TENSOR ALGEBRAS AND HARMONIC ANALYSIS

BY

N. TH. VAROPOULOS

Trinity College, Cambridge, England. Faculté des Sciences (Math.), Orsay, France.
The Institute for Advanced Study, Princeton, N. J., U.S.A.

Introduction

The material presented in this paper is a systematic exposition of the theory of tensor algebras and their applications and connections with harmonic analysis.

We shall not attempt here in the introduction to describe or summarize the methods and results of this paper. We shall instead refer the reader to [13], [28] and [29] for that. We shall also refer the reader to [19], [23], [26], and [27] for background reading relative to the two main problems considered in this paper, namely, the problem of "spectral synthesis" and that of "symbolic calculus".

We would like to point out, however, that none of the above literature is an essential pre-requirement for the understanding of this paper. What is needed instead is a certain familiarity with commutative Banach algebras and in particular regular algebras. One can acquire this in [2]; also we shall have to assume in this paper one or two easy but slightly technical points of harmonic analysis that are very well exposed in [5]. Finally some knowledge of the general theory of the topological tensor product as is to be found in [1] is desirable but not essential provided that the instructions given below as to how this paper should be read are carefully followed.

In Ch. 1 we recall definitions and notations from functional analysis and prove some easy lemmas.

In Ch. 2 we define a tensor algebra in two ways: using functional analytic concepts in § 1 and directly in § 2. The reader who wishes to ignore functional analysis should start reading this paper from Ch. 2, § 2.

In Ch. 3 we develop some of the fundamental topological techniques that allow us to work with tensor algebras. There § 4 is the most crucial paragraph and also the easiest to read. The reader can go directly from Ch. 2, § 2 to Ch. 3, § 4 provided that he is prepared to refer back for definitions.

In Ch. 4 the first link between tensor algebras and group algebras is established and in a first reading it suffices there to read § 1 and § 2.

Ch. 5 is more technical and is developed only for the sake of some specific applications. The reader could skip it altogether in a first reading and proceed directly to Ch. 8 where the second essential link with harmonic analysis is established. One thing that I should like to point out here is that the presentation of the material of Ch. 8 differs from the one I originally gave in [12]; instead the much more elegant formulation of [13] is followed.

The remaining chapters deal with various aspects and applications of the theory and are much more specialized; they can in fact be read more or less independently from one another.

1. Generalities on the tensor product and Banach algebras

§ 1. The tensor product of vector spaces

In this paragraph we shall list some notations and definitions from functional analysis which we intend to adopt.

For arbitrary vector spaces $E_1, E_2, \ldots, E_n$ we denote by $E_1 \otimes E_2 \otimes \ldots \otimes E_n$ their tensor product which is a new vector space, for $\{e_j \in E_j\}_{j=1}^n$ elements of the spaces we denote then by $e_1 \otimes e_2 \otimes \ldots \otimes e_n$ their tensor product [1]. Also for arbitrary linear mappings $\{T_j: E_j \to H_j\}_{j=1}^n$ between vector spaces we denote by:

$$T = T_1 \otimes T_2 \otimes \ldots \otimes T_n : E_1 \otimes E_2 \otimes \ldots \otimes E_n \to H_1 \otimes H_2 \otimes \ldots \otimes H_n$$

the canonically induced mapping on the tensor products.

When the spaces $\{E_j; H_j\}_{j=1}^n$ are normed linear spaces and $\{T_j \in \mathcal{L}(E_j; H_j)\}_{j=1}^n$ are continuous linear mappings we denote by $E = E_1 \hat{\otimes} E_2 \hat{\otimes} \ldots \hat{\otimes} E_n$ the completion of $E_1 \otimes E_2 \otimes \ldots \otimes E_n$ with the projective $\otimes_\pi$ norm; E is then a Banach space [1]. T can then be extended by continuity to:

$$\hat{T} = T_1 \hat{\otimes} T_2 \hat{\otimes} \ldots \hat{\otimes} T_n : E = E_1 \hat{\otimes} E_2 \hat{\otimes} \ldots \hat{\otimes} E_n \to H_1 \hat{\otimes} H_2 \hat{\otimes} \ldots \hat{\otimes} H_n.$$

It is then well known that $\|\hat{T}\| \leqslant \|T_1\| \, \|T_2\| \ldots \|T_n\|$. Also in the case where the spaces $E_1, E_2, \ldots, E_n$ are already Banach spaces it is well known that every element $e \in E$ admits an expansion of the form:

$$(\mathcal{E}) \qquad e = \sum_{j=1}^{\infty} e_1^{(j)} \otimes e_2^{(j)} \otimes \ldots \otimes e_n^{(j)}; \quad e_i^{(j)} \in E_i \quad (1 \leqslant i \leqslant n;\ j = 1, 2, \ldots)$$

such that
$$T_{\mathcal{E}} = \sum_{j=1}^{\infty} \|e_1^{(j)}\| \, \|e_2^{(j)}\| \ldots \|e_n^{(j)}\| < +\infty,$$

where of course $T_\mathcal{E}$ depends on the particular expansion. We can say further that:

$$\|e\|_E = \inf_{\mathcal{E}} T_\mathcal{E}$$

the inf being taken over all possible expansions of e and $\|e\|_E$ being the norm of $e \in E$.

§ 2. Approximate inverse and tensor product

Let B and C be two Banach spaces and let $T: B \to C$ be a continuous linear mapping ($T \in \mathcal{L}(B; C)$). We shall say that a family $\{T_\alpha \in \mathcal{L}(C; B)\}_{\alpha \in A}$ of linear mappings $T_\alpha: C \to B$ which is directed, in the sense that the index set A is a directed set, is an approximating inverse of T if:

(α) $\|T_\alpha\| \leqslant 1 \quad \forall \alpha \in A$

(β) $T_\alpha \circ T \xrightarrow[\alpha \in A]{} I\partial(B)$ for the strong operator topology (i.e. $T_\alpha \circ Tx \xrightarrow[\alpha \in A]{} x$ in B $\forall x \in B$).

(In general for any set X we denote by $I\partial(X)$ the identity mapping on the set X.) We say then that T has an approximating inverse.

Let us observe then at once that if $T: B \to C$ is a linear mapping between two Banach spaces of norm at most one $\|T\| \leqslant 1$ that has an approximating inverse then T is an isometry. Indeed suppose not then for some $b \in B$ we have $\|Tb\|_C < \|b\|_B$ then we have, denoting by $\{T_\alpha\}_{\alpha \in A}$ the approximating inverse of T:

$$\|T_\alpha \circ Tb\|_B \leqslant \|Tb\|_C < \|b\|; \quad T_\alpha \circ Tb \xrightarrow[\alpha \in A]{} b \text{ in } B$$

which gives the required contradiction.

Let us now suppose that $\{T^{(j)}: B_j \to C_j\}_{j=1}^n$ is a family of bounded linear mappings between pairs of Banach spaces each with an approximating inverse $\{T_\alpha^{(j)}\}_{\alpha \in A_j}$ and let us also consider

$$T = T^{(1)} \hat{\otimes} T^{(2)} \hat{\otimes} \ldots \hat{\otimes} T^{(n)}: B_1 \hat{\otimes} B_2 \hat{\otimes} \ldots \hat{\otimes} B_n \to C_1 \hat{\otimes} C_2 \hat{\otimes} \ldots \hat{\otimes} C_n$$

their tensor product; it is then immediate to verify that the family:

$$\{T_\alpha\}_{\alpha \in A} = \{T_{\alpha_1}^{(1)} \hat{\otimes} T_{\alpha_1}^{(2)} \hat{\otimes} \ldots \hat{\otimes} T_{\alpha_n}^{(n)}\}_{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in A_1 \times A_2 \times \ldots \times A_n = A}, \tag{1.2.1}$$

where $A = A_1 \times A_2 \times \ldots \times A_n$ the product space with the product order is an approximating inverse of T. (This is an immediate consequence either of general theorems or of the decomposition ($\mathcal{E}$) in § 1.)

From the above and § 1 we see that

LEMMA 1.2.1. *If $T_j: B_j \to C_j$ $(j=1, 2, \dots n)$ are bounded linear mappings of Banach spaces such that*

(α) $$\|T_j\| \leqslant 1, \quad j=1, 2, \dots n.$$

(β) *T_j has an approximating inverse for $j=1, 2, \dots n$.*

Then $T = T_1 \hat{\otimes} T_2 \hat{\otimes} \dots \hat{\otimes} T_n$ is an isometry.

Let now again B, C be two Banach spaces and let

$$B = B_0 \supset B_1 \supset \dots \supset B_n \supset \dots \bigcap_{n=1}^{\infty} B_n = B_\infty$$

be a nested sequence of closed subspaces such that the canonical injections:

$$i^{(n)}: B_n \to B_{n-1}, \quad n = 1, 2, \dots,$$

have an approximating inverse $\{i_\alpha^{(n)}\}_{\alpha \in A_n}$. Let us also denote by $j_n: B_n \to B$ the canonical injection; it is then clear that the directed family

$$\{j_{\alpha_1, \alpha_2, \dots, \alpha_n} = i_{\alpha_n}^{(n)} \circ i_{\alpha_{n-1}}^{(n-1)} \circ \dots i_{\alpha_1}^{(1)}\}_{(\alpha_1, \alpha_2, \dots, \alpha_n) \in A_1 \times A_2 \times \dots \times A_n}$$

is an approximating inverse of j_n for each $n = 1, 2, \dots$. Let us now make the additional assumption:

(*) For any choice of the sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots) \in A = A_1 \times A_2 \times \dots$ the sequence of mappings $\{j_n \circ j_{\alpha_1, \alpha_2, \dots \alpha_n}\}_{n=1}^{\infty}$ converges in the strong operator topology to a mapping:

$$j_{\boldsymbol{\alpha}} = j_{\alpha_1, \alpha_2, \dots}: B \to B.$$

It is then clear that $\operatorname{Im} j_{\boldsymbol{\alpha}} \subset B_\infty$ $\forall \boldsymbol{\alpha} \in A$ and that the family $\{j_\alpha\}_{\alpha \in A}$ for the product order of $A = A_1 \times A_2 \times \dots$ is an approximating inverse of the canonical injection

$$j_\infty: B_\infty \to B.$$

Therefore it follows that under the hypothesis (*) we can identify $B_n \hat{\otimes} C$ to a closed subspace of $B \hat{\otimes} C$ for $n = 1, 2, \dots \infty$; and obtain a nested sequence

$$B \hat{\otimes} C \supset B_1 \hat{\otimes} C \supset \dots X_\infty = \bigcap_{n=1}^{\infty} (B_n \hat{\otimes} C) \supseteq B_\infty \hat{\otimes} C$$

we claim further that $X_\infty = B_\infty \hat{\otimes} C$. Indeed let $x \in X_\infty \subset B \hat{\otimes} C$ and $\varepsilon > 0$ be arbitrary then for any choice of $\boldsymbol{\alpha} \in A$ we have:

$$x = [j_\alpha \hat{\otimes} I\partial(C)]\,(x) \in B_\infty \hat{\otimes} C;$$

but for a proper choice of $\alpha \in A$ we have $\|x - x_\alpha\|_{B\hat{\otimes}C} \leqslant \varepsilon$ and $B_\infty \hat{\otimes} C$ being a closed subspace our assertion follows.

