NORMAL FAMILIES AND THE NEVANLINNA THEORY

BY

DAVID DRASIN

Purdue University, Lafayette, Ind., U.S.A.

I. Introduction

1. Let $\mathfrak{F}$ be a family of nonconstant holomorphic functions defined in the disc $\Delta = \{|z| < 1\}$. $\mathfrak{F}$ is said to be *normal* if every sequence of functions in $\mathfrak{F}$ either contains a subuniformly convergent subsequence, or contains a subsequence which converges subuniformly to the constant ∞. A family $\mathfrak{F}$ of meromorphic functions is normal when every sequence of functions of $\mathfrak{F}$ has a subsequence which is subuniformly convergent with respect to the chordal metric.

P. Montel [15] first realized the scope and coherence of these families, and used them to give a particularly unified treatment of Picard's great theorems, and Schottky's and Landau's theorems. The fact that these results were so intimately related led A. Bloch to the hypothesis that precisely those properties which reduce a function meromorphic in $\mathbf{C}$ $(=\{|z| < \infty\})$ to a constant, make normal a family of functions meromorphic in Δ.

2. The Nevanlinna theory of meromorphic functions has proved an effective means of studying the value-distribution of a single meromorphic function in $\mathbf{C}$. In particular, a recent paper of W. K. Hayman [8] contains several striking results of this type.

In view of Bloch's observation, Hayman asks [10] whether his results have normal family analogues; the present paper establishes an affirmative answer in the important special case of *holomorphic* functions. Of greater interest, however, is that by using the standard arguments of the Nevanlinna theory we are able to present a unified exposition of the major value-distribution criteria for normal families of holomorphic functions (compare especially the proofs of Theorem 5 here and Theorem 8 of [8]). An extension of a theorem of Montel, valid for families of *meromorphic* functions, is also obtained with little additional effort.

The major problem faced is the handling of what will be referred to as *initial value terms*; that is, terms which depend on the values of the function or its derivatives at the origin (cf. the end of § 3 of Part II). While negligible when considering a single function, they are collectively difficult to manage when families of functions are involved. The proof of Theorem 1 (especially § 5, Part II) shows that these terms are of intrinsic importance.

3. We present the exposition at a somewhat leisurely pace. Part II centers about a new proof of Montel's "critère fondamental" [15, p. 61]; although the result is standard, the proof given here shows many of the ideas used subsequently. This seems to be the only proof which requires F. Marty's well-known necessary and sufficient conditions for normality. We remark that the proof given here can be modified to yield an explicit upper bound for

$$\max_{|z|=r} |f(z)|,$$

where $f \neq 0, 1, \infty$ in Δ (sharp form of Schottky's theorem).

Additional machinery is developed in Part III. In the next part we mimic the procedures used by Milloux [13] and Hayman [8] in their study of a fixed function meromorphic in C to give a new proof of a theorem of C. Chuang [3], [16]; the existing proofs are very complicated. The proof presented here extends easily (Part V) to establish a new result which includes a theorem of Chang and Yang [18] that had answered affirmatively a conjecture of Hayman [10]. Another of Hayman's conjectures is verified in Part VI.

The final part contains a proof of a growth lemma (Lemma 2) which in the generality needed in Parts IV and VI is new.

Although knowledge of the notations $m(r, f)$, $T(r, f)$, $N(r, f)$, $\bar{N}(r, f)$, etc. and elementary properties of the Nevanlinna theory will be freely used, the present methods are elementary.

It would be of interest to extend Theorems 3, 4 and 5 to families of meromorphic functions, but we have not been able to do this.

I wish to thank Professor W. K. Hayman for a reading of the manuscript which detected some errors, and provided many useful suggestions. In addition, I have had several useful conversations with L. Sons, K. V. R. Rao, and A. Obrock.

4. *Notation.* Constant independent of the particular function under consideration will be denoted A, B, ..., M, N ...; these constants will usually depend on the family $\mathfrak{F}$, however. If a constant depends on a parameter p, it will appear as $A(p)$, $B(p)$, ... The same letter is used for different constants when no confusion is to be feared.

The numbering of equations and formulas begins anew in each part. When it is necessary to refer to a numbered formula from a different part, the number of the part is included in the citation. Theorems and lemmas are numbered consecutively, however.

II. Montel's theorems

1. The main result of this part is

THEOREM 1 (Montel). *If every $f \in \mathfrak{F}$ omits the values 0, 1 and ∞, then $\mathfrak{F}$ is normal.*

The derivation presented here parallels that used by R. Nevanlinna in his proof of the second fundamental Theorem (cf. [9]).

Conditions necessary and sufficient for $\mathfrak{F}$ to be normal have been discovered by F. Marty [12] (for a proof, cf. also [1, p. 218]):

LEMMA A. *$\mathfrak{F}$ is normal if and only if to each $r_0 < 1$, corresponds a number $M(r_0) < \infty$ with*

$$\frac{|f'(z)|}{1+|f(z)|^2} < M(r_0),$$

for $|z| < r_0$, independent of $f \in \mathfrak{F}$.

Since the family $\mathfrak{F}$ under consideration consists of holomorphic functions, it is convenient to record a more standard condition:

LEMMA B. *Let $\mathfrak{F}$ be a family of holomorphic functions in Δ. For $f \in \mathfrak{F}$, set*

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

If there is an increasing, finite-valued function $\Sigma(r)$, $0 \leqslant r < 1$, with

$$M(r, f) \leqslant \Sigma(r), \quad r_1 \leqslant r < 1 \tag{1.1}$$

independent of $f \in \mathfrak{F}$, then $\mathfrak{F}$ is normal.

Nevanlinna's inequality for the *Schmiegungsfunktion* ([9, p. 18]) $m(\varrho, f)$ of a *holomorphic* function $f(z)$,

$$\log^+ M(r, f) < \frac{\varrho + r}{\varrho - r} m(\varrho, f), \quad r < \varrho < 1 \tag{1.2}$$

allows (1.2) to be replaced by

$$m(r, f) \leqslant \Sigma(r), \quad r_1 \leqslant r < 1 \tag{1.3}$$

as a sufficient condition for normality of a holomorphic family.

2. For $|\alpha| < 1$, consider the Möbius transformation

$$\varphi_\alpha(z) = \frac{z - \alpha}{\bar{\alpha} z - 1}. \tag{2.1}$$

Then if $f \in \mathfrak{F}$, define f_α by
$$f_\alpha(z) = f(\varphi_\alpha(z)). \tag{2.2}$$
Note that
$$f_\alpha(0) = f(\alpha), \tag{2.3}$$
$$f_\alpha'(0) = f'(\alpha)\varphi_\alpha'(0). \tag{2.4}$$

That much is to be gained from this composition is evident from (2.3) and (2.4): greater flexibility in handling the initial-value terms; that nothing is lost follows from

LEMMA 1. *Suppose there is an $r_0 < 1$ such that corresponding to each f in $\mathfrak{F}$ is an $\alpha = \alpha(f)$, $|\alpha| \leqslant r_0$, with the property that the family $\{f_\alpha\}$ is normal. Then $\mathfrak{F}$ is normal.*

Proof. The functions $\{\varphi_\alpha\}$ and $\{\varphi_\alpha^{-1}\}$ are uniformly bounded sets of holomorphic functions on $|z| < 1$, and thus (by Lemma B) are normal. Since
$$f = f_\alpha \circ \varphi_\alpha^{-1},$$
the lemma follows.

Since much of the discussion in this article centers on a local study of $\mathfrak{F}$, we need a variant of Lemma 1; the proof is evident.

LEMMA 1'. *$\mathfrak{F}$ is normal in a neighborhood of $z_0 \in \Delta$ if there exists a sequence $\alpha_n \to z_0$ such that $\{f_{\alpha_n}\}$ is normal in some neighborhood of the origin.*

Remark. It is clear that Lemmas 1 and 1' actually give necessary and sufficient conditions for normality (resp. local normality) of $\mathfrak{F}$.

3. *The main inequalities.* Because these derivations will be needed in later sections, we temporarily ignore the particular assumptions of Theorem 1 and consider an arbitrary function f meromorphic in Δ. Choose $r_0 < 1$ and $|\alpha| < r_0$ subject only to the restrictions $f(\alpha) \neq 0, \infty$, $f'(\alpha) \neq 0$. A fundamental inequality, due to Nevanlinna [9, pp. 31, 32], when applied to f_α yields (upon letting $q = 2$, $a_1 = 0$, $a_2 = 1$):
$$m(r, f_\alpha) + m\left(r, \frac{1}{f_\alpha}\right) + m\left(r, \frac{1}{f_\alpha - 1}\right) \leqslant 2T(r, f_\alpha) - N_1(r, f_\alpha) + S(r, f_\alpha), \tag{3.1}$$
where
$$N_1(r, f_\alpha) = N\left(r, \frac{1}{f_\alpha'}\right) + 2N(r, f_\alpha) - N(r, f_\alpha'),$$
and
$$S(r, f_\alpha) = 2m\left(r, \frac{f_\alpha'}{f_\alpha}\right) + m\left(r, \frac{f_\alpha'}{f_\alpha - 1}\right) + \log\left|\frac{1}{f_\alpha'(0)}\right| + C,$$
with $C(= \log 144)$ an absolute constant.

Nevanlinna's first fundamental Theorem [9, p. 5] asserts that if $f(\alpha) \neq 0, 1$

$$m\left(r, \frac{1}{f_\alpha}\right) + N\left(r, \frac{1}{f_\alpha}\right) + \log|f(\alpha)| = m\left(r, \frac{1}{f_\alpha - 1}\right) + N\left(r, \frac{1}{f_\alpha - 1}\right) + \log|f_\alpha(0) - 1| + \varepsilon^* = T(r, f_\alpha),$$

where $|\varepsilon^*| < \log 2$. Thus, if

$$N\left(r, \frac{1}{f_\alpha}\right) + N(r, f_\alpha) + N\left(r, \frac{1}{f_\alpha - 1}\right) + \log|f_\alpha(0)\{f_\alpha(0) - 1\}|$$

is added to both sides of (3.1), the first fundamental Theorem and the relation

$$N(r, f_\alpha') - N(r, f_\alpha) = \bar{N}(r, f_\alpha)$$

imply that (3.1) can be written as

$$T(r, f_\alpha) < \bar{N}(r, f_\alpha) + N\left(r, \frac{1}{f_\alpha}\right) + N\left(r, \frac{1}{f_\alpha - 1}\right) - N\left(r, \frac{1}{f_\alpha'}\right) + 2m\left(r, \frac{f_\alpha'}{f_\alpha}\right) + m\left(r, \frac{f_\alpha'}{f_\alpha - 1}\right)$$

$$+ \log\left|\frac{f(\alpha)\{f(\alpha) - 1\}}{f_\alpha'(0)}\right| + C \tag{3.2}$$

whenever

$$f(\alpha) \neq 0, 1, \infty; \quad f'(\alpha) \neq 0. \tag{3.3}$$

Inequality (3.2) (where α satisfies (3.3)) is valid for any meromorphic function $f(z)$. Let us now return to the proof of Theorem 1; the assumptions

$$N(r, f_\alpha) \equiv N\left(r, \frac{1}{f_\alpha}\right) \equiv N\left(r, \frac{1}{f_\alpha - 1}\right) \equiv 0$$

permit (3.2) to be sharpened to

$$m(r, f_\alpha) < 2m\left(r, \frac{f_\alpha'}{f_\alpha}\right) + m\left(r, \frac{f_\alpha'}{f_\alpha - 1}\right) + \log\left|\frac{f(\alpha)\,(f(\alpha) - 1)}{f_\alpha'(0)}\right| + C. \tag{3.4}$$

To handle the terms involving means of logarithmic derivatives in (3.4), we use Nevanlinna's estimate [9, p. 36]:

LEMMA C. *If g is meromorphic in $|z| < 1$, and $\delta < r < R < 1$, and $g(0) \neq 0, \infty$, then*

$$m\left(r, \frac{g'}{g}\right) < 4\log^+ T(R, g) + 6\log\frac{1}{R - r} + 4\log^+\log^+\left|\frac{1}{g(0)}\right| + C,$$

$C = C(\delta)$; if $\delta' > \delta$, we may take $C(\delta') = C(\delta)$.

