ASYMPTOTIC BEHAVIOUR OF SPECTRA OF COMPACT QUOTIENTS OF CERTAIN SYMMETRIC SPACES

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1. Introduction

Let G be a connected noncompact semisimple Lie group with finite centre C, and let K be a maximal compact subgroup of G. Let Γ be a discrete subgroup of G such that the quotient $\Gamma\backslash G$ is compact. Γ acts on the symmetric space G/K by left translations and the quotient space $\Gamma\backslash G/K$ is also compact. Γ is unimodular and so there exists a G-invariant measure $d\tilde{x}$ on the quotient space $\Gamma\backslash G$. We denote by $L_2(\Gamma\backslash G)$ the space of complex valued measurable functions f on G such that (i) $f(\gamma x)=f(x)$ for $\gamma\in\Gamma$, $x\in G$ and (ii) $\int_{\Gamma\backslash G}|\tilde{f}(\tilde{x})|^2\,d\tilde{x}<\infty$, where $\tilde{f}$ is the function induced on $\Gamma\backslash G$ by f by virtue of its Γ-invariance. By $L_2(\Gamma\backslash G/K)$ we denote the subspace of $L_2(\Gamma\backslash G)$ consisting of functions f which in addition to (i) and (ii) satisfy (iii) $f(xk)=f(x)$, $x\in G$, $k\in K$. For short, a function f on G will be said to be *automorphic* (with respect to Γ and K) if $f(\gamma xk)=f(x)$, $\gamma\in\Gamma$, $x\in G$, $k\in K$. We denote by $\mathfrak{D}(G/K)$ the algebra of differential operators on G that commute with left translations by elements of G and right translations by elements of K.

It is well known [1], [2] that $L_2(\Gamma\backslash G/K)$ is the orthogonal direct sum of subspaces $\{H_i\}_{i=0}^{\infty}$ of the following description: (a) Each H_i is finite dimensional. (b) Each function in each H_i is infinitely differentiable. (c) On each H_i, the natural action of each element $D\in\mathfrak{D}(G/K)$ is by scalar multiplication. Thus, given a $D\in\mathfrak{D}(G/K)$ and a function $\varphi\in H_i$, we have $D\varphi=h_i(D)\varphi$, with $h_i(D)\in\mathbf{C}$. The mapping $h_i:\mathfrak{D}(G/K)\to\mathbf{C}$ is obviously a homomorphism of $\mathfrak{D}(G/K)$.

The role of the subspaces H_i in the harmonic analysis of $L_2(\Gamma\backslash G/K)$ is analogous to the role played in the harmonic analysis of functions on a compact group by the

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spaces of representative functions associated to the various irreducible representations of the group. At present not much specific information is known about the spaces H_i.

In this paper we shall be interested in the dimensions of the spaces H_i, and under the assumption that G is a *complex* group we will establish a result about the asymptotic behaviour of the dimensions of H_i in a sense which we shall now describe.

Let ω be the Laplace–Beltrami operator of the symmetric space G/K with respect to its canonical G-invariant Riemannian structure. ω can be regarded as an element of $\mathfrak{D}(G/K)$, so that ω acts on each H_i by the scalar $h_i(\omega)$. It can be shown that $h_i(\omega) \leqslant 0$ and that the number of spaces H_i for which $|h_i(\omega)|$ is less than any given positive real number r is finite. Assume then that the spaces H_i are so numbered that $|h_0(\omega)| \leqslant |h_1(\omega)| \leqslant |h_2(\omega)| \leqslant \dots$, and let us define the function $N(r)$ for $r \geqslant 0$ by

$$N(r) = \sum_{\{i: |h_i(\omega)| \leqslant r\}} \dim H_i. \tag{1.1}$$

It is the asymptotic behaviour of $N(r)$ as $r \to \infty$ that will be investigated. The final result is contained in Theorem 5.9. It connects the asymptotic behaviour of $N(r)$ with certain integrals of the Plancherel measure for spherical functions on G/K and with the volume of $\Gamma\backslash G/K$. Since the Plancherel measure for G/K is known in terms of the root structure of (G, K), we get in this way a description of the asymptotic behaviour of $N(r)$ in terms of intrinsic objects involving G, K and Γ.

Our method uses the analogue on G/K of the classical Gauss–Laplace function $(4\pi t)^{-\frac{1}{2}} \exp -x^2/4t$ on the real line. One defines on G/K, the function $g_t(x)$, $x \in G/K$, $t > 0$ as the fundamental solution of the heat equation $\omega u - \partial u/\partial t = 0$ on $G/K \times [0, \infty)$. g_t determines a convolution operator on $L_2(\Gamma\backslash G/K)$ which turns out to be of trace class. Evaluation of the trace of this operator in two different ways yields an analogue of the classical theta-relation of Jacobi adapted to our context. (It will be observed in the course of the proof that this procedure in our case is nothing more than an application of Mercer's theorem to our set up, and is incidentally also the prenatal version of Selberg's trace formula applied to this situation).

It turns out that when G is complex, the various terms in this formula can be analysed further, using various properties of g_t. This analysis, combined with the Plancherel theorem for spherical functions on G/K and standard Tauberian theorems then yields the desired result. It is to be noted that no knowledge of the fine structure of conjugacy classes of Γ is needed for our method to work.

In his address at the International Congress of Mathematicians in Stockholm, Gelfand announced a result similar to ours, without assuming that G is complex. [3, page 77].

Gelfand's formulation is marginally inaccurate, as will be pointed out below, and in any event, no proof of his result has appeared in the last six years. In the book [4] of Gelfand, Graev and Pyatetski-Shapiro, which appeared in 1966, Gelfand's announced result of [3] is asserted to hold for $G=SL(2,\mathbf{R})$ and $G=SL(2,\mathbf{C})$, and the proof of the result is formally reduced to establishing the existence of a family of functions on G which are to have assorted properties detailed on pages 118–119 of [4]. The existence of such functions is not proved in [4], even for the cases $G=SL(2,\mathbf{R})$ or $SL(2,\mathbf{C})$, nor has the proof appeared elsewhere, to our knowledge. For other G, it is even less obvious that the method of [4] works. In view of these phenomena, Gelfand's announcement of this result must be regarded as less than final, and it seems to us that it would be useful to have an accurate formulation and complete proof of this result. This will be done below assuming that G is complex. Our method is different from the one aspired to in [4].

It follows from our method that $N(r)\sim C_G r^{n/2}$ where $n=\dim G/K$ and C_G is a constant. If Γ was assumed to act freely on G/K, i.e. if $\Gamma\backslash G/K$ were a manifold, this much information could have heen gleaned from the paper of Minakshisundaram and Pleijel [5]. Indeed, were $\Gamma\backslash G/K$ a manifold, our result can be deduced from theirs used in conjunction with the Plancherel theorem for spherical functions on G. However, the elements of Γ which act with fixed points on G/K do seem to cause some problems and their method does not seem to generalize to the case when such elements exist, and in any case it does not relate $N(r)$ to the Plancherel measure of G/K. Thus our contribution here is in obtaining a relation between $N(r)$ and the root structure of G/K, and this without assuming that Γ acts freely on G/K. Moreover, our method is group theoretical and does not involve the detailed estimates of Green's functions that are used in [5].

The assumption that G is complex is used at two points in this paper. First in showing that the functions g_t satisfy a condition of regular growth. (See condition (3.48) below.) Second, in the evaluation of the contributions to the trace formula by the elements of Γ that act with fixed points on G/K. (See § 5.) We have every reason to believe that the condition of regular growth (3.48) is satisfied by g_t in general, even if G is not complex. In fact it can be shown to hold if rank $G/K=1$. The proof will be sketched below. Unfortunately we do not know how to prove it in general. The second use of the hypothesis that G is complex is likely to be more difficult to avoid, as will appear from our analysis below. Nevertheless, with the hope that this hypothesis will be eventually removed, we shall expound matters without the assumption that G is complex except when these two steps are involved.

The paper is organized as follows: § 2 is devoted to notation and preliminary information. We briefly describe the Plancherel formula in it. § 3 defines the functions g_t and deduces various properties of these functions. § 4 and § 5 are devoted to the proof of our result. In § 6 are inserted some remarks about a possibility of realaxing the hypothesis on G and also to outlining possible generalizations of the result.

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2. Preliminaries

Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ be the Lie algebras of G, K respectively and let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$, so that $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are orthogonal complements of one another with respect to the Killing form B on $\mathfrak{g}_0\times\mathfrak{g}_0$. B is negative definite on $\mathfrak{k}_0\times\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0\times\mathfrak{p}_0$. B induces a norm on $\mathfrak{p}_0$ denoted by $|\cdot|$. The space G/K is endowed with the G-invariant Riemannian metric induced by the form B. We denote by $d(x, y)$ the distance between $x, y\in G/K$ in this metric.

Let $\mathfrak{h}_{\mathfrak{p}_0}$ be a maximal abelian subspace of $\mathfrak{p}_0$. Extend $\mathfrak{h}_{\mathfrak{p}_0}$ to a maximal abelian subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and put $\mathfrak{h}_{\mathfrak{k}_0}=\mathfrak{h}_0\cap\mathfrak{k}_0$. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{h}_{\mathfrak{k}}$, $\mathfrak{h}_{\mathfrak{p}}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{h}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{h}_{\mathfrak{k}_0}$, $\mathfrak{h}_{\mathfrak{p}_0}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and we have $\mathfrak{h}_0=\mathfrak{h}_{\mathfrak{k}_0}\oplus\mathfrak{h}_{\mathfrak{p}_0}$. Let Δ be the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix compatible orderings on the duals of the real vector spaces $\mathfrak{h}_{\mathfrak{p}_0}$ and $\mathfrak{h}^*=i\mathfrak{h}_{\mathfrak{k}_0}+\mathfrak{h}_{\mathfrak{p}_0}$. [6, p. 222.] Each root $\alpha\in\Delta$ is real valued on $\mathfrak{h}^*$, so we can order Δ by means of this ordering. Let Δ_+ denote the set of positive roots. Then we have $\Delta_+=P_+\cup P_-$ where $P_+=\{\alpha\in\Delta_+;\ \alpha\not\equiv 0$ on $\mathfrak{h}_{\mathfrak{p}_0}\}$ and $P_-=\{\alpha\in\Delta_+;\ \alpha\equiv 0$ on $\mathfrak{h}_{\mathfrak{p}_0}\}$. Let $\mathfrak{g}^\alpha$ be the root space corresponding to α in $\mathfrak{g}$. Put $\mathfrak{n}=\sum_{\alpha\in P_+}\mathfrak{g}^\alpha$ and $\mathfrak{n}_0=\mathfrak{g}_0\cap\mathfrak{n}$. Then $\mathfrak{n}_0$ is a nilpotent subalgebra of $\mathfrak{g}_0$ and $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{h}_{\mathfrak{p}_0}\oplus\mathfrak{n}_0$ is the Iwasawa decomposition. Let $A_{\mathfrak{p}}$, N be the analytic subgroups of G which correspond to $\mathfrak{h}_{\mathfrak{p}_0}$, $\mathfrak{n}_0$ respectively. Then $G=KA_{\mathfrak{p}}N$, and $A_{\mathfrak{p}}$, N are simply connected. Let $l=\dim A_{\mathfrak{p}}$. For $a\in A_{\mathfrak{p}}$, put $\log a$ for the unique element in $\mathfrak{h}_{\mathfrak{p}_0}$ such that $\exp\log a=a$. Similarly for $x\in G$ put $H(x)$ for the unique element of $\mathfrak{h}_{\mathfrak{p}_0}$ such that $x=k\exp H(x)n$ with $k\in K$, $n\in N$.

Let Λ_0 be the real dual of $\mathfrak{h}_{\mathfrak{p}_0}$ and Λ the complexification of Λ_0. Λ can be regarded as the dual of $\mathfrak{h}_{\mathfrak{p}}$. Indeed we can identify Λ and $\mathfrak{h}_{\mathfrak{p}}$ as follows: $\lambda\in\Lambda\leftrightarrow H_\lambda\in\mathfrak{h}_{\mathfrak{p}}$ where H_λ is defined by $B(H, H_\lambda)=\lambda(H)$ for each $H\in\mathfrak{h}_{\mathfrak{p}}$. Put $\langle\lambda,\mu\rangle=B(H_\lambda, H_\mu)$ for $\lambda,\mu\in\Lambda$, and for $\nu\in\Lambda_0$ put $|\nu|=\langle\nu,\nu\rangle^{\frac{1}{2}}$. Evidently $|\cdot|$ is a norm. Since Λ, Λ_0 are thus identified with $\mathfrak{h}_{\mathfrak{p}}, \mathfrak{h}_{\mathfrak{p}_0}$ respectively, any function on $\mathfrak{h}_{\mathfrak{p}}$ or $\mathfrak{h}_{\mathfrak{p}_0}$ will be regarded as a function on Λ, Λ_0 under this identification.

As usual, we put $\varrho = \frac{1}{2}\sum_{\alpha\in P_+}\alpha$. Then ϱ is positive under the above ordering. Further, since each $\alpha\in\Delta$ is real on $\mathfrak{h}_{\mathfrak{p}_0}$ and purely imaginary on $\mathfrak{h}_{\mathfrak{k}_0}$, ϱ has these properties too. For any linear function ν on $\mathfrak{h}_0$ (or on $\mathfrak{h}$) we shall denote by ν_* the restriction of ν to $\mathfrak{h}_{\mathfrak{p}_0}$. This notation will be employed often below. Let $\Sigma = \{\alpha_* \mid \alpha\in P_+\}$ and let $\Sigma_0 = \{\alpha\in\Sigma \mid \alpha/n \notin \Sigma$ for any integer $n \neq 1\}$. For $\alpha\in\Sigma$, let m_α be the number of elements of P_+ whose restriction to $\mathfrak{h}_{\mathfrak{p}_0}$ is α. Clearly, $\varrho_* = \frac{1}{2}\sum_{\alpha\in\Sigma} m_\alpha\alpha$.

We put
$$\pi = \prod_{\alpha\in\Sigma_0}\alpha \tag{2.1}$$

Clearly π is a polynomial function on $\mathfrak{h}_{\mathfrak{p}_0}$, and $\pi(H_\lambda) = \prod_{\beta\in\Sigma_0}\beta(H_\lambda) = \prod_{\beta\in\Sigma_0}\langle\beta,\lambda\rangle$, $\lambda\in\Lambda_0$.

The Weyl group of (G, K) is the group W of linear tranformations of $\mathfrak{h}_{\mathfrak{p}_0}$ induced by those elements of $\mathrm{Ad}_G(K)$ which map $\mathfrak{h}_{\mathfrak{p}_0}$ into itself. Here Ad_G stands for the adjoint representation of G on $\mathfrak{g}_0$. W is a finite group of linear transformations and preserves the Killing form B. W acts on $A_{\mathfrak{p}}$ by exponentiation, on $\mathfrak{h}_{\mathfrak{p}}$ by complexification and on Λ_0, Λ by duality. We also make it act on various functions on these spaces in the obvious way.

Let $\mathcal{S}(\Lambda_0)$ (resp. $\mathcal{S}(\mathfrak{h}_{\mathfrak{p}_0})$) be the Schwartz space of C^∞ functions on Λ_0 (resp. $\mathfrak{h}_{\mathfrak{p}_0}$) which together with their derivatives decrease rapidly at ∞, and by $\mathcal{J}(\Lambda_0)$ (respectively $\mathcal{J}(\mathfrak{h}_{\mathfrak{p}_0})$) the subspaces of W-invariants in $\mathcal{S}(\Lambda_0)$ ($\mathcal{S}(\mathfrak{h}_{\mathfrak{p}_0})$). These spaces are equipped with their usual topologies.

A function f on G is said to be *spherical* if $f(kxk') = f(x)$; $k, k'\in K$, $x\in G$. For any $\lambda\in\Lambda$, the function defined by

$$\varphi_\lambda(x) = \int_K \exp\ (i\lambda - \varrho)\, H(xk)\, dk \tag{2.2}$$

is called the *elementary spherical function corresponding to* λ. φ_λ satisfies (i) $\varphi_\lambda(e) = 1$ (ii) $\varphi_\lambda(kxk') = \varphi_\lambda(x)$, $k, k'\in K$, $x\in G$ and (iii) the functional equation $\int_K \varphi_\lambda(xky)\,dk = \varphi_\lambda(x)\,\varphi_\lambda(y)$; $x, y\in G$. Further, each φ_λ is an eigenfunction of every operator $D\in\mathfrak{D}(G/K)$ and $\varphi_\lambda \equiv \varphi_{\lambda'}$, if and only if λ and λ' belong to the same orbit of W on Λ. It is known that if $\lambda\in\Lambda_0$ then φ_λ is a positive definite function on G. For all these matters see [6], [7] and [8].