§ 3. The role of the basis problem in tensor products

Let B be a Banach space. We shall say that it has a basis if there exists $\{T_\alpha \in \mathcal{L}(B)\}_{\alpha\in A}$ a directed family of linear operators on B (linear mappings from B to B) of finite rank (i.e. the spaces $T_\alpha(B)$ are all of finite dimension) such that

$$\sup_{\alpha\in A} \|T_\alpha\| < +\infty; \quad T_\alpha \to I\partial(B) \text{ in the strong operator topology.}$$

All known Banach spaces have a basis and it is an open problem whether Banach spaces without a basis exist.

Let now B be an arbitrary Banach space and let $K \subset B$ be a closed subspace and $B \to B/K$ be the canonical projection, let further X be another Banach space with a basis and let us consider the mapping:

$$\tilde{p} = I\partial(X) \hat{\otimes} p : X \hat{\otimes} B \to X \hat{\otimes} B/K;$$

then it is immediate that if we identify $X \otimes K$ to a subspace of $X \hat{\otimes} B$ that $X \otimes K \subset \operatorname{Ker} \tilde{p}$. We claim further that $X \otimes K$ is dense in $\operatorname{Ker} \tilde{p}$ i.e.

$$\overline{X \otimes K} = \operatorname{Ker} \tilde{p}. \tag{1.3.1}$$

Indeed let $\{Z_\alpha \in \mathcal{L}(X)\}_{\alpha\in A}$ be the operators of finite rank such that

$$\sup_{\alpha\in A} \|Z_\alpha\| < +\infty; \quad Z_\alpha \to I\partial X \text{ in the strong operator topology.}$$

Then if we denote by $Y_\alpha = Z_\alpha \hat{\otimes} I\partial(B)$ we see that

$$Y_\alpha \to I\partial(X \hat{\otimes} B) \text{ for the strong operator topology; } Y_\alpha[\operatorname{Ker} \tilde{p}] \subset X \otimes K \tag{1.3.2}$$

the second relation holds because Z_α are of finite rank; and (1.3.2) of course implies our result.

Let now $\{B_j \supset K_j\}_{j=1}^n$ be Banach spaces and closed subspaces and let $\{p_j : B_j \to B_j/K_j\}_{j=1}^n$ be the canonical projections and let us suppose that all the spaces $\{B_j; B_j/K_j\}_{j=1}^n$ have bases; let us further denote by:

$$\mathcal{K}(\{B_j \supset K_j\}_{j=1}^n) = K_1 \otimes B_2 \otimes \ldots \otimes B_n + B_1 \otimes K_2 \otimes \ldots \otimes B_n$$
$$+ \ldots B_1 \otimes B_2 \otimes \ldots \otimes B_{n-1} \otimes K_n \subset B_1 \hat{\otimes} B_2 \hat{\otimes} \ldots \hat{\otimes} B_n \tag{1.3.3}$$

a vector subspace. Then a repeated application of (1.3.1) (or equivalently induction over n) gives at once

$$\overline{\mathcal{K}(\{B_j \supset K_j\}_{j=1}^n)} = \operatorname{Ker}(p_1 \hat{\otimes} p_2 \hat{\otimes} \ldots \hat{\otimes} p_n). \tag{1.3.4}$$

§ 4. Commutative Banach algebras

We start by listing some classical notations on Banach algebras which we shall adopt.

Let R be a semisimple, regular, commutative Banach algebra with 1 and $\mathfrak{M}_R$ as spectrum [2]; for any $r \in R$ we shall denote then by $\hat{r} \in C(\mathfrak{M}_R)$ its Gelfand transform. Let also $E \subset \mathfrak{M}_R$ be a closed subspace; we then denote by:

$$I^R(E) = I(E) = \{r \in R;\ \hat{r}^{-1}(0) \supset E\},$$

$$I_0^R(E) = I_0(E) = \{r \in R;\ \hat{r}^{-1}(0) \text{ is a nhd. of } E\},$$

$$J^R(E) = J(E) = \overline{I_0(E)}.$$

It is then a classical theorem that $I(E)$ is the largest ideal with E as hull, that $I_0(E)$ is the smallest ideal with E as hull and that $J(E)$ is the smallest closed ideal with E as hull. (For any ideal $J \lhd R$ we define hull of $J = h(J) \subset \mathfrak{M}_R$

$$h(J) = \{M \in \mathfrak{M}_R;\quad M \supset J\}) \ [2]$$

We also say that E is a set of spectral synthesis for the algebra R if $J(E) = I(E)$.

Let now R_1 and R_2 be two regular Banach algebras with identity and let $\theta: R_1 \to R_2$ be an isometric algebraic homomorphism that takes the identity of R_1 on the identity of R_2, and identifies R_1 with a closed subalgebra of R_2. θ then induces canonically by transposition a continuous mapping

$$\tilde{\theta}: \mathfrak{M}_{R_2} \to \mathfrak{M}_{R_1};$$

$\tilde{\theta}$ is then onto, for R_1 being regular and being identified to a closed subalgebra of R_2 any of its maximal ideals can be extended to a maximal ideal of R_2.

Let now $E_1 \subset \mathfrak{M}_{R_1}$ be a closed subset and let $E_2 = \tilde{\theta}^{-1}(E_1) \subset \mathfrak{M}_{R_2}$ then we can verify at once that:

$$\theta^{-1}[I^{R_2}(E_2)] = I^{R_1}(E_1); \quad \theta^{-1}[J^{R_2}(E_2)] \supset J^{R_1}(E_1). \tag{1.4.1}$$

We have also

LEMMA 1.4.1. *Let $\theta: R_1 \to R_2$ be as above and such that:*

(α) *It has an approximating inverse $\{\theta_\alpha\}_{\alpha \in A}$.*

(β) *For any $r_2 \in R_2$ we have*

$$\operatorname{supp}[\theta_\alpha(r_2)]^\wedge \xrightarrow[\alpha \in A]{} \tilde{\theta}[\operatorname{supp} \hat{r}_2].$$

In the sense that for any Ω open nhd. of $\tilde{\theta}[\operatorname{supp} \hat{r}_2]$ in $\mathfrak{N}_{R_1}$ there exists $\alpha_\Omega \in A$ such that

$$\alpha \geqslant \alpha_\Omega \Rightarrow \operatorname{supp} [\theta_\alpha(r_2)]^\wedge \subset \Omega.$$

Then for any closed set $E_1 \subset \mathfrak{N}_{R_1}$ as above and $E_2 = \tilde{\theta}^{-1}(E_1)$ we have:

$$\theta^{-1}(J^{R_2}(E_2)) = J^{R_1}(E_1).$$

Proof. Indeed from (1.4.1) we see that it suffices to prove that $\theta^{-1}(J^{R_2}(E_2)) \subset J^{R_1}(E_1)$. So let $\varepsilon > 0$ and $x \in R_1$ be arbitrary such that $\theta(x) \in J^{R_2}(E_2)$; then there exists $y_\varepsilon \in I_0^{R_2}(\mathrm{E}_2)$ such that $\|\theta(x) - y_\varepsilon\|_{R_2} \leqslant \varepsilon/2$.
Also there exists $\alpha \in A$ such that

$$\|\theta_\alpha \circ \theta(x) - x\|_{R_1} \leqslant \varepsilon/2; \quad \operatorname{supp} [\theta_\alpha(y_\varepsilon)]^\wedge \cap E_1 = \emptyset$$

by the conditions of the lemma. But then

$$\theta_\alpha(y_\varepsilon) \in I_0(E_1), \quad \|x - \theta_\alpha(y_\varepsilon)\| \leqslant \varepsilon,$$

and ε being arbitrary this proves that $x \in J^{R_1}(E_1)$ and completes the proof of the lemma.

It follows in particular that if a mapping $\theta: R_1 \to R_2$ satisfies the conditions of Lemma 1.4.1, and if a closed set $E_1 \subset \mathfrak{N}_R$ is not a set of spectral synthesis for the algebra R_1, then the set $E_2 = \tilde{\theta}^{-1}(E_1)$ is not a set of spectral synthesis for the algebra R_2.

Let us finally introduce the following convenient definition:

Definition 1.4.1. We shall say that $\theta: R_1 \to R_2$, an isometric algebraic homomorphism between two regular Banach algebras with identity that takes the identity of R_1 on the identity of R_2, has a local approximating inverse if it has an approximating inverse satisfying the conditions of Lemma 1.4.1.

§ 5. Tensor product of Banach algebras

The statements that follow without proof are almost all trivial and well known [3].

Let $R_1, R_2, \ldots R_n$ be n commutative Banach algebras with identity, then we can give on $R = R_1 \hat{\otimes} \ldots \hat{\otimes} R_n$ a canonical structure of a commutative Banach algebra with identity. If further the $\{R_j\}_{j=1}^n$ are *Banach algebras a *Banach algebra structure can be given on R. If further $\{p_j: R_j \to \tilde{R}_j\}_{j=1}^n$ are continuous Banach algebra homomorphisms from the commutative Banach algebras R_j to the commutative Banach algebras $\tilde{R}_j$ for $j = 1, 2, \ldots n$ then

$$p_1 \hat{\otimes} p_2 \hat{\otimes} \ldots \hat{\otimes} p_n : R = R_1 \hat{\otimes} R_2 \hat{\otimes} \ldots \hat{\otimes} R_n \to \tilde{R} = \tilde{R}_1 \hat{\otimes} \tilde{R}_2 \hat{\otimes} \ldots \hat{\otimes} \tilde{R}_n$$

is also an algebraic homomorphism.

