# $k$-MERSIONS OF MANIFOLDS

BY

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## 1. Introduction and statement of results

This paper contains a generalization of the Smale–Hirsch classification of immersions, and the Phillips classification of submersions.

Let $M^n$ be an $n$-dimensional $C^\infty$ manifold and $W^p$ a $p$-dimensional $C^\infty$ manifold. A $C^\infty$ mapping $f$: $M^n \to W^p$ is called a $k$-mersion if its rank is $\geqslant k$ everywhere. The set of $k$-mersions, endowed with the $C^1$ topology, is denoted $C^\infty(M^n, W^p; k)$. A $k$-regular homotopy between $k$-mersions $f$ and $g$ is a continuous map $G$: $I \to C^\infty(M^n, W^p; k)$ such that $G(0)=f$ and $G(1)=g$.

A $k$-bundle map, $\psi$: $TM^n \to TW^p$ between the tangent spaces of $M^n$ and $W^p$ is a continuous fiber preserving map such that the restriction of $\psi$ to any fiber is a linear map of rank at least $k$. The space of $k$-bundle maps with the compact open topology is denoted $T(M^n, W^p; k)$.

An $n$-mersion is an immersion, and an $n$-regular homotopy is usually called a regular homotopy. In 1958 and 1959, Smale [8], [10], published papers classifying immersions of spheres in euclidean spaces. Smale proved that if $n<p$, the regular homotopy classes of immersions of $S^n$ in $R^p$ are in one to one corresponence with the homotopy classes of sections of $S^n$ into the bundle associated with $TS^n$ whose fiber is the Stiefel manifold $V_{p,n}$ of $n$ frames in $p$-dimensional euclidean space. Smale obtained this classification by proving a stronger result, namely, that the map $d$: $C^\infty(S^n, R^p; n) \to T(S^n, R^p; n)$ defined by $d(f)=df$ is a weak homotopy equivalence if $n<p$.

In 1959, Hirsch [3] extended this result to the case of immersions, $C^\infty(M^n, W^p; n)$, of a $C^\infty$ manifold in another, where $n<p$ and $\partial W^p$ is empty. Poenaru's exposition of this result [8] was the basis of Phillips' thesis, published in 1965 as [7]. Say that a manifold

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is closed if it is compact and without boundary. Phillips proved that if $M$ is not closed and $\partial W$ is empty, then $d\colon C^\infty(M^n, W^p; p) \to T(M^n, W^p; p)$ is a weak homotopy equivalence. Phillips called maps whose rank equalled the dimension of the image space "submersions".

Poenaru's exposition also is the basis of the generalization given here. As in the earlier work, the result will depend upon showing that certain maps are fibrations (i.e., satisfy the covering homotopy property).

THEOREM 1. *Let $M^n$ and $W^p$ be $C^\infty$ manifolds with $\partial W^p = \varnothing$. The mapping $d\colon C^\infty(M^n, W^p; k) \to T(M^n, W^p; k)$ defined by $d(f) = df$ is a weak homotopy equivalence if either $M^n$ is not closed or $k < p$.*

COROLLARY 1. *If $M^n$ is not closed or $k < p$, and if $\partial W^p = \varnothing$, the $k$-regular homotopy classes of $k$-mersions of $M^n$ in $W^p$ are in one-to-one correspondence with the homotopy classes of $k$-bundle maps of $TM^n$ in $TW^p$.*

Denote by $M^*(p, n; k)$ the set of $p \times n$ matrices of rank at least $k$.

COROLLARY 2. *If either $M^n$ is not closed or $k < p$, the $k$-regular homotopy classes of $k$-mersions of $M^n$ in $R^p$ are in one-to-one correspondence with the homotopy classes of sections of $M^n$ into the bundle associated with $TM^n$ whose fiber is $M^*(p, n; k)$.*

The next section will contain some applications of Theorem 1 and its Corollaries. Sections 3, 4, and 5 will introduce some notation and general background. In sections 6–9, the covering homotopy property will be shown to hold for restriction maps

$$i^*\colon C^\infty(V^n, W^p; k) \to C^\infty(U^n, W^p; k)$$

induced by inclusion $U \subset V$, when either $V^n$ is the union of $U^n$ with a handle of index $\lambda < n$, or $k < p$. Section 10 contains a proof of the fact that if $\psi \in T(M^n, W^p; k)$ (where $\partial W^p = \varnothing$), if $k < p$ and $M^n$ is a compact manifold, then $f \in C^\infty(M^n, W^p; k)$ may be found so that $df$ is homotopic to $\psi$ through $k$-bundle maps, and $f$ is an approximation of $\bar{\psi}$, the map covered by $\psi$.

## 2. Applications and examples

Let $M(p, n; k)$ denote that set of $p \times n$ matrices whose rank is $k$. (This space has dimension $k(n+p-k)$.) Recall that $M^*(p, n; k)$ denotes the set of $p \times n$ matrices whose rank is at least $k$.

Let $\pi\colon E \to M^n$ be the bundle over $M$ associated with $TM$, whose fiber is $M^*(p, n; k)$. By Corollary 2 of Theorem 1, if there exists a section of $M$ in $E$, and either $M$ is not closed or $k < p$, then there is a $k$-mersion of $M$ in $R^p$.

When $k=n$, the fiber of $E$ is $V_{p,n}$. From the fact that $\pi_i(V_{p,n})=0$ when $i<p-n$, it follows that when $p=2n$, a section in $E$, and hence, an immersion of $M$ in $R^{2n}$, exists. When $k=p$, the fiber of $E$ is $V_{n,p}$. If $M$ is not closed, it is easily seen that there is no obstruction to a section when $p=1$, i.e., a 1-mersion exists. These examples and many others are discussed in [3] and [7].

Similar computations can be made when $k<\inf(n, p)$ by examining the homotopy groups of the fibers $M^*(p, n; k)$. Now

$$M^*(p, n; k) = M(p, n) - \bigcup_{0\leqslant i<k} M(p, n; i),$$

where $M(p, n)$ is the space of all $p\times n$ matrices. We will have $M^*(p, n; k)$ connected and simply connected since the codimension of $\bigcup_{0\leqslant i<k} M(p, n; i)$ in $M(p, n)$ is greater than 2 when $k<\inf(n, p)$. Let $X=\bigcup_{1\leqslant i<k} M(p, n; i)$. Alexander duality can be applied to show that when $2\leqslant q<pn-2$,

$$H_q(M^*(p, n; k)) = H^{pn-q-2}(X).$$

Thus, if
$$pn-q-2>(k-1)(p+n-(k-1))-1 \tag{2.1}$$

then $H_q(M^*(p, n; k))=0$. Setting $q=n-1$, and taking $p\geqslant 2$, $n\geqslant 2$, there will be a section of $M$ in $E$ if

$$p>k+[(k-1)/(n-(k-1))], \tag{2.2}$$

where $[x]$ denotes the greatest integer in $x$. Thus if $0<k\leqslant 1+n/2$, there is a $k$-mersion of $M^n$ in $R^{k+1}$. More generally, there is a $k$-mersion of $M^n$ in euclidean $k+r$ space if $k<1+(r/(r+1))n$.

Note that when $\pi_q(M^*(p, n; k))=0$ for $1\leqslant q\leqslant n$, all sections in $E$ are homotopic. Setting $q=n$ in (2.1), this is the case when

$$pn-n-1>(k-1)(p+n-(k-1)). \tag{2.3}$$

A result of this is the following.

(2.4) PROPOSITION. *If $p>3/2(n-1)$, all $n-1$ mersions of $M^n$ in $R^p$ are homotopic through $n-1$ mersions. In particular, any two immersions are homotopic through $n-1$ mersions.*

When the fiber of $E$ is a Stiefel manifold, the primary obstruction to a section in $E$ is a characteristic class of the manifold. Cohomology classes giving the primary obstruction to sections when the fiber is $M^*(p, n; k)$ may be defined similarly. Suppose that $M^n$ is compact and oriented. In [2], Chern described the relationship between real Pontrjagin classes and the primary obstruction to a section in the bundle over $M$ with fiber

$M^*(p, n; p-1)$ associated with $TM$, in the case that $n-p$ is even. (By Corollary 2, this is also the obstruction to the existence of a $p-1$ mersion in $R^p$.) The primary obstruction is an element $\mathbf{P}'_{4m} \in H^{4m}(M; Z)$, where $m = \frac{1}{2}(n-p+2)$. The elements $\mathbf{P}'_{4m}$, $m=1, \ldots, [n/4]$, are expressible in terms of polynomials in the Pontrjagin classes $\mathbf{P}_{4t}$, $t \leqslant m$, and conversely, $\mathbf{P}_{4m}$ is a polynomial in the $\mathbf{P}'_{4t}$, $t \leqslant m$. (See [11].) Note that if a section in $E(M^*(p, n; p-1))$ exists, so does a section in $E(M^*(p-2, n; p-3))$, so that

$$\mathbf{P}'_{4m} = \mathbf{P}'_{4(m+1)} = \ldots = \mathbf{P}'_{4[n/4]} = 0.$$

Thus, if there is an $n-1$ mersion of $M^n$ in $R^n$, the Pontrjagin classes of $M$ are zero.

## 3. Definitions and background

Denote the interval $[0, 1]$ by I, and the $m$-fold product of the interval by $I^m$. The set $I^{m-1}$ may be considered a subset of $I^m$ by identifying $q \in I^{m-1}$ with the point $(q, 0)$ of $I^m$. A point of $I^m$ will usually be denoted by $(q, t)$ where $q \in I^{m-1}$ and $t \in I$.

Let $\mathcal{E}$ and $\mathcal{F}$ be spaces and $j$: $\mathcal{F} \to \mathcal{E}$ be a continuous map. Call a pair $(G_0, g)$ of maps such that $G_0$: $I^{m-1} \to \mathcal{F}$, $g$: $I^m \to \mathcal{E}$ and $j \circ G_0(q) = g(q, 0)$ an $m$-covering pair. The map $j$ has the covering homotopy property if, given any $m$-covering pair, $(G_0, g)$ where $m$ is a positive integer, there always exists a mapping $G$: $I^m \to \mathcal{F}$ such that $j \circ G(q, t) = g(q, t)$ and $G \mid I^{m-1} = G_0$.

The map $j$ has the local covering homotopy property if for every $e \in \mathcal{E}$, there exists a neighborhood $\mathcal{A}(e)$ such that $j$ has the covering homotopy property for $\mathcal{A}(e)$. It is well known that $j$: $\mathcal{F} \to \mathcal{E}$ has the covering homotopy property if and only if $j$ has the local covering homotopy property.