Let $L_p(K\backslash G/K)$ be the space of spherical functions belonging to $L_p(G)$. For $f\in L_1(K\backslash G/K)$ we define the Fourier transform $\hat{f}(\lambda)$ by

$$\hat{f}(\lambda) = \int_G f(x)\,\varphi_\lambda(x^{-1})\,dx, \tag{2.3}$$

for those $\lambda\in\Lambda$ for which the integral converges absolutely. $\hat{f}$ is defined at least on

Λ_0 for each $f \in L_1(K\backslash G/K)$ and indeed for all $\lambda \in \Lambda$ for which φ_λ is bounded. It is obvious that $\hat{f}$ is invariant under W. For $f \in L_1(K\backslash G/K)$ let Φ_f be the function on $A_\mathfrak{p}$ defined by

$$\Phi_f(a) = \exp \varrho(\log a) \int_N f(an)\, dn. \text{ (1)} \tag{2.4}$$

Then it is known that $\Phi_f \in L_1(A_\mathfrak{p})$ and for $\lambda \in \Lambda_0$, $\hat{f}(\lambda)$ equals the Euclidean Fourier transform of Φ_f i.e.

$$\hat{f}(\lambda) = \int_{A_\mathfrak{p}} \Phi_f(a) \exp - i\lambda \,(\log a)\, da. \tag{2.5}$$

We shall now describe the Plancherel theorem for spherical functions on G, due to Harish-Chandra. Let dm be the G-invariant measure on G/K normalized by the condition $dx = dm dk$ and let dX be the volume element on $\mathfrak{p}_0$ determined by the metric $|\cdot|$ on $\mathfrak{p}_0$. Let $\Theta(X)$ be the positive function on $\mathfrak{p}_0$ determined by

$$\int_{G/K} f(m)\, dm = \int_{\mathfrak{p}_0} f(\exp X)\, \Theta(X)^2\, dX \tag{2.6}$$

for each compactly supported continuous function f on G/K. The space $\mathcal{C}(K\backslash G/K)$ is defined as the set of functions $f \in C^\infty(G)$ satisfying the following conditions (a) and (b) viz.: (a) $f(kxk') = f(x)$, $x \in G$, $k, k' \in K$. (b) For each left invariant differential operator D on G and each integer $r \geqslant 0$, we have $\tau_{D,r}(f) < \infty$; here $\tau_{D,r}$ is the seminorm defined by

$$\tau_{D,r}(f) = \sup_{x \in G} (1 + |X|^r)\, \Theta(X)\, |(Df)(x)|, \tag{2.7}$$

where as usual $x = k \exp X$, $X \in \mathfrak{p}_0$. Note that C^∞ spherical functions of compact support are contained in $\mathcal{C}(K\backslash G/K)$. The space $\mathcal{C}(K\backslash G/K)$ is topologized by the seminorms τ, given by (2.7) with varying D and r. It is a Frechet space.

The space $\mathcal{C}(K\backslash G/K)$ is precisely what Harish-Chandra calls $I(G)$ in [9], [10]. In the terminology of [11], the space $\mathcal{C}(K\backslash G/K)$ is just the collection of functions in $\mathcal{C}(G)$ which are bi-invariant under K, i.e. $f(kxk') = f(x)$.

The Plancherel theorem for spherical functions on G/K, as formulated and proved by Harish-Chandra in [9], [10] and [11] can now be stated as follows.

THEOREM. *$\mathcal{C}(K\backslash G/K)$ is a commutative topological algebra under convolution. The Fourier transform $\hat{f}$ of a function $f \in \mathcal{C}(K\backslash G/K)$ is defined for $\lambda \in \Lambda_0$ and belongs to $\mathfrak{J}(\Lambda_0)$.*

(1) It is known [9] that the Haar measures da, dn on $A_\mathfrak{p}$, N, can be so chosen that $dx = \exp 2\varrho(\log a)\, dk\, da\, dn$, where dk is the normalized Haar measure of K. This choice is implicitly made here and carried throughout.

The map $f \to \hat{f}$ *is a topological isomorphism of the algebra* $\mathcal{C}(K\backslash G/K)$ *onto the algebra* $\mathfrak{I}(\Lambda_0)$ *(with pointwise operations and the Schwartz topology on* $\mathfrak{I}(\Lambda_0)$ *being understood). Moreover, there exists a function* $c(\lambda)$ *on* Λ_0 *such that* $c(\lambda)^{-1}$ *is a tempered distribution,* $|c(s\lambda)| = |c(\lambda)|$ *for* $s \in W$, *and the inverse of the map* $f \to \hat{f}$ *is given by*

$$f(x) = |W|^{-1} \int_{\Lambda_0} \hat{f}(\lambda)\, \varphi_\lambda(x)\, |c(\lambda)|^{-2}\, d\lambda, \tag{2.9}$$

where $|W|$ *is the order of* W.

It follows from this theorem that if $f \in \mathcal{C}(K\backslash G/K)$ then

$$\int_G |f(x)|^2\, dx = |W|^{-1} \int_{\Lambda_0} |\hat{f}(\lambda)|^2\, |c(\lambda)|^{-2}\, d\lambda. \tag{2.10}$$

Since $\mathcal{C}(K\backslash G/K)$ is dense in $L_2(K\backslash G/K)$, the relation (2.10) can be extended in the usual way to all functions in $L_2(K\backslash G/K)$, so that the Fourier transform $f \to \hat{f}$ can be viewed as a unitary equivalence between $L_2(K\backslash G/K)$ and the Hilbert space of functions on Λ_0 which are invariant under W and are square integrable with respect to the measure $|c(\lambda)|^{-2}\, d\lambda$.

Harish-Chandra's proof of this theorem is contained in [9] and [10] except for two conjectures left unproved there. The first of these conjectures was proved by Gindikin and Karpelevič [12] who also obtained an explicit formula for $c(\lambda)$, described below. The second conjecture was proved recently in [11] by Harish-Chandra. The explicit formula for $c(\lambda)$ is as follows.

$$c(\lambda) = I(i\lambda)/I(\varrho_*), \tag{2.11}$$

where

$$I(\nu) = \prod_{\alpha \in \Sigma} \beta\left(\frac{m_\alpha}{2}, \frac{m_{\alpha/2}}{4} + \frac{\langle \nu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right); \ \nu \in \Lambda \tag{2.12}$$

and $\beta(\cdot, \cdot)$ is the classical beta function. Special cases of this formula had been proved earlier by Harish-Chandra [9] and Bhanu-Murthy [13]. See also Helgason [7] for further explanation of these matters.

When G is a complex group, a number of simplifications occur. Since we shall be using these, it is necessary to spend some time on them. Suppose then that $\mathfrak{g}_0$ carries the complex structure J. Then one knows in this case that $\mathfrak{p}_0 = J\mathfrak{k}_0$. For $\mathfrak{h}_0$ we can take the subalgebra $\mathfrak{h}_{\mathfrak{p}_0} + J\mathfrak{h}_{\mathfrak{p}_0}$ so that $\mathfrak{h}_{\mathfrak{k}_0} = J\mathfrak{h}_{\mathfrak{p}_0}$. Under the multiplication $(a+ib) * X = aX + bJX$ $\mathfrak{g}_0$ becomes a Lie algebra over $\mathbf{C}$ and $\mathfrak{h}_0$ is a Cartan subalgebra. Let Δ' be the set of nonzero roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$. For each $\alpha \in \Delta'$ let us define $H'_\alpha \in \mathfrak{h}_0$ by $B'(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}_0$, where B' is the Killing form of $\mathfrak{g}_0$ regarded as a Lie algebra over $\mathbf{C}$. Then it is known that $H'_\alpha \in \mathfrak{h}_{\mathfrak{p}_0}$, α is real on $\mathfrak{h}_{\mathfrak{p}_0}$ and purely imaginary on $\mathfrak{h}_{\mathfrak{k}_0}$. See for example [6, Chapter VI].

Now regarding $\mathfrak{g}_0$ as a Lie algebra over R, we have of course the constructs $\mathfrak{g}, \mathfrak{h}, \Delta, \Delta_+, P_+$ etc. as earlier. Recall that Δ was ordered by means of a basis of $\mathfrak{h}^* = i\mathfrak{h}_{\mathfrak{k}_0} + \mathfrak{h}_{\mathfrak{p}_0}$, and that $\mathfrak{h}^*$ and $\mathfrak{h}_{\mathfrak{p}_0}$ were ordered compatibly. If $\alpha \in \Delta'$, then $H'_\alpha \in \mathfrak{h}_{\mathfrak{p}_0}$, so the lexicographic ordering of $\mathfrak{h}_{\mathfrak{p}_0}$ also can be used to order Δ'. Let Q be the set of positive elements in Δ' under this ordering. Then the relation between the constructs $\Delta, \Delta_+, P_+, P_-, \Delta', Q$ is known to be as follows [14, p. 513]. For $\alpha \in \Delta'$ let α_+ and α_- be the complex linear functions on $\mathfrak{h}$ defined by

$$\alpha_+(H) = \alpha(H) \quad \alpha_-(H) = \overline{\alpha(H)}; \quad H \in \mathfrak{h}_0. \tag{2.13}$$

Then $\Delta = \{\alpha_+, \alpha_- \mid \alpha \in \Delta'\}$ and $\Delta_+ = \{\alpha_+, \alpha_- \mid \alpha \in Q\}$. Moreover, since each $\alpha \in \Delta'$ is complex linear on $\mathfrak{h}_0$ (when $\mathfrak{h}_0$ is regarded as a complex space) and since $\mathfrak{h}_0 = \mathfrak{h}_{\mathfrak{p}_0} + J\mathfrak{h}_{\mathfrak{p}_0}$, it follows that α cannot vanish identically on $\mathfrak{h}_{\mathfrak{p}_0}$. Hence neither α_+ nor α_- can be identically zero on $\mathfrak{h}_{\mathfrak{p}_0}$, so $P_- = \emptyset$ in the present case, and $\Delta_+ = P_+ = \{\alpha_+, \alpha_- \mid \alpha \in Q\}$. Now let $H \in \mathfrak{h}_{\mathfrak{p}_0}$. Then $\alpha(H)$ is real for $\alpha \in \Delta'$ so (2.13) shows that $\alpha_+(H) = \alpha_-(H) = \alpha(H)$ for $H \in \mathfrak{h}_{\mathfrak{p}_0}$. Thus the set Σ of restrictions to $\mathfrak{h}_{\mathfrak{p}_0}$ of elements in P_+ coincides in this case with the set of restrictions to $\mathfrak{h}_{\mathfrak{p}_0}$ of elements of Q. It is also obvious that $\Sigma = \Sigma_0$ in this case. Thus the function π of (2.1) is actually equal to $\prod_{\alpha \in Q} \alpha_*$ (1) in this case. [See 9, pp. 303–304.]

The elementary spherical functions φ_λ have a simple expression in the complex case. Since $H = KA_{\mathfrak{p}}K$, it is enough to describe φ_λ on $A_{\mathfrak{p}}$. The formula is

$$\varphi_\lambda(a) = \frac{\sum_{s \in W} c(s\lambda) \exp is\lambda(\log a)}{\prod_{\alpha \in Q} (e^{\alpha \log a} - e^{-\alpha \log a})}; \quad a \in A_{\mathfrak{p}}. \tag{2.14}$$

The function $c(\lambda)$ also has the simple form given by

$$c(\lambda) = \pi(\varrho_*)/\pi(i\lambda) \quad \lambda \in \Lambda_0. \tag{2.15}$$

Continuing to assume that G is complex, let $A_{\mathfrak{k}}$ be the analytic subgroup of K corresponding to $\mathfrak{h}_{\mathfrak{k}_0}$. Then $A_{\mathfrak{k}}$ is a maximal torus of K. Let $A = A_{\mathfrak{k}} A_{\mathfrak{p}}$. Then A is a Cartan subgroup of G. A is connected and the exponential map maps $\mathfrak{h}_0$ onto A. Let $h \in A$ and let $H \in \mathfrak{h}_0$ be such that $\exp H = h$. If H' is any other element of $\mathfrak{h}_0$

(1) Recall that α_* denotes the restriction of α to $\mathfrak{h}_{\mathfrak{p}_0}$.

such that $\exp H' = h$, then $\exp (H - H') = e$ so that $H - H'$ belongs to the unit lattice of K, viz. to the lattice $\{X \in \mathfrak{h}_{\mathfrak{k}_0} \mid \alpha(X) \in 2\pi i \mathbf{Z}$ for each $\alpha \in \Delta'\}$. It follows that for any $\alpha \in \Delta'$ we can define a function ξ_α on A by $\xi_\alpha(h) = \exp \alpha(H)$ where H is any element of $\mathfrak{h}_0$ such that $\exp H = h$. Now consider the functional $\varrho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$. Since $\Delta_+ = \{\alpha_+, \alpha_- \mid \alpha \in Q\}$, we see that for $H \in \mathfrak{h}_0$, $\varrho(H) = \frac{1}{2} \sum_{\alpha \in Q} (\alpha_+(H) + \alpha_-(H))$. Now, by (2.13), $\alpha_+(H) = \overline{\alpha_-(H)}$ for $H \in \mathfrak{h}_0$, so $\varrho(H)$ is real for $H \in \mathfrak{h}_0$. On the other hand, each $\alpha \in \Delta_+$ is purely imaginary on $\mathfrak{h}_{\mathfrak{k}_0}$, so ϱ is purely imaginary on $\mathfrak{h}_{\mathfrak{k}_0}$. It follows that ϱ is zero on $\mathfrak{h}_{\mathfrak{k}_0}$. Hence we can unambiguously define a function ξ_ϱ on A by $\xi_\varrho(h) = \exp \varrho(H)$, $h \in A$, where H is any element of $\mathfrak{h}_0$ such that $\exp H = h$.

Now let $D(h)$ be the function on A defined by

$$D(h) = \xi_\varrho(h) \prod_{\alpha \in \Delta_+} (1 - \xi_\alpha(h)^{-1}). \tag{2.16}$$

We claim that in the present case, i.e. when G is complex, $D(h)$ is real and nonnegative. To see this, pick $H \in \mathfrak{h}_0$ so that $\exp H = h$; then $D(h) = \exp \varrho(H) \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha(H)})$. Now, we know that $\varrho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$, so $D(h) = \prod_{\alpha \in \Delta_+} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})$. But $\Delta_+ = \{\alpha_+, \alpha_- \mid \alpha \in Q\}$ so we have

$$D(h) = \prod_{\alpha \in Q} (e^{\frac{1}{2}\alpha_+(H)} - e^{-\frac{1}{2}\alpha_+(H)}) (e^{\frac{1}{2}\alpha_-(H)} - e^{-\frac{1}{2}\alpha_-(H)}). \tag{2.17}$$

The definition of α_+ and α_- shows that $\alpha_+(H) = \overline{\alpha_-(H)}$ for $H \in \mathfrak{h}_0$. Hence each term in the above product is real and nonnegative and hence $D(h)$ is real and nonnegative. We shall use this fact in § 5 below.

3. The Gauss kernel g_t

The main tool we shall use below is a family of functions $\{g_t, t > 0\}$. which play on G/K, a role analogous to the Gaussian functions

$$\frac{1}{\sqrt{4\pi t}} \exp - \frac{x^2}{4t}$$

on the real line. We shall loosely call g_t the Gauss kernel on G/K. Here we shall define and establish some properties of g_t, and also get an integral representation for it.

How should g_t be defined? The classical function

$$\frac{1}{\sqrt{4\pi t}} \exp - \frac{x^2}{4t}$$

is the fundamental solution of the heat equation $\partial^2 u/\partial x^2 = \partial u/\partial t$ on the line. One would thus hope to define g_t as the fundamental solution of the equation $\omega u = \partial u/\partial t$ where ω is the Laplace–Beltrami operator of G/K. One hopes of course that g_t would be a spherical function on G. Though it is not in general possible to write down g_t explicitly, it is fairly easy to guess what its Fourier transform ought to be. Indeed, since it is known that $\omega\varphi_\lambda = -(|\lambda|^2 + |\varrho_*|^2)\varphi_\lambda$, we know that under the Fourier transform, the operator ω goes over into multiplication by $-(|\lambda|^2 + |\varrho_*|^2)$. This makes it plausible that if a spherical fundamental solution g_t of

$$\omega u = \frac{\partial u}{\partial t} \tag{3.1}$$

is being sought, then its Fourier transform $\hat{g}_t(\lambda)$ ought to satisfy

$$-(|\lambda|^2 + |\varrho_*|^2)\,\hat{g}_t(\lambda) = \frac{d}{dt}\,\hat{g}_t(\lambda). \tag{3.2}$$

or

$$\hat{g}_t(\lambda) = C \exp -t(|\lambda|^2 + |\varrho_*|^2). \tag{3.3}$$

The constant C ought to be equal of 1, because formally, g_0 should behave like the Haar measure of K, so its Fourier transform should be identically 1. Thus a candidate for $\hat{g}_t(\lambda)$ would be $\exp -t(|\lambda|^2 + |\varrho_*|^2)$. Now, this function is clearly in $\mathfrak{J}(\Lambda_0)$, and this suggests that via Harish-Chandra's Plancherel theorem, g_t would have the representation

$$g_t(x) = |W|^{-1} \int_{\Lambda_0} \exp -t(|\lambda|^2 + |\varrho_*|^2)\,\varphi_\lambda(x)\,|c(\lambda)|^{-2}\,d\lambda. \tag{3.4}$$

The following proposition justifies all these speculations. Our procedure is to define g_t by (3.4), and then to prove that it is indeed the desired fundamental solution of the heat equation $\omega u = \partial u/\partial t$.