Relative to R the tensor product of the commutative Banach algebras $R_1, R_2, \dots R_n$ the following facts are trivial: $\mathfrak{N}_R$ the spectrum of R can be identified to $\mathfrak{N}_{R_1} \times \mathfrak{N}_{R_2} \times \dots \times \mathfrak{N}_{R_n}$. Also if the algebras $R_1, R_2 \dots R_n$ are regular so is R. Further, and this is not entirely trivial [3], if all the algebras $R_1, R_2, \dots R_n$ are semisimple, and if qua Banach spaces they satisfy the Banach approximation property (in particular if they have bases), then the algebra R is also semisimple.

An important tensor product of algebras arises in group algebras, where for $G_1, G_2, \dots G_n$ compact abelian groups we can identify isometrically and canonically

$$A(G_1 \times G_2 \times \dots \times G_n) = A(G_1) \hat{\otimes} A(G_2) \hat{\otimes} \dots \hat{\otimes} A(G_n).$$

The same is true for general locally compact abelian groups but then the algebras have no identity [1]. (For notations cf. [5] and Ch. 8.)

We shall prove now the following general

THEOREM 1.5.1. *Let $R_1, R_2, \dots R_n$ be semisimple regular commutative Banach algebras with identity and let $E_j \subset \mathfrak{N}_{R_j}$ be closed subsets of spectral synthesis (i.e. $I_j = I^{R_j}(E_j) = J^{R_j}(E_j)$, $j = 1, 2, \dots n$) and let us further suppose that the Banach spaces (R_j; R_j/I_j; $j = 1, 2, \dots n$) have bases. Then the set $E = E_1 \times E_2 \times \dots \times E_n \subset \mathfrak{N}_R$ is a set of spectral synthesis of the algebra $R = R_1 \hat{\otimes} R_2 \hat{\otimes} \dots \hat{\otimes} R_n$.*

Proof. Let $\mathcal{K} = \mathcal{K}(\{R_j \supset I_j\}_{j=1}^n)$ with the notation of (1.3.3). Then $\mathcal{K}$ is an ideal of R and $h(\mathcal{K}) = E$ and also by the very definition of $J(E)$ $\mathcal{K} \subset J(E)$. Therefore using (1.3.4) we see that:

$$\bar{\mathcal{K}} = J(E) = \operatorname{Ker}(p_1 \hat{\otimes} p_2 \hat{\otimes} \dots \hat{\otimes} p_n)$$

($p_j : R_j \to R_j/I_j$ the canonical mapping $j = 1, 2, \dots n$). From this it follows that $R/J(E)$ can be identified to $R_1/I_1 \hat{\otimes} R_2/I_2 \hat{\otimes} \dots \hat{\otimes} R_n/I_n$ which by what we have said above is a simisimple Banach algebra; thus $R/J(E)$ being semisimple it follows that $J(E) = I(E)$ and our theorem is proved.

2. Definition of the tensor algebras

§ 1. The functional analytic approach

Let $\mathcal{K} = \{K_j\}_{j=1}^n$ be compact topological spaces and let $K = K_1 \times K_2 \times \dots \times K_n$ and let us denote by $\{k_j \in K_j\}_{j=1}^n$ and $k = (k_1, k_2, \dots k_n)$ the generic points of $\{K_j\}_{j=1}^n$ and K respectively. We then denote by $\mathbf{C}(K_j)$ the *Banach algebra of complex continuous functions on the space K_j and

$$V = \mathbf{C}(K_1) \hat{\otimes} \mathbf{C}(K_2) \hat{\otimes} \dots \hat{\otimes} \mathbf{C}(K_n)$$

which is then also a commutative *Banach algebra, semisimple and regular with K as the maximal ideal space. We call V the tensor algebra over the spaces $\mathcal{K}=\{K_j\}_{j=1}^n$ or simply "a tensor algebra"; we denote it as $V(\mathcal{K})$ or less accurately simply $V(K)$ or even V when no confusion can arise.

Let now $\mathcal{K}'=\{K'_j\}_{j=1}^n$ be another family of compact spaces and $p=\{p_j: K_j \to K'_j\}_{j=1}^n$ a family of continuous mappings. The p_j induce then, by transposition, Banach algebra homomorphisms

$$\check{p}_j : \mathbf{C}(K'_j) \to \mathbf{C}(K_j)$$

and a homomorphism of the tensor algebras

$$\check{p} = \check{p}_1 \hat{\otimes} \check{p}_2 \hat{\otimes} \dots \hat{\otimes} \check{p}_n : V(\mathcal{K}') \to V(\mathcal{K}).$$

Now the vector space $T=\mathbf{C}(K_1)\otimes\mathbf{C}(K_2)\otimes\dots\otimes\mathbf{C}(K_n)$ can be identified to a dense subspace of $\mathbf{C}(K)$; indeed $\mathbf{C}(K)$ is no other but the completion of T for an appropriate norm on T, namely the injective $\otimes_\varepsilon$ norm on the tensor product so that with standard notations [1] we can write

$$\mathbf{C}(K) = \mathbf{C}(K_1) \hat{\hat{\otimes}} \mathbf{C}(K_2) \hat{\hat{\otimes}} \dots \hat{\hat{\otimes}} \mathbf{C}(K_n).$$

The projective norm $\otimes_\pi$ being bigger than the injective norm $\otimes_\varepsilon$ it follows that we have a canonical norm decreasing linear mapping

$$J = J(\mathcal{K}) : V(\mathcal{K}) \to \mathbf{C}(K)$$

which is also $(1-1)$ since the spaces $\mathbf{C}(K_j)$ satisfy the Banach approximation property [1]. Using then the expansion $\mathcal{E}$ of Ch. 1, § 1, we see that $f \in \operatorname{Im} J \subset \mathbf{C}(K)$ if and only if f admits an expansion $(\mathcal{E})$

$$f(k) = \sum_{j=1}^{\infty} f_j^{(1)}(k_1) f_j^{(2)}(k_2) \dots f_j^{(n)}(k_n) \quad \forall k=(k_1, k_2, \dots k_n) \in K$$

such that

$$\left. \begin{aligned} & f_j^{(i)} \in \mathbf{C}(K_i), \quad i=1,2,\dots n;\ j=1,2,\dots, \\ & T_\varepsilon = \sum_{j=1}^{\infty} \|f_j^{(1)}\|_\infty \|f_j^{(2)}\|_\infty \dots \|f_j^{(n)}\|_\infty < +\infty \end{aligned} \right\} \tag{2.1.1}$$

and that if $f=Jv$ for some $v \in V$ then

$$\|v\|_V = \inf_{\mathcal{E}} T_\varepsilon,$$

i.e. we obtain the norm of v in V as the inf of T_ε over all possible expansions $\mathcal{E}$ of $Jv=f$ as in (2.1.1).

It is also immediate that $J(\mathcal{K})$ can be identified to the Gelfand representation of the algebra $V=V(\mathcal{K})$ i.e. the maximal ideal space of V being identified to K we have $Jv=\hat{v}$ $\forall v\in V$. So the fact that J is $(1-1)$ is equivalent to the semisimplicity of V (cf. Ch. 1, § 5).

§ 2. Elementary definition of a tensor algebra

Here we shall give an elementary definition of a tensor algebra equivalent to the previous one but entirely independent from the general theory of topological tensor products. For the rest of the paper we shall feel free to use either of the two definitions. It will, however, be very easy for the reader to reconstruct any of our subsequent arguments starting from whichever definition he pleases, and in particular to read the rest of this paper ignoring completely the functional analytic language we use.

We denote again as in § 1 by $\mathcal{K}=\{K_j\}_{j=1}^n$ a family of n compact spaces,

$$K=K_1\times K_2\times \dots \times K_n$$

and we denote by $\{k_j\in K_j\}_{j=1}^n$ and $k=(k_1, k_2, \dots k_n)$ the generic points of $\{K_j\}_{j=1}^n$ and K respectively.

We then define $V=V(\mathcal{K})$ as the subspace of $\mathbf{C}(K)$ of those functions $f\in \mathbf{C}(K)$ that admit a decomposition $(\mathcal{E})$

$$f(k)=\sum_{j=1}^{\infty} f_j^{(1)}(k_1)\, f_j^{(2)}(k_2)\dots f_j^{(n)}(k_n) \quad \forall k=(k_1, k_2, \dots k_n)\in K$$

such that

$$\left.\begin{aligned} & f_j^{(i)}\in \mathbf{C}(K_i), \quad i=1,2,\dots n;\ j=1,2,\dots, \\ & T_{\mathcal{E}}=\sum_{j=1}^{\infty}\|f_j^{(1)}\|_\infty \|f_j^{(2)}\|_\infty \dots \|f_j^{(n)}\|_\infty < +\infty \end{aligned}\right\} \tag{2.2.1}$$

and then we norm the space V by setting for every $f\in V$

$$\|f\|_V=\inf_{\mathcal{E}} T_{\mathcal{E}}; \tag{2.2.2}$$

the inf being taken over all possible decompositions of f as in (2.2.1).

V then becomes an algebra of functions, and it is easy to verify that the norm $\|\ \|_V$ as defined in (2.2.2) is a complete norm i.e. that we obtain a Banach algebra. With this definition we must now verify that the spectrum of $V(\mathcal{K})$ can be identified to K, and that $V(\mathcal{K})$ is regular, $V(\mathcal{K})$ also inherits the *algebra structure of $\mathbf{C}(K)$ i.e. $f^*=\bar{f}$ the complex conjugate function. These verifications are easy. Also it is quite immediate that this definition is identical to the previous one of Ch. 2, § 1.

On the basis of this definition we can also define the homomorphism p induced by the mappings $\{p_j: K_j\to K_j'\}_{j=1}^n$. We leave this to the reader.

A point that I would like to make is that in previous publications on the subject both myself and other authors have used systematically the notation $V(G) = \mathbf{C}(G) \hat{\otimes} \mathbf{C}(G)$ (or even $B(G) = \mathbf{C}(G) \hat{\hat{\otimes}} \mathbf{C}(G)$) where G is a compact abelian group (or more accurately the underlying topological space of such a group). The notation $V(G)$ is very convenient for tensor algebras of only two factors and we shall use it when confusion can not arise. For tensor algebras with more than two factors it is of course quite inappropriate. A more accurate notation for $V(G)$ in our previous notations is $V(\mathcal{G})$ where $\mathcal{G} = \{G_1, G_2\}$ with G_1, G_2 two identical copies of G, or $V(G \times G)$. On the other hand, the notation $B(G)$ we shall definitely abandon.