Using this estimate in (3.4) (with $g=f_\alpha$), and noting that $T(R,f_\alpha)=m(R,f_\alpha)$, we obtain

$$T(r,f_\alpha)<12\log^+ T(R,f_\alpha)+18\log\frac{1}{R-r}+8\log^+\log^+\left|\frac{1}{f(\alpha)}\right|+\log|f(\alpha)|$$
$$+4\log^+\log^+\left|\frac{1}{f(\alpha)-1}\right|+\log|f(\alpha)-1|+\log\left|\frac{1}{f_\alpha'(0)}\right|+C,\quad \tfrac{1}{2}<r<R<1. \tag{3.5}$$

Now it is easy to see that if $A>e$,

$$\log|u|+A\log^+\log^+\left|\frac{1}{u}\right|\leqslant \log^+|u|+A\log A,$$

so (3.5) becomes

$$T(r,f_\alpha)<12\log^+ T(R,f_\alpha)+18\log\frac{1}{R-r}+\log^+|f(\alpha)|+\log^+|f(\alpha)-1|$$
$$+\log\left|\frac{1}{f_\alpha'(0)}\right|+C,\quad \tfrac{1}{2}<r<R<1. \tag{3.6}$$

If $|\alpha|<r_0$, (2.1) implies that

$$\left|\frac{1}{\varphi_\alpha'(0)}\right|<S,\quad S=S(r_0);$$

thus it follows at once from (2.4) that

$$\log\left|\frac{1}{f_\alpha'(0)}\right|<\log\left|\frac{1}{f'(\alpha)}\right|+\log S,$$

and (3.6) becomes

$$T(r,f_\alpha)<12\log^+ T(R,f_\alpha)+18\log\frac{1}{R-r}+2\log^+|f(\alpha)|+\log\left|\frac{1}{f'(\alpha)}\right|+C_1+S_1, \tag{3.7}$$

for $\tfrac{1}{2}<r<R<1$, $|\alpha|<r_0<1$, $(S_1=S_1(R_0))$.

Remark. Terms like $2\log^+|f(\alpha)|+\log\left|\dfrac{1}{f'(\alpha)}\right|$ in (3.7) or

$$\log\left|\frac{f(\alpha)\,(f(\alpha)-1)}{f_\alpha'(0)}\right|$$

in (3.2) will be referred to as *initial-value terms* when they appear throughout this article.

4. *Tentative proof of Theorem 1.* Let $\mathfrak{F}_1$ be an infinite denumerable subcollection of the family $\mathfrak{F}$, and apply (3.7) to each member of $\mathfrak{F}_1$. Suppose that there are constants

$M < \infty$ and $r_0 < 1$ and an infinite subset $\mathfrak{F}_2$ of $\mathfrak{F}_1$ with the property that if $f \in \mathfrak{F}_2$, there is an $\alpha = \alpha(f)$, $|\alpha| < r_0$, with

$$2\log^+ |f(\alpha)| + \log \left|\frac{1}{f'(\alpha)}\right| < M. \tag{4.1}$$

At this stage we need a growth lemma which generalizes those of Bureau [2] and Hiong [11]; the proof is deferred to Part VII.

LEMMA 2. *Let $U(r)$ and $\gamma(r)$ be continuous non-decreasing functions of r, $r_1 < r < 1$. If there is an r_0, $r_1 < r_0 < 1$, and $b > 1$ with*

$$U(r) < M + b \log \frac{1}{R-r} + c \log^+ U(R) + \gamma(r), \quad r_0 < r < R < 1, \tag{4.2}$$

then for $r \geq r_0$,

$$U(r) < M_1 + 4\gamma(R) + 2b \log \frac{1}{R-r}, \quad r_0 < r < R < 1, \tag{4.3}$$

where M_1 depends only on M, b and c.

We apply Lemma 2 to (3.7) with $U(r) = T(r, f_\alpha)$ and $\gamma(R) \equiv 0$; then for each $r < 1$, the numbers $T(r, f_\alpha)$ $(= m(r, f_\alpha))$ are bounded independently of $f \in \mathfrak{F}_2$, and the criterion (1.3) coupled with Lemma 1 establishes the normality of $\mathfrak{F}$.

5. *Completion of proof.* If condition (4.1) fails to hold, then, given $M < \infty$, $r_0 < 1$,

$$2\log^+ |f(\alpha)| + \log \left|\frac{1}{f'(\alpha)}\right| > M, \quad |\alpha| \leq r_0,$$

for all but a finite number of $f \in \mathfrak{F}_1$ (it is obviously no longer necessary to assume that $f'(\alpha) \neq 0$). However,

$$\left|2\log^+ |f(\alpha)| + \log\left|\frac{1}{f'(\alpha)}\right| - \log\left\{\frac{1+|f(\alpha)|^2}{|f'(\alpha)|}\right\}\right| = |2\log^+|f(\alpha)| - \log(1+|f(\alpha)|^2)| \leq \log 2,$$

so given $r_0 < 1$,
$$\frac{|f'(z)|}{1+|f(z)|^2} < 1, \quad |z| \leq r_0$$

for all but a finite number of $f \in \mathfrak{F}_1$. Thus, the condition of Lemma A is satisfied, and $\mathfrak{F}$ is normal.

6. *Generalization of a theorem of Montel.*

THEOREM 2. *Let $\mathfrak{F}$ be a family of meromorphic functions in Δ, and suppose that all poles are of multiplicity $\geqslant h$, all zeros of multiplicity $\geqslant k$ and all zeros of $f(z)-1$ of multiplicity $\geqslant l$, with*

$$\frac{1}{h}+\frac{1}{k}+\frac{1}{l}=\mu<1.$$

Then $\mathfrak{F}$ is normal.

Remark. Montel's form of Theorem 2 [15, p. 125] required that the poles have multiplicity divisible by h, the zeros have multiplicity divisible by k, and the zeros of $f(z)-1$ have multiplicity divisible by l, with $1/h+1/k+1/l<1$. Results analogous to Theorem 2 in the special case of holomorphic families are well-known (cf. [17]).

Proof. We start with the basic inequality (3.2), where α is chosen so that $f(\alpha)\neq 0,1,\infty$, $f'(\alpha)\neq 0$. By hypothesis,

$$\bar{N}(r,f_\alpha)\leqslant\frac{1}{h}N(r,f_\alpha)\leqslant\frac{1}{h}T(r,f_\alpha);$$

$$\bar{N}\left(r,\frac{1}{f_\alpha}\right)\leqslant\frac{1}{k}N\left(r,\frac{1}{f_\alpha}\right)\leqslant\frac{1}{k}T\left(r,\frac{1}{f_\alpha}\right)=\frac{1}{k}[T(r,f_\alpha)-\log|f(\alpha)|];$$

$$\bar{N}\left(r,\frac{1}{f_\alpha-1}\right)\leqslant\frac{1}{l}N\left(r,\frac{1}{f_\alpha-1}\right)\leqslant\frac{1}{l}T\left(r,\frac{1}{f_\alpha-1}\right)\leqslant\frac{1}{l}[T(r,f_\alpha)-\log|f(\alpha)-1|]+\log 2,$$

so (3.2) becomes

$$T(r,f_\alpha)<\mu T(r,f_\alpha)+2m\left(r,\frac{f_\alpha'}{f_\alpha}\right)+m\left(r,\frac{f_\alpha'}{f_\alpha-1}\right)$$
$$+\left(1-\frac{1}{k}\right)\log|f(\alpha)|+\left(1-\frac{1}{l}\right)\log|f(\alpha)-1|+\log\left|\frac{1}{f_\alpha'(0)}\right|+C.$$

Upon applying Lemma C and routine manipulations as in §3, we obtain

$$(1-\mu)\,T(r,f_\alpha)<12\log^+T(R,f_\alpha)+18\log\frac{1}{R-r}+8\log^+\log^+\left|\frac{1}{f(\alpha)}\right|$$
$$+\left(1-\frac{1}{k}\right)\log|f(\alpha)|+4\log^+\log^+\left|\frac{1}{f(\alpha)-1}\right|+\left(1-\frac{1}{l}\right)\log|f(\alpha)-1|+\log\left|\frac{1}{f_\alpha'(0)}\right|+C$$
$$<12\log^+T(R,f_\alpha)+18\log\frac{1}{R-r}+\left(2-\frac{1}{k}-\frac{1}{l}\right)\log^+|f(\alpha)|+\log\left|\frac{1}{f'(\alpha)}\right|+C+S, \tag{6.1}$$

$S=S(r_0)$ $(|\alpha|<r_0<1)$, $r_0<r<R<1$.

To prove normality of $\mathfrak{F}$, it suffices to show that each $z_0 \in \Delta$ has a neighborhood in which $\mathfrak{F}$ is normal. With no loss of generality, we take $z_0 = 0$; otherwise consider the family

$$\mathfrak{F}_{z_0} = \{f(z - z_0);\ |z - z_0| < 1 - |z_0|,\ f \in \mathfrak{F}\}.$$

Thus, let $\mathfrak{F}_1$ be a denumerably infinite subcollection of $\mathfrak{F}$. The proof of Theorem 2 will follow from considering two cases.

Case 1. There exist functions $f_n \in \mathfrak{F}_1$, $n = 1, 2, \ldots$, $\alpha_n = \alpha_n(f_n) \neq 0$, $\alpha_n \to 0$, $f_n(\alpha_n) \neq 0, 1, \infty$, $f_n'(\alpha_n) \neq 0$, and $M < \infty$ with

$$\left(2 - \frac{1}{k} - \frac{1}{l}\right) \log^+ |f_n(\alpha_n)| + \log \left| \frac{1}{f_n'(\alpha_n)} \right| < M. \tag{6.2}$$

In this case, Lemma 2 when applied to (6.1) and (6.2) shows that

$$T(r, f_{n,\alpha_n}) < \Sigma(r), \quad \tfrac{1}{2} \leqslant r < 1, \tag{6.3}$$

where $\Sigma(r)$ is an increasing, finite-valued function.

Inequality (6.3) and the auxiliary conditions on the α_n imply local normality of $\mathfrak{F}_1$. We state this as

LEMMA 4. *Let $\alpha_n \to 0$, $\alpha_n \neq 0$, with $f_n(\alpha_n) \neq \infty$ and (6.3) satisfied. Then $\mathfrak{F}_1$ is normal in a neighborhood of the origin.*

Proof. The origin cannot be a limit point of poles. For if x_{n_k} is a pole of f_{n_k} with $x_{n_k} \to 0$ for some subsequence n_k, then $x_{n_k} - \alpha_{n_k} \to 0$. Abbreviate f_{n_k}, x_{n_k}, α_{n_k} by f, x, α. Then if $r > \frac{1}{2}$

$$\Sigma(r) > N(r, f) > \int_{x-\alpha}^{r} \frac{1}{t}\, dt = \log\left(\frac{r}{x - \alpha}\right) \geqslant \log\left(\frac{1}{x - \alpha}\right) - \log 2$$

which is a contradiction to (6.3) since $x - \alpha \to 0$. Thus there is a neighborhood $\{|z| < \delta < \frac{1}{2}\}$ on which the functions f are holomorphic, and (6.3) reduces to the condition

$$m(r, f_\alpha) < \Sigma(\tfrac{1}{2}) \quad 0 \leqslant r < \tfrac{1}{2},$$

sufficient for normality of $\mathfrak{F}_1$ in this neighborhood (cf. Lemma 1′ and (1.3)).

Case 2. There is a $\delta > 0$ with the property that if $|\alpha| < \delta$, $f(\alpha) \neq 0, 1, \infty$, $f'(\alpha) \neq 0$,

$$\left(2 - \frac{1}{k} - \frac{1}{l}\right) \log^+ |f(\alpha)| + \log \left| \frac{1}{f'(\alpha)} \right| > M,$$

for all but finitely many $f \in \mathfrak{F}_1$.