All manifolds and mappings between manifolds will be assumed $C^\infty$ unless otherwise stated. "Smooth" or "differentiable" will mean $C^\infty$. Manifolds may have boundary, unless otherwise stated.

Let $X$ and $W$ be smooth manifolds. Let $T(X, W)$ denote the set of *continuous* maps between $TX$ and $TW$, carrying fibers linearly into fibers. The set $T(X, W)$ will be assumed to carry the compact open topology. The set of all smooth maps from $X$ to $W$ will be denoted $C^\infty(X, W)$. The function $d$: $C^\infty(X, W) \to T(X, W)$ given by $d(f) = df$ gives a one to one correspondence of $C^\infty(X, W)$ with a subset of $T(X, W)$. The $C^1$ topology on $C^\infty(X, W)$ is the topology induced by this correspondence. The special case $k=0$ in Theorem 1 says that $d$ is a weak homotopy equivalence, if $\partial W = \emptyset$ and dim $W > 0$.

A more intuitive description of the $C^1$ topology is given by looking at a particular subbasis. $W$ can be embedded in euclidean space. Given a compact coordinate neighborhood $Y$ in $X$, maps restricted to $Y$ can be viewed as maps of euclidean spaces. A subbasis neighborhood of a map $f$ is given by the set of maps whose restrictions to $Y$ have function values and derivatives close to the values of $f$ and $df$. If $X$ is itself compact, then a norm, $\|f-g\|$ can be defined by taking the sups of all the euclidean norms of $(f-g)$ and $(df-dg)$ over a fixed covering of coordinate neighborhoods. Then the $C^1$ is the metric topology given by $\varrho(f, g)=\|f-g\|$. (See [6].)

It is easy to show that if $X$ is a compact manifold, then $C^\infty(X, W; k)$ is an open subset of $C^\infty(X, W)$.

Some other sets of smooth maps will be of interest The first is the set of embeddings of $X$ in $W$, denoted Emb $(X, W)$. It is a consequence of properties of the $C^1$ topology that Emb $(X, W)$ is an open subset of $C^\infty(X, W)$. Next, define Aut $(W)$ as the set of diffeomorphisms of $W$ onto $W$ with compact support contained in Int $(W)$, i.e., each map is identity on the complement of a compact set contained in the interior of $W$. If $S \subset \operatorname{Int}(W)$, the elements of Aut $(W)$ which are identity on a neighborhood of $W - \operatorname{Int}(S)$ are denoted $\operatorname{Aut}_S(W)$.

A number of proofs have appeared of the following useful lemma, notably those of Palais, [6], and Lima, [4].

(3.1) PALAIS LEMMA. *Let $X$ be a compact manifold. Given a map $b \in \operatorname{Emb}(X, \operatorname{Int}(W))$, there is a neighborhood $\mathcal{B}$ of $b$, $\mathcal{B} \subset \operatorname{Emb}(X, W)$, and a continuous map $\beta$: $\mathcal{B} \to \operatorname{Aut}(W)$ such that $e=\beta(e)\circ b$ for every $e \in \mathcal{B}$ and $\beta(b)=$ identity.*

## 4. The factorization of smooth maps

Let $U=U^n$ be a smooth compact $n$-dimensional manifold with boundary. Using a collaring of $U$, and identifying $\partial U$ with $\partial U \times \{0\}$, it is easy to define a $C^\infty$ structure for $U \cup \partial U \times [0, 2]$ so that this set is a compact neighborhood of $U$, which shall be denoted $N(U)$, or simply by $N$. The lemma that follows gives a useful property of any smooth $f \in C^\infty(U, W^p)$, where $\partial W^p = \emptyset$; namely, that maps in some neighborhood of $f$ can be factored through automorphisms of some compact manifold which contains an imbedded image of $N(U)$.

Dimensional superscripts will be omitted where the meaning is clear. When $E$ is a vector bundle with a Riemannian metric, let $E(a)$ be the bundle of vectors whose length is $\leqslant a$.

(4.1) LEMMA. *Let $U^n$ be a smooth compact manifold with boundary, and $W^p$ be a smooth manifold without boundary. Given $f \in C^\infty(U, W)$, there exists a neighborhood $\mathcal{A}$ of $f$ and a factorization of the maps of $\mathcal{A}$ through automorphisms of some compact $n+p$ dimensional manifold $\mathbf{D}$ containing an imbedded image of $N(U)$; i.e., there exists a smooth imbedding $s$: $N(U) \to \mathbf{D}$, a smooth map $\phi$: $\mathbf{D} \to W$, and a continuous map $\nu$: $\mathcal{A} \to \operatorname{Aut}(\mathbf{D})$ such that the diagram below is commutative.*

$$\begin{tikzcd} & \mathbf{D} \arrow[r, dash, "\nu(g)"] & \mathbf{D} \arrow[ddr, "\phi"] & \\ N(U) \arrow[ur, "s"] & & & \\ U \arrow[u, hook] \arrow[rrr, "g"] & & & W \end{tikzcd}$$

*Proof.* Fix a riemannian metric for $TW$ and let $\exp(z, v)$, $v \in TW_z$, denote the corresponding exponential function. Clearly, there exists a smooth extension $h$ of $f$ to $N$. There is a neighborhood of the zero section of $h(N)$ on which $\exp(z, v)$ is defined and one to one on fibers. Let $E$ be the bundle induced from this neighborhood by $h$, with the induced riemannian metric. The map

$$\zeta(x, v) = (x, \exp(h(x), v)) \tag{4.2}$$

is one to one on fibers, and has rank $n+p$ on $E(\mathcal{E})$ for some $\mathcal{E} > 0$. But $\zeta$ is one to one, for if $\zeta(x, v) = \zeta(x', v')$, then $x = x'$, and so also $v = v'$.

Let $s$: $N \to E$ be the zero section and let $\mathbf{D}$ be any smooth $n+p$ dimensional manifold contained in $E(\mathcal{E})$, containing $s(N)$. Then $\zeta \circ s$ is the graph of $h$, and $\zeta$ maps $\mathbf{D}$ diffeomorphically onto a neighborhood of the graph of $h$ in $N \times W$. Hence, there is a neighborhood of $f$ on which the map $g \to \theta(g)$ given by

$$\theta(g)(x) = \zeta^{-1}(x, g(x))$$

is defined and continuous. Let $\phi = \pi_W \circ \zeta$, where $\pi_W$ is the projection of $N \times W$ on $W$. Then $\phi \circ \theta(g)(x) = g(x)$.

In order to replace $\theta(g)$ by an automorphism $\nu(g)$, note that $\theta(f) = s \mid U$ is an embedding. Applying the Palais lemma, (3.1), there is a neighborhood $\mathcal{B}$ of $s \mid U$ in Emb $(U, \mathbf{D})$ and a continuous map $\beta$: $\mathcal{B} \to \operatorname{Aut}(\mathbf{D})$ such that $e = \beta(e) \circ (s \mid U)$ for every $e \in \mathcal{B}$, and $\beta(s \mid U) =$ identity. Then for a sufficiently small neighborhood $\mathcal{A}$ of $f$, $\nu = \beta \circ \theta$ satisfies the requirements of Lemma (4.1).

## 5. A handlebody decomposition

Suppose that $M^n$ is a connected manifold. If $M^n$ is compact, then a Morse function may be used to obtain a handlebody decomposition of $M$. If $M$ is not compact, then the procedure used in [7] yields such a decomposition. By alternating the addition of a collar with the addition of a handle, it can be assumed that each handle is attached in a coordinate neighborhood of $M$ (see [5]). Thus $M$ is built up from a disk $D^n$ by successively adding a collar, and then adding a handle in a coordinate neighborhood. It will be convenient to change this decomposition slightly. If $Z$ is $Y \cup$ handle of index $\lambda$, we cut out of $Z$ a part of the handle $H$ which is diffeomorphic to (Int $D^\lambda$) $\times D^{n-\lambda}$, so that the set that remains contains an open subset diffeomorphic to $S^{\lambda-1} \times D^{n-\lambda} \times (-1, 0]$. A set obtained from a manifold in this way will be called a $\lambda$-truncated manifold.

Using this procedure of $\lambda$-truncating manifolds, we can obtain a decomposition:

$$D^n = Y_1 \subset X_1 \subset Y_2 \subset X_2 \subset \ldots \subset M^n. \tag{5.1}$$

(5.2) *Definition.* Let $X$ be a smooth manifold and $Y \supset X$ be smooth or $\lambda$-truncated. Then $Y$ is collar retractible with respect to $X$ if, given a neighborhood $S$ of $X$ in $Y$, $Y$ may be deformed into $S$ through diffeomorphisms of $Y$ into $Y$ which hold $X$ fixed.

Hence, $M^n$ is built up from a disk by successive inclusion in a collar retractible neighborhood and addition of a handle.

The proof of Theorem 1 is based on the diagram:

$$\begin{array}{ccc} C^\infty(M, W; k) & \xrightarrow{d} & T(M, W; k) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ C^\infty(Y_2, W; k) & \xrightarrow{d} & T(Y_2, W; k) \\ \downarrow & & \downarrow \\ C^\infty(X_1, W; k) & \xrightarrow{d} & T(X_1, W; k) \\ \downarrow & & \downarrow \\ C^\infty(D^n, W; k) & \xrightarrow{d} & T(D^n, W; k) \end{array} \tag{5.3}$$

The proof that $d$ is a weak homotopy equivalence follows the same outlines originally introduced by Smale in his work on the immersions of spheres, and Poenaru in his exposition using handle bodies. Namely, it must be shown that the $d$ of the bottom row is a weak homotopy equivalence, that all of the vertical restriction maps are fibrations (i.e., satisfy the covering homotopy property), and that $d$ restricted to a fiber is a weak homotopy equivalence in any of the boxes of the diagram. Given these results, using the five lemma and induction, every $d$ map reached after a finite number of steps will be a weak homotopy equivalence. If the diagram is infinite, then the fact that the inverse limit of weak homotopy equivalences of fiber spaces is a weak homotopy equivalence, which is proved in Appendix 1 of [7], yields Theorem 1.

The main step in establishing Theorem 1 is the proof of the local covering homotopy property for the restiction maps on the left side of (5.3). This property will be proved in three cases. These are stated below as theorems. Henceforth the target manifold $W^p$ will be a manifold without boundary.