PROPOSITION 3.1. *For each $t > 0$, define the function $g_t(x)$ on G by*

$$g_t(x) = |W|^{-1} \int_{\Lambda_0} \exp -t(|\lambda|^2 + |\varrho_*|^2)\,\varphi_\lambda(x)\,|c(\lambda)|^{-2}\,d\lambda. \tag{3.5}$$

Then g_t possesses the following properties:

$$g_t \in \mathcal{C}(K\backslash G/K) \tag{3.6}$$

$$\hat{g}_t(\lambda) = \exp -t(|\lambda|^2 + |\varrho_*|^2). \tag{3.7}$$

$$g_t(x) \textit{ satisfies } \omega g_t = \frac{\partial g_t}{\partial t} \tag{3.8}$$

$$g_t * g_s = g_{t+s} \quad t, s > 0. \tag{3.9}$$

(3.10) *For any continuous spherical C^∞ function f with compact support in G, the function $u_t = f * g_t$ satisfies $\omega u_t = \partial u_t/\partial t$, and further $\|u_t - f\|_2 \to 0$ as $t \to 0$. $\|f\|_2$ is the L_2-norm of f.*

$$g_t \geq 0, \tag{3.11}$$

$$\int_G g_t(x)\, dx = 1 \textit{ for each } t > 0. \tag{3.12}$$

Proof. For each $t > 0$, the function $\exp - t(|\lambda|^2 + |\varrho_*|^2)$ is in $\mathcal{J}(\Lambda_0)$, so that the Plancherel theorem of § 2 implies that $g_t \in \mathcal{C}(K\backslash G/K)$ and also that $\hat{g}_t(\lambda) = \exp - t(|\lambda|^2 + |\varrho_*|^2), \lambda \in \Lambda_0$. This proves (3.6), (3.7). Next, if we apply ω to g_t as defined in (3.5), and carry out the differentiation under the integral sign, we see that

$$\begin{aligned}(\omega g_t)(x) &= |W|^{-1} \int_{\Lambda_0} \exp - t(|\lambda|^2 + |\varrho_*|^2)\, (\omega\varphi_\lambda)(x)\, |c(\lambda)|^{-2}\, d\lambda \\ &= -|W|^{-1} \int_{\Lambda_0} (\exp - t(|\lambda|^2 + |\varrho_*|^2))\, (|\lambda|^2 + |\varrho_*|^2)\, \varphi_\lambda(x)\, |c(\lambda)|^{-2}\, d\lambda, \end{aligned} \tag{3.13}$$

where we used $\omega\varphi_\lambda = -(|\lambda|^2 + |\varrho_*|^2)\, \varphi_\lambda$. The differentiation under the integral is valid because $\exp - t(|\lambda|^2 + |\varrho_*|^2) \in \mathcal{J}(\Lambda_0)$ and because $(|\lambda|^2 + |\varrho_*|^2)\, |c(\lambda)|^{-2}$ is tempered. In a like manner we see that

$$\frac{\partial}{\partial t} g_t(x) = -|W|^{-1} \int_{\Lambda_0} (|\lambda|^2 + |\varrho_*|^2) \exp - t(|\lambda|^2 + |\varrho_*|^2)\, \varphi_\lambda(x)\, |c(\lambda)|^{-2}\, d\lambda \tag{3.14}$$

and this proves (3.8). Next, $\widehat{g_t * g_s}(\lambda) = \hat{g}_t(\lambda)\, \hat{g}_s(\lambda) = \exp - (t+s)\, (|\lambda|^2 + |\varrho_*|^2) = \hat{g}_{t+s}(\lambda)$. Since $g_t \in \mathcal{C}(K\backslash G/K)$, and the Plancherel theorem quoted in § 2 guarantees that $f \to \hat{f}$ is a topological isomorphism between $\mathcal{C}(K\backslash G/K)$ and $\mathcal{J}(\Lambda_0)$, this implies that $g_t * g_s = g_{t+s}$, proving (3.9). Now let f be as described in (3.10). Then $f \in \mathcal{C}(K\backslash G/K)$; It is obvious because of (3.8) that the function $u_t = f * g_t$ satisfies $\omega u_t = \partial u_t = \partial u_t/\partial t$. Now, $\widehat{f * g_t}(\lambda) = \hat{f}(\lambda)\, \hat{g}_t(\lambda) = \hat{f}(\lambda) \exp - t(|\lambda|^2 + |\varrho_*|^2)$. So, by the Plancherel theorem, we have

$$\|f * g_t - f\|_2^2 = \int_G |(f * g_t)(x) - f(x)|^2\, dx = |W|^{-1} \int_{\Lambda_0} |\hat{f}(\lambda)|^2\, (\hat{g}_t(\lambda) - 1)^2\, |c(\lambda)|^{-2}\, d\lambda. \tag{3.15}$$

As $t \to 0$, we see that $(\hat{g}_t(\lambda) - 1)^2 \to 0$ boundedly, and certainly $\int |\hat{f}(\lambda)|^2\, |c(\lambda)|^{-2}\, d\lambda < \infty$.

So by Lebesgue's theorem, the integral on the right goes to zero as $t \to 0$. This proves (3.10).

The properties (3.6)–(3.10) are tantamount to asserting that g_t is the fundamental solution of the heat equation $\omega u = \partial u / \partial t$ for the class of L_2 spherical functions. It is well known that fundamental solutions of such parabolic equations are non-negative. and have total integral equal to 1. See for example S. Ito [15] or Nelson [16]. This proves (3.11) and (3.12).

The above proposition shows that for each $t > 0$, the measure $g_t(x)\,dx$ is a non-negative spherical measure of total mass 1. Its Fourier–Stieltjes transform is clearly $\exp - t(|\lambda|^2 + |\varrho_*|^2)$. Now, as $t \to 0$, this transform approaches the function identically 1 on Λ_0. This function is the Fourier–Stieltjes transform of the Haar measure dk of K. By applying Theorem 4.2 of Gangolli [17], we see that for each bounded continuous spherical function f on G, we have

$$\int_G f(x)\,g_t(x)\,dx \to \int_K f(k)\,dk = f(e) \tag{3.16}$$

as $t \to 0$. In particular, if V is any neighbourhood of e in G such that $KVK \subset V$, then

$$\int_{V^c} g_t(x)\,dx \to 0 \quad \text{as } t \to 0, \tag{3.17}$$

where V^c is the complement of V. Of course, this is no more than to say that g_t is an approximate identity. This fact will be used decisively below. The reader who is familiar with the theory of parabolic equations will of course realize that (3.9) can be considerably strengthened, but we shall not need its strongest version.

For future use, it will be necessary to know what the function Φ_{g_t} is (cf. § 2). Since $\hat{g}_t(\lambda) = \exp - t(|\lambda|^2 + |\varrho_*|^2)$, we know from (2.5) that the Euclidean Fourier transform of Φ_{g_t} is precisely $\exp - t(|\lambda|^2 + |\varrho_*|^2)$. That is

$$\int_{A_{\mathfrak{p}}} \Phi_{g_t}(a) \exp - i\lambda(\log a)\,da = \exp - t(|\lambda|^2 + |\varrho_*|^2).$$

It follows that
$$\Phi_{g_t}(a) = (4\pi t)^{-l/2} \exp - \left(t|\varrho_*|^2 + \frac{|\log a|^2}{4t} \right). \tag{3.18}$$

Now let $\lambda \in \Lambda_0$ and let φ_λ be the corresponding elementary spherical function Then we see that

$$\int_G g_t(x)\,\varphi_\lambda(x)\,dx = \int_{A_{\mathfrak{p}}} \Phi_{g_t}(a) \exp i\lambda(\log a)\,da.$$

Since Φ_{g_t} is of the above form, it follows easily that this integral converges absolutely for all $\lambda \in \Lambda$ (and not merely $\lambda \in \Lambda_0$). It is obviously a holomorphic function of λ, and coincides with $\exp -t(\langle \lambda, \lambda\rangle + \langle \varrho_*, \varrho_*\rangle)$ on Λ_0. Hence if follows that for all $\lambda \in \Lambda$ we must have

$$\int_G g_t(x)\, \varphi_\lambda(x)\, dx = \exp - t(\langle \lambda, \lambda\rangle + \langle \varrho_*, \varrho_*\rangle).$$

Now suppose $\lambda \in \Lambda$ is such that φ_λ is positive definite. Then $|\varphi_\lambda(x)| \leqslant \varphi_\lambda(e) = 1$ for all $x \in G$, and since $g_t \geqslant 0$, we see that $|\int_G g_t(x)\,\varphi_\lambda(x)\,dx| \leqslant \int_G g_t(x)\,dx = 1$ for such λ. Moreover, since $\varphi_\lambda(x^{-1}) = \overline{\varphi_\lambda(x)}$ and $g_t(x^{-1}) = g_t(x)$, we see that $\int_G g_t(x)\,\varphi_\lambda(x)\,dx$ is real. Thus $\exp - t(\langle \lambda, \lambda\rangle + \langle \varrho_*, \varrho_*\rangle)$ is real and $\leqslant 1$. Since this is true for each $t > 0$, it follows that $\langle \lambda, \lambda\rangle + \langle \varrho_*, \varrho_*\rangle$ is real and $\geqslant 0$ for all $\lambda \in \Lambda$ for which φ_λ is positive definite. We shall have to use this below.

When G is complex, we can compute g_t explicitly. Since $G = KA_{\mathfrak{p}}K$, and g_t is spherical, we only need to know g_t on $A_{\mathfrak{p}}$.

PROPOSITION 3.2. *Let G be a complex group and suppose $a \in A_{\mathfrak{p}}$. Let q = number of elements in Q, $n = \dim G/K$ and $l = \dim \Lambda_0$. Then*

$$g_t(a) = C(4t)^{-n/2} \exp -t\,|\varrho_*|^2 \prod_{\alpha\in Q} \frac{\alpha(\log a)}{\exp \alpha(\log a) - \exp - \alpha(\log a)} \times \exp -(|\log a|^2/4t), \quad (3.19)$$

where $C = 2^q\, \pi^{-l/2}\, \pi(\varrho_*)^{-1}$.

Proof. Whether or not G is complex, it is known that $|c(\lambda)|^2 = c(\lambda)\, c(-\lambda) = c(s\lambda)\, c(-s\lambda)$, $s \in W$. See [9]. When G is complex, we use this in (2.14) to get

$$\varphi_\lambda(a)\,|c(\lambda)|^{-2} = \frac{\sum_{s\in W} c(-s\lambda)^{-1} \exp is\lambda(\log a)}{\prod_{\alpha\in Q} (\exp \alpha(\log a) - \exp - \alpha(\log a))}. \quad (3.20)$$

Use this in (3.5) to get

$$\begin{aligned} g_t(a) &\prod_{\alpha\in Q} (\exp \alpha(\log a) - \exp - \alpha(\log a)) \\ &= |W|^{-1} \sum_{s\in W} \int_{\Lambda_0} (\exp -t(|\lambda|^2 + |\varrho_*|^2) \exp is\lambda(\log a)\, c(-s\lambda)^{-1}\, d\lambda. \end{aligned} \quad (3.21)$$

On the right side the function $\exp -t(|\lambda|^2 + |\varrho_*|^2)$ is invariant under each $s \in W$, and also under the substitution $\lambda \to -\lambda$. Thus

$$g_t(a) \prod_{\alpha \in Q} (\exp \alpha(\log a) - \exp - \alpha(\log a))$$

$$= \int_{\Lambda_0} \exp - (t(|\lambda|^2 + |\varrho_*|^2) + i\lambda(\log a))\, c(\lambda)^{-1}\, d\lambda$$

$$= \frac{\exp - t\,|\varrho_*|^2}{\pi(\varrho_*)} \int_{\Lambda_0} \exp - (t\,|\lambda|^2 + i\lambda(\log a))\, \pi(i\lambda)\, d\lambda, \tag{3.22}$$

where we used (2.15) for the last step.

This integral can be computed by using the theory of Fourier transforms on Euclidean spaces. We know that

$$\int_{\Lambda_0} \exp - t\,|\lambda|^2 - i\lambda(\log a) d\lambda = (4\,\pi t)^{-l/2} \exp - |\log a|^2/4t.$$

with $l = \dim \Lambda_0$ and also that $\pi(i\lambda) = \prod_{\alpha \in Q} i\lambda(H_{\alpha_*})$. Hence if D_α is the differential operator corresponding differentiation in the direction of H_{α_*}, it follows that

$$\int_{\Lambda_0} (\exp - t\,|\lambda|^2 - i\lambda(\log a))\, \pi(i\lambda)\, d\lambda = (4\,\pi t)^{-l/2} (\prod_{\alpha \in Q} - D_\alpha)\, (\exp - (|\log a|^2/4t)). \tag{3.23}$$

If $\lambda \in \Lambda_0$ and D_λ is the corresponding differential operator, then $D_\lambda \exp - |\log a|/4t = -(2t)^{-1} \lambda(\log a) \exp - |\log a|^2/4t$. Hence we get

$$(\prod_{\alpha \in Q} - D_\alpha)\, (\exp - |\log a|^2/4t) = (2t)^{-q} (\prod_{\alpha \in Q} \alpha_*(\log a) + P(\log a)) \exp - (|\log a|^2/4t)$$

$$= (2t)^{-q} (\pi(\log a) + P(\log a)) \exp - (|\log a|^2/4t), \tag{3.24}$$

where P is a polynomial function on $\mathfrak{h}_{\mathfrak{p}_0}$, whose total degree is less than that of π. We claim that $P \equiv 0$. To prove this, recall that a function ψ on $\mathfrak{h}_{\mathfrak{p}_0}$ is said to be skew if $\psi(sH) = \det s\, \psi(H)$, $s \in W$, $H \in \mathfrak{h}_{\mathfrak{p}_0}$. It is known that π is skew and that π divides each skew polynomial function on $\mathfrak{h}_{\mathfrak{p}_0}$ [9]. A glance at (3.21) shows that the right side of (3.21) is skew, so that the function $(\pi(\log a) + P(\log a)) \exp - (|\log a|^2/4t)$ must be a skew function of $\log a$. Now $\exp - (|\log a|^2/4t)$ is invariant under W. Hence $\pi(\log a) + P(\log a)$ must be skew, hence divisible by π. Therefore P is divisible by π, and being of total degree less than that of π, we must have $P \equiv 0$. Thus

$$(\prod_{\alpha \in Q} - D_\alpha)\, (\exp - |\log a|^2/4t) = (2t)^{-q}\, \pi(\log a) \exp - (|\log a|^2/4t). \tag{3.25}$$

Putting all this together, we see

$$g_t(a)=2^{-l-q}\,\pi^{-l/2}\,\pi(\varrho_*)^{-1}\,\frac{\exp-t|\varrho_*|^2}{t^{(l+2q)/2}}\prod_{\alpha\in Q}\frac{\alpha(\log a)}{e^{\alpha(\log a)}-e^{-\alpha(\log a)}}\times\exp-\frac{|\log a|^2}{4t}. \tag{3.26}$$

Recalling that $l+2q=n=\dim G/K$ we get the result of the proposition.

In the above proof, we have not been very careful to distinguish between regular and singular elements of $\mathfrak{h}_{\mathfrak{p}_0}$. We may carry out the computation assuming that a is regular, and then extend the above formula to all elements of G, observing that the right side of (3.25) is C^∞ on $A_{\mathfrak{p}}$.

Let η be the projection of G onto G/K. Recall that $d(x,y)$ is the metric on G/K.

It is obvious that if $x\in G$ and $x=kak'$ with $a\in A_{\mathfrak{p}}$, then $d(\eta(x),\eta(e))=d(\eta(a),\eta(e))=|\log a|$, since d agrees on $A_{\mathfrak{p}}$ with the euclidean metric on $A_{\mathfrak{p}}$ given by the Killing form,

PROPOSITION 3.3. *Let G be complex and let g_t be as above. Let ω be the Laplace–Beltrami operator of G/K. Then given any real number $r>0$, there exists $t_0>0$ such that for all $0<t\leqslant t_0$ and all $x\in G$ for which $d(\eta(x),\eta(e))\geqslant r$ we have $(\omega g_t)(x)\geqslant 0$.*

Moreover, given any $s>0$, there exists $r_0>0$ such that for all $0<t\leqslant s$ and for all $x\in G$ with $d(\eta(x),\eta(e))\geqslant r_0$, we have $(\omega g_t)(x)\geqslant 0$.

Proof. Let $x\in G$, and suppose $x=kak'$, $k,k'\in K$, $a\in A_{\mathfrak{p}}$. Then $d(\eta(x),\eta(e))=|\log a|$. Further, observe that ωg_t is a spherical function, so $(\omega g_t)(x)=(\omega g_t)(a)$. Hence we only need to prove the proposition for elements in $A_{\mathfrak{p}}$.

We know $g_t(a)$. On the other hand, we also know that g_t satisfies $\omega g_t=\partial g_t/\partial t$. Differentiating the expression for $g_t(a)$ with respect to t we get

$$(\omega g_t)(a)=\frac{(|\log a|^2-2nt-4t^2|\varrho_*|^2)}{4t^2}\,g_t(a) \tag{3.27}$$

from this expression the proposition follows easily, since $g_t(a)\geqslant 0$.

We now need to establish a property of g_t which will be very useful in §§ 4, 5. We first need two lemmas.

Let η be the projection of G onto G/K and let o be the point $\eta(e)\in G/K$. For any point $p\in G/K$ let r denote the distance of p from o. It is obvious that r is a C^∞ function on $G/K-\{o\}$.

LEMMA 3.4. *Let $n=\dim G/K$. If $n>2$, the function $v=r^{-n+2}$ is C^∞ on $G/K-\{o\}$ and satisfies $\omega v\leqslant 0$ on $G/K-\{o\}$. If $n=2$ then the function $v=-\log r$ is C^∞ on $G/K-\{o\}$ and satisfies $\omega v\leqslant 0$ on $G/K-\{o\}$. Here ω is the Laplace–Beltrami operator of G/K.*

Proof. First suppose $n>2$. It is clear that $v=r^{-n+2}$ is C^∞ on $G/K-\{o\}$. We shall compute ωv.

Let $m\in G/K$ and let X be the unique element of $\mathfrak{p}_0$ such that $\operatorname{Exp} X=m$, where Exp is the exponential mapping of $\mathfrak{p}_0$ considered as the tangent space of the Riemannian manifold G/K. Let dm be the invariant Riemannian measure on G/K, and let f be any continuous function with compact support on G/K. Then one knows [18, p. 251] that the following formula holds

$$\int_{G/K} f(m)\,dm=\int_{\mathfrak{p}_0} f(\operatorname{Exp} X)\,\det A_X\,dX. \tag{3.28}$$

Here dX is the Euclidean measure on $\mathfrak{p}_0$ regarded as a Euclidean space with the Killing form B. A_X is the endomorphism of $\mathfrak{p}_0\to\mathfrak{p}_0$ given by

$$A_X=\sum_{j\geqslant 0}(T_X)^j/(2j+1)! \tag{3.29}$$

T_X being the restriction of $(\operatorname{ad} X)^2$ to $\mathfrak{p}_0$. [18, p. 251].