But then the criterion of Lemma A is satisfied in $|\alpha| < \delta$ (by a continuity argument at those points where $f(\alpha) = 0$, 1, or ∞, or $f'(\alpha) = 0$); for example, consider separately the case $|f(\alpha)| < 1$ and $|f(\alpha)| \geqslant 1$. Thus $\mathfrak{F}_1$ is normal in a neighborhood of the origin.

III. A new composition

1. Let $f \in \mathfrak{F}$; then if $|\alpha| < r = |z|$, the function f_α is defined as in [(2.1), Part II]. This composition has been useful in the discussions of Theorems 1 and 2, but a somewhat more complicated composition will also be needed.

We apply this composition to prove Lemmas 5–8; these Lemmas are used throughout the rest of this work. The corollary to Lemma 5 is of crucial importance in Part V. The inequalities derived in Lemmas 6–8 are analogous to Milloux' basic estimates (cf. [9, p. 55], [12, p. 11 *et seq.*]) which are important in the study of value-distribution properties of entire (meromorphic) functions. The Lemmas here require that the family $\mathfrak{F}$ is not normal; thus, they give some insight on Bloch's conjecture.

Fix $r_0 < 1$, and let $r_0 < r < 1$. Then if $|\alpha| \leqslant r_0$, define

$$\psi_\alpha(z) = r^2 \frac{z-\alpha}{r^2 - \bar{\alpha} z}; \tag{1.1}$$

to avoid confusion, the composition of f with ψ_α will always be written in full, but we continue to use the convenient notation f_α for $f \circ \varphi_\alpha$.

While $\psi_\alpha(z)$ depends also on r, it will only be used in terms like $m(r, f\circ\psi_\alpha)$, where the explicit choice of r is obvious (cf. Lemma 5).

Note that ψ_α is a conformal map of $|z| \leqslant r$ onto itself. While $f\circ\psi_\alpha$, $f \in \mathfrak{F}$, will not be defined in all of $|z| < 1$, it is defined if $|z| < r^2(1+r_0)/(r_0 + r^2)$.

LEMMA 5. *Let τ and θ be related by*

$$re^{i\tau} = \psi_\alpha(re^{i\theta}), \quad 0 \leqslant \theta \leqslant 2\pi,$$

where $|\alpha| \leqslant r_0 < r$. Then there exists a constant $K = K(r, r_0)$ with the property that

$$\frac{1}{K} < \frac{d\tau}{d\theta} < K, \quad 0 \leqslant \theta \leqslant 2\pi. \tag{1.2}$$

Proof. This can be established by a more or less direct computation, but the following proof seems more in the spirit of things. As α varies in $|z| < r_0$, the $\{\psi_\alpha\}$ are a collection of uniformly bounded holomorphic functions on each compact subset of $|z| < r^2/r_0$; in particular, the ψ_α are uniformly bounded in a neighborhood of $|z| \leqslant r$, so that

$$\max_{\substack{|\alpha| \leqslant r_0 < r \\ |z| \leqslant r}} |\psi_\alpha'(z)| < K_1 < \infty.$$

If $z = re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$,

$$\left|\frac{d\tau}{d\theta}\right| = |\psi_\alpha'|,$$

and the right inequality in (1.2) follows at once. The proof is completed by noting that $\psi_\alpha^{-1} = \psi_{-\alpha}$.

Remark. If r_0 and r are fixed, $r_0<r<1$, then for $s>r$ we may take $K(s, r_0)=K(r, r_0)$.

COROLLARY. *Let* $|\alpha| \leqslant r_0$. *Then if* g *is meromorphic in* $|z| \leqslant r$ $(r>r_0)$,

$$\frac{1}{K} m(r, g) < m(r, g \circ \psi_\alpha) < K m(r, g). \tag{1.3}$$

Proof.

$$m(r, g \circ \psi_\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g \circ \psi_\alpha(re^{i\tau})| \, d\tau = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| \frac{d\tau}{d\theta} d\theta;$$

thus
$$|m(r, g \circ \psi_\alpha) - m(r, g)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| \left| \frac{d\tau}{d\theta} - 1 \right| d\theta \leqslant K m(r, g),$$

and the other inequality follows similarly.

Remark. It is clear that the constant K of (1.2) and (1.3) may be taken as close to 1 as we please, provided that $r>\eta$ and $|\alpha|<\delta(\eta)$.

2. We return to the study of normal families and apply the composition of $f \circ \psi_\alpha$ to prove three new lemmas. Unfortunately, our proofs of these lemmas fail for meromorphic families, and this inadequacy restricts our remaining theorems to families of holomorphic functions.

The first lemma is a qualitative improvement of Lemma C:

LEMMA 6. *If* $\mathfrak{F}$ *is a family of functions holomorphic in* Δ, *and* $\mathfrak{F}$ *is not normal, then there is an* $r_0<1$ *with*

$$m\left(r, \frac{f'}{f}\right) < A + B \log^+ T(R, f) + C \log \frac{1}{R-r}, \quad \tfrac{1}{2}(1+r_0) < r < R < 1 \tag{2.1}$$

for f *in an infinite subfamily* $\mathfrak{F}_1$ *of* $\mathfrak{F}$.

Proof. The assumptions imply that there is an $r_0<1$ such that for all but a finite number of $f \in \mathfrak{F}$ there exist $\alpha=\alpha(f)$, $|\alpha|<r_0$, with $|f(\alpha)| \geqslant 1$.

Let $\mathfrak{F}_1$ denote this subfamily of $\mathfrak{F}$, and apply Lemma C and (1.3) with $g=f \circ \psi_{-\alpha}$, $f \in \mathfrak{F}_1$. Then

$$m\left(r, \frac{f'}{f}\right) < K m\left(r, \frac{f'}{f} \circ \psi_{-\alpha}\right) < K\left\{4 \log^+ T(r', f \circ \psi_{-\alpha}) + 6 \log \frac{1}{r'-r} + D\right\},$$
$$\tfrac{1}{2}(1+r_0) < r < r' < r^2(1+r_0)/(r_0+r^2). \tag{2.2}$$

Since f is holomorphic and the circle $|\psi_{-\alpha}(z)|=r'$ is contained in

$$|z| = r^2(r'-r_0)/(r^2-r_0 r') = r_1, \tag{2.3}$$

we obtain (using (1.2), Part I)

$$T(r', f\circ\psi_{-\alpha}) \leqslant \log^+ M(r', f\circ\psi_{-\alpha}) \leqslant \log^+ M(r_1, f) \leqslant \frac{R+r_1}{R-r_1} T(R,f), \quad r_1 < R < 1;$$

thus
$$\log^+ T(r', f\circ\psi_{-\alpha}) \leqslant \log\frac{1}{R-r_1} + \log^+ T(R,f) + A, \tag{2.4}$$

with A independent of f and α if, say, $r>\frac{1}{2}$. Now given $R>r$, choose r' (which determines r_1 from (2.3)) so that $1/(r'-r)=1/(R-r_1)$. It is then easy to see that there is a constant $A=A(r_0)$ with the property that $1/(r'-r)=1/(R-r_1)\leqslant A(R-r)$, and (2.1) is a consequence of (2.2) and (2.4).

Remark. The estimate (2.1) should be compared to that of Lemma C; while it is less precise, it allows the elimination of the term involving $\log^+\log^+|1/g(0)|$. The assumption that $\mathfrak{F}$ was not normal was essential, as can be seen by considering the functions $\{e^{n(z-1)};\ n=1,2,\ldots\}$.

In applications, we often use a local version of Lemma 6; we state (and omit the routine proof):

LEMMA 6'. *Let $\mathfrak{F}$ be a family of functions holomorphic in $|z|<\delta$ $(\delta>0)$ which is not normal in any neighborhood of $z_0=0$. Then there are constants $A=A(\delta)$, $B=B(\delta)$, $C=C(\delta)$ with the property that*

$$m(r, f'/f) < A + B\log^+ T(R,f) + C\log 1/(R-r)$$

when $\frac{1}{2}\delta<r<R<\delta$, for f in an infinite subfamily of $\mathfrak{F}$.

3. The techniques used in the proof of Lemmas 6 and 6' allow further applications.

LEMMA 7. *Let $\mathfrak{F}$ be a family of functions holomorphic in Δ (resp. $|z|<\eta$) which is not normal (resp. not normal in any neighborhood of the origin). Then there is an $r_1<1$ and a constant K with the property that if $r>r_1$ (resp. $r>\eta/2$)*

$$m\left(r,\frac{1}{f}\right) \leqslant K m(r,f)$$

for an infinite number of $f\in\mathfrak{F}$.

Proof. Find $r_0<1$ such that to each f of an infinite subfamily $\mathfrak{F}_1$ of $\mathfrak{F}$ corresponds an $\alpha=\alpha(f)$, $|\alpha|<r_0$ with

$$|f(\alpha)|\geqslant 1,$$

and $f(\alpha)\neq 0,\infty$ (resp., find a sequence $\alpha_n\to 0$ with $|\alpha_n|<\eta/3$, $n=1,2,\ldots$, $|f_n(\alpha_n)|\geqslant 1$, $f_n(\alpha_n)\neq 0,\infty$). Let $r_1=\frac{1}{2}(1+r_0)$. Then if $r>r_1$ (resp. $r>\eta/2$), we see from Lemma 5 that if $f\in\mathfrak{F}_1$

$$m\left(r,\frac{1}{f}\right) < Km\left(r,\frac{1}{f}\circ\psi_{-\alpha}\right) \leqslant KT\left(r,\frac{1}{f}\circ\psi_{-\alpha}\right) = K\{T(r,f\circ\psi_{-\alpha}) - \log|f(\alpha)|\}$$
$$< KT(r, f\circ\psi_{-\alpha}) = Km(r, f\circ\psi_{-\alpha}) < K^2 m(r,f).$$

4. Lemma 7 permits a generalization of Lemma 6; our result is similar to Theorem 1 of [7], except that we have been able to eliminate the dependence of our estimate on the initial value terms.

LEMMA 8. *Let $\mathfrak{F}$ be a family of functions holomorphic in Δ which is not normal and let $k \geqslant 1$ be a fixed integer. Then there exists $r_0 < 1$, $r = r_0(k)$, such that to each fixed ϱ, $r_0 < \varrho < 1$, correspond constants $A = A(\varrho)$, $B = B(\varrho)$, $C = C(\varrho)$ and an infinite subfamily $\mathfrak{F}_1$ of $\mathfrak{F}$ with*

$$m\left(r, \frac{f^{(k)}}{f}\right) < A + B \log^+ T(R, f) + C \log \frac{1}{R-r}, \quad f \in \mathfrak{F}_1,$$

for $r_0 < r < R < \varrho$.

Remark. Note that the subfamily depends on ϱ; however the number r_0 depends only on the full family $\mathfrak{F}$, as is clear from the proof.

Proof. We use mathematical induction. When $k = 1$, the lemma is a consequence of Lemma 6 [in fact Lemma 6 is slightly stronger, since ϱ may be taken to equal 1]. Now let $k > 1$ be fixed.

Case 1. There exists $r_1 < 1$ and an infinite subcollection $\mathfrak{F}_1$ of $\mathfrak{F}$ with the property that to each $f \in \mathfrak{F}_1$ corresponds an $\alpha = \alpha(f)$, $|\alpha| < r_1$, with

$$|f^{(k-1)}(\alpha)| \geqslant 1.$$

Then the proof of Lemma 6 yields that if $f \in \mathfrak{F}_1$,

$$m\left(r, \frac{f^{(k)}}{f^{(k-1)}}\right) < A + B \log^+ T(R, f^{(k-1)}) + C \log \frac{1}{R-r}, \quad r_0 = \tfrac{1}{2}(1 + r_1) < r < R < 1. \tag{4.1}$$

To estimate the right side of (4.1), note that standard manipulations and the induction hypothesis yield that

$$\begin{aligned} T(r, f^{(k-1)}) \leqslant m\left(r, \frac{f^{(k-1)}}{f}\right) + T(r, f) &\leqslant A + B \log^+ T(r, f) + C \log \frac{1}{R-r} + T(r, f) \\ &\leqslant A + C \log \frac{1}{R-r} + CT(r, f). \end{aligned} \tag{4.2}$$

We use (4.2) in (4.1) together with the obvious estimate

$$m\left(r, \frac{f^{(k)}}{f}\right) \leqslant m\left(r, \frac{f^{(k)}}{f^{(k-1)}}\right) + m\left(r, \frac{f^{(k-1)}}{f}\right)$$

and the induction hypothesis once more to complete the proof of the lemma under the assumptions of Case 1.