(5.4) THEOREM. *Let $V^n$ be a compact manifold or $\lambda$-truncated manifold, $\lambda \leqslant n$, and let $U^n \subset \operatorname{Int}(V^n)$ be a smooth compact manifold. Suppose that $\partial W^p = \varnothing$ If $V^n$ is collar retractible* *with respect to* $U^n$, *then* $i^*$: $C^\infty(V, W; k) \to C^\infty(U, W; k)$ *has the local covering homotopy property, where* $i^*$ *is induced by the inclusion on* $U$ *in* $V$.

(5.5) THEOREM. *Suppose that the smooth compact manifold* $V^n$ *is the union of a* $\lambda$-*truncated* $U^n \subset V^n$ *with a handle,* $V^n = U^n \cup D^\lambda \times D^{n-\lambda}$. *Suppose that* $\partial W^p = \varnothing$. *If* $\lambda < n$, *then the map* $i^*$: $C^\infty(V, W; k) \to C^\infty(U, W; k)$ *induced by the inclusion map has the local covering homotopy property.*

(5.6) THEOREM. *Suppose that* $V^n = U^n \cup D^n$, *i.e.,* $V^n$ *is* $U^n$ *with a handle of index* $n$ *attached. Let* $\partial W^p = \varnothing$. *Then, if* $k < p$, $i^*$: $C^\infty(V, W; k) \to C^\infty(U, W; k)$ *induced by inclusion has the local covering homotopy property.*

Note that since the handles of the decomposition (5.1) are added in coordinate neighborhoods, it will suffice to prove (5.5) and (5.6) when $U^n \subset V^n \subset R^n$.

Much of the proof of these theorems will be based on [7] and [8]. The main difficulties raised by generalization appear in the proof of (5.6).

The other facts required for Theorem 1 are stated here without proofs, since they may be proved exactly as are the corresponding facts in [7]. (The proof of (5.10) follows the same outline as the corresponding fact in [7], substituting (5.5) and (5.6) for the analagous statements about submersions.)

(5.7) THEOREM. *Let* $D^n$ *denote an n-dimensional disk. The map*

$$d: C^\infty(D^n, W^p; k) \to T(D^n, W^p; k)$$

*defined by* $d(f) = df$, *is a weak homotopy equivalence, when* $W^p$ *is a manifold with empty boundary.*

(5.8) THEOREM. *Suppose that either* $U^n$ *is smooth,* $U^n \subset \operatorname{Int}(V^n)$, *where* $V^n$ *is* $\lambda$-*truncated, or that* $U^n$ *is* $\lambda$-*truncated and* $V^n = U^n \cup D^\lambda \times D^{n-\lambda}$ *is smooth. Then the map* $j^*$: $T(V, W; k) \to T(U, W; k)$ *induced by inclusion is a fibration.*

(5.9) THEOREM. *Given* $U^n \subset \operatorname{Int}(V^n)$, $V$ *collar retractible with respect to* $U$, $\partial W = \varnothing$ *then the restriction maps* $i^*$: $C^\infty(V, W; k) \to C^\infty(U, W; k)$ *and* $j^*$: $T(V, W; k) \to T(U, W; k)$ *are homotopy equivalences.*

(5.10) THEOREM. *Let* $U^n$ *be a* $\lambda$-*truncated manifold and* $V^n = U^n \cup D^\lambda \times D^{n-\lambda}$. *Let* $i^*$: $C^\infty(V, W^p; k) \to C^\infty(U, W^p; k)$ *and* $j^*$: $T(V, W^p; k) \to T(U, W^p; k)$ *be the restriction maps induced by the inclusion of* $U$ *in* $V$, *where* $\partial W^p = \varnothing$. *Then if* $\lambda < n$ *or* $k < p$, *for each* $f \in C^\infty(U, W; k)$,

$$d \mid i^{*-1}f\colon\ i^{*-1}(f) \to j^{*-1}(df)$$

*is a weak homotopy equivalence.*

## 6. Some extension lemmas

Throughout this section, we will write $U$ and $V$ for $U^n$ and $V^n$. Also, it will be assumed that either $U$ is smooth and $V$ is either smooth or $\lambda$-truncated, or that $U$ is $\lambda$-truncated, $1 \leqslant \lambda \leqslant n$, and $V = U \cup D^\lambda \times D^{n-\lambda}$. Recall that the boundary, $\partial W^p$, of the target manifold is always assumed empty.

The local covering homotopy property states that given $f \in C^\infty(U, W; k)$, there is a neighborhood $\mathcal{A}$ of $f$ such that for each pair $(G_0, g)$ with $g: I^m \to \mathcal{A}$, $G_0: I^m \to C^\infty(V, W; k)$, and $G_0(q) \mid U = g(q, 0)$, there is a continuously varying set of extensions $G(q, t)$ of $g(q, t)$ to $V$ such that $G(q, 0) = G_0(q)$. The theorems (5.4), (5.5), and (5.6) are proved by finding a set of hypotheses for $\mathcal{A}$ which make possible the explicit construction of a lift of any given covering pair for $\mathcal{A}$. The two lemmas of this section contribute to the construction in all three cases. In fact, (5.4) and (5.5) are easily proved in the next section, using these lemmas.

The first states that for any small enough $\mathcal{A}$, there is a neighborhood $\hat{U}$ of $U$ in $V$ and a lift to $\hat{U}$ of $(G_0 \mid \hat{U}, g)$. The second shows that a covering pair $(G_0, g)$ for $\mathcal{A}$ can always be at least partially lifted; i.e., there is an $\varepsilon > 0$ and a lift over the interval $[0, \varepsilon]$. Both of these extension lemmas are "general", that is, they require no hypotheses on the size of $k$ or, on the index of the handle, when $V$ is $U \cup$ handle.

It will be convenient to use some special notation for certain neighborhoods of $U$ in $V$. Let $\dot{U}$ be the boundary of $U$ in $V$. If $U$ is smooth, $\dot{U} = \partial U$, and if $U$ is $\lambda$-truncated, $\dot{U}$ is $S^{\lambda-1} \times D^{n-\lambda} \times \{0\}$. Let $\hat{U} = U \cup \dot{U} \times [0, 1]$ and $\hat{U}_\tau = U \cup \dot{U} \times [0, \tau]$, $0 < \tau \leqslant 1$.

First note that the results of section 4 can easily be extended to a $\lambda$-truncated manifold $U$. For if $f \in C^\infty(U, W; k)$, let $h$ be an extension of $f$ to a smooth neighborhood $N$ of $U$, $h \in C^\infty(N, W; k)$ and define $\mathbf{D}$ and the factorization just as before.

Thus, for $U$ smooth or $\lambda$-truncated there is a neighborhood $\mathcal{A}$ of $f$ and a factorization as in (4.1), i.e., if $g \in \mathcal{A}$, then $g(x) = \phi \circ \nu(g) \circ s(x)$, $x \in U$. Note that the map $\alpha \to \phi \circ \alpha \circ s$ defines a continuous map from Aut $(\mathbf{D})$ to $C^\infty(N, W)$ mapping the identity onto $h \in C^\infty(N, W; k)$. Hence there is a neighborhood $\Gamma$ of the identity in Aut $(\mathbf{D})$ mapping into $C^\infty(N, W; k)$. Denote by $\Gamma'$ a neighborhood contained in $\Gamma$ such that if $\gamma_1, \dots, \gamma_{10} \in \Gamma'$, then $\gamma_1^{\pm 1} \dots \gamma_{10}^{\pm 1} \in \Gamma$. We can assume that $\mathcal{A}$ has been chosen with $\nu(\mathcal{A}) \subset \Gamma'$. This will be a very useful property. For example, if $\beta \in \Gamma'$ and $\beta \mid s(U)$ is the identity, then $\phi \circ \nu(g) \circ \beta \circ s$ gives another extension of $g$ to $N$ belonging to $C^\infty(N, W; k)$.

(6.1) LEMMA. *Given $f \in C^\infty(U^n, W^p; k)$, ($\partial W^p = \emptyset$), let $\mathcal{A}$ be a neighborhood of $f$ for which a factorization $g(x) = \phi \circ \nu(g) \circ s(x)$ is defined as in* (4.1). *Suppose that $\Gamma$ and $\Gamma'$ are defined as above and $\nu(\mathcal{A}) \subset \Gamma'$. Given an m-covering pair $(G_0, g)$ with $g: I^m \to \mathcal{A}$, there is a number $\tau > 0$ and a map $\alpha: I^{m-1} \to \Gamma'$ such that*

$$\begin{aligned} &(a)\quad \alpha(q)\,|\,s(U) = \textit{identity}, \\ &(b)\quad \phi\circ\nu(g(q,0))\circ\alpha(q)\circ s\,|\,\hat{U}_\tau = G_0(q)\,|\,\hat{U}_\tau. \end{aligned} \tag{6.2}$$

The proof of this lemma will be given at the end of the section. It is purely technical, and the reader is advised to omit it from a first reading in order to preserve the continuity of the main argument.

It is immediate from (6.2) that $\phi\circ\nu(g(q,t))\circ\alpha(q)\circ s\,|\,\hat{U}_\tau$ is a lift to $\hat{U}_\tau$ of $(G_0\,|\,\hat{U}_\tau, g)$.

The next lemma is suggested by examination of formulas (6.2). If $\nu(g(q,t))\circ\alpha(q)$ could be altered so that on a neighborhood of $\dot{U}\times\{\tau\}$, $\nu(g(q,t))\circ\alpha(q) = \nu(g(q,0))\circ\alpha(q)$, then the extensions $G(q,t)$ of $g(q,t)$ could be defined on all of $V$ simply by setting $G(q,t) = G_0(q)$ on $V - \hat{U}_\tau$.

(6.3) LEMMA. *Suppose that the hypotheses of* (6.1) *hold for* $\mathcal{A}$, *and, given an m-covering pair* $(G_0, g)$, *that* $\tau > 0$ *and* $\alpha\colon I^{m-1}\to\Gamma'$ *have been found satisfying* (6.2), (*a*) *and* (*b*). *Then there is an* $\varepsilon > 0$ *and a map* $\mu\colon I^{m-1}\times[0,\varepsilon]\to\Gamma'$ *such that*

(*a*) $\mu(q,0) = \textit{identity}$,

(*b*) $\mu(q,t)\circ s\,|\,U = \alpha(q)^{-1}\circ\nu(q,0)^{-1}\circ\nu(q,t)\circ s\,|\,U$,

(*c*) $\mu(q,t)\,|\,s(\dot{U}\times[\tau/2,\tau]) = \textit{identity}$.