3. The tensor algebra homomorphisms

§ 1. The space V′ and the mapping $\tilde{\omega}$

Let us suppose that $\mathcal{G} = \{G_j\}_{j=1}^n$ is a family of n compact abelian groups (possibly finite), or more accurately the underlying spaces of such groups, which we shall also denote by G_j. Let us denote by h_{G_j} the normalized Haar measures of these groups ($\|h_{G_j}\| = 1$) and $L^\infty(G_j)$ the L^∞ spaces formed with these measures, let finally $\pi_j: \mathbf{C}(G_j) \to L^\infty(G_j)$ be the isometric canonical embedding ($j = 1, 2, \ldots n$). We shall then consider

$$V'(\mathcal{G}) = L^\infty(G_1) \hat{\otimes} L^\infty(G_2) \hat{\otimes} \ldots \hat{\otimes} L^\infty(G_n)$$

which is a Banach space.

Just as in Ch. 2, § 1, we can identify $L^\infty(G_1) \otimes L^\infty(G_2) \otimes \ldots \otimes L^\infty(G_n)$ with a subspace of $L^\infty(G)$ where $G = G_1 \times G_2 \times \ldots \times G_n$, and the projective norm on $L^\infty(G_1) \otimes L^\infty(G_2) \otimes \ldots \otimes L^\infty(G_n)$ being larger than the norm of $L^\infty(G)$ we obtain a norm decreasing mapping

$$J' = J'(\mathcal{G}): V'(\mathcal{G}) \to L^\infty(G)$$

which is $(1-1)$ and which identifies $V'(\mathcal{G})$ to a subspace of $L^\infty(G)$. The fact that J' is $(1-1)$ is a consequence of the Banach approximation property which is valid for the spaces $L^\infty(G_j)$. Indeed for $x \in V'(\mathcal{G})$ $x \neq 0$ we can then find $F_j \in (L^\infty(G_j))'$ = the dual space of $L^\infty(G_j)$, such that $F_1 \otimes F_2 \otimes \ldots \otimes F_n(x) \neq 0$. But the unit ball of $L^1(G_j)$ being dense in the unit ball of $(L^\infty(G_j))'$ we can also suppose that $F_j \in L^1(G_j)$ so that $F_1 \otimes F_2 \otimes \ldots \otimes F_n \in L^1(G)$, and this proves the result.

Let us also observe that the space V' can be defined directly, independently from the general theory of the topological tensor product, in a manner analogous to the one used in Ch. 2, § 2, as a subspace of $L^\infty(G)$. Such a definition is based of course in the injection J' and the decomposition $(\mathcal{E})$ of Ch. 1, § 1. We leave the details to the reader.

We now define

$$\tilde{\omega} = \pi_1 \hat{\otimes} \pi_2 \hat{\otimes} \dots \hat{\otimes} \pi_n : V(\mathcal{G}) \to V'(\mathcal{G})$$

and we see that $\|\tilde{\omega}\| \leqslant \|\pi_1\| \, \|\pi_2\| \dots \|\pi_n\| \leqslant 1$ (cf. Ch. 1, § 1).

Let now H be some compact abelian group and let us, being consistent with our previous notations, denote by $\pi: \mathbf{C}(H) \to L^\infty(H)$ the canonical identification ($L^\infty(H) = L^\infty(H, h_H)$). Our next task is to show that π has an approximating inverse and to construct it explicitly.

Let $\{e_\alpha \in \mathbf{C}(H)\}_{\alpha \in A}$ be a directed family such that

$$e_\alpha \geqslant 0; \ \|e_\alpha\|_{L^1} = \int_H e_\alpha \, dh_H = 1, \qquad \operatorname{supp} e_\alpha \xrightarrow[\alpha \in A]{} 0_H. \tag{3.1.1}$$

The limit of the support being taken in the sense that for any Ω nhd. of 0_H there exists $\alpha_\Omega \in A$ such that

$$\alpha \geqslant \alpha_\Omega \Rightarrow \operatorname{supp} e_\alpha \subset \Omega.$$

$\{e_\alpha\}_{\alpha \in A}$ is then an approximating identity of $L^1(H)$ and when H is metrizable it can even be chosen as a sequence.

Let us then define:

$$\pi_\alpha : L^\infty(H) \to \mathbf{C}(H); \quad \pi_\alpha(f) = f * e_\alpha \quad \forall f \in L^\infty(H), \, \alpha \in A.$$

It is then immediate to verify that $\{\pi_\alpha\}_{\alpha \in A}$ is an approximating inverse of π.

Now returning to our mappings $\pi_j : \mathbf{C}(G_j) \to L^\infty(G_j)$ ($j = 1, 2, \dots n$) and to

$$\tilde{\omega} = \pi_1 \hat{\otimes} \pi_2 \hat{\otimes} \dots \hat{\otimes} \pi_n$$

we see that we can construct an approximating inverse of $\tilde{\omega}$ using the approximating inverses of the π_j's and (1.2.1). The approximating inverse $\{\tilde{\omega}_\alpha\}_{\alpha \in A}$ of $\tilde{\omega}$ that we obtain that way is realized as follows:

For n families $\{e_\alpha^{(j)} \in \mathbf{C}(G_j)\}_{\alpha \in A_j}$ ($j = 1, 2, \dots n$) satisfying (3.1.1) we define

$$\omega_\alpha : L^\infty(G) \to \mathbf{C}(G) \quad (G = G_1 \times G_2 \times \dots \times G_n)$$

$$\omega_\alpha(f) = f * h_\alpha; \ h_\alpha = e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \dots \otimes e_{\alpha_n}^{(n)}; \ f \in L^\infty(G),$$

where $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in A = A_1 \times A_2 \times \dots \times A_n$ with the product order; $\tilde{\omega}_\alpha$ is then defined by

$$J \circ \tilde{\omega}_\alpha = \omega_\alpha \circ J', \quad \alpha \in A. \tag{3.1.2}$$

Since $\tilde{\omega}$ has an approximating inverse it is an isometry.

§ 2. Tensor algebras on group spaces

Let $\mathcal{K}=\{K_j\}_{j=1}^n$ be a family of n metrizable compact spaces and let $\mathcal{G}=\{G_j\}_{j=1}^n$ be a family of n metrizable compact groups or more accurately the underlying topological spaces of these groups, and let further $p=\{p_j:K_j\to G_j\}_{j=1}^n$ be continuous mappings that are onto, they induce then as explained in Ch. 2, § 1, $\check{p}:V(\mathcal{G})\to V(\mathcal{K})$ an algebra homomorphism. We shall show in this paragraph that $\check{p}$ has an approximating local inverse (Definition 1.4.1) and therefore in particular it is isometric.

First of all we can define [4]

$$p_j^{-1}:G_j\to K_j,\quad j=1,2,\dots n,$$

Borel inverses of the p_j; i.e. mappings such that

(α) p_j^{-1} is a Borel function from G_j to K_j, $j=1,2,\dots n$.

(β) For each $g\in G_j$ we have $p_j\circ p_j^{-1}(g)=g$, $j=1,2,\dots n$.

The $\{p_j^{-1}\}_{j=1}^n$ induce then n mappings:

$$\bar{p}_j:\mathbf{C}(K_j)\to L^\infty(G_j);\ \bar{p}_j(f)=f\circ p_j^{-1};\quad \forall f\in\mathbf{C}(K_j),\quad j=1,2,\dots n,$$

where of course the L^∞ is taken with respect to the Haar measure of G_j. We observe now that the composed mappings:

$$\bar{p}_j\circ\check{p}_j=\pi_j:\mathbf{C}(G_j)\to L^\infty(G_j),\quad j=1,2,\dots n, \tag{3.2.1}$$

are no other than the canonical identifications of $\mathbf{C}(G_j)$ to subspaces of $L^\infty(G_j)$.

Let us now define

$$\bar{p}=\bar{p}_1\hat{\otimes}\bar{p}_2\hat{\otimes}\dots\hat{\otimes}\bar{p}_n:V(\mathcal{K})\to V'(\mathcal{G})$$

we deduce then from (3.2.1) that $\bar{p}\circ\check{p}=\tilde{\omega}$. Let then $\{\tilde{\omega}_\alpha\}_{\alpha\in A}$ be the approximating inverse constructed in (3.1.2), then it follows that $\{\check{p}_\alpha=\tilde{\omega}_\alpha\circ\bar{p}\}_{\alpha\in A}$ is an approximating inverse of $\check{p}$. It remains to show that $\{\check{p}_\alpha\}_{\alpha\in A}$ satisfies the conditions of Definition 1.4.1. Towards that we observe that $(\check{p})^\sim$ the transposed mapping of $\check{p}$ is none other than

$$(\check{p})^\sim=p=p_1\times p_2\times\dots\times p_n:K=K_1\times K_2\times\dots\times K_n\to G=G_1\times G_2\times\dots\times G_n.$$

It is also clear that for any $f\in V(\mathcal{K})$ the function $J'\circ\bar{p}(f)\in L^\infty(G)$ is zero (i.e. zero a.e.) outside $p[\operatorname{supp} J(f)]$. Therefore

$$\operatorname{supp}[\omega_\alpha\circ J'\circ\bar{p}(f)]\ \xrightarrow[\alpha\in A]{}\ p[\operatorname{supp} J(f)]$$

by the very definition of ω_α. But $\check{p}_\alpha=\tilde{\omega}_\alpha\circ\bar{p}$ therefore $J\circ p_\alpha=\omega_\alpha\circ J'\circ\bar{p}$ (cf. (3.1.2)) and from that our assertion follows (cf. end of Ch. 2, § 1).