Case 2. For each $\varrho < 1$, the inequality

$$|f^{(k-1)}(z)| \leqslant 1, \quad |z| < \tfrac{1}{2}(1 + \varrho) = \varrho_1$$

holds for a subcollection $\mathfrak{F}_1$ of $\mathfrak{F}$ (which depends on ϱ) such that $\mathfrak{F}-\mathfrak{F}_1$ is finite. We choose such a ϱ (which in turn determines ϱ_1). The choice of ϱ is arbitrary, but it is convenient to require that $\varrho>\varrho_0\geqslant\frac{3}{4}$; here ϱ_0 (<1) is determined by the condition that $\mathfrak{F}$ is not normal in $|z|<\varrho_0$. Having chosen ϱ, it remains fixed for the remainder of the proof.

An application of Cauchy's formula

$$f^{(k)}(z)=\frac{1}{2\pi i}\int_{|\zeta|=\varrho_1}\frac{f^{(k-1)}(\zeta)}{(\zeta-z)^2}\,d\zeta$$

yields that if $|z|<\varrho$,

$$|f^{(k)}(z)|\leqslant 4(1-\varrho)^{-2}, \tag{4.3}$$

so that in $|z|<\varrho$

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leqslant A(\varrho)\cdot\left|\frac{1}{f(z)}\right|,$$

and thus if $r<\varrho$ and $f\in\mathfrak{F}_1$

$$m\left(r,\frac{f^{(k)}(z)}{f(z)}\right)\leqslant A+m\left(r,\frac{1}{f(z)}\right). \tag{4.4}$$

A natural (though incorrect) line of proof would be to use Lemma 7 to improve (4.4) to

$$m\left(r,\frac{f^{(k)}(z)}{f(z)}\right)\leqslant A+BT(r,f(z)),$$

but this estimate is not as good as that demanded in the statement of Lemma 8.

However, we shall show that the assumptions

$$\mathfrak{F}_1 \textit{ not normal in } |z|<\varrho \tag{4.5}$$

for some $\varrho<1$ (we assume that $\varrho>\varrho_0>\frac{3}{4}$)

$$\max_{f\in\mathfrak{F}_1} M(\varrho,f^{(k)})\leqslant A \tag{4.6}$$

(here, A is given by (4.3)) imply the stronger estimate

$$m\left(r,\frac{1}{f}\right)<B, \tag{4.7}$$

if $\frac{3}{4}<r<\varrho$, for an infinite subfamily of $\mathfrak{F}_1$. Since $\mathfrak{F}$ is not normal in Δ, it fails to be normal in $|z|<\varrho$, $\varrho>\varrho_0$, for some $\frac{3}{4}\leqslant\varrho_0<1$; thus (4.7) when combined with (4.4) completes the proof of Lemma 8.

Turning to the proof of (4.7), note that from (4.6) follows at once that there is an M, independent of $f\in\mathfrak{F}_1$, for which

$$f(z)=A(z)+B\prod_{j=1}^{r}(z-b_j),\quad f\in\mathfrak{F}_1,$$

where $r\leqslant k-1$ and $|A(z)|<M$, $|z|\leqslant\varrho$.

Let the b_j $(b_j = b_j(f))$ be so numbered that

$$|b_1| < 4, \ldots, |b_t| < 4, \quad t = t(f),$$

whereas $|b_{t+1}| \geqslant 4, \ldots, |b_r| \geqslant 4$. With this convention, we write

$$f(z) = A(z) + B(z) \prod_{j=1}^{t} (z - b_j),$$

$|A(z)| < M$.

If there were a constant N so that for an infinite subcollection $\mathfrak{F}_2 \subset \mathfrak{F}_1$ the corresponding functions $B(z)$ satisfied

$$M(\varrho, B(z)) < N, \tag{4.8}$$

then the functions in $\mathfrak{F}_2$ would be uniformly bounded in $|z| \leqslant \varrho$, and normal. Since $\mathfrak{F}$ (and thus $\mathfrak{F}_1$) is assumed not normal, it follows that for any N, (4.8) fails for infinitely many $f \in \mathfrak{F}_1$.

In fact, a stronger conclusion holds: given any N, $1 < N < \infty$, the inequality

$$\min_{|z| \leqslant \varrho} |B(z)| > N \tag{4.9}$$

must hold for an infinite subfamily $\mathfrak{F}_2$ of $\mathfrak{F}_1$.

To see this, note that $$B(z) = B \prod_{j=t+1}^{r} (z - b_j),$$

so that if z and ζ are in Δ, $$|B(\zeta)| \geqslant 3^{-k-1} |B(z)|.$$

Choose z for which $|B(z)| = M(\varrho, B(z))$; this and the fact that the numbers $M(\varrho, B(z))$ are unbounded yield (4.9).

From the representation

$$f(z) = B(z) \{A(z) B(z)^{-1} + \prod_{j=t+1}^{r} (z - b_j)\} = B(z) f_1(z) \tag{4.10}$$

and the fact that $m(r, 1/B(z)) = 0$, $r < \varrho$, for $f \in \mathfrak{F}_2$, follows that

$$m\left(r, \frac{1}{f}\right) \leqslant m\left(r, \frac{1}{f_1}\right), \quad f \in \mathfrak{F}_2, \tag{4.11}$$

and it suffices to prove (4.7) for f_1 instead of f.

We note first that there is an absolute constant C with the property that if $f \in \mathfrak{F}_2$

$$T(r, f_1) = m(r, f_1) \leqslant C. \tag{4.12}$$

It is easy to see from an elementary compactness argument, or the Boutroux–Cartan lemma, that there is an absolute constant K_1 with the property that for each f_1 is a $\beta = \beta(f_1)$, with

$$|\beta| < \tfrac{1}{2}, \quad \prod_{j=1}^{t} (\beta - b_j) > e^{K_1}.$$

Then, using Lemma 6, if $\frac{3}{4} < r \leqslant \varrho$,

$$m\left(r, \frac{1}{f_1}\right) < Km\left(r, \frac{1}{f_1} \circ \psi_{-\beta}\right) < KT\left(r, \frac{1}{f_1} \circ \psi_{-\beta}\right) \leqslant K\{T(r, f_1 \circ \psi_{-\beta}) + K_1\}$$
$$\leqslant K^2 T(r, f_1) + KK_1 \leqslant C,$$

with C an absolute constant.

This estimate, with (4.4), (4.11), and (4.12), completes the proof of Lemma 8.

Lemma 8 leads to the standard comparison between $T(r, f^{(k)})$ and $T(r, f)$, valid for a family which is not normal.

COROLLARY. *Let $\mathfrak{F}$ be a family of functions holomorphic in Δ which is not normal, and let k be a fixed integer $\geqslant 1$. Then there exists $r_0 < 1$ such that to each fixed ϱ, $r_0 < \varrho < 1$ corresponds $A = A(\varrho)$, $B = B(\varrho)$, $C = C(\varrho)$ and an infinite subfamily $\mathfrak{F}_1$ of $\mathfrak{F}$ with*

$$T(r, f^{(k)}) \leqslant A + BT(R, f) + C \log \frac{1}{R - r}, \quad f \in \mathfrak{F}_1,$$

if $r_0 < r < R < \varrho$.

Proof. The standard estimate

$$T(r, f^{(k)}) \leqslant m\left(r, \frac{f^{(k)}}{f}\right) + T(r, f)$$

and the lemma yield that

$$T(r, f^{(k)}) \leqslant A + B \log \frac{1}{R - r} + T(r, f) + C \log^+ T(R, f).$$

However, we have $\log^+ T(R, f) < T(R, f)$, and the lemma is proved.

Remark. Let $M < \infty$ and k be fixed positive numbers and $\mathfrak{F}$ be a family of functions each of which can be written as

$$g(z) + h(z),$$

$g(z)$ a polynomial of degree $< k$ and $|h(z)| < M$. While $\mathfrak{F}$ need not be normal, it is clear that it is quasi-normal [15, p. 66]; consider the family $\{nz\}$, for example. The discussion used in Case 2 of Lemma 8 shows that this example is somewhat typical: if $\mathfrak{F}$ is not normal, then there is an $r_0 < 1$ and M such that if $r_0 < r < \varrho (< 1)$, the inequality

$$m\left(r, \frac{1}{f}\right) < M,$$

holds for an infinite subcollection of $\mathfrak{F}$. In fact, it is not hard to see that, given $\varepsilon>0$, $\delta>0$ and $\varrho<1$, the stronger estimate

$$m\left(r, \frac{1}{f}\right) < \varepsilon,$$

$\delta \leqslant r \leqslant \varrho$ holds for an infinite subfamily of $\mathfrak{F}$. To see this, one uses (4.11), the fact that $1/f$ will be large only near the b_j, a "small arcs" lemma [6, p. 322] applied to f_1, and the manipulations used at the close of the proof of Lemma 8. We omit the details.

IV. A theorem on nonvanishing families of holomorphic functions

1. We prove

THEOREM 3. *Let $\mathfrak{F}$ be a family of nonvanishing holomorphic functions in Δ, and $a_0(z)$, ..., $a_{k-1}(z)$ fixed holomorphic functions. Let $\mathcal{G}$ consist of the functions*

$$g(z)=f^{(k)}(z)+a_{k-1}(z)f^{(k-1)}(z)+\ldots+a_0(z)f(z), \quad f\in\mathfrak{F}, \tag{1.1}$$

and assume that the equation $g(z)=1$ has no solutions for $z\in\Delta$. Then $\mathfrak{F}$ is normal.

Theorem 3 is due to Chuang [3], [16, p. 41]; the special case $g(z)=f^{(k)}(z)$ was considered in [14]. The requirement that $\mathfrak{F}$ be nonvanishing is almost best possible: consider the functions $g(z)$ obtained from the family $\mathfrak{F}=\{f_n\}$, $n=2, 3, \ldots$, with $k=1$, $f_n(z)=nz$ and $a_0(z)\equiv 0$. Then each $f\in\mathfrak{F}$ has a single simple zero, the equation $g(z)=1$ has no solution for $g\in\mathcal{G}$, but $\mathfrak{F}$ is not normal. In Part V, it will be seen that the condition that $\mathfrak{F}$ be nonvanishing can be weakened if one has the opposite situation in which all zeros of functions in $\mathfrak{F}$ occur with high multiplicity.

An important fact needed is contained in

LEMMA 9. *If $\mathcal{G}$ is normal, then $\mathfrak{F}$ is normal.*

The proof is deferred to § 3.

2. *Proof of Theorem 3.* Let $\mathfrak{F}_1$ be an arbitrary denumerable subcollection from $\mathfrak{F}$; we will show that $\mathfrak{F}_1$ contains an infinite subfamily which converges normally in $|z|<\varrho$, where ϱ is any fixed number <1. For convenience, we assume that

$$\varrho \geqslant \tfrac{1}{2}\max(1+r_0, 1+r_1, 1+r_2, 1+r_3), \tag{2.1}$$

where the constants r_0, r_1, r_2 and r_3 will be determined below; they depend only on $\mathfrak{F}_1$. $\mathcal{G}_1$ will denote the subfamily of $\mathcal{G}$ which corresponds to $\mathfrak{F}_1$; g_α will be related to f_α by (1.1), etc. Since some of these arguments will be used in Part V, we proceed in somewhat greater generality than needed for present purposes.