The proof is omitted, since the proof of the corresponding fact in [7], can be adapted to fit the present case.

Using $\mu$, a lift over $I^{m-1}\times[0,\varepsilon]$ can be formulated (see (7.4)).

*Proof of Lemma (6.1).* Suppose given a covering pair $(G_0, g)$ for $\mathcal{A}$. Write $\nu(q,t)$ for $\nu(g(q,t))$. First we have:

(6.4) SUBLEMMA. *There is a* $\sigma$, $0<\sigma\leqslant 1$, *and a map* $\lambda\colon I^{m-1}\to\operatorname{Emb}(\hat{U}_\sigma, \operatorname{Int}(\mathbf{D}))$, *with*

(*a*) $\lambda(q)\,|\,U = s\,|\,U$

(*b*) $\phi\circ\nu(q,0)\circ\lambda(q) = G_0(q)\,|\,\hat{U}_\sigma$.

*Proof.* Let $\lambda(q)$ be the map given by

$$\lambda(q)(x) = \nu(q,0)^{-1}\circ\zeta^{-1}(x, G_0(q)(x)),\quad x\in\hat{U},$$

where $\zeta$ is the map defined in (4.2). For a fixed $q$, there is a $\sigma$ such that $\lambda(q)$ is an embedding on $\hat{U}_\sigma$. By a standard compactness argument, there is a $\sigma$ which works for all $q\in I^{m-1}$. Clearly (*a*) and (*b*) hold for this $\sigma$ and $\lambda$.

Now $s\,|\,\hat{U}_\sigma$ embeds $\hat{U}_\sigma$ in $\operatorname{Int}(\mathbf{D})$ so that, applying the Palais Lemma, we may define a neighborhood $\mathcal{B}$ of $s\,|\,\hat{U}_\sigma$ and a map $\beta\colon\mathcal{B}\to\operatorname{Aut}(\mathbf{D})$ as in (4.5). If $\lambda(I^{m-1})$ were contained in $\mathcal{B}$, then $\alpha(q) = \beta(\lambda(q))$ would satisfy (6.2). In the next sublemma, it will be shown that the $\lambda(q)$ can be replaced by embeddings $\lambda'(q)$ which are so "close" to $s\,|\,\hat{U}_\sigma$ that they will be in $\mathcal{B}$.

Let $\mathbf{D}$ be embedded in some euclidean space $R^m$. There is a neighborhood $T$ of $\mathbf{D}$ in $R^m$, an open set $F \subset \operatorname{Int}(\mathbf{D})$ with $\lambda(q)(\hat{U}_\sigma) \subset \operatorname{Int}(F)$, $q \in I^{m-1}$, and a smooth map $r: T \to \mathbf{D}$ such that $r \mid F = \text{identity}$. Now $r$ induces a continuous map

$$r^*: C^\infty(\hat{U}_\sigma, T) \to C^\infty(\hat{U}_\sigma, \mathbf{D})$$

with $r^*(s) = s$. In fact, if $g \in C^\infty(\hat{U}_\sigma, T)$ and $g(x) \in F$, $r^*(g)(x) = g(x)$.

Given coordinate neighborhoods and maps for $\dot{U}$, define coordinates for $\dot{U} \times [-1, 1]$, using these maps $\times$ the identity on $[-1, 1]$. Let $\varrho$ be a metric for $C^\infty(\hat{U}, W)$ defined using these coordinates. Define restrictions of $\varrho$, $\varrho \mid \hat{U}_r$, $0 < r \leqslant 1$, in the obvious way, and denote the restriction of $\varrho$ to $\hat{U}_r$ by $\varrho_r$. Then there is an $\varepsilon > 0$ such that if $g \in \operatorname{Emb}(\hat{U}_\sigma, T)$ and $\varrho_\sigma(g, s) < \varepsilon$, then $r^*(g) \in \mathcal{B}$. Thus it suffices to show:

(6.5) SUBLEMMA. *There is a $\tau$, $0 < \tau < \sigma$, and a map $\lambda': I^{m-1} \to \operatorname{Emb}(\hat{U}_\sigma, T)$ with*

(*a*) $\varrho_\sigma(\lambda'(q), s) < \varepsilon, \quad q \in I^{m-1}$

(*b*) $\lambda'(q) \mid \hat{U}_\tau = \lambda(q) \mid \hat{U}_\tau, \quad q \in I^{m-1}$

*Proof*. Let $e(q) = \lambda(q) - s$. Then for every $q$, $e(q)(x) = 0$ if $x \in U$. By the usual compactness argument, for some $\sigma_1$, $0 < \sigma_1 < \sigma$, $\varrho_{\sigma_1}(e(q), 0) < \varepsilon/2$ for every $q \in I^{m-1}$.

Now let $c$: $[0, \sigma] \to [0, \sigma_1]$ be a $C^\infty$ map which is the identity on $[0, \sigma_1/2]$ and such that $0 < c'(t) \leqslant 1$. Define

$$e_1(q)(x) = \begin{cases} 0 & x \in U \\ e(q)(y, c(t)) & x = (y, t) \in \dot{U} \times [0, \sigma]. \end{cases}$$

Let $\lambda'(q) = s + e_1(q)$ and let $\tau = \sigma_1/2$. Then $\lambda'(q) \mid \hat{U}_\tau = \lambda(q) \mid \hat{U}_\tau$. This completes the proof of (6.1).

## 7. Covering homotopy property in the collar retractible and $\lambda < n$ case

(7.1) LEMMA. *Let $V^n$ be a compact manifold or $\lambda$-truncated manifold, and $U^n \subset V^n$ be a smooth compact manifold. Suppose that $V$ is collar retractible with respect to $U$. Let $f \in C^\infty(U, W^p; k)$. If $\mathcal{A}$ is a neighborhood of $f$ satisfying the hypotheses of* (6.1), *then $i^*$* (*induced by inclusion*) *has the covering homotopy property for $\mathcal{A}$.*

*Proof*. Given an $m$-covering pair, $(G_0, g)$ for $\mathcal{A}$, choose $\tau$, $\alpha$, and $\mu$, $\varepsilon$, as in (6.1) and (6.3). Denote $\nu(g(q, t))$ by $\nu(q, t)$. There is a deformation $\delta$: $[0, 1] \to \operatorname{Emb}(V, V)$ with $\delta(0) =$ identity, $\delta(t) \mid U = \text{identity}$, and $\delta(1)(V) \subset \hat{U}_\tau$. Define, for $t \leqslant \varepsilon$,

$$G(q, t) = \begin{cases} \phi \circ \nu(q, 0) \circ \alpha(q) \circ \mu(q, t) \circ s \circ \delta(t/\varepsilon)(x), & x \in \delta^{-1}(t/\varepsilon)(\hat{U}_\tau) \\ G_0(q) \circ \delta(t/\varepsilon)(x) & \text{otherwise.} \end{cases} \tag{7.2}$$

If $\varepsilon \leqslant t \leqslant 1$, let

$$G(q, t) = \phi \circ \nu(q, t) \circ \nu^{-1}(q, \varepsilon) \circ \nu(q, 0) \circ \alpha(q) \circ \mu(q, \varepsilon) \circ s \circ \delta(1). \tag{7.3}$$

It is easy to check that $G$ satisfies all of the requirements.

Note that Theorem (5.4) is immediate from (7.1).

When $V = U \cup$ handle of index $\lambda < n$, the situation is not quite as simple. We do at least have a map

$$G'\colon I^{m-1} \times [0, \varepsilon] \to C^\infty(V, W; k)$$

defined by

$$\begin{aligned} G'(q, t)(x) &= \phi \circ \nu(q, 0) \circ \alpha(q) \circ \mu(q, t) \circ s(x) \quad & x \in \hat{U}_\tau \\ G'(q, t)(x) &= G_0(q)(x) & x \in V - \hat{U}_\tau. \end{aligned} \tag{7.4}$$

The problem is to change $G'$ so that an extension can be lifted all of the way across $[0, 1]$. The idea behind the change is the following. Near $\partial \mathbf{D}$, all of the automorphisms of $\mathbf{D}$ are the identity. If there were a deformation $\xi\colon [0, \varepsilon] \to \operatorname{Emb}(\hat{U}_\tau, \mathbf{D})$ such that $U$ was held fixed (i.e., $\xi(t) \mid U = s \mid U$) while $\dot{U} \times \{2\tau/3\}$ was pulled out to the boundary of $\mathbf{D}$, then for any $\omega \in \operatorname{Aut}(\mathbf{D})$, $\phi \circ \omega \circ \xi(\varepsilon)(x) = \phi \circ \omega \circ s(x)$ for $x$ in $U$, and $\phi \circ \omega \circ \xi(\varepsilon)(x) = \phi \circ \xi(\varepsilon)(x)$ for $x$ near $\dot{U} \times \{2\tau/3\}$. The use of $\xi$ would thus move all of the maps $G'(q, \varepsilon)$ to maps which all agree near $\dot{U} \times \{2\tau/3\}$. Using this, it would not be hard to paste together an extension $G$.

Now there always exist such deformations $\xi$, but the difficulty will lie in showing the existence of a deformation such that rank of at least $k$ is preserved in the construction. It is in assuring this property that the hypothesis $\lambda < n$ (and later, $k < p$) is used. All of this will be made precise in the definition and lemmas that follow.

We will want to discuss embeddings of $\hat{U}_\tau$ in a larger manifold containing a factoring manifold $\mathbf{D}$. First note that in section 4, the extension $h$ could have been assumed defined and of rank $\geqslant k$ on a neighborhood $N' = N \cup \partial N \times I$. Similarly, the diffeomorphism could have been defined on $\mathbf{D}' = \mathbf{D} \cup \partial \mathbf{D} \times I$, and $\phi$ could be considered as a mapping on $\mathbf{D}' \supset \mathbf{D}$. All of the automorphisms $\nu(g)$ may be assumed to lie in $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$.

(7.5) *Definition.* Suppose given a map $f \in C^\infty(U^n, W^p; k)$, a factorization of maps in a neighborhood of $f$ through elements of $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$, and a neighborhood $\Gamma$ of the identity in $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$. We say that positioning deformations exist for $\mathbf{D}$ and $\Gamma$ if, given any $\tau$, $0 < \tau \leqslant 1$, there is a map $\xi\colon [0, 1] \to \operatorname{Emb}(\hat{U}_\tau, \mathbf{D}')$ with

(a) $\xi(0) = s \mid \hat{U}_\tau$.