§ 3. Some trivial cases

Case 1: The subspaces. Let $\mathcal{K}=\{K_j\}_{j=1}^n$ $\mathcal{K}'=\{K_j'\}_{j=1}^n$ be two families of compact spaces such that $K_j\subset K_j'$ $j=1,2,\dots n$ and let $p=\{p_j:K_j\to K_j'\}_{j=1}^n$ be the canonical injections. Then $\check{p}_j$ can be identified to the quotient mapping

$$\mathbf{C}(K_j')\to\mathbf{C}(K_j')/I(K_j)=\mathbf{C}(K_j);\ \ I(K_j)=I^{\mathbf{C}(K_j')}(K_j).$$

From that it follows that $\check{p}$ is onto, also for $f\in V(\mathcal{K}')$ we have $J\circ\check{p}(f)=J(f)\,|_K$ the restriction on $K=K_1\times K_2\times\dots\times K_n$. So $V(\mathcal{K})$ can be identified isometrically with the quotient algebra and $\check{p}$ with the quotient mapping

$$\check{p}:V(\mathcal{K}')\to V(\mathcal{K}')/I(K)=V(\mathcal{K}).$$

Case 2: Finite codimension. Here we shall prove a lemma which we shall need later. Let X, A, B be arbitrary compact spaces and let $\{x_j,x_j'\in X\}_{j=1}^p$ be $2p$ distinct points of X. We shall consider the subspaces

$$\Lambda_n=\{f\in\mathbf{C}(X\times A);\ f(x_j,\alpha)=f(x_j',\alpha)\ \forall\alpha\in A,\ 1\leqslant j\leqslant n\}\subset\mathbf{C}(X\times A),$$

$$M_n=\{f\in\mathbf{C}(X\times A)\hat{\otimes}\mathbf{C}(B);\ Jf(x_j,\alpha,\beta)=Jf(x_j',\alpha,\beta),\ \forall\alpha\in A,\beta\in B,1\leqslant j\leqslant n\}\subset\mathbf{C}(X\times A)\hat{\otimes}\mathbf{C}(B)$$

for all $1\leqslant n\leqslant p$. Let then

$$\lambda_n:\Lambda_n\to\mathbf{C}(X\times A);\ \mu_n=\lambda_n\hat{\otimes}(I\partial B)^{\vee}:\Lambda_n\hat{\otimes}\mathbf{C}(B)\to\mathbf{C}(X\times A)\hat{\otimes}\mathbf{C}(B).$$

It is then quite clear that $\operatorname{Im}(\mu_n)\subset M_n$ for $n=1,2,\dots p$. We shall in fact prove

LEMMA 3.3.1. $\operatorname{Im}\mu_n=M_n,\quad n=1,2,\dots p.$

Proof. We first prove the lemma for $n=1$. We fix $F(x)\in\mathbf{C}(X)$ such that $F(x_1)=0$, $F(x_1')=1$ but arbitrary otherwise, then every element $f\in\mathbf{C}(X\times A)$ admits a unique decomposition

$$f=f^{(1)}+f^{(2)};\ f^{(1)}\in\Lambda_1\quad f^{(2)}=F(x)\,\zeta_f(\alpha),\ \zeta=\zeta_f\in\mathbf{C}(A).\tag{3.3.1}$$

It suffices to set $\zeta_f(\alpha)=f(x_1',\alpha)-f(x_1,\alpha)$. Now let $f\in M_1$ then we can write

$$Jf(x,\alpha,\beta)=\sum_{j=1}^{\infty}g_j(x,\alpha)\,h_j(\beta),\ g_j\in\mathbf{C}(X\times A),\ h_j\in\mathbf{C}(B);\ \sum_{j=1}^{\infty}\|g_j\|_\infty\,\|h_j\|_\infty<+\infty$$

decomposing then each g_j as in (3.3.1) we see that

$$Jf(x,\alpha,\beta)-\sum_{j=1}^{\infty}g_j^{(1)}(x,\alpha)\,h_j(\beta)=F(x)\sum_{j=1}^{\infty}\zeta_{gj}(\alpha)\,h_j(\beta)\tag{3.3.2}$$

and substituting $x=x_1, x_1'$ in (3.3.2) and subtracting the two equations so obtained we obtain that

$$\sum_{j=1}^{\infty} \zeta_{0j}(\alpha)\, h_j(\beta)=0 \quad \forall \alpha \in A \quad \forall \beta \in B$$

and this together with (3.3.2) proves that $f \in \operatorname{Im} \mu_1$ which is our lemma for $n=1$.

Now for any n $(1 \leqslant n \leqslant p)$ there exists a unique compact space X_n such that we can identify Λ_n with $\mathbf{C}(X_n \times A)$, X_n is the space we obtain from X by identifying x_j with x_j' for $1 \leqslant j \leqslant n$. Using that fact we see our lemma for arbitrary n can be proved by induction on n. Indeed suppose it holds for $n=1, 2, \ldots r-1<p$ and let $f \in M_r \subset M_{r-1}$, the inductive hypothesis gives then that $f=\mu_{r-1}(f')$ for some $f' \in \Lambda_{r-1} \hat{\otimes} \mathbf{C}(B)=\mathbf{C}(X_{r-1} \times A) \hat{\otimes} \mathbf{C}(B)$ and since we must have $Jf'(x_r, \alpha, \beta)=Jf'(x_r', \alpha, \beta)$ $\forall \alpha, \beta$ the first part of the proof applied to f' gives our result.

§ 4. The mapping d: $D_\infty \to \mathbf{T}$

In this section we shall study a particular homomorphism of tensor algebras which will play a central role for the rest of the theory. This section will be independent of the heavy Borel cross-section theorem we used in § 2.

We fix some notations first. Let $\omega=1, 2, \ldots \aleph_0$ be either a positive integer or $\aleph_0$ the countable cardinal; for any space X we shall denote then by X^ω the cartesian product of X with itself ω-times; in this context we shall always denote $\aleph_0$ simply by ∞ so that we shall write X^∞ rather than $X^{\aleph_0}$. For any value of ω it is then clear that $(X^\infty)^\omega = X^{\aleph_0 \omega} = X^\infty$ since $\aleph_0 \omega = \aleph_0$, when in the future we say that we identify $(X^\infty)^\omega$ with X^∞ we shall mean that we identify the index set $\aleph_0 \omega$ with $\aleph_0$ in any fashion whatsoever, such an identification is then unique up to permutation of the index set of the product.

Let now $\mathbf{Z}(2)$ be the group of two elements, let $\mathbf{R}$ and $\mathbf{T}$ be the real line and the circle group ($\mathbf{T}=\mathbf{R}\,(\operatorname{mod} 2\pi)$) and let us denote by $D_\infty=(\mathbf{Z}(2))^\infty$; we have then for any ω, $D_\infty^\omega = D_\infty$. Let us finally denote by $\mathbf{I}$ the unit interval $[0, 1]$ and let

$$p: \mathbf{R} \to \mathbf{T};\; p(r)=\exp(2\pi i r) \in \mathbf{T} \quad \forall r \in \mathbf{R}$$

the exponential mapping, and also:

$$\delta: D_\infty \to \mathbf{I};\; \delta(\alpha)=\sum_{j=1}^{\infty} \frac{\alpha_j}{2^j}, \quad \forall \alpha=(\alpha_1, \alpha_2, \ldots) \in D_\infty$$

the canonical mapping induced by the binary expansion of the real numbers $r \in \mathbf{I}$, and let us denote finally by $d=p \circ \delta: D_\infty \to \mathbf{T}$ the composed mapping.

We see that δ and d are onto and such that if we remove from D_∞ a countable set they become $(1-1)$. From this it follows that the Borel inverse

$$d^{-1}: \mathbf{T} \to D_\infty; \quad d \circ d^{-1} = I\partial\mathbf{T}, \quad d^{-1} \text{ Borel mapping}$$

can be explicitly constructed and is well defined up to a countable set of points. Also we see that d identifies the Haar measure of D_∞ to the Haar measure of $\mathbf{T}$. For any ω, d then induces

$$d_\omega: D_\infty(=D_\infty^\omega) \to \mathbf{T}^\omega$$

the cartesian product mapping $d \times d \times \ldots \times d$ after identification of D_∞ with $D_\infty^\omega = (D_\infty)^\omega$ as explained above. d_ω is then defined only up to possible permutation of the index set of the product $[\mathbf{Z}(2)]^{\aleph_0}$ which defines D_∞; the properties of d_ω which we shall use, however, will be independent of such a permutation, and thus the order of the index set can be fixed in any fashion whatsoever.

It follows now that d_ω is onto, and that its Borel inverse d_ω^{-1} can be constructed by taking the cartesian product ω times of d^{-1}, also d_ω is $(1-1)$ if we remove from D_∞ a set of Haar measure zero and it identifies the Haar measure of D_∞ to the Haar measure of $\mathbf{T}^\omega$.

For any $n \geq 1$ we consider now $\mathcal{D}_n = \{D_\infty^{(j)}\}_{j=1}^n$, $\mathcal{T}_n = \{\mathbf{T}_j^\omega\}_{j=1}^n$ the families that consist of n identical copies of D_∞ and $\mathbf{T}^\omega$ respectively and $\partial_\omega^n = \{d_\omega: D_\infty^{(j)} \to \mathbf{T}_j^\omega\}_{j=1}^n$ the family of n mappings identical to d_ω.

It then follows from Ch. 3, § 2, that the tensor algebra homomorphism induced by ∂_ω^n

$$\check{\partial}_\omega^n: V(\mathcal{T}_n) = \mathbf{C}(\mathbf{T}^\omega) \hat{\otimes} \ldots \hat{\otimes} \mathbf{C}(\mathbf{T}^\omega) \to V(\mathcal{D}_n) = \mathbf{C}(D_\infty) \hat{\otimes} \ldots \hat{\otimes} \mathbf{C}(D_\infty)$$

is isometric and has a local approximating inverse. Observe that here since we know explicitly the Borel inverse of d_ω the proof of these facts becomes much simpler; the reader is advised to reconstruct the proof directly observing that using d_ω we can in fact identify $L^\infty(D_\infty)$ and $L^\infty(\mathbf{T}^\omega)$. For further reference we denote by

$$d_\omega^n = d_\omega \times d_\omega \times \ldots \times d_\omega: D_\infty \times D_\infty \times \ldots \times D_\infty \to \mathbf{T}^\omega \times \mathbf{T}^\omega \times \ldots \times \mathbf{T}^\omega$$

$$\check{d}_\omega^n: \mathbf{C}(\mathbf{T}^\omega \times \mathbf{T}^\omega \times \ldots \times \mathbf{T}^\omega) \to \mathbf{C}(D_\infty \times D_\infty \times \ldots \times D_\infty)$$

the cartesian product of d_ω n times and its transposed mapping.

§ 5. The subalgebra $V(\mathcal{T}_n) \subset V(\mathcal{D}_n)$

In this paragraph we shall study further the mapping $d: D_\infty \to \mathbf{T}$ and the induced algebra homomorphism.

Let us denote by $\Delta = \{p/2^s; p \in \mathbf{Z}; s \in \mathbf{Z}\}$ the set of diadic rationals of $\mathbf{R}$ and $\Gamma = p(\Delta) = (t_1, t_2, \ldots t_n, \ldots)$ their image on $\mathbf{T}$ denumerated in any manner whatsoever.