Unless $\mathcal{G}_1$ has a normally convergent subfamily (in which case so would $\mathfrak{F}_1$), there must exist an $r_0<1$ and an infinite subfamily $\mathcal{G}_2$ of $\mathcal{G}_1$ such that to each $g \in \mathcal{G}_2$ corresponds an $\alpha(=\alpha(g))$, $|\alpha|<r_0$, with

$$\left|\frac{g(\alpha)-1}{g'(\alpha)}\right|<1. \tag{2.2}$$

Note also that from the definition of g_α, there exists an absolute constant A with

$$\begin{aligned}\log^+ T(R', g_\alpha) &\leqslant \sum_{j=1}^{k} \log^+ T(R', a_{j,\alpha}) + \sum_{j=1}^{k} \log^+ T(R', f^{(j)}{}_\alpha) + A \\ &= L_\alpha(R') + \sum_{j=1}^{k} \log^+ T(R', f^{(j)}{}_\alpha) + A \end{aligned} \tag{2.3}$$

where we have let $L_\alpha(R') = \Sigma \log^+ T(R', a_{j,\alpha})$.

Since $\mathcal{G}_2$ may be assumed to contain no convergent subfamily, there must exist $r_1<1$ such that both Lemma 8 and its corollary may be applied to an infinite subfamily $\mathcal{G}_3$ of $\mathcal{G}_2$. The corollary is used in (2.3) to yield that if $r_1<R'<R<\varrho$

$$\log^+ T(R', g_\alpha) \leqslant L_\alpha(R') + A \log \frac{1}{R-R'} + B \log^+ T(R, f_\alpha) + C.$$

We use this last estimate and Lemma 8 itself in the fundamental inequality (3.2) of Part II, with f_α replaced by g_α, and $R' = \frac{1}{2}(r+R)$. Then if ϱ is a fixed number less than 1 (chosen to satisfy (2.1)), it follows that there is an $r_1<\varrho$ (we assume $r_1>r_0$) such that

$$\begin{aligned}T(r, g_\alpha) < N\left(r, \frac{1}{g_\alpha}\right) + N\left(r, \frac{1}{g_\alpha - 1}\right) - N\left(r, \frac{1}{g_\alpha'}\right) + L_\alpha(R) + A \log^+ T(R, f_\alpha) \\ + B \log \frac{1}{R-r} + \log\left|\frac{g(\alpha)\{g(\alpha)-1\}}{g'(\alpha)}\right| + C, \end{aligned} \tag{2.4}$$

when $r_1<r<R<\varrho$, if g belongs to an infinite subfamily $\mathcal{G}_3$ of $\mathcal{G}_2$. In (2.4) we must assume that $g(\alpha) \neq 0, 1$, and $g'(\alpha) \neq 0$, but if $g(z) \equiv 0$ or 1, or $g'(z) \equiv 0$ for infinitely many $g \in \mathcal{G}_2$, then $\mathcal{G}_2$—and thus $\mathfrak{F}_2$—would be normal. We also note that if α is variable, but $|\alpha|<r_0<1$, and $R<\varrho$, then $L_\alpha(R)<A$, $A=A(r_0, \varrho)$.

The next goal is to revise (2.4) to get a relation involving only f_α, $f \in \mathfrak{F}_3$. Since

$$m\left(r, \frac{1}{g_\alpha}\right) + N\left(r, \frac{1}{g_\alpha}\right) = T(r, g_\alpha) - \log|g(\alpha)|,$$

(2.4) becomes

$$\begin{aligned}m\left(r, \frac{1}{g_\alpha}\right) < N\left(r, \frac{1}{g_\alpha - 1}\right) - N\left(r, \frac{1}{g_\alpha'}\right) + A \log^+ T(R, f_\alpha) + B \log \frac{1}{R-r} \\ + \log\left|\frac{g(\alpha)-1}{g'(\alpha)}\right| + C, \quad r_1<r<R<\varrho.\end{aligned}$$

Also $$T\left(r, \frac{1}{f_\alpha}\right) = m\left(r, \frac{1}{f_\alpha}\right) + N\left(r, \frac{1}{f_\alpha}\right) \leqslant m\left(r, \frac{g_\alpha}{f_\alpha}\right) + m\left(r, \frac{1}{g_\alpha}\right) + N\left(r, \frac{1}{f_\alpha}\right).$$

It is clear from Lemma 8 that (since $\varrho < 1$ continues to be fixed) there are infinite subfamilies $\mathcal{F}_4$ and $\mathcal{G}_4$ of $\mathcal{F}_3$ and $\mathcal{G}_3$ for which

$$m\left(r, \frac{g_\alpha}{f_\alpha}\right) \leqslant \sum_{j=1}^{k} m(r, a_{j,\alpha}) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}{}_\alpha}{f_\alpha}\right) + C \leqslant H_\alpha(r) + A \log^+ T(R, f_\alpha) + B \log \frac{1}{R-r} + C \quad (2.5)$$

if $r_2 < r < R < \varrho$ (we assume with no loss of generality that $r_1 < r_2 < \varrho$). Here $H_\alpha(r) = \sum_1^k m(r, a_{j,\alpha})$. If α is variable, but $|\alpha| < r_0 < 1$, then it is easy to find a fixed function H_0 with $H_\alpha(r) < H_0(r)$. Upon combining these manipulations, we obtain

$$T\left(r, \frac{1}{f_\alpha}\right) \leqslant H_0(r) + N\left(r, \frac{1}{f_\alpha}\right) + N\left(r, \frac{1}{g_\alpha - 1}\right) - N\left(r, \frac{1}{g_\alpha'}\right)$$
$$+ A \log^+ T(R, f_\alpha) + B \log \frac{1}{R-r} + C + \log\left|\frac{g(\alpha)-1}{g'(\alpha)}\right|, \quad (2.6)$$

$r_2 < r < R < 1$. Since $|\alpha| < r_0$ and ϱ is fixed, the constants A, B and C depend only on the family $\mathcal{F}_2$.

The particular assumptions of Theorem 3 allow (2.6) to be simplified to

$$T\left(r, \frac{1}{f_\alpha}\right) \leqslant H_0(r) + A \log^+ T(R, f_\alpha) + B \log \frac{1}{R-r} + C + \log\left|\frac{g(\alpha)-1}{g'(\alpha)}\right|, \quad (2.7)$$

$r_2 < r < R < \varrho$.

Finally, it follows from Lemma 7, since $\mathcal{F}$ is a family of nonvanishing holomorphic functions, that unless $\mathcal{F}_4$ is normal there is an $r_3 < \varrho$ (we suppose that $r_2 < r_3 < \varrho$) and K such that

$$T(r, f_\alpha) \leqslant KT\left(r, \frac{1}{f_\alpha}\right) \quad r_3 < r < \varrho, \quad (2.8)$$

for $f \in \mathcal{F}_5$, an infinite subset of $\mathcal{F}_4$.

Put (2.8) in (2.7), and choose α (and r_0) in accord with (2.2). If we apply Lemma 2 to the functions

$$U(r) = T(\varrho r, f_\alpha), \quad \gamma(r) = H_0(\varrho r)$$

(where ϱ is determined by (2.1)), it follows that the functions $T(r, f_a)$, $f \in \mathcal{F}_5$, are uniformly bounded in $r \leqslant \varrho$. Thus $\mathcal{F}$ is normal in $|z| < \varrho$, and since ϱ can be chosen arbitrarily near 1, a standard diagonalization argument completes the proof of Theorem 3.

3. *Proof of Lemma 9.* Let $\{g_n\} \in \mathcal{G}$, $g_n \to g_0$ normally. We show that each point $z_0 \in \Delta$ has a neighborhood on which the corresponding family $\{f_n\}$ is normal.

Case 1. $g_0(z_0) \neq 0$. It is no loss of generality to suppose that $z_0 = 0$. Then z_0 has a neighborhood $|z| < 2\delta < 1/2$ on which $|g(z)| > \varepsilon$ for an infinite subfamily $\mathcal{G}_1 \subset \{g_n\}$. But if $\delta < r < 2\delta$, a local version of (2.5) shows that unless $\mathfrak{F}_1$ were normal in a neighborhood of the origin,

$$m\left(r, \frac{1}{f}\right) \leqslant m\left(r, \frac{g}{f}\right) + m\left(r, \frac{1}{g}\right) \leqslant H_0(r) + A \log^+ T(R, f) + B \log \frac{1}{R-r} + C$$

$$\leqslant A \log^+ T(R, f) + B \log \frac{1}{R-r} + C + H_0(\tfrac{1}{2}),$$

since by assumption we have $m(r, 1/g) = O(1)$. Apply Lemmas 2 and 8 to $U(r) = m(r/2\delta, 1)$, $\gamma(r) = 0$ to deduce (as in the last section) that $\mathfrak{F}_1$ is normal.

Case 2. $g_0(z_0) = 0$. We first recall some elementary facts from the theory of linear differential equations. If $h_1(z), \ldots, h_k(z)$ are a linearly independent set of solutions of the homogeneous equation

$$L(h) = a_0(z) h(z) + \ldots + h^{(k)}(z) = 0,$$

then the solutions of $L(h) = g(z)$ are given [5, p. 87] (variation of parameters) by

$$h(z) = \sum_{m=1}^{k} \{\alpha_m + \beta_m(z)\} h_m(z). \tag{3.1}$$

In (3.1) the α_m are arbitrary constants, and

$$\beta_m(z) = \int_0^z \left\{ \frac{W_m(h_1, \ldots, h_k)(t)}{W(h_1, \ldots, h_k)(t)} \right\} g(t)\, dt, \tag{3.2}$$

where $W(h_1, \ldots, h_k)$ is the Wronskian determinant of $h_1, \ldots, h_k$, and $W_m(h_1, \ldots, h_k)$ is the Wronskian of $h_1, \ldots, h_{m-1}, h_{m+1}, \ldots, h_k$. From the derivation of (3.1) it follows that the $\beta_m(z)$ satisfy the constraints

$$\left.\begin{aligned}
&\beta_1'(z) h_1(z) + \beta_2'(z) h_2(z) + \ldots + \beta_k'(z) h_k(z) = 0\\
&\quad\vdots\\
&\beta_1'(z) h_1^{(k-2)}(z) + \beta_2'(z) h_2^{(k-2)}(z) + \ldots + \beta_k'(z) h_k^{(k-2)}(z) = 0\\
&\beta_1'(z) h_1^{(k-1)}(z) + \beta_2'(z) h_2^{(k-1)}(z) + \ldots + \beta_k'(z) h_k^{(k-1)}(z) = g(z).
\end{aligned}\right\} \tag{3.3}$$

Returning to the discussion of Lemma 9, we see from (3.1) that since $L(f_n) = g_n$, there are constants $\alpha_{n,m}$ $(m = 1, \ldots, k)$ and functions $\beta_{n,m}(z)$ $(m = 1, \ldots, k)$ determined by (3.2) with $g = g_n$, for which

$$f_n(z) = \sum \{\alpha_{n,m} + \beta_{n,m}(z)\} h_m(z).$$

Further, given $\varepsilon > 0$, one can find $\delta > 0$ and $M < \infty$ with the property that

$$\left|\frac{W_m(t)}{W(t)}\right| < M, \quad |g_n(t)| < \varepsilon \quad (n > n_0, \ m = 1, 2, \ldots, k)$$

if $|t| < \delta$, since g_0 is a cluster function of $\{g_n\}$ and $g_0(0) = 0$. Thus

$$|\beta_{n,m}(z)| < M\varepsilon = \varepsilon^*, \quad |z| < \delta, \ n > n_0.$$

Let $\mathfrak{F}^*$ be the class of corresponding combinations $\{\Sigma\, \alpha_{n,m} h_m\}$, as determined by (3.1). Thus, corresponding to $F \in \mathfrak{F}^*$ there is an $f_n \in \mathfrak{F}$ with

$$|F(z) - f_n(z)| < k\varepsilon^* \max_{|z|<\delta} |h_m(z)| < \varepsilon_1, \quad (|z| < \delta_1(\varepsilon_1)). \tag{3.4}$$

It is clear that $\mathfrak{F}_1 = \{f_n\}$ is normal if the corresponding functions $F(z) \in \mathfrak{F}^*$ are normal, so Lemma 9 is a consequence of the local version of

LEMMA 10. *Let $h_1(z), \ldots, h_k(z)$ be fixed linearly independent functions holomorphic in Δ. Let $\mathfrak{F}^*$ be a family of functions holomorphic in Δ such that if $f \in \mathfrak{F}^*$,*

$$f(z) = \sum_1^k \alpha_i h_i(z)$$

for suitable constants $\alpha_i = \alpha_i(f)$. Suppose that there exists an M with the property that for each $F \in \mathfrak{F}^*$ corresponds a g, $g = g(F)$, holomorphic in Δ, $|g(z)| < M$, such that the equation

$$F(z) + g(z) = 0$$

has no solutions in Δ. Then $\mathfrak{F}^*$ is normal.