(b) $\xi(t)(x) = s(x)$, for $x$ in a neighborhood of $\hat{U}_{\tau/2} \cup \dot{U} \times \{\tau\}$.

(c) $\xi(1)(\hat{U}_\tau) \cap \partial \mathbf{D} = \xi(1)(\dot{U} \times \{2\tau/3\} \cup \dot{U} \times \{5\tau/6\})$.

(d) If $\omega \in \Gamma$, then $\phi \circ \omega \circ \xi(t) \in C^\infty(\hat{U}_\tau, W; k)$, $0 \leqslant t \leqslant 1$.

(7.6) LEMMA. *Suppose that maps in a neighborhood of $f$ are factored through elements of* $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$, *and that there is a neighborhood $\Gamma$ of the identity in* $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$ *such that positioning deformations exist for $\mathbf{D}$ and $\Gamma$. Then the local covering homotopy property holds at $f$.*

*Proof.* Let $\mathcal{A}$ be a neighborhood of $f$ with $\nu(\mathcal{A})\subset\Gamma'$. Given a covering pair $(G_0, g)$ for $\mathcal{A}$, define $\tau$, $\alpha$, and $\mu$, $\varepsilon$, as in (6.1) and (6.3). Write $\nu(q, t)$ for $\nu(g(q, t))$. Let $\xi$ be a positioning deformation for $\tau$. Define, if $t\leqslant\varepsilon$,

$$G(q,t)=\begin{cases}\phi\circ\nu(q,0)\circ\alpha(q)\circ\mu(q,t)\circ\xi(t/\varepsilon)\,(x), & x\in\hat{U}_\tau\\ G_0(q)\,(x) & x\in V-\hat{U}_\tau.\end{cases}\tag{7.7}$$

If $\varepsilon\leqslant t\leqslant 1$, define

$$G(q,t)=\begin{cases}\phi\circ\nu(q,t)\circ\nu^{-1}(q,\varepsilon)\circ\nu(q,0)\circ\alpha(q)\circ\mu(q,\varepsilon)\circ\xi(1)\,(x) & \text{when } x\in\hat{U}_{2\tau/3},\\ G(q,\varepsilon)\,(x), & \text{when } x\in V-\hat{U}_{2\tau/3}.\end{cases}\tag{7.8}$$

It is easily verified that $G$ is well defined and continuous at $t=\varepsilon$. Clearly $G(q,0)=G_0(q)$, and, by the definition of $\nu$ and $\mu$, $G(q,t)\,|\,U=g(q,t)$.

(7.9) LEMMA. *Let $U^n$ be $\lambda$-truncated, $\lambda<n$. Suppose given $f\in C^\infty(U^n, W^p; k)$. There exists a factoring manifold $\mathbf{D}$ for $f$, and a neighborhood $\Gamma$ of the identity in* $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$, $(\mathbf{D}'=\mathbf{D}\cup\partial\mathbf{D}\times I)$ *such that positioning deformations exist for $\mathbf{D}$ and $\Gamma$.*

The existence of such deformations has been cited in [7]. An explicit construction will not be given here, but we will indicate the idea. Take $\hat{U}_2$ and cut off a neighborhood of the "corners" to get a smooth manifold $U'$. Let $N$ and $N'$ be collarings of $U'$, and construct $\mathbf{D}$ and $\mathbf{D}'$ as in section 4. Let $\Gamma$ be a neighborhood of the identity in $\operatorname{Aut}_{\mathbf{D}}(\mathbf{D}')$ with $\phi\circ\gamma\circ s\in C^\infty(N', W; k)$, $\gamma\in\Gamma$. *Any deformation $\xi$ occurring in $s(N')$ will preserve rank*, i.e., if $\xi(t)(U')\subset s(N')$, then (7.5d) holds.

Let us denote a disk of dimension $r$ and radius $s$ by $D^r_s$. Let $D^r=D^r_1$. Note that the parts of $N$ and $N'$ surrounding $S^{\lambda-1}\times D^{n-\lambda}\times I$ are $S^{\lambda-1}\times D^{n-\lambda}_2\times I$ and $S^{\lambda-1}\times D^{n-\lambda}_3\times I$. Also, $S^{\lambda-1}\times\partial D^{n-\lambda}_2\times I\subset\partial N$. Choosing any radial direction in $D^{n-\lambda}$, given $\tau$, then $\dot{U}\times\{2\tau/3\}\cup\dot{U}\times\{5\tau/6\}$ can be deformed to $\partial N$ holding $U_{\tau/2}$ fixed. Note that the hypothesis $\lambda<n$ is used to assure that some radial direction exists. Composing $s$ with this deformation gives the desired deformation.

## 8. Covering homotopy property when $\lambda=n$ and $k<p$

The only case that remains is $\lambda=n$ and $k<p$. In this case, we can assume that $V^n$ is a disk and $U^n$ is a disk whose center has been hollowed out. For convenience of notation, we can assume that $V=D^n_3$, $U=D^n_3-\operatorname{Int}(D^n_2)$, and $\dot{U}=S^n_2$. In this case $U$ is smooth. Let

$f \in C^\infty(U, W^p; k)$. There is an extension $h$ of $f$ to a neighborhood $N' = D_3^n - \mathrm{Int}\,(D_1^n)$ with $h \in C^\infty(N', W^p; k)$. Define $E = h^\#(TW)$ as before.

The lemmas of section 6 still hold, but it is not possible to define positioning deformations (as in section 7) by deforming the handle to $\partial D$ through the neighborhood $s(N')$ of $s(U)$. Any deformation to $\partial \mathbf{D}$ must move up along fibers of $E$. It is easy to find a deformation satisfying (a), (b), and (c) of (7.5), but it is difficult to satisfy (d), i.e., find a $\xi$ so $\phi \circ \omega \circ \xi$ still has rank at least $k$.

In treating the case of immersions of a manifold in a manifold of higher dimension, Poenaru (1) pointed out that rank could be preserved by using positioning deformations which move the handle to $\partial \mathbf{D}$ through the normal bundle of $h$. The sort of generalization that one would like here would be to take a $k$-dimensional subbundle of $E$ contained in $h^\#(dh(TN'))$, (i.e., $\{(x, v) \in E \mid v \in dh(TN'_x)\}$), define its orthogonal complement as a "$k$-normal bundle", and deform the handle through this. Lemma (8.2) below will show that this is a useful concept. First we introduce some convenient notation.

(8.1) *Definition.* Given $e \in C^\infty(X^n, W^p; k)$, a set of $k$ sections, $\sigma_1, \ldots, \sigma_k$ of $X^n$ in $TX^n$ is called independent with respect to $e$ on $X$ if the vectors $de_x(\sigma_1(x)), \ldots de_x(\sigma_k(x))$ are linearly independent for each $x \in X$. Note that if $X^n \subset R^n$ and $W^p \subset R^p$, a $k$-tuple of tangent vectors in $TX_x$ may be represented as the $k$ columns of a matrix in $M(n, k)$, the set of $n \times k$ matrices, and a $k$-tuple of sections may be considered to be an element $\sigma \in C(X, M(n, k))$, the set of continuous maps from $X$ to $M(n, k)$. We will denote $\sigma = (\sigma_1, \ldots, \sigma_k)$. Sometimes we shall refer to $\sigma$ as a $k$-frame field.

(8.2) LEMMA. *Let $X^n \subset R^n$ be a compact $\lambda$-truncated manifold, where $\lambda \leqslant n$. Suppose $W^p \subset R^p$. Let $e \in C^\infty(X, W; k)$, where $k < p$. Suppose that there is a smooth $\sigma = (\sigma_1, \ldots, \sigma_k)$ defined on a neighborhood of $\dot{X}$ in $X$ which is independent with respect to e, and that the orthogonal complement of the bundle spanned by $e^\#(de \circ \sigma_1), \ldots, e^\#(de \circ \sigma_k)$ is trivial. Then there is a factoring manifold $\mathbf{D}$ and a neighborhood $\Gamma$ of the identity in* Aut $(\mathbf{D})$ *such that positioning deformations exist for $\mathbf{D}$ and $\Gamma$.*

The proof may be found in section 9.

(8.3) COROLLARY. *Under the hypotheses of* (8.2), *the local covering homotopy property holds at e.*

*Proof.* The corollary is immediate from (7.6).

Unfortunately, it is not generally possible to find a $k$-frame field $\sigma$ which is independent

(1) The referee has pointed out that Thom presented this idea in the Seminaire Bourbaki, 1957–1958.

with respect to a given function $h$ on a neighborhood of $\dot{U}$, although such a $k$-frame field may be defined locally, in a neighborhood of any given point. However, it will be our aim to reduce the problem to the case of (8.2). The procedure will be a little complicated, but the basic picture to keep in mind is the following; any covering pair, $(G_0, g)$, could be immediately lifted over most of $V$ by defining $G(q, t)$ to be $G_0(q)$ on a ball, say $D^n_{2-b}$, of radius a little smaller than 2. To lift over the remaining spherical shell, we will break up the shell into little handles, each of which is contained in a region for which a $k$-frame field is defined.

Identify $\dot{U} \times [-1, 1]$ with a bicollaring of $\dot{U}$ in $V$ in such a way that $\dot{U} \times \{0\} = \dot{U}$, $\dot{U} \times [-1, 0] \subset U$, and each $x \times [-1, 1]$ maps into a radial ray. Let $x_1, \ldots, x_n$ be polar coordinates with $x_1$ the radial coordinate. Choose a covering, $B_1, \ldots, B_m$ of $\dot{U} = S^{n-1}_2$ by $(n-1)$-dimensional neighborhoods, each of which is the intersection of $\dot{U}$ with a convex set, such that $f$ maps $B_i \times [-c, 0]$ into a coordinate neighborhood $W_i$ in $W$, and such that there is a $k$-frame field $\mathfrak{X}_i = (\partial/\partial x_{i_1}, \ldots, \partial/\partial x_{i_k})$ independent with respect to $f$ on $B_i \times [-c, 0]$.