Then we have

$$\operatorname{Card}(d^{-1}(t))=1 \quad t\in \mathbf{T}\setminus\Gamma;\ d^{-1}(t_n)=\{t_n^+;\,t_n^-\}\in D_\infty \quad \forall t_n\in\Gamma.$$

Let us then denote for any n

$$C_n=\{f\in \mathbf{C}(D_\infty);\ f(t_\nu^+)=f(t_\nu^-),\quad \nu=1,2,\dots n\};$$

it follows that we have a nested sequence of closed subspaces

$$\mathbf{C}(D_\infty)=C_0\supset C_1\supset\dots\supset\bigcap_{n=1}^{\infty}C_n=C_\infty=C(\mathbf{T}). \tag{3.5.1}$$

We shall prove that these subspaces satisfy the condition (*) of Ch. 1, § 2.

Towards that we first construct a double entry sequence $\mathfrak{J}=\{I_{m,n}\}_{m,n=1}^{\infty}$ of closed arcs (intervals) of $\mathbf{T}$ of length $\leqslant\pi/2$ that satisfy the following conditions:

(α) The center of $I_{m,n}$ is t_n, and its end points $\notin\Gamma\ \forall n, m$

(β) $\forall$ fixed n $I_{1,n}\supsetneqq I_{2,n}\supsetneqq\dots\supsetneqq I_{m,n}\supsetneqq\dots\supset\bigcap_m I_{m,n}=\{t_n\}$

(γ) Two distinct arcs I_{α_1,α_2} and $I_{\beta_1,\beta_2}\in\mathfrak{J}$ of our sequence are either disjoint or one is strictly contained in the other.

Such a double sequence $\mathfrak{J}$ can easily be constructed in successive steps of n (inductively on n) i.e. we first construct $\{I_{m,1}\}_{m=1}^{\infty}$ satisfying (α) and (β), then $\{I_{m,2}\}_{m=1}^{\infty}$ satisfying (α) and (β) and also (γ) with the already constructed sequence ect.

Using now the family $\mathfrak{J}$ we can define for every $n\geqslant 1$

$$i_m^{(n)}: C_{n-1}\to C_n,\quad m=1,2,\dots,$$

by defining for every $f\in\mathbf{C}(D_\infty)$ a new function $i_m^{(n)}(f)\in\mathbf{C}(D_\infty)$ from the conditions:

(γ) $[i_m^{(n)}(f)](\alpha)=f(\alpha)$ if $d(\alpha)\notin I_{m,n}$, $\alpha\in D_\infty$.

(δ) If $d(\alpha)\in I_{m,n}$, $[i_m^{(n)}(f)](\alpha)=\lambda(d(\alpha))$ where $\lambda\in\mathbf{C}(I_{m,n})$ and is such that $\lambda\circ p$ is a linear function on any closed interval of the set $p^{-1}(I_{m,n})\subset\mathbf{R}$.

It is then easy to see that $\{i_m^{(n)}\}_{m=1}^{\infty}$ is the required approximating inverse of the canonical injection

$$i^{(n)}: C_n\to C_{n-1}$$

and that it satisfies the condition (*) of Ch. 1, § 2, for all $n\geqslant 1$.

Let now A, B be two arbitrary compact spaces and let us define for each $n\geqslant 1$

$$\Lambda_n=\{f\in\mathbf{C}(D_\infty\times A);\quad f(t_\nu^+,\alpha)=f(t_\nu^-,\alpha)\quad\forall\alpha\in A,\quad \nu=1,2,\dots n\} \tag{3.5.2}$$

We have then again a sequence

$$\mathbf{C}(D_\infty \times A) = \Lambda_0 \supset \Lambda_1 \supset \dots \bigcap_{n=1}^{\infty} \Lambda_n = \mathbf{C}(\mathbf{T} \times A) \tag{3.5.3}$$

It follows then from the above considerations that the sequence (3.5.3) also satisfies the condition (*) of Ch. 1, § 2. This can either be verified directly on the line of our proof for the sequence (3.5.1), or it can be deduced from the above as follows. For each $n \geqslant 1$ there exists X_n a compact space such that we can identify Λ_n with $\mathbf{C}(X_n \times A)$ and C_n with $\mathbf{C}(X_n)$, we can take as X_n the space D_∞ after identification of t_ν^+ with t_ν^- for $\nu = 1, 2, \dots n$. Then $\Lambda_n = C_n \hat{\hat{\otimes}} \mathbf{C}(A)$ and we can set as the approximating inverse of the injection

$$\Lambda_n \to \Lambda_{n-1}$$

the family $\{i_m^{(n)} \hat{\hat{\otimes}} I\partial \mathbf{C}(A)\}_{m=1}^{\infty}$ for $n = 1, 2, \dots$ [$T_1 \hat{\hat{\otimes}} T_2$ for two mappings $T_j : A_j \to B_j$ $(j = 1, 2)$ is the mapping $T_1 \otimes T_2$ extended to the completion $A_1 \hat{\hat{\otimes}} A_2$]. We leave the verification of (*) Ch. 1, § 2, to the reader. We can now prove

THEOREM 3.5.1. *Let A, B be two compact spaces, then in the diagram:*

$$\begin{array}{ccc} \mathbf{C}(\mathbf{T} \times A) \hat{\otimes} \mathbf{C}(B) & \xrightarrow[(d \times I\partial A)^\vee \hat{\otimes} (I\partial B)^\vee = \varphi]{} & \mathbf{C}(D_\infty \times A) \hat{\otimes} \mathbf{C}(B) \\ \downarrow & & \downarrow J \\ \mathbf{C}(\mathbf{T} \times A \times B) & \xrightarrow[(d \times I\partial A \times I\partial B)^\vee = \theta]{} & \mathbf{C}(D_\infty \times A \times B) \end{array}$$

We have $\operatorname{Im}[J \circ \varphi] = \operatorname{Im} \theta \cap \operatorname{Im} J$.

Proof. Indeed let $x \in \mathbf{C}(D_\infty \times A) \hat{\otimes} \mathbf{C}(B)$ be such that $Jx \in \operatorname{Im} \theta$. Then using Ch. 3, § 3, Case 2, and our previous notations [(3.5.2) and (3.5.3)] we see that $x \in \Lambda_n \hat{\otimes} \mathbf{C}(B)$ for all $n \geqslant 1$ and since the sequence (3.5.3) satisfies (*) of Ch. 1, § 2, it follows that $x \in \mathbf{C}(\mathbf{T} \times A) \hat{\otimes} \mathbf{C}(B)$ [or more accurately that there exists some $x' \in \mathbf{C}(\mathbf{T} \times A) \hat{\otimes} \mathbf{C}(B)$ such that $x = \varphi x'$]. This proves our theorem.

Now a successive application of Theorem 3.5.1 for different spaces A and B yields the following

THEOREM 3.5.2. *For any $n \geqslant 1$ and any compact space A in the diagram*

$$\Delta_n(A) \left\{ \begin{array}{ccc} \mathbf{C}(\mathbf{T}^n) \hat{\otimes} \mathbf{C}(A) & \xrightarrow[\check{d}_n \hat{\otimes} (I\partial A)^\vee = \varphi_n]{} & \mathbf{C}(D_\infty) \hat{\otimes} \mathbf{C}(A) = V_n(A) = V_n \\ & & \downarrow J \\ \mathbf{C}(\mathbf{T}^n \times A) & \xrightarrow[(d_n \times I\partial A)^\vee = \theta_n]{} & \mathbf{C}(D_\infty \times A) = W_n(A) = W_n \end{array} \right.$$

we have $\operatorname{Im}[J \circ \varphi_n] = \operatorname{Im} J \cap \operatorname{Im} \theta_n$.

Let us now for the purpose of the proof of our next theorem introduce the following notation, for any compact group K and any $n \geqslant 1$ let us denote

$$K^{\infty} \xrightarrow[\xi_n = \xi_n(K)]{} K^n \xrightarrow[\zeta_n = \zeta_n(K)]{} K^{\infty}, \quad \zeta_n \circ \xi_n = \eta_n, \tag{3.5.4}$$

$$(k_1, k_2, \dots k_n \dots) \xrightarrow[\xi_n]{} (k_1, k_2, \dots k_n) \xrightarrow[\zeta_n]{} (k_1, k_2, \dots k_n, 0_K, 0_K, \dots).$$

The canonical projection, and injection of the space of the first n coordinates of the infinite product. Observe then that $\check{\eta}_n \xrightarrow[n \to \infty]{} I\partial \mathbf{C}(K^{\infty})$ for the strong operator topology ($\check{\eta}_n$ is the transposed mapping). We can now prove:

THEOREM 3.5.3. *For any compact space A in the diagram*

$$(\Delta_\infty) \left\{ \begin{array}{ccc} \mathbf{C}(\mathbf{T}^{\infty}) \hat{\otimes} \mathbf{C}(A) & \xrightarrow[\check{d}_\infty \hat{\otimes} (I\partial A)^{\vee} = \varphi]{} & \mathbf{C}(D_\infty) \hat{\otimes} \mathbf{C}(A) = V \\ & & \big\downarrow J \\ \mathbf{C}(\mathbf{T}^{\infty} \times A) & \xrightarrow[(d_\infty \times I\partial A)^{\vee} = \theta]{} & \mathbf{C}(D_\infty \times A) = W \end{array} \right.$$

we have $\operatorname{Im}[J \circ \varphi] = \operatorname{Im} J \cap \operatorname{Im} \theta$.

Proof. Together with the diagram (Δ_∞) let us for every $n \geqslant 1$ consider the diagram $\Delta_n(A) = \Delta_n$ of Theorem 3.5.2 with the same space A.

Now the mapping ζ_n applied to $\mathbf{T}^\infty$ and $(D_\infty)^\infty = D_\infty (K = \mathbf{T}$ or D_∞, $\zeta_n(\mathbf{T}): \mathbf{T}^n \to \mathbf{T}^\infty$, $\zeta_n(D_\infty): (D_\infty)^n = D_\infty \to (D_\infty)^\infty = D_\infty)$ induces by transposition a mapping from the diagram (Δ_∞) to the diagram (Δ_n) (i.e. from the spaces of the diagram Δ_∞ to the spaces of the diagram Δ_n) and conversely the mapping ξ_n for $K = \mathbf{T}$ and D_∞ induces a mapping back from (Δ_n) to (Δ_∞).