We can of course regard the Euclidean coordinates of X as giving a coordinate system on G/K, and it follows that in this coordinate system, the invariant measure dm is given by $\det A_X dX$. Now for any $X\in\mathfrak{p}_0$ let $|X|=r$ and put $\tilde X=X/|X|$ for $X\neq 0$, so that $X=r\tilde X$. Then $\tilde X$ is a vector on the unit sphere in $\mathfrak{p}_0$. It follows that $\tilde X$ can be described by angular coordinates $\theta_1,\dots,\theta_{n-1}$, so that $r,\theta_1,\dots,\theta_{n-1}$ are the polar coordinates of X. Note that $\tilde X$ depends only on $\theta_1,\dots,\theta_{n-1}$. These polar coordinates of X, when regarded as a coordinate system on G/K are precisely the geodesic polar coordinates with pole at $o\in G/K$.

We shall compute the measure $\det A_X\,dX$ in these coordinates. For any $Y\in\mathfrak{p}_0$, it is well known that $(\operatorname{ad} Y)^2$ maps $\mathfrak{p}_0\to\mathfrak{p}_0$. and if T_Y is the restriction of $(\operatorname{ad} Y)^2$ to $\mathfrak{p}_0$, then T_Y is symmetric with respect to the Killing form B and has nonnegative eigenvalues. Now since $X=r\tilde X$ it is obvious that $T_X=r^2T_{\tilde X}$. Let $t_1^2,t_2^2\dots t_n^2$ be the eigenvalues of $T_{\tilde X}$. Then the eigenvalues of T_X are $r^2t_1^2,\dots,r^2t_n^2$. Hence the eigenvalues of A_X are $\sum_{j\geqslant 0}(rt_i)^{2j}/(2j+1)!$ with $i=1,\dots,n$. This power series is just the power series of $\sinh rt_i/rt_i$. Thus A_X has these eigenvalues, and we find that $\det A_X=\prod_{i=1}^n \sinh rt_i/rt_i$. Note that t_i are real, nonnegative. Since t_i are defined by means of $\tilde X$, which depends only on $\theta_1,\dots,\theta_{n-1}$, they are functions only of $\theta_1,\dots,\theta_{n-1}$, On the other hand, it is clear that $dX=r^{n-1}\,dr\,d\theta_1,d\theta_2\dots d\theta_{n-1}$. If follows that in polar coordinates the invariant measure dm is given by $r^{n-1}\prod_{i=1}^n\sinh rt_i/rt_i\,dr\,d\theta_1\dots d\theta_{n-1}$.

Now let $g_{ij}(r,\theta_1,\dots,\theta_{n-1})$ be the functions which define the components of the Riemannian metric tensor at the point whose coordinates are $r,\theta_1,\dots,\theta_{n-1}$, and let

$g = \det g_{ij}$. g is positive. It is well known that the Riemannian measure is given by $\sqrt{g}\, dr\, d\theta_1 \dots d\theta_{n-1}$. Comparing this with the above, we get

$$\sqrt{g} = r^{n-1} \prod_{i=1}^{n} \frac{\sinh\, rt_i}{rt_i}. \tag{3.30}$$

Now, due to the classical lemma of Gauss, we know that the geodesics emanating from o intersect the sphere of radius r around o at right angles. Hence it follows that the metric form ds^2 of G/K must have the expression

$$ds^2 = dr^2 + \sum_{i,j=1}^{n-1} g_{ij}\, d\theta_i\, d\theta_j \tag{3.31}$$

and thus the Laplace–Beltrami operator ω is given in geodesic polar coordinates by

$$\omega = \frac{\partial^2}{\partial r^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} \frac{\partial}{\partial r} + \frac{1}{\sqrt{g}} \sum_j \frac{\partial}{\partial \theta_j} \left(\sum_i g^{ij} \sqrt{g} \frac{\partial}{\partial \theta_i} \right), \tag{3.32}$$

where g^{ij} is the inverse matrix of g_{ij}. [18, p. 278].

Let us now apply ω to the function $v = r^{-n+2}$. Then (3.32) shows that

$$\omega v = \left(\frac{d^2}{dr^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} \frac{d}{dr} \right) (r^{-n+2}). \tag{3.33}$$

Because $(1/\sqrt{g})(\partial \sqrt{g}/\partial r) = (\log \sqrt{g})\, \partial/\partial r$ and $\sqrt{g}$ is given by (3.30) with t_i depending only on the $\theta_1, \dots, \theta_{n-1}$, it is easy to compute $(1/\sqrt{g})\, \partial \sqrt{g}/\partial r$. The result is

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} = \sum_{i=1}^{n} t_i \left(\coth\, rt_i - \frac{1}{rt_i} \right) + \frac{n-1}{r}. \tag{3.34}$$

Using this in (3.33) there results

$$\omega v = -\frac{n-2}{r^{n-1}} \sum_{i=1}^{n} t_i \left(\coth\, rt_i - \frac{1}{rt_i} \right). \tag{3.35}$$

When $n = 2$, a similar computation with $v = -\log r$ results in

$$\omega v = -\frac{1}{r} \sum_{i=1}^{2} t_i \left(\coth\, rt_i - \frac{1}{rt_i} \right). \tag{3.36}$$

Now it is easily seen that $\coth x - 1/x \geqslant 0$ for $x > 0$. This implies that $\omega v \leqslant 0$, (We have slurred over the possibility that some of the t_i's could be zero. Note that if one of them say t_1 is zero at some point $r, \theta_1, \dots, \theta_{n-1}$, then a simple continuity argument can be used to show that $\omega v \leqslant 0$ at such a point.) This finishes the proof of Lemma 3.4.

Let V be a connected open set in G/K with a smooth boundary ∂V, and suppose that u, v are C^2 functions defined on an open neighbourhood of $V + \partial V$.

A straightforward generalization of the classical Green's formula yields

$$\int_V [v(\omega u) - u(\omega v)]\, dp = \int_{\partial V} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\sigma, \tag{3.37}$$

where $\partial/\partial n$ is differentiation along the outward pointing normal to ∂V (angles being understood in the sense of the Riemannian geometry of G/K) $d\sigma$ stands for the element of surface area on ∂V in the Riemannian structure on ∂V which it inherits from the Riemannian structure of G/K. Of course, dp is the Riemannian invariant measure of G/K.

Moreover, if we let $v = 1$ in this formula we get

$$\int_V (\omega u)\, dp = \int_{\partial V} \frac{\partial u}{\partial n}\, d\sigma. \tag{3.38}$$

These two formulas will now be used.

LEMMA 3.5. *Let f be a function of class C^2 defined on an open set U in G/K. Suppose that $f \geqslant 0$ on U and that $(\omega f)(y) \geqslant 0$ for each $y \in U$, ω is the Laplace–Beltrami operator of G/K. Let $m_0 \in U$ be any point of U and suppose $\delta > 0$ is a real number such that the ball of radius δ centred at m_0 is contained in U. Then there exists a constant C_δ depending only on δ and the dimension n of G/K such that*

$$f(m_0) \leqslant C_\delta \int_{B_\delta(m_0)} f(m)\, dm, \tag{3.39}$$

where $B_\delta(m_0) = \{m \mid m \in G/K, d(m_0, m) \leqslant \delta\}$, and dm is the invariant measure on G/K.

Proof. Let o be the point $\eta(e)$ of G/K. Because the metric d and the measure dm are both invariant under G, we can assume for the purpose of this lemma that m_0 is the point o, and f is C^2 in a neighbourhood U of o containing the ball of radius δ around o, and satisfies $f \geqslant 0$, $\omega f \geqslant 0$ on U. For any $r > 0$ let B_r be the ball of radius r around o and S_r be the sphere of radius r around o. Thus $S_r = \partial B_r$. Let ε, R be two numbers such that $0 < \varepsilon < R \leqslant \delta$ and let V be the *annular* region bounded by S_ε and S_R. We shall now apply the formula (3.37) to V, by setting $u = f$ and $v = r^{-n+2}$ in that formula. (We assume $n > 2$. A similar argument with $v = -\log r$ works when $n = 2$). It is obvious that the boundary of V consists of S_R and S_ε in opposed orientation. We get

$$\int_V [v(\omega f) - f(\omega v)]\, dp = \int_{S_R} \left(v\frac{\partial f}{\partial n} - f\frac{\partial v}{\partial n}\right) d\sigma_R + \int_{S_\varepsilon} \left(v\frac{\partial f}{\partial n} - f\frac{\partial v}{\partial n}\right) d\sigma_\varepsilon, \tag{3.40}$$

where $d\sigma_R$ is the surface element on S_R, $d\sigma_\varepsilon$ on S_ε and $\partial/\partial n$ is *outward* normal differentiation. Now, v is constant on S_R and S_ε, equals R^{-n+2} on S_R, and equals ε^{-n+2} on S_ε. Further we know that geodesics emanating from o intersect spheres around o at right angles. This means that on S_R, $\partial/\partial n$ is just $\partial/\partial r$. While, on S_ε, $\partial/\partial n$ is just $-\partial/\partial r$. Using all this we get

$$\begin{aligned}\int_V [v(\omega f) - f(\omega v)]\, dp = \frac{1}{R^{n-2}} \int_{S_R} \frac{\partial f}{\partial r}\, d\sigma_R + \frac{n-2}{R^{n-1}} \int_{S_R} f\, d\sigma_R \\ - \frac{1}{\varepsilon^{n-2}} \int_{S_\varepsilon} \frac{\partial f}{\partial r}\, d\sigma_\varepsilon - \frac{n-2}{\varepsilon^{n-1}} \int_{S_\varepsilon} f\, d\sigma_\varepsilon.\end{aligned} \tag{3.41}$$

Now, (3.38) shows that

$$\int_{S_R} \frac{\partial f}{\partial r}\, d\sigma_R = \int_{B_R} \omega f\, dp = \int_V \omega f\, dp + \int_{B_\varepsilon} \omega f\, dp.$$

So,

$$\begin{aligned}\int_V \left[\left(v - \frac{1}{R^{n-2}} \right) (\omega f) - f(\omega v) \right] dp = \frac{1}{R^{n-2}} \int_{B_\varepsilon} (\omega f)\, dp + \frac{n-2}{R^{n-1}} \int_{S_R} f\, d\sigma_R \\ - \frac{1}{\varepsilon^{n-2}} \int_{S_\varepsilon} \frac{\partial f}{\partial r}\, d\sigma_\varepsilon - \frac{n-2}{\varepsilon^{n-1}} \int_{S_\varepsilon} f\, d\sigma_\varepsilon.\end{aligned} \tag{3.42}$$

At each point of V_0 we have $v = r^{-n+2} \geqslant R^{-n+2}$ so the expression $(v - 1/R^{n-2})(\omega f)$ is $\geqslant 0$ by our hypothesis. Moreover, $f \geqslant 0$, and $\omega v \leqslant 0$ by Lemma 3.4. Hence we see that

$$0 \leqslant \frac{1}{R^{n-2}} \int_{B_\varepsilon} (\omega f)\, dp + \frac{n-2}{R^{n-1}} \int_{S_R} f\, d\sigma_R - \frac{1}{\varepsilon^{n-2}} \int_{S_\varepsilon} \frac{\partial f}{\partial r}\, d\sigma_\varepsilon - \frac{n-2}{\varepsilon^{n-1}} \int_{S_\varepsilon} f\, d\sigma_\varepsilon. \tag{3.43}$$

This is true for each $0 < \varepsilon < R$. Now let $\varepsilon \to 0$. The first term on the right goes to zero as $\varepsilon \to 0$ because ωf is continuous and volume $(B_\varepsilon) \to 0$ as $\varepsilon \to 0$. The third term is majorized by $(\sup_{S_\varepsilon}|\partial f/\partial r|)\,A(\varepsilon)/\varepsilon^{n-2}$ where $A(\varepsilon)$ is the area of S_ε. Now it is well known that as $\varepsilon\to 0$, $A(\varepsilon)/\varepsilon^{n-1}$ approaches a finite limit Ω_n equal to the area of the unit sphere in n-dimensional euclidean space. Since f is of class C^2 it follows that $(\sup_{S_\varepsilon}|\partial f/\partial r|)\,A(\varepsilon)/\varepsilon^{n-2}\to 0$ as $\varepsilon\to 0$. Hence the third term $\to 0$. The fourth term obviously approaches $(n-2)\,\Omega_n f(o)$ as $\varepsilon\to 0$. The result is

$$0\leqslant\frac{n-2}{R^{n-1}}\int_{S_R} f\,d\sigma_R-(n-2)\,\Omega_n f(o). \tag{3.44}$$

If $n=2$, we can carry out the above analysis with $v=-\log r$ and we get the inequality

$$0\leqslant\frac{1}{R}\int_{S_R} f\,d\sigma_R-\Omega_2 f(o). \tag{3.45}$$

Thus in any event, we have

$$f(o)\leqslant\frac{1}{\Omega_n R^{n-1}}\int_{S_R} f\,d\sigma_R. \tag{3.46}$$

This holds for each $R\leqslant\delta$. Since $dm=d\sigma_R\,dR$ we can integrate this inequality to get

$$\int_{B_\delta} f(m)\,dm=\int_0^\delta\int_{S_R} f\,d\sigma_R\,dR\geqslant\Omega_n f(o)\int_0^\delta R^{n-1}dR\geqslant\Omega_n\frac{\delta^n}{n}f(o) \tag{3.47}$$

so our proposition follows with $C_\delta=(\Omega_n\delta^n)^{-1}n$.

For any $x\in G$ and $\delta>0$ let $U_\delta(x)=\{x'\in G\,|\,d(\eta(x),\eta(x'))\leqslant\delta\}$. It is obvious that U_δ is a compact neighbourhood of x in G.

Proposition 3.6. *Given $r>0$, there exists $t_0>0$ and a constant $C_r>0$ such that for each $x\in G$ for which $d(\eta(x),\eta(e))\geqslant 3r$ and for each $0<t\leqslant t_0$, we have*

$$g_t(x)\leqslant C_r\int_{U_r(x)} g_t(z)\,dz. \tag{3.48}$$

Further, given any $t>0$, there exist $r>0$ and a constant $C_r>0$ such that for each $x\in G$ for which $d(\eta(x),\eta(e))\geqslant 3r$ we have

$$g_t(x)\leqslant C_r\int_{U_r(x)} g_t(z)\,dz. \tag{3.49}$$

Proof. Since $g_t(xk)=g_t(x)$, we can regard g_t as a function, say $\tilde{g}_t$ on G/K. Given $r>0$ there exists $t_0>0$ such that for all $0<t\leqslant t_0$, and all $x\in G$ with $d(\eta(x),\eta(e))\geqslant 3r$,

one has $(\omega g_t)(x) \geqslant 0$. (Cf. the first half of Proposition 3.3.) Let x be such a point of G and let $m_0 = \eta(x)$, $o = \eta(e)$. If $B_r(m_0)$ is the ball of radius r around m_0 in G/K, we see that the function $\tilde{g}_t$ satisfies all the hypotheses of Lemma 3.5 in the ball $B_r(m_0)$. Hence from the lemma,

$$\tilde{g}_t(m_0) \leqslant C_r \int_{B_r(m_0)} \tilde{g}_t(m)\, dm. \tag{3.50}$$

Under the projection map, $U_r(x)$ projects onto $B_r(m_0)$, and by our convention, the Haar measure dz of G is normalized by the relation $dz = dm\, dk$. Finally, since g_t is spherical, we see that

$$\int_{B_r(m_0)} \tilde{g}_t(m)\, dm = \int_{U_r(x)} g_t(z)\, dz.$$

Hence then we get (3.48). The second half of the proposition follows in the same way upon using the second half of Proposition 3.3.

The reader who is familiar with Selberg's paper [1] will recognize that the above proposition means that g_t satisfies the condition of regular growth of Selberg. The technique that we have used above gives us a handy condition on a function $f \geqslant 0$ which will guarantee that f satisfies a condition of regular growth, and should be useful in other contexts as well.

4. The theta relation

We begin by making a few conventions. Suppose H is a closed unimodular subgroup of G. Then the coset space $H\backslash G$ (or G/H) has a G-invariant measure, say $d\bar{x}$.

Let f be a continuous function on G such that $f(hx) = f(x), h \in H, x \in G$. Then f induces a continuous function $\bar{f}$ on $H\backslash G$ in the obvious way. We shall have to consider integrals of the form $\int_{H\backslash G} \bar{f}(\bar{x})\, d\bar{x}$ quite often. For typographical convenience, we shall omit the bars and write $\int_{H\backslash G} f(x)\, dx$ for this integral bearing in mind its precise meaning. Similar remarks apply to integrals of the form $\int_{G/H} f(x)\, dx$, or $\int_{H_1\backslash G/H_2} f(x)\, dx$.

In the situation of § 1, when Γ is a discrete subgroup of G with compact quotient $\Gamma\backslash G$, we shall normalize the invariant measure $d\bar{x}$ on $\Gamma\backslash G$ in such a way that if f is a continuous function with compact support on G, then $\int_{\Gamma\backslash G} (\sum_{\gamma \in \Gamma} f(\gamma x))\, d\bar{x} = \int_G f(x)\, dx$.