Now choose $\mathcal{A}$ to be a neighborhood of $f$ such that each $g$ in $\mathcal{A}$ satisfies the same conditions, i.e.,

$$\begin{aligned}&\text{(a)}\quad g(B_i \times [-c, 0]) \subset W_i,\\ &\text{(b)}\quad \mathfrak{X}_i \text{ is independent with respect to } g \text{ on } B_i \times [-c, 0].\end{aligned} \tag{8.4}$$

We shall prove the covering homotopy property for $\mathcal{A}$. Most of this section will be devoted to proving the following lemma.

(8.5) LEMMA. *Suppose given $\mathcal{A}$ as above and a covering pair $(G_0, g)$ with $g: I^m \to \mathcal{A}$. There exists a manifold $V'$, where $U \subset V' \subset V$, and $V' = V$ minus the interiors of a finite number of disks, $D^n(1), \ldots, D^n(r)$, such that*

(a) *setting $\bar{G}_0(q) = G_0(q) \mid V'$, there is a lifting $\bar{g}$ over $V'$ of $(\bar{G}_0, g)$, and*
(b) *for each $i$, $1 \leqslant i \leqslant r$, and each $(q, t) \in I^m$ there exists a $k$-frame field $\sigma^i(q, t)$ defined in a neighborhood of $D^n(i)$ and independent with respect to $\bar{g}(q, t)$ on $\partial D^n(i)$.*

Throughout the remainder of this section, let $(G_0, g)$, $g: I^m \to \mathcal{A}$ be a *fixed* covering pair.

By (8.4) and the compactness of $G_0(I^{m-1})$, there is a number $b$, $0 < b \leqslant 1$, such that, for every $q$ in $I^{m-1}$,

(a) $G_0(q)(B_i \times [-c, 2b]) \subset W_i$
(b) $\mathfrak{X}_i$ is independent with respect to $G_0(q)$ on $B_i \times [-c, 2b]$.

Now, define
$$\tilde{g}(q, t)(x) = \begin{cases} g(q, t)(x) & x \in U \\ G_0(q)(x) & x \in V - \dot{U}_b. \end{cases}$$

Then it is only necessary to lift $\tilde{g}(q,t)\,|\,S_2^{n-1}\times[-c,0]\cup[b,2b])$ over the shell $S_2^{n-1}\times[-c,2b]$ with, of course, the value $G_0(q)$ at $(q, 0)$.

To do this, a handlebody decomposition is used, which may be obtained as follows. Take a triangulation of $S_2^{n-1}$ so fine that each simplex is contained in the interior of some $B_i$. We can assume that the triangulation is so fine that if a simplex $\Delta\subset\operatorname{Int}(B_i)$ then the convex span of the vertices of $\Delta$ is contained in $B_i\times[0, b/2)$. Let $L$ be the euclidean polygon determined by associating to each simplex of the triangulation the convex set spanned by its vertices. Now the complex $L\times I\cong L\times[0, b]$ can be subdivided to form a simplicial complex $K$ with no additional vertices. Let $K_0=L\times(\{0\}\cup\{b\})$, and let $\Delta_1, \dots, \Delta_r$ be the simplexes of $K-K_0$, arranged so that $\dim(\Delta_i)\leqslant\dim(\Delta_{i+1})$. We can start with a manifold neighborhood of $K_0$, and build up to a manifold neighborhood of $K$ by adding a handle of index $i$ for each $i$-simplex. To be precise, by the results of Cairns, [1], there are smooth compact manifolds, $N^*(\Delta_i)$, with $\Delta_i$ a continuous deformation retract of its neighborhood manifold, such that if $\Delta_i\subset\operatorname{Int}(B_j\times[-c, 2b])$, so is $N^*(\Delta_i)$. Furthermore, if $K_i=K_0\cup\Delta_1\cup\dots\cup\Delta_i$, there are smooth neighborhoods $N^*(K_0)\subset\dots\subset N^*(K_r)$, and

(a) $N^*(K_i)=N^*(K_{i-1})\cup N^*(\Delta_i)$,

(b) $N^*(K_i)=N^*(K_{i-1})\cup\text{handle } H_i$ whose index $=\dim\Delta_i$,

(c) $H_i$ is attached to $N^*(K_0)\cup\text{handles } H_j$ such that $\Delta_j\subset\partial\Delta_i$ (i.e., the image of the attaching map for $H_i$ lies in $N^*(K_0)\cup$ the handles which contain the faces of $\Delta_i$.),

(d) $N^*(K_i)\subset S_2^{n-1}\times(-c, 2b)$.

Moreover, $N^*(K)$ can be deformed diffeomorphically along radii onto $S_2^{n-1}\times[-c, 2b]$ in such a way that $N^*(K_0)$ is carried onto $S_2^{n-1}\times([-c, 0]\cup[b, 2b])$. Since points move radially, points of $B_i\times[-c, 2b]$ remain in this set.

Using this diffeomorphism, we may assume that the maps $\tilde{g}$ have domain $N^*(K_0)$ and the maps $G_0$ have domain $N^*(K)=N^*(K_r)$. In order to use the results of section 7, we $\lambda_i$-truncate the manifolds $N^*(K_i)$ with $\lambda<n$, by cutting a disk $D^{\lambda_i}\times D^{n-\lambda_i}$ out of the handle $H_i$. Call the remaining manifold $N'(K_i)$. Suppose that $H_{j_1}$ is the first handle of index $n$. Then we have

$$N^*(K_0)\subset N'(K_1)\subset N^*(K_1)\subset\dots\subset N^*(K_{j_1-1}). \tag{8.6}$$

Now $N^*(K_{j_1-1})$ is the set $V'$. Note that $V'$ is built up from $N^*(K_0)$ by a succession of operations of the form of an inclusion in a $\lambda$-truncated manifold followed by adding a handle of index $\lambda$. Hence, by section 7, any covering pair could be lifted over $V'$, which is equal to $N^*(K)$ minus the interiors of a finite number of disks. However, (8.2) can not be applied to complete a lift over $V$ unless the lift over $V'$ is defined with care. To clarify the problem, consider an individual $n$-simplex $\Delta$. Now $N^*(\Delta)$ is contained in some $B_i \times (-c, 2b)$. Let $N^*(\partial\Delta) = N^*(\Delta) \cap V'$. Note that $N^*(\partial\Delta)$ is the union of the submanifold $N^*(K_0) \cap N^*(\Delta)$, which will be denoted $N_0^*$, with handles of index less than $n$. We must lift $\tilde{g}$ over these handles to a map $\bar{g}$ in such a way that the hypotheses of (8.2) are satisfied for each $\bar{g}(q, t)$. Namely, for each $(q, t)$ in $I^m$,

(1) $\bar{g}(q, t)$ maps $N^*(\partial\Delta)$ into a coordinate neighborhood in $W$,

(2) there exists a map $\bar{\sigma}(q, t) \in C^\infty(N^*(\partial\Delta), M(n, k))$, independent with respect to $\bar{g}(q, t)$,

(3) the $k$-normal bundle of $\bar{g}(q, t)$ determined by $\bar{\sigma}(q, t)$ is trivial.

Since, for each $(q, t)$ in $I^m$, $g(q, t)$ maps $N_0^*$ into the coordinate neighborhood $W_i$, and $G_0(q)$ maps $N^*(\partial\Delta)$ into $W_i$, (1) can be satisfied trivially. For (2), it would suffice to find a $k$-frame field $\bar{\sigma}(q, t)$ independent with respect to $\bar{g}(q, t)$, for such a map can be approximated by a smooth field which is still independent with respect to $\bar{g}(q, t)$. This will be done as follows; let $\Sigma(q) = \chi_i | N^*(\partial\Delta)$, and $\sigma(q, t) = \chi_i | N_0^*$. Note that inclusion induces a map $j^*$: $C(N^*(\partial\Delta), M(n, k)) \to C(N_0^*, M(n, k))$, and $(\Sigma, \sigma)$ is an $m$-covering pair for these spaces.

Lemma (8.7) will show that there are lifts $\bar{g}$ and $\bar{\sigma}$ of $(G_0 | N^*(\partial\Delta), g)$ and $(\Sigma, \sigma)$, with $\bar{\sigma}$ independent with respect to $\bar{g}$ (i.e., each $\bar{\sigma}(q, t)$ is independent with respect to $\bar{g}(q, t)$). But this will also imply (3), since the $k$-normal bundle determined by $\bar{g}(q, t)$ and $\bar{\sigma}(q, t)$ is equivalent to the $k$-normal bundle given by $\bar{g}(q, 0)$ and $\bar{\sigma}(q, 0)$, and this is trivial since $G_0(q)$ and $\chi_i(q)$ determine an extension of this bundle over $N^*(\Delta)$, which is a contractible space.

(8.7) LEMMA. *Suppose that either*

(a) *$X^n$ is smooth, $Y^n$ is $\lambda$-truncated, $X^n \subset Y^n \subset R^n$, with $Y$ homeomorphic to $X \cup S^{\lambda-1} \times D^{n-\lambda} \times [0, 1]$ (where $S^{\lambda-1} \times D^{n-\lambda} \times \{0\} \subset \partial X$), or*

(b) *$X^n$ is $\lambda$-truncated, $Y^n = X^n \cup$ handle of index $\lambda$, $\lambda < n$, and $X \subset Y \subset R^n$.*

*Assume $W_*^p \subset R^p$. Let $(\mathcal{G}, g)$ be an $m$-covering pair, $g$: $I^m \to C^\infty(X, W_*; k)$ and $(\Sigma, \sigma)$ be an $m$-covering pair, $\sigma$: $I^m \to C(X, M(n, k))$, which is independent with respect to $(\mathcal{G}, g)$. Then there exist lifts $\bar{g}$ of $(\mathcal{G}, g)$ and $\bar{\sigma}$ of $(\Sigma, \sigma)$ such that $\bar{\sigma}$ is independent with respect to $\bar{g}$.*

This lemma will be proved in section 9.