So now let $x \in \mathbf{C}(D_\infty) \hat{\otimes} \mathbf{C}(A) = V$ be such that $Jx \in \operatorname{Im} \theta \subset W$ then

$$x_n = [\check{\zeta}_n(D_\infty) \hat{\otimes} (I\partial A)^{\vee}](x) \in V_n \quad \text{and} \quad Jx_n \in \operatorname{Im} \theta_n \subset W_n,$$

thus an application of Theorem (3.5.2) to the diagram (Δ_n) gives that $x_n \in \operatorname{Im} \varphi_n$ thus

$$\tilde{x}_n = [\check{\xi}_n(D_\infty) \hat{\otimes} (I\partial A)^{\vee}](x_n) = [\check{\eta}_n(D_\infty) \hat{\otimes} (I\partial A)^{\vee}](x) \in \operatorname{Im} \varphi.$$

But from our previous remark $\tilde{x}_n \to x$, so, $\operatorname{Im} \varphi$ being closed in $\mathbf{C}(D_\infty) \hat{\otimes} \mathbf{C}(A)$ (cf. Ch. 3, § 4), our result follows.

Applying Theorem (3.5.3) twice we obtain

THEOREM 3.5.4. *For any $\omega = 1, 2, \ldots \infty$; in the diagram*

$$\begin{array}{ccc} \mathbf{C}(\mathbf{T}^{\omega})\hat{\otimes}\mathbf{C}(\mathbf{T}^{\omega}) & \xrightarrow[\check{\partial}^2_{\omega}]{} & \mathbf{C}(D_{\infty})\hat{\otimes}\mathbf{C}(D_{\infty}) \\ & & \downarrow J \\ \mathbf{C}(\mathbf{T}^{\omega}\times\mathbf{T}^{\omega}) & \xrightarrow[\check{d}^2_{\omega}]{} & \mathbf{C}(D_{\infty}\times D_{\infty}) \end{array}$$

we have $\operatorname{Im}[J\circ\check{\partial}^2_{\omega}] = \operatorname{Im} J \cap \operatorname{Im}\check{d}^2_{\omega}$.

We have, for simplicity, in all the preceding considerations restricted our attention to the tensor product of only two factors, but of course Theorem 3.5.4, for instance, can be generalized to any number of factors, using a simple inductive technique. We shall, however, have no use for theorems of the above type for more than two factors.

4. The embedding of a tensor algebra in a group algebra

§ 1. Definitions and classical results

Let G be a locally compact abelian group and let $\hat{G}$ be its character group. We shall introduce here a number of definitions and well-known propositions.

(i) $K \subset G$ a compact set is called a Kronecker set if for any $f \in \mathbf{C}(K)$ with $|f| \equiv 1$ and $\varepsilon > 0$ we can find $\chi \in \hat{G}$ such that $\sup_{k\in K}|f(k) - \chi(k)| < \varepsilon$

(ii) $K \subset G$ a compact subset is called a K_p set for $p \geq 2$ some natural prime if:

$$\{\chi|_K;\, \chi \in \hat{G}\} = \{f \in \mathbf{C}(K);\, f^p \equiv 1\}$$

i.e. the restrictions of the characters on K coincides with all the $\mathbf{Z}(p)(\subset \mathbf{C}(\mathbf{T}))$ valued continuous functions on K.

(iii) A set $K \subset G$ that is either a Kronecker or a K_p set for some prime p will be called a $\mathcal{K}$-set of G.

(iv) A subset $K \subset G$ is called independent (resp. p-independent for p some natural prime) if for any choice of $\{k_j \in K\}_{j=1}^J$ and $\{n_j \in \mathbf{Z}\}_{j=1}^J$ we have

$$\sum_{j=1}^{J} n_j k_j = 0_G \Leftrightarrow n_j = 0 \quad (\text{resp. } n_j \equiv 0 \ (\operatorname{mod} p)), \quad j = 1, 2, \ldots, J.$$

(v) It is well known and trivial that a Kronecker set of G is independent and a K_p subset of G is p-independent; and that for all $g \in K =$ a K_p set of G we have $\operatorname{ord} g = p$.

(vi) Let G be an arbitrary locally compact non discrete abelian group, then there exists K a Cantor subset of G which is either a Kronecker set or a K_p set for some prime.

Note: Cantor set means that it is perfect metrizable and totally disconnected i.e. topologically homeomorphic to D_{∞}.

Now quite generally for E any closed subset of G a locally compact group, we denote by $A(E)\subsetneq \mathbf{C}(E)$ the algebra of restrictions of functions $f\in A(G)$ on the set E. $A(E)$ can then be identified to $A(G)/I(E)$ and as such is assigned canonically with a norm and a *Banach algebra structure. It will always be considered as a Banach algebra with the above canonical quotient norm.

(vii) We say that $E\subset G$ a compact subset of the locally compact group G is a Helson set if $A(E)=\mathbf{C}(E)$.

(viii) Let G be a compact group and K a Kronecker set of G then $A(K)=\mathbf{C}(K)$ isometrically i.e. $\|\ \|_{A(K)}=\|\ \|_{\mathbf{C}(K)}$.

(ix) Let G be a compact group and K a K_p set for some prime p, then $A(K)=\mathbf{C}(K)$ and $\|f\|_{A(K)}\leqslant k_p\|f\|_{\mathbf{C}(K)}$ for all $f\in\mathbf{C}(K)$, where $1\leqslant k_p\leqslant 2$.

For proofs and comments on the above definitions we refer the reader to the standard literature on abstract harmonic analysis e.g. [5].

The rest of this chapter, and indeed the whole motivation of tensor algebras, rests on the following simple observation.

Let G_1, G_2 be two compact groups and $E_1\subset G_1$, $E_2\subset G_2$ two compact Helson subsets i.e. $A(E_j)=\mathbf{C}(E_j)$ $(j=1, 2)$. Let also $E=E_1\times E_2\subset G_1\times G_2$ then we have

$$A(E)=A(E_1)\hat{\otimes}A(E_2)=\mathbf{C}(E_1)\hat{\otimes}\mathbf{C}(E_2).$$

In other words $A(E)$ is a tensor algebra. We leave the verification of this to the reader (observe that $A(G)=A(G_1)\hat{\otimes}A(G_2)$ Chapter 1, § 5) since in the next paragraph we shall examine in detail a much more general case of the above phenomenon.

§ 2. The basic embedding theorem

Let $H_1, H_2, \dots, H_n$ be arbitrary compact subsets of the compact abelian group G, and let us denote by

$$s: H=H_1\times H_2\times\dots\times H_n\to\tilde{H}=H_1+H_2+\dots+H_n\subset G \tag{4.2.1}$$

$$s(h_1, h_2, \dots, h_n)=h_1+h_2+\dots+h_n$$

the group addition mapping, and let us also make the technical hypothesis:

($\mathcal{H}$): The algebra $\mathcal{A}(H)=A(H_1)\hat{\otimes}A(H_2)\hat{\otimes}\dots\hat{\otimes}A(H_n)$ is semisimple.

We have already observed [Chapter 1, § 5] that ($\mathcal{H}$) is satisfied in particular when the Banach spaces $A(H_j)$ $(j=1, 2, \dots, n)$ satisfy the Banach approximation property, so ($\mathcal{H}$)

is certainly verified when $A(H_j)=\mathbf{C}(H_j)$ which is the only case we shall need for applications.

When the hypothesis $(\mathcal{H})$ is verified we can identify $\mathcal{A}(H)$ to an algebra of functions on H the carrier space of $\mathcal{A}$. So under the hypothesis $(\mathcal{H})$ we can define a linear norm decreasing mapping:

$$\lambda: A(G) \to \mathcal{A}(H)$$

by defining for all $\chi \in \hat{G}$

$$\lambda(\chi) = \chi(h_1)\chi(h_2) \dots \chi(h_n), \quad h_j \in H_j, \ j=1, 2, \dots, n,$$

and extending by linearity. It is then clear that with the identification $\mathcal{A}(H) \subset \mathbf{C}(H)$ we have for all $f \in A(G)$ $(f|_{\tilde{H}}) \circ s = \lambda(f)$ which shows that $\lambda(I(\tilde{H}))$ and that λ induces

$$\dot{\lambda}: A(\tilde{H}) \to \mathcal{A}(H)$$

a norm decreasing $(1-1)$ algebraic homomorphism which, of course, is no other than the one defined by $(\dot{\lambda})^{\check{}} = s$ [cf. Chapter 1, § 4].

We shall now study conditions under which

$$\dot{\lambda} \text{ is isometric and onto} \tag{4.2.2}$$

and it identifies the two algebras $\mathcal{A}(H)$ and $A(\tilde{H})$. Towards that let $K_1, K_2, \dots K_n \subset G$ be disjoined compact sets such that $K^* = K_1 \cup K_2 \cup \dots \cup K_n$ is a $\mathcal{K}$-set of G, let also $r_1, r_2, \dots, r_n$ be positive integers and let us suppose that the sets $\{H_j\}_{j=1}^n$ satisfy

$$H_j \subset \left\{ \sum_{\alpha=1}^{r_j} \varepsilon_\alpha k_\alpha; \ \varepsilon_\alpha = \pm 1, k_\alpha \in K_j \right\}. \tag{4.2.3}$$

We claim then that (4.2.2) is satisfied.

Indeed let us introduce the notation

$$f^{(j)} = (\mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes f \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}) \in \mathcal{A}(H); \ \forall f \in A(H_j), \ j=1, 2, \dots n,$$

where $\mathbf{1}$ is the function identically equal to 1 and f is placed on the jth place of the product; and let us also denote by B and $\mathcal{B}$ the unit balls of the Banach spaces $A(\tilde{H})$ and $\mathcal{A}(H)$ respectively.

To show now that $\dot{\lambda}$ is isometric and onto it suffices to show:

$$\overline{\dot{\lambda}(B)} = \mathcal{B}; \tag{4.2.4}$$

where the bar indicates of course the topological closure in $\mathcal{A}(H)$.