The existence of the spaces H_i described in the introduction is due to Tamagawa [2]. Briefly, the commutative convolution algebra $L_1(K\backslash G/K)$ acts on $L_2(\Gamma\backslash G/K)$ via the representation $f \to T_f$, where $T_f g = g * f$, $g \in L_2(\Gamma\backslash G/K)$. For each $f \in L_1(K\backslash G/K)$, T_f

turns out to be a compact operator. It follows that $L_2(\Gamma\backslash G/K)$ can be decomposed into an orthogonal direct sum of finite dimensional subspaces $\{H_i\}_{i=0}^{\infty}$ such that each H_i is an eigenspace of each of the commuting family of compact operators $\{T_f, f \in L_1(K\backslash G/K)\}$. The restriction of any T_f to H_i is a scalar equal to $h_i(f)$ say. Then for $g \in H_i$ and $f \in L_1(K\backslash G/K)$ one has $g * f = h_i(f)\, g$. Choosing f to be C^∞ with compact support, one sees immediately that g must be C^∞. Thus all the functions in H_i are differentiable. Finally, it is not hard to show that if a function g on G is an eigenfunction of each operator T_f then g must also be an eigenfunction of each $D \in \mathfrak{D}(G/K)$. It follows that each $D \in \mathfrak{D}(G/K)$ acts on H_i by a scalar, say $h_i(D)$, so that $Dg = h_i(D)\, g$, for $g \in H_i$, $D \in \mathfrak{D}(G/K)$. Now suppose that $g \in H_i$ and $g \not\equiv 0$. Choose x_0 so that $g(x_0) \neq 0$ and define $g^\#$ by

$$g^\#(x) = g(x_0)^{-1} \int_K g(x_0\, kx)\, dk. \tag{4.1}$$

Then $g^\#$ is an *elementary* spherical function on G and $g^\#$ does not depend on the choice of x_0, [2, p. 370]. Since $g^\#$ is an integral of *left* translates of g, we see that if D is a left invariant differential operator on G, then $(Dg)^\# = D(g^\#)$. But $(Dg)^\# = (h_i(D)\,g)^\# = h_i(D)\, g^\#$ for $D \in \mathfrak{D}(G/K)$. Thus $g^\#$ is an eigenfunction of each $D \in \mathfrak{D}(G/K)$ with eigenvalue $h_i(D)$. If g_1, g_2 are two functions in H_i, and $g_1^\#$, $g_2^\#$ the corresponding elementary spherical functions, then we have $Dg_1^\# = h_i(D)\, g_1^\#$, $Dg_2^\# = h_i(D)\, g_2^\#$ for $D \in \mathfrak{D}(G/K)$. Thus $g_1^\#$, $g_2^\#$ have the same system of eigenvalues relative to $\mathfrak{D}(G/K)$. Hence by [6, Corollary 7.3, p. 438] we have $g_1^\# = g_2^\#$. Thus all nonzero $g \in H_i$ give the *same* elementary spherical function $g^\#$. We shall label it φ_{λ_i} with $\lambda_i \in \Lambda$. Each φ_{λ_i} is positive definite [2, p. 380, Proposition 10]. As a consequence let us note that the scalar $h_i(\omega)$, which is equal to $-(\langle \lambda_i, \lambda_i\rangle + \langle \varrho_*, \varrho_*\rangle)$ (this being the eigenvalue of ω applied to φ_{λ_i}) must be nonpositive, in accordance with a result of § 3. All this identifies the spaces H_i with the spaces which Tamagawa calls $\mathfrak{M}(\omega)$ in [2, Theorem 2]. We shall say that the space H_i *lies over* the elementary spherical function φ_{λ_i}.

At this juncture it is not clear that the function $N(r)$ talked about in § 1 is finite valued. However, let us pretend that it is, and ask how one might investigate its asymptotic behaviour, as $r \to \infty$. The classical procedure is to investigate the behaviour of its Laplace transform $L(t) = \int_0^\infty \exp -tr\, dN(r)$ as $t \to 0$. One must first show that $L(t)$ exists. To see how this might be done, notice that since the measure $dN(r)$ is a collection of point masses of size $\dim H_i$ at the points $-h_i(\omega)$ on the real axis, $L(t)$ can be formally written as $\sum_{i=0}^\infty \exp t\, h_i(\omega) \dim H_i$. If we show that this series converges, then we shall not only be able to show that $L(t)$ exists, but also, incidentally show that $N(r)$ is finite valued, as we will find below.

The operator ω acts on H_i as the scalar $h_i(\omega)$. Thus, formally, $(\exp t\,h_i(\omega))\dim H_i$ is the trace of the operator $e^{t\omega}$ on H_i, and the sum $\sum_i \dim H_i \exp t\,h_i(\omega)$ is seen to be formally equal to the trace of $e^{t\omega}$ on $L_2(\Gamma\backslash G/K)$. This discussion suggests that we should look to showing that $e^{t\omega}$ exists in some sense and has a trace on $L_2(\Gamma\backslash G/K)$. Now the formal operators $e^{t\omega}$ exist in a precise way on $L_2(K\backslash G/K)$ and indeed form the semi-group associated with the heat equation $\omega u = \partial u/\partial t$ on G/K. Moreover, $e^{t\omega}f = f * g_t$ for nice f in $L_2(K\backslash G/K)$ as we saw in § 3. Thus $e^{t\omega}$ is an integral operator whose kernel is just $g_t(y^{-1}x)$. This raises the hope that on $L_2(\Gamma\backslash G/K)$ we may be able to realize the analogous operator by means of a kernel which is just $g_t(y^{-1}x)$ "periodized" by "wrapping it" around $\Gamma\backslash G/K$ along the orbits of Γ on G/K. That is to say, by means of the kernel

$$G_t(y,x) = \sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x). \tag{4.2}$$

In what follows, we shall justify this hope, by showing that this last series converges and defines a continuous function $G_t(y,x)$ on $\Gamma\backslash G/K \times \Gamma\backslash G/K$. The integral operator on $L_2(\Gamma\backslash G/K)$ whose kernel is G_t will then turn out to have all the properties that were hoped for. Writing down the trace of this operator in two different ways then yields the theta relation.

LEMMA 4.1. *Let P, Q be compact subsets of G and let $r>0$ be a given real number. There exist only a finite number of elements $\gamma\in\Gamma$ such that $d(\eta(y^{-1}\gamma x),\eta(e))\leqslant r$ for some $x\in P$, $y\in Q$,*

Proof. The condition $d(\eta(y^{-1}\gamma x),\eta(e))\leqslant r$ means that $y^{-1}\gamma x\in U_r(e)=\{x' \mid d(\eta(x'),\eta(e))\leqslant r\}$ so that $\gamma\in y\,U_r(e)\,x^{-1}\subset QU_r(e)\,P^{-1}$. This last set is compact, so it contains only a finite number of elements of Γ. It follows that at most this finite number of elements of Γ can satisfy the condition $d(\eta(y^{-1}\gamma x),\ \eta(e))\leqslant r$ for some $x\in P$, $y\in Q$.

LEMMA 4.2. *Let P be a compact subset of G and let $\delta>0$ be a given real number. Then there exists an integer M depending only on P,δ, such that for any $x\in P$, $y\in G$, at most M of the sets $\{U_\delta(y^{-1}\gamma x),\gamma\in\Gamma\}$ can have a nonempty intersection.*

Proof. Fix $x\in P, y\in G$ and suppose that m (possibly infinite) of the sets $\{U_\delta(y^{-1}\gamma x),\gamma\in\Gamma\}$ have a nonempty intersection. Let $\{U_\delta(y^{-1}\gamma_i x)\}_{i=1}^m$ be these sets and suppose z belongs to their intersection. For any $1\leqslant i\leqslant m$, we have

$$d(\eta(y^{-1}\gamma_i x),\eta(y^{-1}\gamma_1 x))\leqslant d(\eta(y^{-1}\gamma_i x),\eta(z))+d(\eta(y^{-1}\gamma_1 x),\eta(z))\leqslant\delta+\delta\leqslant 2\delta. \tag{4.3}$$

So,
$$d(\eta(x^{-1}\gamma_1^{-1}\gamma_i x),\eta(e))=d(\eta(y^{-1}\gamma_i x),\eta(y^{-1}\gamma_1 x))\leqslant 2\delta. \tag{4.4}$$

This means that $x^{-1}\gamma_1^{-1}\gamma_i x \in U_{2\delta}(e)$, so $\gamma_1^{-1}\gamma_i \in x\,U_{2\delta}(e)\,x^{-1} \subset P U_{2\delta}(e)\,P^{-1}$. This last set is compact and can contain at most a finite number, say M, of elements of Γ. Thus for each i, $\gamma_1^{-1}\gamma_i$ must be one of these M elements. This proves that $m \leqslant M$ and the lemma is proved.

PROPOSITION 4.3. *For any $t>0$, let g_t be the Gauss kernel defined in § 3. Then the series $\sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x)$ converges uniformly for $x, y \in G$, to a continuous function $G_t(y,x)$ on $G\times G$.*

Proof. Since $\Gamma\backslash G$ is compact we can find a compact subset P of G such that $\bigcup_{\gamma\in\Gamma}\gamma P = G$. It is enough to show that the series converges uniformly for $(x,y)\in P\times P$. Because of Proposition 3.3, there is an $r>0$ such that if $d(\eta(x),\eta(e)) \geqslant r$ then $(\omega g_t)(x) \geqslant 0$. Fix such an $r>0$. There is a finite subset F_r of Γ such that for $\gamma\in\Gamma - F_r$, we have $d(\eta(y^{-1}\gamma x),\eta(e)) \geqslant 3r$ for $(x,y)\in P\times P$, by Lemma 4.1. We only need to show that $\sum_{\Gamma - F_r} g_t(y^{-1}\gamma x)$ converges uniformly. Now, for $\gamma \in \Gamma - F_r$, we have, by (3.48) of Proposition 3.6,

$$g_t(y^{-1}\gamma x) \leqslant C_r \int_{U_r(y^{-1}\gamma x)} g_t(z)\,dz. \tag{4.5}$$

Also, because of Lemma 4.2, for any $(x,y)\in P\times P$, the sets $\{U_r(y^{-1}\gamma x), \gamma\,\varepsilon\,\Gamma\}$ cover G at most M times. Hence

$$\sum_{\gamma\in\Gamma - F_r} g_t(y^{-1}\gamma x) \leqslant C_r M \int_G g_t(z)\,dz$$

and since $\int_G g_t(z)\,dz = 1$, we have proved that $\sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x)$ converges uniformly on $G\times G$ to a continuous function $G_t(y,x)$.

It is obvious that $G_t(\gamma y k, \gamma' x k') = G_t(y,x)$ for any $(x,y)\in G\times G$, $(\gamma,\gamma')\in\Gamma\times\Gamma$, and $(k,k')\in K\times K$. Thus G_t may be regarded as a continuous function on $\Gamma\backslash G/K \times \Gamma\backslash G/K$.

Let R_t be the integral operator on $L_2(\Gamma\backslash G/K)$ whose kernel is $G_t(y,x)$. Thus, given $f\in L_2(\Gamma\backslash G/K)$,

$$(R_t f)(x) = \int_{\Gamma\backslash G/K} f(y)\,G_t(y,x)\,dy. \tag{4.6}$$

Since $G_t(y,x)$ is continuous and $\Gamma\backslash G/K$ is compact, it is clear that R_t has finite trace equal to $\int_{\Gamma\backslash G/K} G_t(x,x)\,dx$. On the other hand, we can identify the eigenfunctions and the eigenvalues of R_t directly.

PROPOSITION 4.4. *Let $f\in H_i$, and let φ_{λ_i}, $\lambda_i\in\Lambda$ be the elementary spherical function that underlies H_i. Then $R_t f = (\exp - t(\langle\lambda_i,\lambda_i\rangle + \langle\varrho_*,\varrho_*\rangle))\,f$.*

Proof.

$$(R_t f)(x) = \int_{\Gamma\backslash G/K} f(y)\, G_t(y,x)\, dy = \int_{\Gamma\backslash G/K} f(y) \sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x)\, dy$$

$$= \int_{G/K} f(y)\, g_t(y^{-1}x)\, dy = \int_G f(y)\, g_t(y^{-1}x)\, dy = (f * g_t)(x) = (T_{g_t} f)(x). \quad (4.7)$$

Now, we know that since $g_t \in L_1(K\backslash G/K)$, the operator T_{g_t} acts by some scalar on H_i. Hence we must have $R_t f = T_{g_t} f = c_i f$ for some scalar c_i. Let us identify this scalar. To do this, map each side of the equation $T_{g_t} f = c_i f$ by the map $f \to f^{\#}$ described above. Since the map $f \to f^{\#}$ is defined by averaging *left* translates of f, it is obvious that it commutes with *right* convolution by any function. Thus $(T_{g_t} f)^{\#} = (f * g_t)^{\#} = f^{\#} * g_t$. So we have $f^{\#} * g_t = c_i f^{\#}$. Now $f^{\#}$ is exactly φ_{λ_i}, so c_i is to be determined by $\varphi_{\lambda_i} * g_t = c_i \varphi_{\lambda_i}$. Let us compute $\varphi_{\lambda_i} * g_t$.

$$(\varphi_{\lambda_i} * g_t)(x) = \int_G \varphi_{\lambda_i}(xy^{-1})\, g_t(y)\, dy = \int_G \int_K \varphi_{\lambda_i}(xky^{-1})\, dk\, g_t(y)\, dy$$

$$= \int_G \varphi_{\lambda_i}(x)\, \varphi_{\lambda_i}(y^{-1})\, g_t(y)\, dy = \varphi_{\lambda_i}(x) \int_G \varphi_{\lambda_i}(y^{-1})\, g_t(y)\, dy$$

$$= \varphi_{\lambda_i}(x)\, \hat{g}_t(\lambda_i) = (\exp - t(\langle \lambda_i, \lambda_i\rangle + \langle \varrho_*, \varrho_*\rangle))\, \varphi_{\lambda_i}(x). \quad (4.8)$$

Here we used, successively, the following facts: $g_t(yk) = g_t(y)$, $\int_K \varphi_\lambda(xky)\, dk = \varphi_\lambda(x)\, \varphi_\lambda(y)$, and $\hat{g}\ (\lambda) = \exp - t(\langle \lambda, \lambda\rangle + \langle \varrho_*, \varrho_*\rangle)$.

This computation shows $c_i = \exp - (t\langle \lambda_i, \lambda_i\rangle + \langle \varrho_*, \varrho_*\rangle)$.

COROLLARY 4.5. *The series* $\sum_{i=0}^{\infty} \exp t\, h_i(\omega) \dim H_i$ *converges.*

Proof. $h_i(\omega) = -(\langle \lambda_i, \lambda_i\rangle + \varrho_*, \varrho_*\rangle)$. Thus the series is just the trace of R_t on $L_2(\Gamma\backslash G/K)$. Since this operator has finite trace, the series must converge.

It follows immediately from the convergence of this series that $|h_i(\omega)|$ can be less than a given $r > 0$ for only finitely many indices i. We can thus assume the spaces H_i so arranged that $0 = |h_0(\omega)| \leqslant |h_1(\omega)| \leqslant \dots$, and that for any p, $\bigoplus_{i=0}^{p} H_i$ has finite dimension. Clearly, this means that $N(r)$ is finite valued.

Finally, let for each i, $\{\varphi_{ij}\}_{j=1}^{\dim H_i}$ be an orthonormal basis of H_i. Then we may apply Mercer's theorem to the positive kernel $G_t(y,x)$ on $\Gamma\backslash G/K \times \Gamma\backslash G/K$ to get

$$G_t(y,x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\dim H_i} \exp t\, h_i(\omega)\, \varphi_{ij}(x)\, \overline{\varphi_{ij}(y)}$$

and this expansion converges uniformly.

The above results can be summarized as follows.

PROPOSITION 4.6. *For any $r>0$, there exist only a finite number of indices i such that $|h_i(\omega)|\leqslant r$. The function $N(r)$ defined by*

$$N(r)=\sum_{\{i,\,|h_i(\omega)|\leqslant r\}}\dim H_i \tag{4.9}$$

is finite valued. For any $t>0$, the series $\sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x)$ converges uniformly for $(x,y)\in G\times G$ to a continuous function $G_t(y,x)$. $G_t(y,x)$ admits the expansion

$$G_t(y,x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\dim H_i}\exp t\,h_i(\omega)\,\varphi_{ij}(x)\,\overline{\varphi_{ij}(y)}, \tag{4.10}$$

where $\{\varphi_{ij}\}_{i=1}^{\dim H_i}$ is an orthonormal basis of H_i. The Laplace transform $L(t)$ of $N(r)$, $L(t)=\int_0^\infty \exp-\operatorname{tr} dN(r)$ exists for each $t>0$, and indeed,

$$L(t)=\sum_{i=0}^{\infty}(\exp t\,h_i(\omega))\dim H_i=\int_{\Gamma\backslash G/K}G_t(x,x)\,dx=\int_{\Gamma\backslash G/K}\sum_{\gamma\in\Gamma}g_t(x^{-1}\gamma x)\,dx. \tag{4.11}$$

The analogue of formula (4.11) when applied to $G=$ Real line, $\Gamma=\mathbf{Z}$ yields precisely Jacobi's theta relation for the function

$$\frac{1}{\sqrt{4\pi t}}\exp-\frac{x^2}{4t}.$$

5. Asymptotic behaviour of $N(r)$

We must study the behaviour of $L(t)$ as $t\to 0$.

Let $\Gamma_C=\Gamma\cap C$ where C is the centre of G. Since C is finite and contained in K, Γ_C is finite and contained in K. Let s be the order of Γ_C. Next, let Γ_E be the set of those elements in $\Gamma-\Gamma_C$ whose action on G/K leaves fixed some point of G/K.

If $\gamma\in\Gamma_E$ and γ' is any element of Γ which is conjugate in G to γ, then $\gamma'\in\Gamma_E$. For, if $\gamma'=x\gamma x^{-1}$ and if γ leaves yK fixed, then γ' leaves xyK fixed. Thus Γ_E is a union of classes of G-conjugate elements of Γ.

Finally let Γ_H be the complement in Γ of the union of Γ_C and Γ_E. Clearly Γ_H is also a union of classes of G-conjugate elements in Γ. Further, no element of Γ_H can have a fixed point on G/K.

Let $A=\{x^{-1}\gamma x\mid x\in G,\ \gamma\in\Gamma_H\}$.