This does not yet show that there is a lift to all of $V'$ however, because a simplex $\Delta$ with dim $(\Delta) < n$ may be contained in the boundaries of several $n$-simplexes, so that when a lift is taken over the handle $H$ corresponding to $\Delta$, the $\sigma$-maps of all of these $n$-simplexes must be considered. However, simple changes in the proof of (8.7) will yield:

(8.8) COROLLARY. *Suppose given $X^n \subset Y^n \subset R^n$, as in Lemma (8.7), and $W^p_*$ an open subset of $R^p$. Given m-covering pairs $(\Sigma(1), \sigma(1)), \ldots, (\Sigma(w), \sigma(w))$, such that each pair is independent with respect to an m-covering pair $(\mathcal{G}, g)$, there exist lifts $\bar{g}$ of $(\mathcal{G}, g)$ and $\bar{\sigma}(i)$ of $(\Sigma(i), \sigma(i))$ such that $\bar{\sigma}(i)$ is independent with respect to $\bar{g}$, $i = 1, \ldots, w$.*

Finally, the reduction to (8.2) can be completed. Let $\Delta_{j_1}, \ldots, \Delta_r$ be the $n$-simplexes of $K$. For each $j$, $j_1 \leqslant j \leqslant r$, choose a $B_i \times (-c, 2b)$ containing $N^*(\Delta_j)$. For $\chi_i$ the corresponding $k$-frame, let

$$\Sigma(j)(q) = \chi_i | N^*(\partial\Delta_j),$$

$$\sigma(j)(q, t) = \chi_i | N^*(K_0) \cap N^*(\partial\Delta_j).$$

Denote the neighborhood $W_i$ by $W(j)$.

(8.9) LEMMA. *There exist a lift $\bar{g}$ of $(G_0 | V', g)$ over $V'$, and lifts $\bar{\sigma}(j)$ of $(\Sigma(j), \sigma(j))$ over $N^*(\partial\Delta_j)$, $j = j_1, \ldots, r$, such that $\bar{\sigma}(j)$ is independent with respect to $\bar{g}$ on $N^*(\partial\Delta_j)$.*

*Proof.* (By induction.) Let $g^0(q, t) = g(q, t)$, and $\sigma(j)^0(q, t) = \sigma(j)(q, t)$, $j = j_1, \ldots, r$. Suppose that lifts $g^i$ of $(G_0, g)$ to $N^*(K_i)$ and $\sigma(j)^i$ of $(\Sigma(j), \sigma(j))$ to $N^*(K_i) \cap N^*(\partial\Delta_j)$, $j = j_1, \ldots, r$, have been found which satisfy the induction hypotheses:

(a) $g^i(q, t)$ maps $N^*(K_i) \cap N^*(\partial\Delta_j)$ into $W(j)$, $j = j_1, \ldots, r$,

(b) for each $j = j_1, \ldots, r$, $\sigma(j)^i(q, t)$ is independent with respect to $g^i(q, t)$ on $N^*(K_i) \cap N^*(\partial\Delta_j)$.

Now $K_{i+1} = K_i \cup \Delta_{i+1}$. Let $\Delta_{n_1}, \ldots, \Delta_{n_w}$ be the set of all $n$-simplexes of $K$ whose boundaries contain $\Delta_{i+1}$.

Let $W_* = W(n_1) \cap \ldots \cap W(n_w)$. By (a) above, $g^i(q, t)$ maps $N^*(\partial\Delta_{i+1})$ into $W_*$. By (b), $\sigma(n_1)^i(q, t), \ldots, \sigma(n_w)^i(q, t)$ are all independent with respect to $g^i(q, t)$ on $N^*(\partial\Delta_{i+1})$. Then by Corollary (8.8), the induction step can be made.

## 9. Proofs of (8.2) and (8.7)

For convenience, we restate Lemma (8.2) below.

(9.1) RESTATEMENT OF LEMMA (8.2). *Let $X^n$ be a compact $\lambda$-truncated manifold, $\lambda \geqslant n$, and $X^n \subset R^n$. Suppose that $W^p \subset R^p$. (Recall that always $\partial W^p = \emptyset$.) Let $e \in C^\infty(X, W; k)$, where $k < p$. Suppose that there is a smooth k-frame field, $\sigma = (\sigma_1, \ldots, \sigma_k)$ defined on a neighborhood of $\dot{X}$ in $X$ which is independent with respect to e, and that the orthogonal complement of the bundle spanned by $e^\#(de \circ \sigma_1), \ldots, e^\#(de \circ \sigma_k)$ is trivial. Then there is a factoring manifold* **D** *and a neighborhood* $\Gamma$ *of the identity in* $\mathrm{Aut}_{\mathbf{D}}(\mathbf{D}')$, $(\mathbf{D}'=\mathbf{D}\cup\partial\mathbf{D}\times I)$, *such that positioning deformations exist for* $\mathbf{D}$ *and* $\Gamma$.

*Proof.* For the purpose of proving the above, we need only consider a neighborhood of $\dot{X}$ in $X$, and hence we might as well assume that $X$ is this neighborhood, and $\sigma=(\sigma_1,\dots,\sigma_k)$ is defined on all of $X$. Let $U'$, $N$, and $N'$ be defined as in section 8. There is a smooth extension $\sigma'$ of $\sigma$ to $N'$ and a smooth extension $h$ of $e$ to $N'$ such that $dh\circ\sigma'\in C^\infty(N', M(p,k;k))$. Let $E=h^{\#}(TR^p)=N'\times R^p$. Then the $k$ smooth vector fields, $h^{\#}(dh_x\circ\sigma_1'(x))$, $\dots$, $h^{\#}(dh_x\circ\sigma_k'(x))$ span a trivial $k$-dimensional subbundle of $E$. Let $B$ be the orthogonal complement of this bundle. Since $B|X$ is trivial and $X$ is a defoımation retract of $N'$, $B$ is trivial. Hence there is a smooth section $b$ of $N'$ into $B$ given by $b(x)=(x,\theta(x))$, where $\theta\colon N'\to R^p$, $\theta(x)=(\theta_1(x),\dots,\theta_p(x))$, and $\|\theta(x)\|=1$, $x\in N'$. For $a\colon N'\to R^1$, denote $b_a(x)=(x, a(x)\theta(x))$.

Choose $\mathbf{e}'>0$ so that $\zeta(x,v)=(x,h(x)+v)$ is a diffeomorphism mapping $E(\mathbf{e}')$ into $w$. Now the map $\phi$ (see (4.1)) for $N'\times R^p$ is just $h(x)+v$, and $\phi\circ b_a(x)=h(x)+a(x)\theta(x)$.

By examination of $d(\phi\circ b_a)$, there is an $\mathbf{e}$, $0<\mathbf{e}\leqslant\mathbf{e}'$, such that for any $a\colon N\to R^1$ with $|a(x)|\leqslant\mathbf{e}$, $x\in N'$, $\sigma'$ is independent with respect to $\phi\circ b_a$, and so $\phi\circ b_a\in C^\infty(N', R^p; k)$. Now define $\mathbf{D}$ as follows. Let

$$\mathbf{D}_1=\{(x,v)\in E(\mathbf{e})\mid x\in U'\cup\partial U'\times[0,1],\ \|v\|^2\leqslant(\mathbf{e}/3)^2\},$$

$$\mathbf{D}_2=\{(x,v)\in E(\mathbf{e})\mid x=(y,z)\in\partial U'\times[1,2],\ \|v\|^2+(\mathbf{e}/3)^2(z-1)^2\leqslant(\mathbf{e}/3)^2\}$$

and let $\mathbf{D}=\mathbf{D}_1\cup\mathbf{D}_2$. In particular, the subset of $\mathbf{D}$ "over" $\dot{U}\times[-1,1]$ has the form $\dot{U}\times[-1,1]\times D^p_{\mathbf{e}/3}$, with $\dot{U}\times[-1,1]\times\partial D^p_{\mathbf{e}/3}\subset\partial\mathbf{D}$. $\mathbf{D}'$ can be defined in a similar way, as a manifold whose "height" over $\dot{U}\times[-1,1]$ is $2\mathbf{e}/3$.

We claim that there is a neighborhood $\Gamma$ of the identity in $\mathrm{Aut}_{\mathbf{D}}(\mathbf{D}')$ such that if $\gamma\in\Gamma$, then $\sigma'$ is independent with respect to $\phi\circ\gamma\circ b_a$, for *all* $b_a$ with $|a(x)|\leqslant\mathbf{e}$, $x\in N'$; i.e., $\phi\circ\gamma\circ b_a\in C^\infty(N', W; k)$. This follows because the matrix for $d(\phi\circ\gamma\circ b_a)\circ\sigma'$ has the form $Q+(c_1(a(x))\theta',\dots,c_k(a(x))\theta')$ where $Q$ is close to $dh\circ\sigma'$, and $\theta'$ is a column vector close to $\theta$. Now $\theta$ is a column vector orthogonal to $dh\circ\sigma'$, and if the neighborhood $\Gamma$ is chosen small enough, $\theta'$ will be independent of the $k$ columns of $Q$, so that $Q+(c_1(a(x))\theta',\dots,c_k(a(x))\theta')$ will have rank $k$.

Finally, given $\tau$, $0<\tau\leqslant1$, let $r\colon R^1\to R^1$ be a $C^\infty$ map which is zero outside of $[\tau/2+\delta,\tau-\delta]$ for some $\delta$, $0<\delta<\tau/4$, with $r(2\tau/3)=r(5\tau/6)=\mathbf{e}/3$, and such that $r$ has a single maximum $m$ with $\mathbf{e}/3<m<2\mathbf{e}/3$. Let

$$a(x)=\begin{cases}0 & x\in X\\ r(z) & x=(y,z)\in\dot{X}\times[0,\tau].\end{cases}$$

Let $\xi(t)(x)=(x, ta(x)\theta(x))$. Clearly, $\mathbf{D}$, $\Gamma$, and $\xi$ satisfy (7.5). This completes the proof of (8.2).

In the proof of (8.7), the lifts of a covering pair $(\mathcal{G}, g)$ will be given by functions of the form in (7.2) and (7.3), and (7.7) and (7.8), respectively. Corresponding formulas for lifts of $(\Sigma, \sigma)$ will be developed. First, we restate the lemma in local form.

(9.2) LEMMA. *Suppose given $X^n$ and $Y^n$ as in* (8.7). *Suppose $W^p_*$ is an open subset of $R^p$. Given any map $e\in C^\infty(X, W_*; k)$ and a map $\chi\in C(X, M(n, k))$ independent with respect to e, there are neighborhoods $\mathcal{A}$ of e and $\Lambda$ of $\chi$ such that given a covering pair $(\mathcal{G}, g)$, $g: I^m\to\mathcal{A}$, and a covering pair $(\Sigma, \sigma)$, $\sigma: I^m\to\Lambda$, then lifts $\bar{g}$ and $\bar{\sigma}$ can be found with $\bar{\sigma}(q, t)$ independent with respect to $\bar{g}(q, t)$, for every $(q, t)$ in $I^m$.*

*Proof.* Define the neighborhoods $N(X)$, and $N'(X)$ as before. In order to present a unified proof for both cases, the letter $F$ will be used to denote the set in which a deformation will take place. In the case (a), let $F=Y$, $\hat{F}_\tau=\hat{X}_\tau\cap F$. Let $\mathfrak{h}$ be an extension of $e$ to $N'$, and $\mathfrak{X}$ an extension of $\chi$ to $F=Y$, which is independent with respect to $\mathfrak{h}$ on $F$. Note that any covering pair $(\Sigma, \sigma)$ with $\Sigma(q)$ defined on $F$ and $\sigma(q, t)$ defined on $X$ can be lifted over $F$. Call the lift $\Sigma'(q, t)$.

