.5 applies, so there is a $*$-homomorphism π: $M_3\to M_3$ such that $\pi(X)=\varphi(X)$ for $X\in\mathcal{A}$.

Because M_3 is simple π must be faithful, and an obvious dimension argument shows that $\pi(M_3)=M_3$; thus π is a $*$-automorphism of M_3. It is well-known that a $*$-automorphism of $L(\mathfrak{H})$ is unitarily implemented, and we conclude that there is a unitary matrix $U\in M_3$ such that $\varphi(X)=UXU^{-1}$ for all $X\in\mathcal{A}$. But we now have an absurdity, by A.3.4, and thus φ could not have been completely contractive.

References

[1]. ARVESON, W. B., On subalgebras of C^*-algebras. *Bull. Amer. Math.Soc.*, 75 (1969), 790–794.

[2]. BROWN, A. & HALMOS, P. R., Algebraic properties of Toeplitz operators. *J. Reine Angew. Math.*, 213 (1963), 89–102.

[3]. COBURN, L. A., The C^*-algebra generated by an isometry. *Bull. Amer. Math. Soc.*, 73 (1967), 722–726.

[4]. DIXMIER, J., *Les C^*-algèbres et leurs representations*. Gauthier-Villars, Paris, 1964.

[5]. —— *Les algèbres d'operateurs dans l'espace Hilbertien*. Gauthier-Villars, Paris, 1957.

[6]. DUNFORD, N. & SCHWARTZ, J. T., *Linear operators, part I*. Interscience, New York, 1958.

[7]. GLIMM, J., On a certain class of operator algebras. *Trans. Amer. Math. Soc.*, 95 (1960), 318–340.

[8]. —— A Stone-Weierstrass theorem for C^*-algebras. *Ann. of Math.*, 72 (1960), 216–244.

[9]. HALMOS, P. R., *A Hilbert space problem book*. Van Nostrand, Princeton, 1967.

[10]. HELSON, H., *Lectures on invariant subspaces*. Academic Press, New York, 1964.

[11]. HOFFMAN, K., *Banach spaces of analytic functions*. Prentice-Hall, Englewood Cliffs, 1962.

[12]. IONESCU-TULCEA, A. & C., On the lifting property (1), *J. Math. Anal. Appl.*, 3 (1961), 537–546.

[13]. KADISON, R. V., A generalized Schwarz inequality and algebraic invariants for operator algebras. *Ann. of Math.*, 56 (1952), 494–503.

[14]. —— The trace in finite operator algebras. *Proc. Amer. Math. Soc.*, 12 (1961), 973–977.

[15]. KELLEY, J., NAMIOKA, I., and co-authors, *Linear topological spaces*. Van Nostrand, Princeton, 1963.

[16]. PHELPS, R. R., *Lectures on Choquet's theorem*. Van Nostrand, Princeton, 1966.

[17]. RICKART, C. E., *Banach algebras*. Van Nostrand, Princeton, 1960.

[18]. RIESZ, F. & SZ.-NAGY, B., *Functional analysis*. Frederick Ungar, New York, 1955.

[19]. RUDIN, W., Boundary values of continuous analytic functions. *Proc. Amer. Math. Soc.*, 7 (1956), 808–811.

[20]. SARASON, D. E., On spectral sets having connected complement. *Acta Sci. Math. (Szeged)*, 26 (1965), 289–299.

[21]. —— A note on the Volterra operator. *J. Math. Anal. Appl.*, 12 (1965), 244–246.

[22]. —— Generalized interpolation in H^∞. *Trans. Amer. Math. Soc.*, 127 (1967), 179–203.

[23]. STINESPRING, W. F., Positive functions on C^*-algebras. *Proc. Amer. Math. Soc.*, 6 (1955), 211–216.

[24]. STØRMER, E., Positive linear maps of operator algebras. *Acta Math.*, 110 (1963), 233–278.

[25]. SZ.-NAGY, B., *Extensions of linear transformations in Hilbert space which extend beyond this space*. Appendix to [18].

[26]. SZ.-NAGY, B., & FOIAS, C., Sur les contractions de l'espace de Hilbert VIII. *Acta Sci. Math. (Szeged)*, 25(1964), 38–71.

[27]. TAKESAKI, M., A note on the cross-norm of the direct product of operator algebra. *Kodai Math. Seminar Reports*, 10 (1958), 137–140.

[28]. TOPPING, D., UHF algebras are singly generated. *Math. Scand.*, 22 (1968), 224–226.

[29]. ROSENBERG, A., The number of irreducible representations of simple rings with no minimal ideals. *Amer. J. Math.* 75 (1953), 523–530.

[30] ARVESON, W. B., Unitary invariants for compact operators. *Bull. Amer. Math. Soc.*, 76 (1970) (to appear).

Received April 9, 1969