LEMMA 5.1. *A is closed, and $K\cap A$ is empty.*

Proof. Suppose $x_j \in G$, $\gamma_j \in \Gamma_H$ are such that $x_j^{-1}\gamma_j x_j$ converges in G to z say. Since $\Gamma\backslash G$ is compact, we can assume by passing to a subsequence that there exist $\beta_j \in \Gamma$ such that $\beta_j x_j$ converges in G to an element $x \in G$. Then $\beta_j \gamma_j \beta_j^{-1} = (\beta_j x_j) x_j^{-1} \gamma_j x_j (\beta_j x_j)^{-1} \to xzx^{-1}$. But $\gamma_j \in \Gamma_H$, and $\beta_j \gamma_j \beta_j^{-1}$ is an element in Γ which is conjugate to γ_j, so $\beta_j \gamma_j \beta_j^{-1} \in \Gamma_H$, as was remarked above. Since Γ is discrete, it follows that $xzx^{-1} \in \Gamma_H$. Thus $z = x^{-1}\gamma x$ for some $\gamma \in \Gamma_H$, and so $z \in A$, and A is closed.

To prove the second part, suppose an element $k \in K$ is also in A. Then $k = x^{-1}\gamma x$ for some $x \in G$ and some $\gamma \in \Gamma_H$. Then $\gamma = xkx^{-1}$. It is obvious that γ leaves fixed the element xK of G/K. This contradicts $\gamma \in \Gamma_H$.

COROLLARY 5.2. *There exists $r > 0$ such that $U_{3r}(e) \cap A = \emptyset$.*

Proof. Since K is compact, A is closed, and $K \cap A$ is empty, there is an open set B in G which contains K, and misses A. Then $\eta(B)$ is open in G/K, contains $\eta(e)$ and misses $\eta(A)$. Since G/K carries the topology of the metric d, it follows that there is an $r > 0$, such that the ball of radius $3r$ around $\eta(e)$ in G/K is contained in $\eta(B)$, and therefore is disjoint from $\eta(A)$. This means exactly that $U_{3r}(e) \cap A = \emptyset$.

Let F be a fundamental domain for Γ in G/K. The expression

$$\int_{\Gamma\backslash G/K} \sum_{\gamma \in \Gamma} g_t(x^{-1}\gamma x)\, dx$$

on the right side of (4.11) can then be written

$$\sum_{\gamma \in \Gamma} \int_F g_t(x^{-1}\gamma x)\, dx. \tag{5.1}$$

(Since $g_t \geqslant 0$ and the series converges uniformly on compact subsets of $G \times G$, there is no problem about any rearrangements that we shall do.) We break up this sum into three parts, over Γ_C, Γ_E, Γ_H respectively.

Let $J_C(t) = \sum_{\gamma \in \Gamma_C} \int_F g_t(x^{-1}\gamma x)\, dx$, and let $J_E(t)$, $J_H(t)$ be defined similarly. We shall now estimate each of these.

$J_C(t)$ is simple to compute. For any $\gamma \in \Gamma_C$, $g_t(x^{-1}\gamma x) = g_t(\gamma)$ since γ commutes with x, and $g_t(\gamma) = g_t(e)$ since $\gamma \in K$ and g_t is spherical. Thus each term in $J_C(t)$ is equal to $g_t(e)$ volume (F). Hence $J_C(t) = s\, g_t(e)$ volume (F), where $s =$ order of Γ_C.

LEMMA 5.3. *$J_H(t) \to 0$ as $t \to 0$.*

Proof. Let P be a compact in G which covers F under the projection map of G onto G/K. Then, since $g_t \geqslant 0$, it is clear that

$$J_H(t) \leqslant \sum_{\gamma \in \Gamma_H} \int_P g_t(x^{-1}\gamma x)\, dx. \tag{5.2}$$

Now choose $r>0$ such that $U_{3r}(e) \cap A = \emptyset$ as in Corollary 5.2. Then choose t_0 so small that for $t \leqslant t_0$ we have $(\omega g_t)(x) \geqslant 0$ whenever $d(\eta(x), \eta(e)) \geqslant r$ according to Proposition 3.3. Let $\gamma \in \Gamma_H$ and $x \in P$. Then $x^{-1}\gamma x \in A$, and since $A \cap U_{3r}(e) = \emptyset$, we see that $(\omega g_t)(y) \geqslant 0$ whenever $y \in U_r(x^{-1}\gamma x)$. Thus by Proposition 3.6, wee see that for $x \in P$, $\gamma \in \Gamma_H$,

$$g_t(x^{-1}\gamma x) \leqslant C_r \int_{U_r(x^{-1}\gamma x)} g_t(z)\, dz. \tag{5.3}$$

Now, if $y \in U_r(x^{-1}\gamma x)$, then $d(\eta(y), \eta(x^{-1}\gamma x)) \leqslant r$ so,

$$\begin{aligned} d(\eta(y), \eta(e)) &\geqslant d(\eta(x^{-1}\gamma x), \eta(e)) - d(\eta(y), \eta(x^{-1}\gamma x)) \\ &\geqslant 3r - r = 2r, \quad \text{because } U_{3r}(e) \cap A = \emptyset. \end{aligned} \tag{5.4}$$

Hence $y \in U_r(x^{-1}\gamma x) \Rightarrow y \notin U_r(e)$. So the union of the sets $\{U_r(x^{-1}\gamma x), x \in P, \gamma \in \Gamma_H\}$ is contained in the complement of $U_r(e)$. Further, there exists an integer M so that for $x \in P$, at most M of the sets $\{U_r(x^{-1}\gamma x), \gamma \in \Gamma_H\}$ can have a nonempty intersection. Therefore for $x \in P$

$$\sum_{\gamma \in \Gamma_H} g_t(x^{-1}\gamma x) \leqslant C_r M \int_{G - U_r(e)} g_t(z)\, dz. \tag{5.5}$$

Now, $U_r(e)$ is a spherical neighbourhood of e in G, so by (3.17), we see that as $t \to 0$ the right side of (5.5) goes to 0. Thus the integrand in $\int_P \sum_{\gamma \in \Gamma_H} g_t(x^{-1}\gamma x)\, dx$ goes to zero uniformly on P as $t \to 0$. This shows that $J_H(t) \to 0$ as that $t \to 0$.

Remark. It can be shown by more delicate considerations that $J_H(t)$ goes to zero as $t \to 0$ like $\exp - (c/t)$ where c is a constant.

We now have to deal with the term $J_E(t)$ and study its behaviour as $t \to 0$.

Following Selberg [1] we may write $J_E(t)$ as

$$J_E(t) = \sum_{\{\gamma\} \subset \Gamma_E} \int_F \sum_{\gamma \in \{\gamma\}} g_t(x^{-1}\gamma x)\, dx = \sum_{\{\gamma\} \subset \Gamma_E} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma \tag{5.6}$$

where $\{\gamma\}$ stands for the conjugacy class of γ in Γ, Γ_γ is the centralizer of γ in Γ, G_γ is the centralizer of γ in G. The sum $\sum_{\{\gamma\} \subset \Gamma_E}$ means that the summation is over those G-conjugacy classes of γ which are contained in Γ_E.

$d\bar{x}_\gamma$ stands for the invariant measure on $G_\gamma \backslash G$, so normalized that $d\bar{x}_\gamma\, dx_\gamma = dx$ where dx_γ is a Haar measure on G_γ. The volume of $\Gamma_\gamma \backslash G_\gamma$ is to be computed in the measure on $\Gamma_\gamma\backslash G_\gamma$ induced by this Haar measure on G_γ. It is clear that the choice of this Haar measure on G_γ does not effect the product

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma,$$

even though it may effect each factor in this product. Note also that, the integral $\int_{G_\gamma\backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma$ depends only on the G-conjugacy class of γ i.e. only on $\{\gamma\}$. We saw above that Γ_E is a union of classes of elements which are conjugate in G. The following proposition is well known, but we include a proof, as it is not readily available in the literature.

PROPOSITION 5.4. *Γ_E is a finite union of classes of elements which are conjugate in G.*

Proof. Let $\gamma \in \Gamma_E$ and suppose γ leaves $x^{-1}K$ fixed. Then $x\gamma x^{-1}$ leaves K fixed, so $x\gamma x^{-1} \in K$. Let F_γ be the G-conjugacy class of γ, $F_\gamma = \bigcup_{x\in G}\{x^{-1}\gamma x\}$. Then F_γ intersects K. Thus the G-conjugacy class of each element γ in Γ_E intersects K. The proposition is therefore evidently contained in the following lemma.

LEMMA 5.5. *Let G be a locally compact separable group and suppose Γ is a discrete subgroup in G such that $\Gamma\backslash G$ is compact. Let K be a compact subset of G. Then only finitely many G-conjugacy classes of elements of Γ can intersect K.*

Proof. Suppose that F_x is the G-conjugacy class of an element $x \in G$. If the lemma were false, there exists a sequence of elements $\gamma_n \in \Gamma$ such that for each n, $F_{\gamma_n} \neq F_{\gamma_j}$ for any $j < n$ and $F_{\gamma_n} \cap K \neq \emptyset$. Let $K_n \in F_{\gamma_n} \cap K$. Since K is compact, we can assume, by passing to a subsequence if necessary, that k_n converges to an element k_∞. Now $k_n \in F_{\gamma_n}$, so $k_n = y_n^{-1}\gamma_n y_n$ for some $y_n \in G$. Because $\Gamma\backslash G$ is compact, we can assume, by passing to a subsequence if necessary, that there exist $\beta_n \in \Gamma$ such that $\beta_n y_n$ converges in G, to an element y say. Then $\beta_n\gamma_n\beta_n^{-1} = (\beta_n y_n)\, y_n^{-1}\gamma_n y_n(\beta_n y_n)^{-1} \to yk_\infty y^{-1}$. But $\beta_n\gamma_n\beta_n^{-1} \in \Gamma$ and Γ is discrete. So the sequence $\beta_n\gamma_n\beta_n^{-1}$ is eventually constant, i.e. for $n >$ a large n_0, $\beta_n\gamma_n\beta_n^{-1} = \beta_{n+1}\gamma_{n+1}\beta_{n+1}^{-1}$. However, this means that $F_{\gamma_n} = F_{\gamma_{n+1}}$ for $n > n_0$. This contradicts the choice of F_{γ_n}, and the proposition is proved.

Returning to $J_E(t)$, the proposition above implies that $J_E(t)$ is a *finite* sum of terms of the type

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma.$$

We shall now compute each term of this sum.

Fix $\gamma \in \Gamma_E$. We wish to compute $\int_{G_\gamma \backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma$. We shall utilize the invariant integral on G which was introduced by Harish-Chandra. First note that $\int_{G_\gamma \backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma$ depends only on the G-conjugacy class of γ, so in computing this, we may replace γ by any element conjugate to γ in G.

If $\gamma \in \Gamma_E$ leaves the point xK of G/K fixed, then $x^{-1}\gamma x$ leaves the point K of G/K fixed. Hence $x^{-1}\gamma x \in K$. Thus each element γ of Γ_E is conjugate in G to some element of K. Hence we can assume that $\gamma \in K$ when computing $\int_{G_\gamma \backslash G} g_t(x^{-1}\gamma x)$. Now let $A_{\mathfrak{k}}$ be the analytic subgroup of K whose Lie algebra is $\mathfrak{h}_{\mathfrak{k}_0}$. Then because G is *complex*, we see that $A_{\mathfrak{k}}$ is a maximal torus of K, and the subgroup $A = A_{\mathfrak{k}} A_{\mathfrak{p}}$ is a Cartan subgroup of G whose Lie algebra is just $\mathfrak{h}_0$. Note that A is connected. It is well known that any element in K is K-conjugate to an element of $A_{\mathfrak{k}}$. Hence we may further assume that $\gamma \in A_{\mathfrak{k}}$. Thus our problem is to compute $\int_{G_\gamma \backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma$ for an element $\gamma \in A_{\mathfrak{k}}$ such that γ is not in the centre of G.

Let $h \in A$. Put $h = h_- h_+$ with $h_- \in A_{\mathfrak{k}}$, $h_+ \in A_{\mathfrak{p}}$.

Let H be any element in $\mathfrak{h}_0$ such that $\exp H = h$. Then $H = H_- + H_+$ with $H_- \in \mathfrak{h}_{\mathfrak{k}_0}$, $H_+ \in h_{\mathfrak{p}_0}$.

Let f be any function in $\mathcal{C}(K\backslash G/K)$. Harish-Chandra has shown that if h is a regular element of A and $f \in \mathcal{C}(K\backslash G/K)$ then $\int_{G_h \backslash G} f(x^{-1}hx)\, dx$ converges [11]. (Harish-Chandra makes this assertion for the space $\mathcal{C}(G)$ defined in [11]. However, it is clear that $\mathcal{C}(K\backslash G/K)$ is just the space of those functions in $\mathcal{C}(G)$ which are biinvariant under K).

For a regular element $h \in A$, G_h has A as its connected component of the identity, and A has finite index in G_h. In the following we will regard the case $G_h = A$. This modifies the integrals involved only by a constant factor and does not effect the results. (1)

The integral in question will now be $\int_{A\backslash G} f(x^{-1}hx)\, dx$. Such integrals occur frequently. For example it is known [14, pp. 502, 503] that

$$\int_{A\backslash G} f(x^{-1}hx)\, dx = \int_{K\times N} f(k^{-1}n^{-1}hnk)\, dk\, dn. \tag{5.7}$$

Since f is biinvariant under K, we see that

$$\int_{A\backslash G} f(x^{-1}hx)\, dx = \int_N f(n^{-1}hn)\, dn. \tag{5.8}$$

Now for any regular $h \in A$, the mapping $n \to h^{-1}n^{-1}hn$ of N to N is a topological isomorphism, whose Jacobian can be computed. See for example [6, Chapter X, Proposition 1.13]. Using this we see that

(1) I am grateful to Dr. Garth Warner for bringing this to my attention.

$$\int_N f(n^{-1}hn)\,dn = \Big|\prod_{\alpha\in P_+}(1-\xi_\alpha(h)^{-1})\Big|^{-1}\int_N f(hn)\,dn. \tag{5.9}$$

Now since $P_+ = \Delta_+$ here, it is clear that

$$\prod_{\alpha\in P_+}(1-\xi_\alpha(h)^{-1}) = \exp - \varrho(H)\,D(h);\ \ |\exp - \varrho(H)| = \exp - \varrho(H_+)$$

since $\varrho(H_-)$ is purely imaginary. See § 2. Moreover $|D(h)| = D(h)$ by our remarks at the end of § 2. Hence

$$\int_N f(n^{-1}hn) = D(h)^{-1}\exp\varrho(\log h_+)\int_N f(hn)\,dn = D(h)^{-1}\exp\varrho(\log h_+)\int_N f(h_- h_+ n)\,dn$$

$$= D(h)^{-1}\exp\varrho(\log h_+)\int_N f(h_+ n)\,dn = D(h)^{-1}\Phi_f(h_+), \tag{5.10}$$

where for the last step but one we used that $h_-\in K$ and $f(kx) = f(x)$ for $k\in K$, $x\in G$.

It follows that for regular h,

$$D(h)\int_{A\backslash G} f(x^{-1}hx)\,d\bar{x}_h = \Phi_f(h_+). \tag{5.11}$$

Now the left side of this equation is precisely what Harish-Chandra calls $F_f(h)$ in [11], [19] (Note that in the present case each root α is complex in the terminology of [19] so that $\varepsilon_R(h) = 1$ in the notation of [19]). Further, it follows from [19, Lemma 40] that in the present case, F_f is a C^∞ function on A. (Since each root in the present case is complex, so that the set of singular imaginary roots is empty.) Since the function $\Phi_f(h_+)$ is obviously a C^∞ function of h, and it agrees with F_f on the regular elements we have the result that $F_f(h) = \Phi_f(h_+)$ in the present case.