For (b), a little more detail is needed. Here, let $F=X\cup S^{\lambda-1}\times D^{n-\lambda}_3\times[-1, 1]$, $\hat{F}_\tau=X\cup S^{\lambda-1}\times D^{n-\lambda}_3\times[-1, \tau]$. Again we choose $\mathfrak{h}$ and $\mathfrak{X}$ with $\mathfrak{X}$ independent with respect to $\mathfrak{h}$ on $F$, and also stipulate that when $(x, y, z)\in S^{\lambda-1}\times D^{n-\lambda}_3\times[-1, 1]$ and $\|y\|\geqslant 1$, then $\chi(x, y, z)=\chi(x, y/\|y\|, z)$.

Also note that given a covering pair $(\Sigma, \sigma)$ we could first define a lift of $(\Sigma|\hat{X}_1, \sigma)$, and then extend the lift to a cube of maps defined on $F$, using the obvious retraction of $F$ onto $\hat{X}_1$. The resulting map will be denoted $\Sigma'(q, t)$, and will be used later in the proof.

In the rest of the proof, both of the cases are treated together.

There is a $\delta>0$ such that if $\varrho(\mathfrak{f}, \mathfrak{h})<\delta$ and $\sup_{x\in F}\|\chi(x)-\eta(x)\|<\delta$, then $\eta(x)$ is independent with respect to $\mathfrak{f}$ on $F$. Let

$$\Lambda=\{\eta\in C(X, M(n, k))\,|\,\sup_{x\in X}\|\eta(x)-\chi(x)\|<\delta/2\}.$$

Set
$$\mathcal{B}=\{\mathfrak{f}\in C^\infty(F, W_*; k)\,|\,\varrho(\mathfrak{f}, \mathfrak{h})<\delta\}.$$

Use $\mathfrak{h}$ to obtain a factorization of maps near $e$ through any factoring manifold $\mathbf{D}$, and suppose that the neighborhood $\Gamma$ of the identity in $\mathrm{Aut}_{\mathbf{D}}(\mathbf{D}')$ is so small that if $\gamma\in\Gamma$, then $\phi\circ\gamma\circ s\in C^\infty(N', W_*; k)$ and $\phi\circ\gamma\circ s\,|\,F\in\mathcal{B}$. Let the neighborhood $\mathcal{A}$ of $e$ be chosen so that $\nu(\mathcal{A})\subset\Gamma'$, using the notation of section 6.

Now suppose that we are given covering pairs $(\mathcal{G}, g)$ with $g$: $I^m\to\mathcal{A}$, and $(\Sigma, \sigma)$ with

$\sigma: I^m \to \Lambda$. Let $\Sigma': I^m \to C(F, M(n, k))$ be defined as above, in the remarks about the individual cases. Clearly, there is a $\tau' > 0$ such that for every $(q, t) \in I^m$,

$$\sup_{x \in \hat{F}_{\tau'}} \|\chi(x) - \Sigma'(q, t)(x)\| < \delta.$$

Hence, for every $\gamma \in \Gamma$ and each $(q, t) \in I^m$,

$$d(\phi \circ \gamma \circ s)_x \circ \Sigma'(q, t)(x) \in M(p, n; k), \quad x \in \hat{F}_{\tau'}. \tag{9.3}$$

Choose $\tau$, $0 < \tau \leqslant \tau'/2$ and $\alpha$ as in Lemma (6.1). Find $\mu$ and $\varepsilon > 0$ as in Lemma (6.3).

Let $\beta$ represent either a collar retraction (for case (a)) as in (7.1), i.e. $\beta: [0, 1] \to$ Emb $(F, F)$, or, (for case (b)) let $\beta$ be a deformation so that $\xi = s \circ \beta$ is a positioning deformation (see (7.5)). If $0 \leqslant t \leqslant \varepsilon$, define

$$\bar{\sigma}(q, t)(x) = \begin{cases} (d\beta(t/\varepsilon)(x))^{-1} \circ \Sigma'(q, t)(\beta(t/\varepsilon)(x)), & x \in \hat{X}_\tau \cap Y \\ \Sigma'(q, t(2\tau - z)/\tau)(x), & x = (y, z) \in \dot{X} \times [\tau, 2\tau] \cap Y \\ \Sigma(q)(x), & x \in Y - \hat{X}_{2\tau}. \end{cases} \tag{9.4}$$

If $\varepsilon \leqslant t \leqslant 1$, define

$$\bar{\sigma}(q, t)(x) = \begin{cases} (d\beta(1)(x))^{-1} \circ \Sigma'(q, t)(\beta(1)(x)), & x \in \hat{X}_\tau \cap Y \\ \Sigma'(q, t(2\tau - z)/\tau)(x), & x = (y, z) \in \dot{X} \times [\tau, 2\tau] \cap Y \\ \Sigma(q)(x), & x \in Y - \hat{X}_{2\tau}. \end{cases} \tag{9.5}$$

Clearly $\bar{\sigma}$ is well defined and continuous, $\bar{\sigma}(q, 0) = \Sigma(q)$, and $\bar{\sigma}(q, t) \,|\, X = \Sigma'(q, t) \,|\, X = \sigma(q, t)$.

Define the lift $\bar{g}$ of $(\mathcal{G}, g)$ using the formulas (7.2) and (7.3) in the collar retractible case, and formulas (7.7) and (7.8) in the handle case. Then for each $x \in Y$, $d\bar{g}(q, t)_x \circ \bar{\sigma}(q, t)(x)$ has one of the three forms below:

$$d\bar{g}(q, t)_x \circ \bar{\sigma}(q, t)(x) = \begin{cases} d(\phi \circ \gamma \circ s)_y \circ \Sigma'(q, t)(y), & y = \beta(x) \in \hat{F}_\tau \\ d\mathcal{G}(q)_x \circ \Sigma'(q, t)(x), & x \in \hat{F}_{2\tau} - \hat{F}_\tau \\ d\mathcal{G}(q)_x \circ \Sigma(q)(x), & x \in Y - \hat{F}_{2\tau}. \end{cases}$$

By (9.3), $\bar{\sigma}$ is independent with respect to $\bar{g}$. This completes the proof of (8.7).

## 10. An approximation theorem

Let $M^n$ be a smooth compact manifold, and $W^p$ be a smooth manifold with empty boundary. If $\phi \in T(M, W; k)$, let $\bar{\phi}$ denote the map of $M$ to $W$ covered by $\phi$. Fix a metric $\varrho$ for $W$, and, for $f$, $g$ in $C(M, W)$, say that $f$ is an $\varepsilon$-approximation to $g$ if $\sup_{x \in M} \varrho(f(x), g(x)) < \varepsilon$. With this notation, we have an approximation theorem, similar to (5.7) of [4].

(10.1) THEOREM. *If there exists* $\phi \in T(M, W; k)$, *and* $k < p$, *then, given* $\varepsilon > 0$, *there is a k-mersion* $f \in C^\infty(M, W; k)$ *such that df is homotopic to* $\phi$ *through k-bundle maps, and f is an* $\varepsilon$*-approximation to* $\phi$.

*Proof*. Choose a $C^\infty$ triangulation of $M$ so fine that each simplex $\Delta_i$ is contained in a coordinate neighborhood of $M$, and also maps, under $\bar{\phi}$, into the interior of some convex coordinate neighborhood, call it $W_i$, in $W$, of diameter less than $\varepsilon$.

The proof is by induction, with induction hypotheses:

(a) A smooth map $f^i$ of rank $\geq k$ is defined on a neighborhood $N_i$ of the $i$-skeleton in $M$, and $df^i$ is homotopic to $\phi | N_i$ through $k$-bundle maps.

(b) If $\Delta$ belongs to the $i$-skeleton and is a face of $\Delta_j$, then $f^i(\Delta) \subset W_j$.

Clearly, (a) and (b) can be satisfied on the 0-skeleton.

Suppose that the hypotheses hold for the $i$-skeleton. Since $f^i(\Delta_r) \subset W_r$, $f^i$ must be an $\varepsilon$-approximation to $\phi$.

Now let $\Delta$ be an $i+1$ simplex, and $\bar{\Delta}$ be a standard $i+1$ simplex imbedded in $R^n$. Then there is a diffeomorphism mapping a neighborhood of $\bar{\Delta}$ onto a neighborhood of $\Delta$. Therefore, as in section 8, we may map smooth manifolds $N^*(\partial\bar{\Delta})$ and $N^*(\bar{\Delta})$ diffeomorphically to submanifolds $U$ and $V$ of $M$, so that $U$ is contained in the domain of $f^i$, and $V = U \cup$ handle. Let $W_*$ be the intersection of all $W_j$ such that $\Delta$ is a face of $\Delta_j$. Clearly we can assume that $f^i(U) \subset W_*$. Consider the diagram:

$$\begin{array}{ccc} C^\infty(V, W_*; k) & \xrightarrow{d} & T(V, W_*; k) \\ \downarrow & & \downarrow \\ C^\infty(U, W_*; k) & \xrightarrow{d} & T(U, W_*; k) \end{array}$$

Since $df^i | U$ is homotopic to $\phi | U$ through $k$-bundle maps, by the covering homotopy property, there is an element $\psi$ lying in the fiber over $df^i$, which is homotopic to $\phi$. But then the fiber over $f^i | U$ must contain an element $f'$ which is mapped by $d$ to the homotopy class of $\psi$. Hence, $df'$ is homotopic to $\phi$, and $f'$ is an $\varepsilon$-approximation of $\bar{\phi}$, by the definition of $W_*$. The map $f'$ agrees with $f^i$ on some neighborhood of the $i$-skeleton. Continuing the process for the other $i+1$ simplexes, and using a possibly smaller neighborhood of the $i$-skeleton, we eventually can define $f^{i+1}$ which satisfies the induction hypotheses on a neighborhood of the $i+1$ skeleton.

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