Proof. For $F \in \mathfrak{F}^*$, let $j = j(F)$ be determined as follows: j is the least integer such that $|\alpha_j| \geq |\alpha_i|$, $i \neq j$. Write

$$F(z) = \alpha_j \sum_1^k \frac{\alpha_i}{\alpha_j} h_i(z) \quad j = j(F).$$

$\mathfrak{F}^*$ will consist of uniformly bounded holomorphic functions and would thus be normal unless $\alpha_j \to \infty$ for an infinite subfamily $\mathfrak{F}_1^*$ of $\mathfrak{F}^*$ and some fixed $j \in \{1, \ldots, k\}$. Assume, with no loss of generality, that $j = 1$. For $F \in \mathfrak{F}_1^*$,

$$F = \alpha_1 h_1 + \sum_2^k \frac{\alpha_i}{\alpha_1} h_i = \alpha_1 \{h_1 + \sum_2^k \beta_i h_i\}. \tag{3.5}$$

Note that $|\beta_i| \leq 1$, $i = 2, \ldots, k$. Let K be a compact subset of Δ. Since the β_i range over a compact subset of $\mathbf{C}^{k-1}$, one can choose a convergent subsequence for $i = 2, \ldots, k$. Let

$$h_1 + \sum_2^k \beta_i^* h_i = F^*$$

be the limit of the corresponding subfamily of $\mathfrak{F}_1^*$, the convergence being normal in Δ. F^* cannot be identically zero, as $\{h_1, ..., h_m\}$ is linearly independent. But then F^* never vanishes on the interior of K. For (3.5) may be rewritten as

$$F/\alpha_1 = h_1 + \sum_2^k \beta_i h_i.$$

Recall that $F = F + g - g$, with $F + g$ never vanishing on Δ. Since $|g(z)| < M$ and $\alpha_1 \to \infty$, g/α_1 tends uniformly to zero. Thus $(F+g)/\alpha_1$ tends normally to F^*; since $(F+g)$ is never 0 and $F^* \not\equiv 0$, Hurwitz' theorem ensures that F^* is never 0 on the interior of K.

Thus if K_1 is a compact subset of the interior of K, there exists $\eta > 0$ such that

$$|h_1 + \sum_2^k \beta_i h_i| > \eta, \quad z \in K_1$$

for an infinite subfamily of $\mathfrak{F}_1^*$. Since $\alpha_1 \to \infty$, we see at once from (3.5) that ∞ is a cluster function of $\mathfrak{F}^*$, and Lemma 10 is established.

V. An extension of Theorem 3

1. The assumption throughout Part IV that $\mathfrak{F}$ be a nonvanishing family can be weakened only in special cases, as might be expected from the discussion in § 1, Part IV. We prove

THEOREM 4. *Let $\mathfrak{F}$ be a family of functions holomorphic in Δ. Suppose the zeros of each f in $\mathfrak{F}$ are of multiplicity $\geq m$, and the zeros of $g(z)-1$ are of multiplicity $\geq p$, where*

$$\frac{k+1}{m} + \frac{k+1}{p} = \tau < 1,$$

and
$$g(z) = f^{(k)}(z) + a_{k-1}(z) f^{(k-1)}(z) + \dots + a_0(z) f(z), \tag{1.1}$$

with $a_0(z), ..., a_{k-1}(z)$ holomorphic in Δ. Then $\mathfrak{F}$ is normal.

Remark. If j of the functions $a_0(z), ..., a_{k-1}(z)$ are identically zero, the proof shows that it is sufficient to assume that

$$\frac{k+1}{m} + \frac{k-j+1}{p} < 1.$$

COROLLARY. (Normal family analogue of a theorem of W. K. Hayman [8].) *Let $\mathcal{G}$ be a family of holomorphic functions in Δ such that the equation*

$$g'(z) g(z)^n = 1$$

(where n is a fixed integer, $n \geq 2$) has no solutions. Then $\mathcal{G}$ is normal.

Proof. Let $f(z)=\left(\frac{1}{n+1}\right)g(z)^{n+1}$, $k=1$ and $a_0(z)=0$ in the statement of Theorem 4. Then $\mathcal{F}$, the family of such f, is normal, and hence so is $\mathcal{G}$.

Remark. This corollary seems to have been discovered first by L. Yang and K. Chang [18].

The proof parallels that used in obtaining Theorem 3. As usual, $\mathcal{G}$ will denote the family of functions obtained by (1.1) from f in $\mathcal{F}$.

2. We need a preliminary result:

LEMMA 11. *If $\mathcal{G}$ is normal then $\mathcal{F}$ is normal.*

Proof. It again suffices to work locally and we modify the argument of § 3, Part IV. Let $g_n\to g_0$ normally, and let $z_0\in\Delta$ (which we take to be $z=0$).

Case 1. $g_0(z_0)\neq 0$. Then the discussion in Case 1 of the proof of Lemma 9 may be applied immediately, since z_0 cannot be a limit point of the zeros of the corresponding f_n (due to the assumption of multiplicity, zeros of f yield zeros of g).

Case 2. $g_0(z_0)=0$ but for some $\delta>0$,

$$n\left(r,\frac{1}{f_n}\right)=0,\quad r<\delta$$

for infinitely many f_n. Then the discussion of Case 2 of the proof of Lemma 9 may be applied once more.

Case 3. $g_0(z_0)=0$, and there exist z_n with $z_n\to 0$, $f_n(z_n)=0$, and thus a sequence $z_n\to 0$ with $g_n(z_n)=0$. Let

$$h_1(z),\ h_2(z),\ \dots,\ h_k(z)$$

be linearly independent solutions of the differential equation $g(z)=0$. Then from (3.1), Part IV, we deduce

$$f_n(z)=f_{n,h}(z)+\sum_{m=1}^{k}\left\{\int_{z_n}^{z}\frac{W_m(t)}{W(t)}g_n(t)\,dt\right\}h_m(z);\tag{2.1}$$

here, $f_{n,h}(z)=\Sigma\,\alpha_{n,m}h_m(z)$ is a solution to the homogeneous equation $g(z)=0$. We claim that in fact

$$f_{n,h}\equiv 0,\quad n>n_0\tag{2.2}$$

so that (2.1) becomes simply

$$f_n(z)=\sum\left\{\int_{z_n}^{z}\frac{W_m(t)}{W(t)}g_n(t)\,dt\right\}h_m(z).\tag{2.3}$$

Let us complete the proof of Lemma 11, deferring temporarily the justification of (2.2). Choose $\delta>0$, and then determine M with

$$|g_n(z)| < M, \quad \left|\frac{W_m(t)}{W(t)}\right| < M, \quad |h_m(z)| < M, \quad |z| < 2\delta, \tag{2.4}$$

$m = 1, 2, \ldots, k$, $n > n_0$. Then there is an absolute constant M^* with

$$|f_n(z)| < M^*,$$

$|z| < \delta$, and $\{f_n\}$ is normal in a neighborhood of the origin.

It remains to show (2.2). Write (2.1) as (for $f_n = f$, $f_{n,h} = f_h$, $g_n = g$):

$$f(z) = f_h(z) + \sum_{m=1}^{k} \beta_m(z)\, h_m(z). \tag{2.5}$$

Note that
$$\beta_m(z_n) = 0 \quad m = 1, 2, \ldots, k. \tag{2.6}$$

Further it follows from (3.3) of Part IV that

$$f'(z_n) = f_h'(z_n) + \sum \beta_m(z_n) h_m'(z_n).$$

The assumption that the zeros of f have multiplicity $\geqslant m$ implies that $f'(z_n) = 0$, and this together with (2.6) allows us to deduce that

$$f_h'(z_n) = 0;$$

continuing, it follows that $f_h''(z_n) = \ldots = f_h^{(k-1)}(z_n) = 0$. Thus, from the uniqueness of solution of the homogeneous form of (1.1) in a neighborhood of z_n, we get (2.2).

3. *Proof of Theorem 4.* Let $\mathfrak{F}_1$ be a denumerably infinite subcollection from $\mathfrak{F}$; we show $\mathfrak{F}_1$ is normal in a neighborhood of $z_0 = 0$. Theorem 4 is a consequence of Theorem 3 unless z_0 is a limit point of the zeros of $\mathfrak{F}_1$.

Note that if $f(\alpha) \neq 0$,

$$N\left(r, \frac{1}{f_\alpha}\right) - N^*\left(r, \frac{1}{g_\alpha'}\right) \leqslant (k+1)\int_0^r \frac{\bar{n}\left(t, \frac{1}{f_\alpha}\right)}{t}\, dt \leqslant \frac{k+1}{m} N\left(r, \frac{1}{f_\alpha}\right) \leqslant \frac{k+1}{m} T\left(r, \frac{1}{f_\alpha}\right)$$
$$= \frac{k+1}{m}\{T(r, f_\alpha) - \log |f(\alpha)|\}; \tag{3.1}$$

here, $N^*(r, 1/g_\alpha')$ counts the zeros of g_α' which are also zeros of the corresponding f_α. Also

$$N\left(r, \frac{1}{g_\alpha - 1}\right) - N^{**}\left(r, \frac{1}{g_\alpha'}\right) \leqslant \frac{1}{p} N\left(r, \frac{1}{g_\alpha - 1}\right) \leqslant \frac{1}{p}\{T(r, g_\alpha) - \log|g(\alpha) - 1| + A\}, \tag{3.2}$$

where $N^{**}(r, 1/g_\alpha')$ counts the zeros of g_α' which arise from zeros of $g_\alpha - 1$. We use (3.1) and (3.2) in (2.6), Part IV. Since only the behavior of $\mathfrak{F}_1$ in a neighborhood of the origin is relevant, we take $\varrho = \frac{1}{2}$ in (2.1) of Part IV, and insist that $|\alpha| < \frac{1}{4}$. From Lemma 8 and the obvious inequality $T(r, f_\alpha^{(k)}) \leqslant m(r, f_\alpha^{(k)}/f_\alpha) + T(r, f_\alpha)$ it follows that

$$T(r, g_\alpha) \leqslant (k+1)\, T(r, f_\alpha) + A \log^+ T(R, f_\alpha), \quad r < R < \varrho.$$

Thus, if f is in an infinite subfamily $\mathcal{F}_2$ of $\mathcal{F}_1$,

$$\{1-\tau\}\,T(r,f_\alpha)\leqslant A\log^+ T(R,f_\alpha)+B\log\frac{1}{R-r}+C$$
$$+\log\left|\{f(\alpha)\}^{1-(k+1)/m}\{g(\alpha)-1\}^{(p-1)/p}\{g'(\alpha)\}^{-1}\right|,\quad \delta<r<R<\tfrac{1}{2},\qquad (3.3)$$

(with these restrictions on α and r, the functions $H_\alpha(r)$ are uniformly bounded, and are absorbed in the constant C). The numbers A, B and C now depend only on the family $\mathcal{F}_1$. It is clear from (3.3) that Lemma 2 will yield that $\mathcal{F}$ is normal in a neighborhood of the origin if there exists a sequence α_n, such that $\alpha_n\to 0$ with

$$\left|f(\alpha)^{(m-k-1)/m}\{g(\alpha)-1\}^{(p-1)/p}g'(\alpha)^{-1}\right|<M,\qquad (3.4)$$

$\alpha=\alpha_n$, $f=f_n\in\mathcal{F}_2$ corresponding to $g=g_n\in\mathcal{G}_2$.