Now suppose h is not regular and h does not lie in the centre of G. Let $\mathfrak{z}_0$ be the centralizer of h in $\mathfrak{g}_0$. Then $\mathfrak{z}_0$ is reductive in $\mathfrak{g}_0$ and $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{z}_0$. As in [11], let $P_{\mathfrak{z}_0}$ be the set of positive roots of $\mathfrak{z}_0$ with respect to $\mathfrak{h}_0$ (the ordering being understood to be the one induced by the ordering of $\mathfrak{h}_0$ fixed before). It is obvious that $P_{\mathfrak{z}_0} = \{\alpha_-, \alpha_+ \mid \alpha\in Q,\ \xi_\alpha(h) = 1\}$, so the cardinality of $P_{\mathfrak{z}_0}$ is even, equal to $2q(h)$ say. Also, since h is not central in G, we must have $P_{\mathfrak{z}_0}\subsetneqq P_+$. So, since $2q$ is the cardinality of P_+, we have $q(h) < q$. We know that via the Cartan–Killing form, $\mathfrak{h}$ can be identified with its dual. For any root α, let $\bar{H}_\alpha$ be the element of $\mathfrak{h}$ which corresponds to α under this identification. Then $\bar{H}_\alpha\in\mathfrak{h}_{\mathfrak{p}_0} + i\mathfrak{h}_{\mathfrak{k}_0}$. In the usual manner, we can regard $\bar{H}_\alpha$ as a differential operator on G. Let $\varpi_{\mathfrak{z}_0}$ be the differential operator $\prod_{\alpha\in P_{\mathfrak{z}_0}}\bar{H}_\alpha$ and let $P_{\mathfrak{g}_0/\mathfrak{z}_0}$ be the complement of $P_{\mathfrak{z}_0}$ in P_+. Then $\xi_\alpha(h)\neq 1$ for any $\alpha\in P_{\mathfrak{g}_0/\mathfrak{z}_0}$. According to [11, Lemma 28], we have

$$(\varpi_{\mathfrak{z}_0} F_f)(h) = c_0 \xi_\varrho(h) \prod_{\alpha \in P_{\mathfrak{g}_0/\mathfrak{z}_0}} (1 - \xi_\alpha(h)^{-1}) \int_{G_h \backslash G} f(x^{-1} h x)\, d\bar{x}_h, \tag{5.12}$$

for all $f \in \mathcal{C}(K\backslash G/K)$ (Note that $\mathcal{C}(K\backslash G/K) \subset \mathcal{C}(G)$), where c_0 is a constant $\neq 0$. Thus

$$\int_{G_h \backslash G} f(x^{-1} h x)\, dx = (c_0 \xi_\varrho(h) \prod_{\alpha \in P_{\mathfrak{g}_0/\mathfrak{z}_0}} (1 - \xi_\alpha(h)^{-1}))^{-1} (\varpi_{\mathfrak{z}_0} F_f)(h). \tag{5.13}$$

Now, $\bar{H}_\alpha \in \mathfrak{h}_{\mathfrak{p}_0} + i\mathfrak{h}_{\mathfrak{k}_0}$. Let $\bar{H}_\alpha = \bar{H}_\alpha^+ + i\bar{H}_\alpha^-$ with $\bar{H}_\alpha^+ \in \mathfrak{h}_{\mathfrak{p}_0}$, $\bar{H}_\alpha^- \in \mathfrak{h}_{\mathfrak{k}_0}$. We have seen above that for $f \in \mathcal{C}(K\backslash G/K)$, we get $F_f(h) = \Phi_f(h_+)$. It follows that the differential operator $\bar{H}_\alpha^-$ annihilates F_f. Hence $(\varpi_{\mathfrak{z}_0} F_f)(h) = (\varpi_{\mathfrak{z}_0}^+ \Phi_f)(h_+)$ where $\varpi_{\mathfrak{z}_0}^+$ is the differential operator $\prod_{\alpha \in P_{\mathfrak{z}_0}} \bar{H}_\alpha^+$. Therefore,

$$\int_{G_h \backslash G} f(x^{-1} h x)\, d\bar{x}_h = (c_0 \xi_\varrho(h) \prod_{\alpha \in P_{\mathfrak{g}_0/\mathfrak{z}_0}} (1 - \xi_\alpha(h)^{-1}))^{-1} (\varpi_{\mathfrak{z}_0}^+ \Phi_f)(h_+). \tag{5.14}$$

Let us note that this formula is valid whether h is regular or singular. If h is regular, we have of course that $\mathfrak{z}_0 = \mathfrak{h}_0$, so $P_{\mathfrak{z}_0}$ is empty, and $P_{\mathfrak{g}_0/\mathfrak{z}_0} = P_+$. The differential operator $\varpi_{\mathfrak{z}_0}$ is to be construed as 1 in this case.

We are interested in the value of the left side of (5.14) with h equal to an element belonging to a conjugacy class in Γ_E. So put $h = \gamma$ in (5.14). We have seen that we may assume that $\gamma \in A_{\mathfrak{k}}$. Then $\gamma_+ = e$, and we see from (5.14) that in order to compute $\int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, d\bar{x}_\gamma$, it would be necessary to compute $(\varpi_{\mathfrak{z}_0}^+ \Phi_f)(e)$. Indeed

$$\int_{G_\gamma \backslash G} f(x^{-1} \gamma x)\, d\bar{x}_\gamma = (c_0 \xi_\varrho(\gamma) \prod_{\alpha \in P_{\mathfrak{g}_0/\mathfrak{z}_0}} (1 - \xi_\alpha(\gamma)^{-1}))^{-1} (\varpi_{\mathfrak{z}_0}^+ \Phi_f)(e). \tag{5.15}$$

Now set $f = g_t$ in this, as we may do because $g_t \in \mathcal{C}(K\backslash G/K)$. We shall now compute $(\varpi_{\mathfrak{z}_0}^+ \Phi_{g_t})(e)$. We have seen that for $h \in A_{\mathfrak{p}}$ with $\exp H = h$, $H \in \mathfrak{h}_{\mathfrak{p}_0}$, we have

$$\Phi_{g_t}(h) = (4\pi t)^{-l/2} (\exp - t\,|\varrho_*|^2) \exp - (|H|^2/4t). \tag{5.16}$$

Now, if H_0 is an element of $\mathfrak{h}_{\mathfrak{p}_0}$ regarded as a differential operator, it is obvious that $(H_0)(\exp - |H|^2/4t) = -\frac{1}{2} t \langle H_0, H\rangle \exp - (|H|^2/4t)$. Using this inductively, we see that $(\varpi_{\mathfrak{z}_0}^+)(\exp - |H|^2/4t)$ is of the form

$$\left(\frac{P_1(H)}{t^{2q_0}} + \frac{P_2(H)}{t^{2q_0-1}} + \dots + \frac{L}{t^{q_0}}\right) \exp - (|H|^2/4t) \tag{5.17}$$

where $2q_0$ (1) is the cardinality of $P_{\mathfrak{z}_0}$; $P_1(H), \dots P_{q_0-1}(H)$ are polynomial functions on

(1) Thus q_0 depends on γ.

$\mathfrak{h}_{\mathfrak{p}_0}$ without constant term and L is a constant. Setting $H=0$ in this expression we see that $(\varpi_{\mathfrak{z}_0}^+)(\exp - |H|^2/4t)|_{H=0}$ equals L/t^{q_0}, with L a constant. L depends only on $P_{\mathfrak{z}_0}$ in fact. Note that all this is valid whether γ is regular or not. If γ is regular, $q_0=0$. Since Φ_{g_t} is given by (5.16), it is obvious that

$$(\varpi_{\mathfrak{z}_0}^+ \Phi_{g_t})(e) = (4\pi t)^{-l/2} (\exp - t|\varrho_*|^2)\, Lt^{-q_0} = L(4\pi)^{-l/2} t^{-(l+2q_0)/2} \exp - t|\varrho_*|^2.$$

Recall that γ is not central in G, so that $q_0 < q$, where $2q=$ cardinality of P_+. Therefore $l+2q_0 < l+2q = n = \dim G/K$. It follows that $\lim_{t\to 0} t^{n/2}(\varpi_{\mathfrak{z}_0}^+\Phi_{g_t})(e)=0$. In view of (5.15), this implies that $\lim_{t\to 0} t^{n/2}\int_{G_\gamma\backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma = 0$ whenever γ is a noncentral element of G lying in $A_{\mathfrak{k}}$. whether γ is regular or not.

We remarked above that $J_E(t)$ is a *finite* sum of terms of the type

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} g_t(x^{-1}\gamma x)\, d\bar{x}_\gamma$$

and that for the evaluation of the integrals appearing here, it is enough to assume that $\gamma\in A_{\mathfrak{k}}$; since every element of Γ_E is noncentral in G, the above considerations imply that each term of $J_E(t)$ is $o(t^{-n/2})$ as $t\to 0$. Since there are only finitely many such terms in $J_E(t)$, we have proved the following result.

PROPOSITION 5.6. *Let G be a complex group. Then* $\lim_{t\to 0} t^{n/2} J_E(t) = 0$, $n=\dim G/K$.

We now consider the behaviour of $J_C(t)$ as $t\to 0$. Since $J_C(t) = s \operatorname{vol}(\Gamma\backslash G/K)\, g_t(e)$, we must consider the behaviour of $g_t(e)$ as $t\to 0$.

PROPOSITION 5.7. $t^{n/2} g_t(e)$ *approaches a finite limit* C_G *as* $t\to 0$, *where* $n=\dim G/K$.

Proof. If G is complex, this is obvious because

$$g_t(e) = 2^q \pi^{-l/2} \pi(\varrho_*)^{-1} 2^{-n} t^{-n/2} \exp - t|\varrho_*|^2.$$

Thus $C_G = 2^q \pi^{-l/2}\pi(\varrho_*)^{-1} 2^{-n}$.

If G is not complex, we have no explicit formula for g_t and we need to proceed somewhat differently.

By (3.5), we have

$$g_t(e) = |W|^{-1} \exp - t|\varrho_*|^2 \int_{\Lambda_0} (\exp - t|\lambda|^2)\, |c(\lambda)|^{-2}\, d\lambda. \tag{5.18}$$

Now introduce polar coordinates in Λ_0; then if $|\lambda| = r$ and $\lambda' = \lambda/r$, we may regard r, λ' as the polar coordinates of λ, with λ' varying over the unit sphere Ω of Λ_0. Let $d\lambda'$ be the surface element on this unit sphere, and let us define for any $r>0$,

$m_0(r)=|W|^{-1}\int_\Omega |c(r,\lambda')|^{-2}d\lambda'$, where $c(r,\lambda')$ is just the function $c(\lambda)$ expressed in polar coordinates. It is obvious that

$$g_t(e)=\exp -t|\varrho_*|^2\int_0^\infty e^{-tr^2}m_0(r)\,r^{l-1}\,dr, \tag{5.19}$$

where $l=\dim\Lambda_0$. We will investigate the behaviour of $m_0(r)$ as $r\to\infty$. The result of the proposition will then follow from the Tauberian theorem.

Recall our definition of Σ_0 in § 2. Let $\alpha\in\Sigma_0$ and let $\mathfrak{g}_0^\alpha$ be the subalgebra of $\mathfrak{g}_0$ generated by the root vectors in $\mathfrak{g}_0$ corresponding to those roots of $\mathfrak{g}_0$ whose restriction to $\mathfrak{h}_{\mathfrak{p}_0}$ coincides with α or $-\alpha$. Let $\mathfrak{k}_0^\alpha=\mathfrak{g}_0^\alpha\cap\mathfrak{k}_0$, $\mathfrak{p}_0^\alpha=\mathfrak{g}_0^\alpha\cap\mathfrak{p}_0$. Then $\mathfrak{g}_0^\alpha$ is semisimple, and $\mathfrak{g}_0^\alpha=\mathfrak{k}_0^\alpha+\mathfrak{p}_0^\alpha$ is a Cartan decomposition. The symmetric space S^α corresponding to the pair $(\mathfrak{g}_0^\alpha,\mathfrak{k}_0^\alpha)$ is of rank one. Let $\mathfrak{h}_{\mathfrak{p}_0}^\alpha=\mathfrak{h}_{\mathfrak{p}_0}\cap\mathfrak{p}_0^\alpha$. Then $\mathfrak{h}_{\mathfrak{p}_0}^\alpha$ is one dimensional, and is a maximal abelian subspace of $\mathfrak{p}_0^\alpha$. For any $\lambda\in\Lambda_0$, let λ_α be its restriction to $\mathfrak{h}_{\mathfrak{p}_0}^\alpha$. Then if c_α is Harish-Chandra's function for the space S^α, so that $|c_\alpha(\lambda_\alpha)|^{-2}d\lambda_\alpha$ is the Plancherel measure for S^α, the following relation holds between c and the various c_α, $\alpha\in\Sigma_0$. See Gindikin and Karpelevič [12], Helgason [7].

$$c(\lambda)=\prod_{\alpha\in\Sigma_0}c_\alpha(\lambda_\alpha). \tag{5.20}$$

Let $\Sigma_0^\alpha,\Sigma^\alpha$ be the analogues of Σ_0,Σ for the space S^α. Then Σ_0^α consists of one element say θ^α. Then $2\theta^\alpha$ is the only other possible element of Σ^α. Let p^α, q^α be their multiplicities. Then $p^\alpha>0$, $q^\alpha\geqslant 0$. Denote the Killing form of $\mathfrak{g}_0^\alpha$ by $\langle\cdot,\cdot\rangle_\alpha$. Then by (2.11), (2.12) applied to S^α, we have

$$c_\alpha(\lambda_\alpha)=I_\alpha(i\lambda_\alpha)/I_\alpha(\varrho_*^\alpha), \tag{5.21}$$

where ϱ_*^α is the analogue of ϱ_* for the space S^α, and

$$I_\alpha(\nu)=\beta\left(\frac{p^\alpha}{2},\frac{\langle\nu,\theta^\alpha\rangle_\alpha}{\langle\theta^\alpha,\theta^\alpha\rangle_\alpha}\right)\beta\left(\frac{q^\alpha}{2},\frac{p^\alpha}{2}+\frac{\langle\nu,2\theta^\alpha\rangle_\alpha}{\langle 2\theta^\alpha,2\theta^\alpha\rangle_\alpha}\right). \tag{5.22}$$

The second factor is to be interpreted as 1 if $q^\alpha=0$.

Now let $\lambda\in\Lambda_0$, and suppose $|\lambda|=r$. Pur $\lambda'=\lambda/r$. Then $\lambda_\alpha=r\lambda'_\alpha$, so

$$I_\alpha(i\lambda_\alpha)=\beta\left(\frac{p^\alpha}{2},ir\frac{\langle\lambda'_\alpha,\theta^\alpha\rangle_\alpha}{\langle\theta^\alpha,\theta^\alpha\rangle_\alpha}\right)\beta\left(\frac{q^\alpha}{2},\frac{p^\alpha}{4}+ir\frac{\langle\lambda'_\alpha,2\theta^\alpha\rangle_\alpha}{\langle 2\theta^\alpha,2\theta^\alpha\rangle_\alpha}\right). \tag{5.23}$$

We have the asymptotic formula, with a,b real

$$\lim_{|b|\to\infty}\Gamma(a+ib)\exp\left(\tfrac{1}{2}\pi|b|\right)|b|^{(1-a)/2}=(2\pi)^{\frac{1}{2}}. \tag{5.24}$$

See e.g. [20, p. 151], where $\Gamma(x)$ is the classical gamma function. Since $\beta(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, we deduce that if x,y,z are real,

$$\lim_{|y|\to\infty}\beta(z,x+iy)\,|y|^z=\Gamma(z). \tag{5.25}$$

It follows that

$$|I_\alpha(i\lambda_\alpha)|^2\,r^{(p^\alpha+q^\alpha)}\,R_\alpha(\lambda'_\alpha)\to\Gamma\left(\frac{p_\alpha}{2}\right)^2\Gamma\left(\frac{q^\alpha}{2}\right)^2\quad\text{as } r\to\infty, \tag{5.26}$$

with

$$R_\alpha(\lambda'_\alpha)=\left|\frac{\langle\lambda'_\alpha,\theta^\alpha\rangle_\alpha}{\langle\theta^\alpha,\theta^\alpha\rangle_\alpha}\right|^{p^\alpha}\left|\frac{\langle\lambda'_\alpha,2\theta^\alpha\rangle_\alpha}{\langle 2\theta^\alpha,2\theta^\alpha\rangle_\alpha}\right|^{q^\alpha}. \tag{5.27}$$

We have $c_\alpha(\lambda_\alpha)=I_\alpha(i\lambda_\alpha)/I_\alpha(\varrho^\alpha_*)$, and $c(\lambda)=I(i\lambda)/I(\varrho^*)=\prod_{\alpha\in\Sigma_0}I_\alpha(i\lambda_\alpha)/\prod_{\alpha\in\Sigma_0}I_\alpha(\varrho^\alpha_*)$. It follows from this and from (5.26) that if $c(r,\lambda')$ is the expression of $c(\lambda)$ in polar coordinates, we get

$$|c(r,\lambda')|^2\,|I(\varrho_*)|^2\prod_{\alpha\in\Sigma_0}r^{(p^\alpha+q^\alpha)}\,R(\lambda')\to\prod_{\alpha\in\Sigma_0}\Gamma\left(\frac{p^\alpha}{2}\right)^2\Gamma\left(\frac{q^\alpha}{2}\right)^2=B_G\quad\text{say, as } r\to\infty. \tag{5.28}$$

Note that the convergence here is uniform in λ', because the limit relation (5.27) involves only $|y|$.

A moment's reflection shows that if $\mathfrak{n}^\alpha_0$ is the nilpotent part of the Iwasawa decomposition of $\mathfrak{g}^\alpha_0$, so that $\mathfrak{g}^\alpha_0=\mathfrak{k}^\alpha_0+\mathfrak{h}^\alpha_{\mathfrak{p}_0}+\mathfrak{n}^\alpha_0$, we must have $p^\alpha+q^\alpha=\dim\mathfrak{n}^\alpha_0=$ cardinality of P^α_+, where P^α_+ is the analogue of P_+, for S^α. A scrutiny of [12, p. 964] shows that $\mathfrak{n}^\alpha_0$ is just $\mathfrak{Z}^+_\alpha$ in the notation of that paper, and further that the cardinality of P_+ is the sum of the cardinalities of P^α_+ as α ranges over Σ_0. Thus $\sum_{\alpha\in\Sigma_0}(p^\alpha+q^\alpha)$ is equal to cardinality of $P_+=\dim\mathfrak{n}_0$, and this last equals $n-l$ where $n=\dim G/K$, $l=\dim\mathfrak{h}_{\mathfrak{p}_0}=\dim\Lambda_0$. Thus,

$$|c(r,\lambda')^2|\,I(\varrho_*)|^2\,r^{n-l}R(\lambda')\to B_G \tag{5.29}$$

or

$$\frac{|c(r,\lambda')|^{-2}}{r^{n-l}}\to|I(\varrho_*)|^2R(\lambda')\,B_G^{-1}. \tag{5.30}$$

Since the convergence is uniform in λ', we see immediately that

$$\frac{m_0(r)}{r^{n-l}}=\frac{\int_\Omega|c(r,\lambda')|^{-2}d\lambda'}{r^{n-l}}\to C'_G\quad\text{as } r\to\infty. \tag{5.31}$$

where $C'_G=I(\varrho_*)^2B_G^{-1}\int_\Omega R(\lambda')\,d\lambda'$. Since $R(\lambda')$ is >0 on Ω except possibly for a set of measure zero, it is clear that $C'_G>0$.

Now

$$g_t(e)=\exp -t\,|\varrho_*|^2\int_0^\infty e^{-tr^2}m_0(r)\,r^{l-1}\,dr. \tag{5.32}$$

A change of variable shows that

$$\int_0^\infty e^{-tr^2} m_0(r)\, r^{l-1}\, dr = \tfrac{1}{2}\int_0^\infty e^{-tz}\, m_0(\sqrt{z})\, z^{l/2-1}\, dz,$$

and because of (5.31), it is obvious that

$$\frac{1}{2}\,\frac{m_0(\sqrt{z})\, z^{l/2-1}}{z^{n/2-1}} \to \tfrac{1}{2} C'_G = C''_G$$

say. It follows from the Tauberian theorem for the Laplace transform that

$$\frac{t^{n/2}}{\Gamma(n/2+1)} \int_0^\infty e^{-tr^2} m_0(r)\, r^{l-1}\, dr \to C''_G \quad \text{as } t \to 0. \tag{5.33}$$

This proves the proposition, with $C_G = \Gamma(n/2+1)\, C''_G$.

It can be checked that if G is complex, this complicated expression for C_G collapses to the one mentioned at the beginning of the proof of this proposition.

Collecting the results of Proposition 5.3, 5.6, 5.7 and taking into account $L(t) = J_C(t) + J_E(t) + J_H(t)$, we get

PROPOSITION 5.8. $\lim_{t\to 0} t^{n/2} L(t) = s \operatorname{vol}(\Gamma \backslash G/K)\, C_G$,

where C_G is a constant >0, and s is the order of the centre of G.

It follows, since $L(t) = \int_0^\infty e^{-tr}\, dN(r)$ that $N(r)/r^{n/2} \to C_G/\Gamma(n/2+1)$ as $r \to \infty$.

Now for any $r \geqslant 0$ put

$$M(r) = |W|^{-1} \int_{|\lambda|^2 + |\varrho_*|^2 \leqslant r} |c(\lambda)|^{-2}\, d\lambda.$$

It is understood that $M(r) = 0$ if $r \leqslant |\varrho_*|^2$. Clearly

$$g_t(e) = |W|^{-1} \int_{\Lambda_0} (\exp - t(|\lambda|^2 + |\varrho_*|^2))\, |c(\lambda)|^{-2}\, d\lambda,$$

so
$$g_t(e) = \int_{|\varrho_*|^2}^\infty e^{-tr}\, dM(r) = \int_0^\infty e^{-tr}\, dM(r).$$

Since we know that $t^{n/2} g_t(e) \to C_G$ as $t \to 0$, we see that $t^{n/2} \int_0^\infty e^{-tr}\, dM(r) \to C_G$ as $t \to 0$, hence $M(r)/r^{n/2} \to C_G/\Gamma(n/2+1)$ as $r \to \infty$, by the Tauberian theorem for Laplace transforms.

Putting these facts together we see that

$$N(r)/M(r) \to s \operatorname{vol}(\Gamma\backslash G/K) \quad \text{as } r \to \infty. \tag{5.34}$$

We write this as $N(r) \sim s \operatorname{vol}(\Gamma\backslash G/K)\, M(r)$. The results obtained above can be summarized as follows. For the sake of clarity we also incorporate some results of Tamagawa in the statement.

THEOREM 5.9. *Let G be a complex connected semisimple Lie group and K a maximal compact subgroup of G. Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Let $L_2(\Gamma\backslash G/K)$ be the space of measurable functions on G that i) satisfy $f(\gamma x k) = f(x)$, $\gamma \in \Gamma$, $x \in G$, $k \in K$, and ii) are square integrable with respect to the invariant measure on $\Gamma\backslash G$ when considered as functions on that space. Let $\mathfrak{D}(G/K)$ be the algebra of left invariant differential operators on G which commute with right translations by elements of K.*

Then (Tamagawa) $L_2(\Gamma\backslash G/K)$ is an orthogonal Hilbert space sum of subspaces $\{H_i\}_{i=0}^{\infty}$ such that i) each H_i is finite dimensional ii) H_i consists of C^∞ functions iii) the action of $\mathfrak{D}(G/K)$ on each H_i is by scalars. Cf. [2].

If $\omega \in \mathfrak{D}(G/K)$ is the Laplace–Beltrami operator of G/K, and $h_i(\omega)$ the scalar by which ω acts on H_i, then $h_i(\omega) \leqslant 0$ for each i, and for any $r > 0$, the number of indices i such that $|h_i(\omega)| \leqslant r$ is finite.

Let $N(r)$ be the function defined for $r \geqslant 0$ by

$$N(r) = \sum_{\{i;\, |h_i(\omega)| \leqslant r\}} \dim H_i. \tag{5.35}$$

Then $N(r)$ is finite valued. Finally, if $c(\lambda)$ is Harish-Chandra's function for the symmetric space G/K, so that $|c(\lambda)|^{-2}\, d\lambda$ is the Plancherel measure of G/K, then

$$N(r) \sim s \operatorname{vol}(\Gamma\backslash G/K)\, M(r) \quad \text{as } r \to \infty, \tag{5.36}$$

where $s =$ order of the intersection of Γ with the centre of G and $\operatorname{vol}(\Gamma\backslash G/K)$ stands for the volume of $\Gamma\backslash G/K$, the function $M(r)$ is defined by

$$M(r) = \int_{|\lambda|^2 + |\varrho_*|^2 \leqslant r} |c(\lambda)|^{-2}\, d\lambda, \tag{5.37}$$

where ϱ_* is half the sum of the positive roots of the symmetric space G/K, and the integral is over the dual of a Cartan subalgebra of the symmetric space G/K.

Moreover, both the functions $N(r)$ and $M(r)$ behave asymptotically as $C''_G r^{n/2}$ when $r \to \infty$, where $n = \dim G/K$, and C''_G is a constant > 0, depending only on G.

Let $f \in L_2(\Gamma\backslash G/K)$. Then f can be expanded as a Fourier series

$$f = \sum_{i=0}^{\infty} \sum_{j=0}^{\dim H_i} a_{ij} \varphi_{ij}$$

converging in $L_2(\Gamma\backslash G/K)$. It is obvious that the coefficients a_{ij} are the Fourier coefficients of f with respect to the orthonormal system φ_{ij}. One then has the Parseval relation

$$\|f\|^2 = \sum_i \sum_j |a_{ij}|^2. \tag{5.38}$$

Suppose now that g is a continuous spherical function of compact support in G. Then we can periodize g with respect to Γ as follows, and define a function g_Γ by

$$g_\Gamma(x) = \sum_{\gamma\in\Gamma} g(\gamma x). \tag{5.39}$$

It is obvious that $g_\Gamma \in L_2(\Gamma\backslash G/K)$. Moreover, it can be shown that if b_{ij} are the Fourier coefficients of g_Γ with respect to the orthonormal system φ_{ij}, then, in fact b_{ij} are independent of j, (equal to b_i say) for $1 \leqslant j \leqslant \dim H_i$. Indeed, $b_i = \hat{g}(\lambda_i)$ where $\hat{g}$ is the Fourier transform of g defined in § 2, and φ_{λ_i} is the elementary spherical function that underlies H_i. We thus have

$$g = \sum_{i=0}^{\infty} b_i \Phi_i, \quad \Phi_i = \sum_{j=1}^{\dim H_i} \varphi_{ij}, \quad \|g\|^2 = \sum_{i=0}^{\infty} |b_i|^2 \dim H_i. \tag{5.40}$$

Let $L_2^{\#}(\Gamma\backslash G/K)$ be the closed subspace of $L_2(\Gamma\backslash G/K)$ that is generated by such g_Γ. In an obvious way, we may regard the sequence $\{\lambda_i\}_{i=0}^{\infty}$ as the dual object on which the Fourier transform of a function in $L_2^{\#}(\Gamma\backslash G/K)$ may be defined and the Parseval relation above shows that the sequence $\{\dim H_i\}_{i=0}^{\infty}$ can be interpreted as the Plancherel measure that is appropriate to the harmonic analysis of functions in $L_2^{\#}$, i.e. of functions in $L_2(\Gamma\backslash G/K)$ which arise as periodizations of spherical functions on G/K. While we will not labour this point of view, it does give rise to an interesting way of looking at formula (5.36). Namely, $N(r)$ is now just the total Plancherel measure of the region of the dual object $\{\lambda_i\}_{i=0}^{\infty}$ where $|\lambda_i|^2 + |\varrho_*|^2 \leqslant r$. Indeed $N(r) = \sum_{\{i:|\lambda_i|^2+|\varrho_*|^2\leqslant r\}} \dim H_i$. On the other hand, Λ_0/W is the natural dual object for the harmonic analysis of $L_2(K\backslash G/K)$, on which the Plancherel measure of G/K, viz. $|c(\lambda)|^{-2}d\lambda$ lives, and $M(r)$ is just the total Plancherel measure of the region in Λ_0/W where $|\lambda|^2 + |\varrho_*|^2 \leqslant r$. Thus the result $N(r) \sim s \operatorname{vol}(\Gamma\backslash G/K)\, M(r)$ shows that the Plancherel measures of $L_2^{\#}(\Gamma\backslash G/K)$ and of $L_2(K\backslash G/K)$ look alike when viewed from infinity. Nature, as it were, allows the subgroup Γ to modify the behaviour of the Plancherel measure of $L_2(K\backslash G/K)$ only in a very simple way.

Before passing to the next section, it may be worthwhile to remark here that our technique also enables us to get results about the asymptotic behaviour of the eigenfunctions that constitute H_i, much in the spirit of [5]. Thus fix $x \in G$ and consider the expansion

$$\sum_i \exp - t(|\lambda_i|^2 + |\varrho_*|^2) \sum_{j=1}^{\dim H_i} |\varphi_{ij}(x)|^2 = \sum_{\gamma \in \Gamma} g_t(x^{-1}\gamma x) \tag{5.41}$$

which ensues from (4.10). If for fixed $x \in G$, we let

$$N_x(r) = \sum_{\{i:\, |\lambda_i|^2 + |\varrho_*|^2 \leqslant r\}} \sum_{j=1}^{\dim H_i} |\varphi_{ij}(x)|^2,$$

the left side of (5.41) is just the Laplace transform of $N_x(r)$. The right side can be split into two sums $J_1(x,t)$ and $J_2(x,t)$. Here J_1 runs over all elements $\gamma \in \Gamma$ such that $x^{-1}\gamma x \in K$. This is clearly a finite sum, since $x^{-1}\Gamma x$ is also a discrete subgroup of G. J_2 runs over the remaining elements of Γ. The technique Lemma 5.3 shows that $J_2(x,t) \to 0$ as $t \to 0$, while $J_1(x,t) = n(x)\, g_t(e)$, where $n(x)$ is the cardinality of $x^{-1}\gamma x \cap K$. Clearly $n(x) \geqslant s =$ order of the intersection of Γ with the centre of G. Indeed $n(x) = s$ unless the element xK in G/K is a fixed point of some $\gamma \in \Gamma$. One finds thus that $\lim_{t\to 0} t^{n/2} \int_0^\infty e^{-tr}\, dN_x(r)$ exists and is equal to $n(x)\, C_G$. In other words $N_x(r) \sim n(x)\, C_G\, r^{n/2}/\Gamma(n/2+1)$ as $r \to \infty$, $n = \dim G/K$. If Γ acts freely on G/K, then $n(x) = s$ for every $x \in G$ and substantially the result of [5] is recovered.

We saw above that underlying each subspace H_i, there is an elementary spherical function φ_{λ_i}, which is positive definite. Now, it is known that for $\lambda \in \Lambda_0$, φ_λ is positive definite, but the converse is not correct. Indeed for the element $-i\varrho_* \in \Lambda$, it is obvious that $\varphi_{-i\varrho_*} = 1$ which is positive definite, yet $i\varrho_* \notin \Lambda_0$. In formulating his result on p. 77 of [3], Gelfand overlooks this complication entirely, and assumes implicitly that the only φ_{λ_i} that can occur are those that arise from $\Gamma_i \in \Lambda_0$. This is not accurate, since the constant 1 occurs, and we see no reason why, similarly, other positive definite elementary spherical functions which correspond to elements of $\Lambda - \Lambda_0$ should not occur. This assumption makes Gelfand's formulation marginally inaccurate. The point is of some relevance, since one does not yet know *precisely* which $\lambda \in \Lambda$ give rise to positive definite elementary spherical functions.

The reader should note that once the theta-relation $L(t) = \sum_{\gamma\in\Gamma} \int_{\Gamma\backslash G/K} g_t(x^{-1}\gamma x)\, dx$ is at hand, one can guess readily at the result one wants to prove. Namely, experience from similar problems in analytic number theory shows that in the analysis of such sums, one expects the term arizing from $\gamma = e$ to predominate. In our situation this term is easily computed. It is just $\mathrm{vol}\,(\Gamma\backslash G/K)\, g_t(e)$. One also knows that the point mass at eK has as its Fourier transform the Plancherel measure of G/K. Thus, once a theta-relation or a formal analogue of it can be surmised, it is possible to arrive at the conjecture outlined in [3]. As in other similar problems, it is the proof that is tedious, but crucial, and is likely to introduce unexpected constants.

The hypothesis that Γ acts without fixed points on G/K is never made in [3]. Without this hypothesis, one cannot avoid having to deal with terms like $J_E(t)$ which involve the elements of Γ that have fixed points on G/K. We cannot think of any method of dealing with these terms that would avoid using Harish-Chandra's invariant integrals. No mention of these terms is made by Gelfand in [3]. It would be very interesting indeed to see an approach which surmounts this difficulty without using the invariant integral.

6. Concluding remarks

Our method relies heavily on the fact that the condition (3.48) of regular growth holds for g_t. This condition is used first to prove that $\sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x)$ converges uniformly, and then again in showing that $J_H(t)\to 0$ as $t\to 0$. So far as the convergence of $\sum_{\gamma\in\Gamma} g_t(y^{-1}\gamma x)$ is concerned, it would be enough to show that g_t has a majorant which is in $L_1(G/K)$ and which satisfies (3.48). Using a technique of Gårding, this can in fact be demonstrated, so that one does in fact have the theta relation in general. However, for the purpose of estimating $J_H(t)$, it is definitely not enough to know that (3.48) holds for a majorant of g_t, for, in showing $J_H(t)\to 0$, crucial use is made of the fact that g_t approaches the Haar measure of K as $t\to 0$, i.e. that g_t is an approximate identity in $L_1(K\backslash G/K)$. In order to salvage this part of the argument, it would be necessary to majorize g_t by an approximate identity in $L_1(K\backslash G/K)$. We do not know how to do this in general.

However, if rank $G/K=1$, it is possible to show that for each $t>0$, there is some $r>0$ so that $(\omega g_t)(a)\geqslant 0$ for $|\log a|\geqslant r$, and also that for each $r>0$ there is a $t_0>0$ so that for all $t\leqslant t_0$ and $|\log a|\geqslant r$, one has $(\omega g_t)(a)\geqslant 0$. The proof of this uses the explicit knowledge of the elementary spherical functions available in this case, and also the fact that spaces of rank 1 have been classified. At the time of writing, we do not have a proof which does not use the classification.

However, it is to be noted that even when g_t has the above property, so that the convergence of $\sum_\gamma g_t(y^{-1}\gamma x)$ and the estimation of $J_H(t)$ can be carried out as above, it is not at all trivial to carry out the evaluation of $J_E(t)$, even in the case rank $G/K=1$. If Γ is assumed to act freely, the problem is of course, absent. In general, when rank $G/K>1$, we cannot conceive of a method of estimating $J_E(t)$ which does not rely on Harish-Chandra's study of the invariant integral.

It should be suspected that in general g_t would have the property (3.48).

In the complex case, one could attempt some extensions of our result. For example, instead of $L_2(\Gamma\backslash G/K)$ one could study the subspace of functions in $L_2(\Gamma\backslash G)$

which transform under K on the right according to some class δ of irreducible representations of K. Some experimentation indicates that a similar result holds, the constant $s \operatorname{vol}(\Gamma\backslash G/K)$ now being replaced by $s \operatorname{degree}(\delta) \operatorname{vol}(\Gamma\backslash G)$. Of course, one does not count the eigenvalues of the Laplace–Beltrami operator any more, but the eigenvalues of the restriction of the Casimir operator of G to those functions which transform under right translations by K according to δ. But we have no proof of these conjectures.

When this paper was almost completed we have come across a paper of McKean and Singer [21] dealing with the problem of recovering information about a compact Riemannian manifold M from data involving the spectral features of its Laplace operator. Bearing in mind the classic example of the vibrating string or membrane, one notices that if one thought of a "solid membrane" in the shape of M set in vibration, then the eigenvalues $\{\lambda_n\}_{n\geqslant 0}$ of the Laplace operator would be characteristic of the normal modes of vibration, and the amplitudes of these normal modes. Following the picturesque sensory terminology of Kac [22], one says that a topological or geometric invariant of M is *audible* if it can be recovered from the spectral data $\{\lambda_n\}$. Even though $\Gamma\backslash G/K$ is not a manifold in our situation, we can nevertheless extend this terminology to our context as well. Now, the function $N(r)$ and its Laplace transform $L(t)$ are objects involving only the spectrum of the Laplace–Beltrami operator ω, and any information about $\Gamma\backslash G/K$ which can be recovered from them may be termed audible. Our result shows of course, that $\operatorname{vol}(\Gamma\backslash G/K)$ is audible, but on further consideration one sees that some characteristics of Γ are likewise audible. For example, the coefficients of the various powers of t that enter into $J_E(t)$ give us information regarding regular and singular points among those elements of Γ which have a fixed point on G/K. To mention but one instance of this, consider the coefficient of $t^{-l/2}$ in $J_E(t)$. A closer look at our analysis shows that one can read off from this coefficient the number of G-conjugacy classes of elements of Γ which have a fixed point on G/K and which consists of regular elements in G. Thus we may say that this invariant of Γ is audible. It is at any rate evident that some information is to be gleaned from combining the methods of this paper and [21], and we hope to return to this question in a future paper.

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