Suppose that f_n has a zero z_n where $z_n\to 0$. Then z_n is a zero of g and g'. Given z_n, determine y_n by: $|z_n-y_n|$ is minimized subject to $|g_n'(y_n)|=1$; if such a y_n fails to exist for infinitely many n, the corresponding functions g_n' would be uniformly bounded in Δ, and consequently these g_n would have a convergent subfamily.

Let $|z_n-y_n|=\delta_n$; then if $|z-z_n|<\delta_n$,

$$|g_n(z)|<\int_{z_n}^{z}|g_n'(z)|\,|dz|<\delta_n.\qquad (3.5)$$

Suppose that there were an $\eta>0$ with the property that $\delta_{n_k}>\eta$ for a subsequence δ_{n_k}. It then follows at once from the definition of δ_{n_k}, the fact that $z_{n_k}\to 0$, and (3.5) that

$$|g_{n_k}(z)|\leqslant 1,\quad |z|<\eta/2,$$

and so G—and thus $\mathcal{F}$—is normal in a neighborhood of z_0.

Hence, we assume that $\delta_n\to 0$. In this case, (2.3) shows that $f_n(y_n)\to 0$. Also,

$$|g_n(y_n)/g_n'(y_n)|\leqslant\int_{z_n}^{y_n}|g_n'(t)|\,dt\leqslant|z_n-y_n|=\delta_n.$$

Thus, the expression (3.4) in fact tends to zero as $n\to\infty$ with $\alpha=y_n$, and the theorem is proved.

4. *An open question.* J. Clunie [4] has proved that an entire function f such that $f'f=1$ has no solutions is constant. Bloch's hypothesis thus makes it likely that the corollary to Theorem 4 is valid when $n=1$, and a similar generalization of Theorem 4 itself may be true. Note that Lemma 11 is valid under these hypotheses. Clunie's proofs were different from those of [8], and it seems that the methods used here are not deep enough to resolve this conjecture.

VI. A new criterion for normality

1. Confirming a conjecture of Hayman [10] we prove

THEOREM 5. *Let $\mathcal{F}$ be a family of functions holomorphic in Δ, and for a fixed $n \geqslant 3$ and $a \neq 0$ suppose that*

$$f' - af^n = b, \quad f \in \mathcal{F}$$

has no solutions in Δ. Then $\mathcal{F}$ is normal.

If $f \in \mathcal{F}$, we follow Hayman [8] and determine h by

$$h = a \frac{f^n}{f' - b - af^n};$$

$\mathcal{H}$ denotes the family of these functions. The hypothesis of Theorem 5 implies that $\mathcal{H}$ is a family of *holomorphic* functions.

2. *A preliminary lemma.*

LEMMA 12. *If $\mathcal{H}$ is normal then so is $\mathcal{F}$.*

Proof. Standard manipulations yield that if $h(\alpha) \neq -1$ then

$$nT(r, f_\alpha) = T(r, f_\alpha^n) = T\left(r, \frac{f'_\alpha - b}{a} \cdot \frac{h_\alpha}{h_\alpha + 1}\right)$$

$$\leqslant T(r, f_\alpha) + m\left(r, \frac{f'_\alpha}{f_\alpha}\right) + T(r, h_\alpha) - \log|h(\alpha) + 1| + K(a, b); \qquad (2.1)$$

thus
$$T(r, f_\alpha) \leqslant \frac{1}{n-1}\left[m\left(r, \frac{f'_\alpha}{f_\alpha}\right) + T(r, h_\alpha) - \log|h(\alpha) + 1| + K(a, b)\right]. \qquad (2.2)$$

Let $\mathcal{F}_1$ be a denumerably infinite subcollection of $\mathcal{F}$, and consider the associated family $\mathcal{H}_1 \subset \mathcal{H}$. Let $\mathcal{H}_1$ have cluster function h^*. We consider three cases.

Case 1. $h^* \not\equiv -1, \infty$. Let $\{h_n\} = \mathcal{H}_2 \subset \mathcal{H}_1$ tend to h^*. There exist $M < \infty$, $r_0 < 1$, $\alpha = \alpha(h)$, $|\alpha| < r_0$, with

$$-\log|h(\alpha) + 1| < M \qquad (2.3)$$

for an infinite subfamily $\mathcal{H}_3 \subset \mathcal{H}_2$. For all but a finite number of functions in $\mathcal{H}_3$, the manipulations of (2.1) are permitted. Also, for any $r < 1$,

$$T(r, h_{n,\alpha}) \to T(r, h^*_\alpha), \quad n \to \infty, \qquad (2.4)$$

and since we assume $\mathcal{F}_1$ is not normal, the estimate derived in Lemma 6 may be applied to $m(r, f'_\alpha/f_\alpha)$, for f in an infinite subfamily $\mathcal{F}_4$ of $\mathcal{F}_3$.

The estimate (2.2) with the above modifications calls for application of Lemma 2 in full force; take $U(r)=T(r,f_{n,\alpha})$, $\gamma(r)=(1/(n-1))\,T(r,h_{n,\alpha})$ (if necessary, increase the coefficient of $\log(1/(R-r))$ to satisfy the hypothesis of Lemma 2). Then if $r>r_0>|\alpha|$,

$$T(r,f_{n,\alpha})\leqslant A+BT(R,h_{n,\alpha})+C\log\frac{1}{R-r},\quad r_0<r<R<1, \tag{2.5}$$

for an infinite subset of the h_n. This, when coupled with (2.4) and the criterion (1.3) of Part II, yields the lemma in this case.

Case 2. $h^*\equiv -1$. Then for each $r_0<1$, and $\varepsilon>0$

$$\left|\frac{f'(z)-b}{af(z)^n}\right|<\varepsilon,\quad |z|<r_0,$$

for infinitely many $f\in\mathfrak{F}$.

It is easy to see that $\mathfrak{F}$ is normal whenever the following is true: if

$$|f(z_1)|\leqslant 1,\quad |z_1|<r_0,\quad f\in\mathfrak{F}, \tag{2.6}$$

then there is an $\eta=\eta(r_0)>0$ with the property that if

$$|f(z_2)|\geqslant 2,\quad |z_2|<r_0, \tag{2.7}$$

then

$$|z_2-z_1|\geqslant\eta. \tag{2.8}$$

Thus, suppose z_1 and z_2 are chosen in accord with (2.6) and (2.7); we may as well assume that $|z_1-z_2|$ is minimized subject to these conditions. Let γ denote the line segment from z_1 to z_2; then γ lies in $|z|<r_0$, so that if z is on γ,

$$|f'(z)|\leqslant\varepsilon|af(z)^n|+\beta\leqslant K\varepsilon+\beta,\quad K=K(a,n),\ \beta=|b|,$$

since $|f(z)|\leqslant 2$. But then

$$1\leqslant|f(z_2)-f(z_1)|\leqslant\int_\gamma|f'(z)|\,|dz|\leqslant(\beta+K\varepsilon)|z_1-z_2|,$$

which proves (2.8).

Case 3. $h^*\equiv\infty$. In this case, the estimate (2.5) is meaningless. But if $h^*\equiv\infty$, then for each $r_0<1$, $\varepsilon>0$,

$$\left|\frac{f'(z)-b}{af(z)^n}\right|<1+\varepsilon,\quad |z|<r,$$

for infinitely many $f\in\mathfrak{F}$, and the argument used in Case 2 may be reapplied to verify directly that $\mathfrak{F}$ is normal.

3. *Proof of Theorem 5.* Let $\mathfrak{F}_1$ be a denumerably infinite subcollection of $\mathfrak{F}$, and $\mathcal{H}_1$ the corresponding subcollection of $\mathcal{H}$. We show that $\mathcal{H}_1$ is locally normal, say, in a neighborhood of the origin; to obtain greatest flexibility with the initial-value terms, both compositions φ_α and ψ_β will be used, with $|\alpha|+|\beta|$ small. In particular, we suppose initially that $|\alpha|<\frac{1}{4}$. The precise choices of α and β will be made later.

We apply the fundamental inequality [(3.1), Part II] to the functions h_α, using Lemma 6′ to estimate the terms involving means of logarithmic derivatives. Thus, if $\mathcal{H}_1$ were not normal in any neighborhood of the origin, there would exist δ_1 and an infinite subfamily $\mathcal{H}_2$ of $\mathcal{H}_1$ whose elements satisfy

$$m(r,h_\alpha)+m\left(r,\frac{1}{h_\alpha}\right)+m\left(r,\frac{1}{h_\alpha+1}\right)\leqslant 2T(r,h_\alpha)-N\left(r,\frac{1}{h_\alpha'}\right)+A\log^+T(R,h_\alpha)$$
$$+B\log\frac{1}{R-r}+C+\log\left|\frac{1}{h'(\alpha)}\right|,\quad h\in\mathcal{H}_2, \tag{3.1}$$

where A, B, C are independent of r if

$$\delta_1<r<R<1. \tag{3.2}$$

The remark at the end of § 1 of Part III implies that given $\varepsilon>0$, $\delta_2<\delta_1/2$ may be chosen with the property that

$$(1-\varepsilon)\,m\left(r,\frac{1}{h_\alpha+1}\circ\psi_\beta\right)\leqslant m\left(r,\frac{1}{h_\alpha+1}\right) \tag{3.3}$$

whenever $|\beta|<\delta_2$ and r satisfies (3.2). Similarly, since h is holomorphic,

$$T(r,h_\alpha\circ\psi_\beta)>(1-\varepsilon)\,T(r,h_\alpha),\quad \beta<\delta_2,\ r>\delta_1. \tag{3.4}$$

The particular choice of ε will be determined later. Add

$$N\left(r,\frac{1}{h_\alpha}\right)+\log|h(\alpha)|+(1-\varepsilon)\left\{N\left(r,\frac{1}{h_\alpha+1}\circ\psi_\beta\right)+\log|(h_\alpha+1)\circ\psi_\beta(0)|\right\}$$

to each side of (3.1), noting (3.3) and (3.4):

$$(1-\varepsilon)^2T(r,h_\alpha)\leqslant\bar{N}\left(r,\frac{1}{h_\alpha}\right)+(1-\varepsilon)\,N\left(r,\frac{1}{h_\alpha+1}\circ\psi_\beta\right)+A\log^+T(R,h_\alpha)+B\log\frac{1}{R-r}+C$$
$$+(1-\varepsilon)\log|(h_\alpha+1)(-\beta)|+\log\left|\frac{h(\alpha)}{h'(\alpha)}\right|,\quad |\beta|<\delta_2,\ \delta_1<r<R<1. \tag{3.5}$$

The hypothesis of the theorem ensures that all zeros of h_α are of multiplicity at least n; i.e.,

$$\bar{N}\left(r,\frac{1}{h_\alpha}\right)\leqslant\frac{1}{n}N\left(r,\frac{1}{h_\alpha}\right)\leqslant\frac{1}{n}T\left(r,\frac{1}{h_\alpha}\right)=\frac{1}{n}\{T(r.\,h_\alpha)-\log|h(\alpha)|\}. \tag{3.6}$$

Similarly, we see that

$$N\left(r, \frac{1}{h_\alpha+1}\circ\psi_\beta\right) \leqslant N\left(r, \frac{1}{f'_\alpha - b}\circ\psi_\beta\right) \leqslant T(r, (f'_\alpha - b)\circ\psi_\beta) - \log|(f'_\alpha - b)(-\beta)|$$

$$\leqslant T(r, f_\alpha\circ\psi_\beta) + m\left(r, \frac{f'_\alpha}{f_\alpha}\circ\psi_\beta\right) - \log|(f'_\alpha - b)(-\beta)| + A. \tag{3.7}$$

Note also that under the assumptions on β,

$$T(r, f_\alpha\circ\psi_\beta) < (1+\varepsilon)\,T(r, f_\alpha), \tag{3.8}$$

$$m\left(r, \frac{f'_\alpha}{f_\alpha}\circ\psi_\beta\right) < (1+\varepsilon)\,m\left(r, \frac{f'_\alpha}{f_\alpha}\right). \tag{3.9}$$

Since we are presuming that $\mathcal{F}_2$ has no normally convergent subfamily in any neighborhood of the origin, the estimate obtained in Lemma 6′ may be applied to the logarithmic derivative term in (3.7). Upon incorporating this with (3.6)–(3.9) in (3.5) it follows that for an infinite subfamily $\mathcal{F}_3$ of $\mathcal{F}_2$ (and a corresponding subfamily $\mathcal{H}_3$),

$$\left[(1-\varepsilon)^2 - \frac{1}{n}\right]T(r, h_\alpha) \leqslant (1+\varepsilon)\left\{T(r, f_\alpha) + A\log^+ T(\varrho, f_\alpha) + B\log\frac{1}{\varrho - r}\right\}$$

$$+ A\log^+ T(r, h_\alpha) + B\log\frac{1}{R-r} + C + (1-\varepsilon)\log\left|\frac{(h_\alpha+1)(-\beta)}{(f'_\alpha - b)(-\beta)}\right|$$

$$+\log\left|\frac{h(\alpha)^{1-(1/n)}}{h'(\alpha)}\right|, \quad \delta_1 < r < \varrho,\ R < 1,\ |\beta| < \delta_2. \tag{3.10}$$

It is now possible to make the choice of α and β more precise. Determine an infinite subfamily $\mathcal{H}_4 \subset \mathcal{H}_3$ as follows. There exists an infinite subcollection $\{h_n\} \subset \mathcal{H}_3$ and $\alpha_n = \alpha_n(h_n)$ with $\alpha_n \to 0$, such that (with $h = h_n$, $\alpha = \alpha_n$, $n = 1, 2, \ldots$)

$$\left|\frac{h(\alpha)^{1-(1/n)}}{h'(\alpha)}\right| < 1; \tag{3.11}$$

if this were not so, then it is easy to see that in some neighborhood of the origin, the criterion of Lemma A is fulfilled, and so $\mathcal{H}_3$—and thus $\mathcal{F}_3$—would be normal. For these functions f and h—and the corresponding choice of α—we note that

$$\left|\frac{h_\alpha+1}{f'_\alpha - b}(-\beta)\right| = \left|\frac{1}{f'_\alpha - b - f^n_\alpha}(-\beta)\right|$$

so that one can find (for perhaps another infinite subsequence of $\{f_n\}$, $f \in \mathcal{F}_4$) β_n corresponding to f_{n,α_n} with $\beta_n \to 0$ and

$$\left|\frac{h_{\alpha_n}+1}{f'_{\alpha_n}-b}(-\beta_n)\right| < M \quad (n\to\infty) \tag{3.12}$$

unless the family $\{f'-b-f^n\}, \quad f\in\mathfrak{F}_4,$

tends to 0 normally in some neighborhood of the origin. But it is easy to see that the argument of § 2 [that (2.8) is a consequence of (2.6) and (2.7)] may be applied to show that $\mathfrak{F}_4$ would also be normal in some neighborhood of the origin in that case.

Thus, assume that (3.11) and (3.12) are satisfied for an infinite subfamily $\mathfrak{F}_5$ of $\mathfrak{F}_4$. Then if $f\in\mathfrak{F}_5$ and h is the corresponding function in $\mathcal{H}_5$, (3.10) becomes

$$\left[(1-\varepsilon)^2-\frac{1}{n}\right]T(r,h_\alpha)\leqslant(1+\varepsilon)\left\{T(r,f_\alpha)+A\log^+T(\varrho,f_\alpha)+B\log\frac{1}{\varrho-r}\right\}$$
$$+A\log^+T(R,h_\alpha)+B\log\frac{1}{R-r}+C_1,\quad \delta_1<r<\varrho,R<1. \tag{3.13}$$

It remains to majorize $T(r,f_\alpha)$ in terms of $T(r,h_\alpha)$. In (2.1), consider $f_\alpha\circ\psi_\beta$ in place of f_α. Then unless the functions in $\mathcal{H}_5$ tend uniformly to -1 or ∞ in a neighborhood of the origin (in which case $\mathcal{H}_5$ is normal), it is possible to associate with an infinite subcollection $f_{n,\alpha_n}\in\mathfrak{F}_5$ a β_n with $\beta_n\to 0$ and $-\log|h_\alpha(\beta)+1|<M$. Then (2.2) becomes, after noting (3.8) and (3.9),

$$T(r,f_\alpha)\leqslant\frac{(1+\varepsilon)^2}{n-1}\left[m\left(r,\frac{f'_\alpha}{f_\alpha}\right)+T(r,h_\alpha)+M\right], \tag{3.14}$$

for $\delta_1<r<1$, $f\in\mathfrak{F}_5$. Still assuming $\mathfrak{F}_5$ has no convergent subfamily, we note from Lemma 6′ that

$$m\left(r,\frac{f'_\alpha}{f_\alpha}\right)<A\log^+T(R,f_\alpha)+B\log\frac{1}{R-r}+C,\quad \delta_1<r<R<1, \tag{3.15}$$

if $f\in\mathfrak{F}_6$, an infinite subfamily of $\mathfrak{F}_5$. Thus, Lemma 2 and (3.15) simplify (3.14) (with R in place of r) to

$$T(R,f_\alpha)\leqslant AT(\varrho,h_\alpha)+B\log\frac{1}{\varrho-R}+C,\quad \delta_1<R<\varrho<1. \tag{3.16}$$

Use (3.16) in (3.15). Upon choosing R so that $1/(\varrho-R)=1/(R-r)$, (3.14) becomes improved from (3.16) to

$$T(r,f_\alpha)\leqslant\frac{(1+\varepsilon)^2}{n-1}T(r,h_\alpha)+A\log^+T(R,h_\alpha)+B\log\frac{1}{R-r}+C,$$

for $\delta_1<r<R<1$, $f\in\mathfrak{F}_6$. Use this estimate in (3.13) with $\varrho=R$:

$$\left[(1-\varepsilon)^2-\frac{1}{n}-\frac{(1+\varepsilon)^2}{n-1}\right]T(r,h_\alpha)\leqslant A\log^+ T(R,h_\alpha)+B\log\frac{1}{R-r}+C; \tag{3.17}$$

choose $\varepsilon<\frac{1}{4}$, and apply Lemmas 2 and 12 to deduce the normality of $\mathcal{H}_6$, and the theorem follows.

VII. Proof of Lemma 2

We prove

LEMMA 2. *Let $U(r)$ and $\gamma(r)$ be continuous non-decreasing functions of r, $r_1<r<1$. If there is an r_0, $r_1<r_0<1$, and $b>1$ with*

$$U(r)<M+b\log\frac{1}{R-r}+c\log^+ U(R)+\gamma(r),\quad r_0<r<R<1, \tag{1.1}$$

then
$$U(r)<M_1+4\gamma(R)+2b\log\frac{1}{R-r},\quad r_0<r<R<1, \tag{1.2}$$

where M_1 depends only on M, b and c.

Proof. If $k\geqslant 1$, an argument due to E. Borel (cf. [9, p. 38]) yields that for each fixed $r>r_0$,

$$U(\varrho')<U(\varrho)+k\log 2,\quad \varrho'=\varrho+\exp\{-U(\varrho)/k\}, \tag{1.3}$$

for $\varrho>r$, save for perhaps a set of values of ϱ which can be enclosed in a finite or infinite number of intervals of length at most

$$\frac{2}{\exp(U(r)/k)}. \tag{1.4}$$

Now let $r_0\leqslant r<R$. If
$$\frac{2}{\exp(U(r)/k)}<R-r, \tag{1.5}$$

there is a ϱ, with $r<\varrho<R$, for which (1.3) holds. The definition of ϱ' in (1.3) yields that

$$\log\frac{1}{\varrho'-\varrho}=\frac{1}{k}U(\varrho). \tag{1.6}$$

Now let $k=2b$, and in (1.1) replace r by ϱ and R by ϱ'. From (1.3) we have that

$$c\log^+ U(\varrho')<c\log^+ U(\varrho)+c\log^+(k\log 2)+c\log 2<c\log^+ U(\varrho)+c(\log b+3\log 2),$$

and this and (1.6) allow (1.1) to be rewritten as

$$U(\varrho)<2M+2c\log^+ U(\varrho)+2c(\log b+3\log 2)+2\gamma(\varrho). \tag{1.7}$$

It is easy to see that unless $U(\varrho)\leqslant 16c^2$ for $r_0<\varrho<1$, we have for some $r_1<1$

$$\log^+ U(\varrho)<\sqrt{U(\varrho)}<\frac{1}{4c}U(\varrho),\quad r_1<\varrho<1.$$

We handle both cases simultaneously by rewriting (1.7) as

$$U(\varrho) < 16c^2 + 4M + 4c(\log b + 3 \log 2) + 4\gamma(\varrho),$$

and since $r < \varrho < R$ and U and γ increase, we find an absolute constant A with

$$U(r) < A + 4\gamma(R). \tag{1.8}$$

All of this depended on the assumption (1.5). But if (1.5) fails, then

$$U(r) < 2b \log \frac{2}{R-r}; \tag{1.9}$$

the possibilities that led to (1.8) and (1.9) may be considered together, leading to

$$U(r) < \{16c^2 + 4M + 4c(\log b + 3 \log 2)\} + 4\gamma(R) + 2b \log \frac{1}{R-r}, \quad r_0 < r < R < 1.$$

Remark 1. It is possible to make the conclusion of Lemma 2 independent of R by setting $R = \frac{(r+1)}{2}$.

Remark 2. It is easy to see that $4\gamma(R)$ in (1.2) may be replaced by $K\gamma(R)$, $K > 1$, by suitably changing the constant M_1 and the coefficient of $\log \frac{1}{R-r}$ in (1.2).

References

[1]. Ahlfors, L., *Complex Analysis*. New York, 1966.

[2]. Bureau, F., Mémoire sur les fonctions uniformes à point singulier essentiel isolé. *Mem. Soc. Roy. Sci. Liège*, 17 (1932), 44.

[3]. Chuang, C. T., Sur les fonctions holomorphes dans le cercle unité, *Bull. Soc. Math. France*, 68 (1940), 11–41.

[4]. Clunie, J., On a result of Hayman. *J. London Math. Soc.*, 42 (1967), 389–392.

[5]. Coddington, E. A. & Levinson, N., *Theory of Ordinary Differential Equations*. New York, 1955.

[6]. Edrei, A. & Fuchs, W. H. J., Bounds for the number of deficient values of certain classes of meromorphic functions. *Proc. London Math. Soc.* 12 (1962), 315–344.

[7]. Hayman, W. K., On Nevanlinna's second fundamental theorem and extensions. *Rend. Circ. Mat. Palermo* (2), 2 (1953), 346–392.

[8]. —— Picard values of meromorphic functions and their derivatives. *Ann. of Math.*, 70 (1959), 9–42.

[9]. —— *Meromorphic Functions*. Oxford, 1963.

[10]. —— *Research Problems in Function Theory*. London, 1967.

[11]. Hiong, K., Sur les fonctions holomorphes dans le cercle-unité admettant un ensemble de valeurs déficientes. *J. Math. Pures Appl.*, 34 (1955), 303–335.

[12]. Marty, F., Recherches sur la répartition des valeurs d'une fonction méromorphe. *Ann. Fac. Sci. Univ. Toulouse* (3), 23 (1931), 183–261.

[13]. Milloux, H., *Les Fonctions Méromorphes et leurs Dérivées*. Paris, 1940.

[14]. MIRANDA, C., Sur un nouveau critère de normalité pour les familles des fonctions holomorphes. *Bull. Soc. Math. France*, 63 (1935), 185–196.
[15]. MONTEL, P., *Leçons sur les Familles Normales de Fonctions Analytiques* .... Paris, 1927.
[16]. VALIRON, G., *Sur les Valeurs Exceptionelles des Fonctions Méromorphes et de leurs Dérivées.* Paris, 1937.
[17]. YANG, L., Sur les valeurs quasi-exceptionelles des fonctions holomorphes. *Sci. Sinica*, 13 (1964), 829–885.
[18]. YANG, L. & CHANG, K., Un nouveau critère et quelques applications. *Sci. Sinica*, 14 (1965), 1262.

Received March 22, 1968