Taking now into account the way the elements of the form $f^{(j)}$ $(f \in A(H_j); \ j=1, 2, \dots n)$

generate the algebra $A(H)$ we see that (4.2.4) will follow if we show that for any $\varepsilon > 0$ any $j = 1, 2, \dots n$ and any $f \in A(H_j)$ there exists $f_\varepsilon^{(j)} \in A(G)$ such that

$$\|f_\varepsilon^{(j)}\|_{A(G)} \leqslant \|f\|_{A(H_j)} + \varepsilon; \quad \|f^{(j)} - \lambda(f_\varepsilon^{(j)})\|_{A(H)} \leqslant \varepsilon. \tag{4.2.5}$$

Towards that let $f_\varepsilon \in A(G)$ be such that $\|f_\varepsilon\|_{A(G)} < \|f\|_{A(H_j)} + \varepsilon \quad f_\varepsilon|_{H_j} = f$ then

$$f_\varepsilon = \sum_{\chi \in \hat{G}} \alpha_\chi \chi; \quad \sum_{\chi \in \hat{G}} |\alpha_\chi| = \|f_\varepsilon\|_{A(G)}.$$

Let us then form $\tilde{f}_{\varepsilon,\eta} = \sum_{\chi \in \hat{G}} \alpha_\chi \chi_\eta$ where $\chi_\eta \in \hat{G}$ is arbitrary subject only to the conditions

$$\sup_{k \in K_j} |\chi(k) - \chi_\eta(k)| \leqslant \eta; \quad \sup_{k \in K_i} |1 - \chi_\eta(k)| \leqslant \eta, \quad i \neq j;$$

we can always choose a χ_η for an arbitrary χ and η by the hypothesis on K^*. It is then quite clear from (4.2.3) that for a small enough η if we set $f_\varepsilon^{(j)} = \tilde{f}_{\varepsilon,\eta}$ our condition (4.2.5) is satisfied (observe that for any $E = \bar{E} \subset G$ and $\varepsilon > 0$ these exists $\eta > 0$ such that for all $\chi, \psi \in \hat{G}$ we have $\|\chi|_E - \psi|_E\|_\infty < \eta \Rightarrow \|\chi|_E - \psi|_E\|_{A(E)} < \varepsilon$ (cf. (5.1.4) and [12], (1)).) We have in fact proved the fundamental

THEOREM 4.2.1. *Let G be any compact abelian group and $K_1, K_2, \dots, K_n \subset G$ compact disjoined subsets such that $K_1 \cup K_2 \cup \dots \cup K_n$ is a $\mathcal{K}$-set of G; let $r_1, r_2, \dots, r_n$ be positive integers and $H_j \subset r_j(K_j - K_j)$ arbitrary compact subsets. Then we can identify canonically and isometrically the algebra $A(H_1) \hat{\otimes} \dots \hat{\otimes} A(H_n)$ with the algebra $A(H_1 + H_2 + \dots + H_n)$ provided that the hypothesis $(\mathcal{H})$ holds.*

We can now deduce a series of important corollaries.

THEOREM 4.2.2. *Let G be a compact abelian group and let $\mathcal{K} = \{K_j \subset G\}_{j=1}^n$ be compact disjoint subsets of G and let us denote*

$$K^* = \bigcup_{j=1}^n K_j; \quad \tilde{K} = \sum_{j=1}^n K_j = \left\{ \sum_{j=1}^n k_j;\ k_j \in K_j,\ j = 1, 2, \dots, n \right\} \subset G.$$

Then:

(α) *If K^* is a Kronecker set of G the algebra $A = A(\tilde{K})$ can be identified canonically and isometrically with the algebra $V(\mathcal{K})$.*

(β) *If K^* is a K_p set of G the algebra $A = A(\tilde{K})$ can be identified canonically and topologically with the algebra $V = V(\mathcal{K})$ so that we have:*

$$2^n \|\ \|_V \geqslant \|\ \|_A \geqslant \|\ \|_V$$

for the A-norm and the V-norm of two identified elements.

It suffices to apply Theorem 4.2.1 and [Chapter 4, § 1, (viii), (ix)].

THEOREM 4.2.3. *For any infinite compact abelian group G there exists E a compact subset such that we can identify topologically and algebraically $A(E)$ to the algebra $V(D_\infty)$.*

Indeed it suffices to observe that if $K\subset G$ is a $\mathcal{K}$-Cantor set of G and if we split it into two disjoint relatively open subsets K_1, K_2 and set $E=K_1+K_2$ we have

$$A(E)\cong \mathbf{C}(K_1)\hat{\otimes}\mathbf{C}(K_2)\cong \mathbf{C}(D_\infty)\hat{\otimes}\mathbf{C}(D_\infty)=V(D_\infty).$$

Then any identification of K_1 with D_∞ and of K_2 with D_∞ fixes the identification of $A(E)$ with $V(D_\infty)$ and the transposed identification of $E=K_1+K_2$ with $D_\infty\times D_\infty$.

From that identification of $E=K_1+K_2$ with $D_\infty\times D_\infty$ it follows at once that if $\tilde{D}\subset D_\infty\times D_\infty$ is a compact subset which is not of spectral synthesis for $V(D_\infty)$ it corresponds to $\tilde{E}\subset E\subset G$ a subset which is not of spectral synthesis for $A(E)$ and thus a fortiori is not a set of spectral synthesis for $A(G)$, i.e. is not a set of spectral synthesis of the group G.

§ 3. The embedding in $P+Q$

In this paragraph we shall give a slightly less canonical condition under which we can identify $A(H)$ and $A(\tilde{H})$. (We preserve all the notations of the previous paragraph.) To simplify our writing we shall suppose that $n=2$. Let us introduce the following

Definition 4.3.1. Let $H_1, H_2\subset G$ be two compact disjoined subsets of the compact group G; and let m_1, m_2 be two elements of the set $(2, 3, ..., n, ..., \infty)$ where ∞ is a "new" symbol. We shall then say that $f\in\mathbf{C}(H_1\cup H_2)$ is an $\{m_1, m_2\}$-function if we have $f^{m_i}(h_i)=1$ $\forall h_i\in H_i$ $(i=1,2)$, where we interpret $f^\infty(x)=|f(x)|$ conventionally. We shall also say that the pair $\{H_1, H_2\}$ is an $\{m_1, m_2\}$ pair if for every $f\in\mathbf{C}(H_1\cup H_2)$ $\{m_1, m_2\}$-function and every $\varepsilon>0$ we can find some $\chi\in\hat{G}$ such that $|f(h)-\chi(h)|\leqslant\varepsilon$ for all $h\in H_1\cup H_2$.

We shall now prove (with the notations of the preceding paragraph)

THEOREM 4.3.1. *If $n=2$ and $\{H_1, H_2\}$ is an $\{m_1, m_2\}$ pair for some m_1 and m_2 then $\dot{\lambda}$ is onto and identifies $A(H_1+H_2)$ with $\mathbf{C}(H_1)\hat{\otimes}\mathbf{C}(H_2)$ topologically.*

Proof. Indeed it is well known that under our hypothesis $A(H_i)=\mathbf{C}(H_i)$ $(i=1, 2)$ [5]. Also it is clear by definition 4.3.1 that the mapping s (4.2.1) is $(1-1)$ and that it identifies $H=H_1\times H_2$ with $\tilde{H}=H_1+H_2$.

To prove the theorem we first observe that for any $f_i\in\mathbf{C}(H_i)$ such that $f^{m_i}(h_i)=1$ $\forall h_i\in H_i$ $(i=1,2)$ (with the same convention $f^\infty(x)=|f(x)|$ as in definition 4.3.1), and any $\varepsilon>0$ there exists $\psi\in A(\tilde{H})$ such that

$$\|f_1\hat{\otimes}f_2-\dot{\lambda}(\psi)\|_A\leqslant\varepsilon,\quad \|\psi\|_{A(\tilde{H})}\leqslant 1.$$

Indeed by our definition of an $\{m_1, m_2\}$ pair we can set $\psi = \chi|_{\tilde{H}}$ for some $\chi \in \hat{G}$. The theorem then follows upon observing that for any $f \in C(H_1) \hat{\otimes} C(H_2) = A(H)$ we have an expansion:

$$f = \sum_{\alpha=1}^{\infty} \lambda_\alpha f_1^{(\alpha)} \otimes f_2^{(\alpha)}; \quad \sum_{\alpha=1}^{\infty} |\lambda_\alpha| \leq 4 \|f\|_{A(H)}, \quad (f_i^{(\alpha)}(h_i))^{m_i} = 1, \quad i = 1, 2, \ \alpha = 1, 2, \ldots.$$

Relative to $\{m_1, m_2\}$ pairs we shall prove the following

THEOREM 4.3.2. *Let $P_1, P_2 \subset G$ be two arbitrary perfect subsets of the compact metrizable abelian group G, then we can find two Cantor sets H_1, H_2 and $g_1, g_2 \in G$ two points such that*

$$H_1 \subset g_1 + P_1; \ H_2 \subset g_2 + P_2$$

and such that $\{H_1, H_2\}$ is an $\{m_1, m_2\}$ pair for some m_1 and m_2.

Proof. For $i = 1, 2$, let us define m_i as the smallest positive integer m, if such an integer exists, such that for $P' \subset P_i$ some perfect subset and $g \in G$ some point we have $mp' = g$ $\forall p' \in P'$. If such an integer does not exist set $m_i = +\infty$.

Let also when $m_i < +\infty$ P_i' be some Cantor set such that $P_i' \subset g_i + P_i$ for some $g_i \in G$ and $m_i p_i' = 0$ $\forall p_i' \in P_i'$, such a P_i' exists by the definition of m_i; when $m_i = +\infty$ set $P_i' = P_i$ $(i = 1, 2)$.

It then follows by the definition of m_i and P_i' that:

For arbitrary $\{X_\alpha^{(1)} \subset P_1'\}_{\alpha=1}^{N_1}; \ \{X_\alpha^{(2)} \subset P_2'\}_{\alpha=1}^{N_2}$

perfect subsets and arbitrary

$$\{-m_1 < m_\alpha^{(1)} < m_1\}_{\alpha=1}^{N_1} \ \{-m_2 < m_\alpha^{(2)} < m_2\}_{\alpha=1}^{N_2}$$

integers that are not all zero we can find points

$$x_\alpha^{(i)} \in X_\alpha^{(i)}, \quad i = 1, 2, \ \alpha = 1, 2, \ldots, N_i,$$

such that $$\sum_{\alpha=1}^{N_1} m_\alpha^{(1)} x_\alpha^{(1)} + \sum_{\alpha=1}^{N_2} m_\alpha^{(2)} x_\alpha^{(2)}$$

(when $m_i = \infty$ we set of course conventionally $-m_i = -\infty$). It is also evident that for each $x \in P_i'$ the order of x (ord) x divides m_i (with an obvious convention or $m_i = \infty$).

Using now the standard technique of constructing an independent and a $\mathcal{K}$-set in a compact group [5; 5.2.4] and staying well inside P_1' and P_2', which we can do because of the italicized statement above, we see that we can construct our two sets H_1 and H_2 satisfying the conditions of Theorem 4.3.2 as the intersection of a decreasing sequence of open sets with P_1' and P_2' respectively.

Combining now Theorem 4.3.1 and Theorem 4.3.2 and taking into account the fact that for arbitrary $E=\bar{E}\subset G$ and $\gamma\in G$ we have $A(\gamma+E)\cong A(E)$ we see that we have proved

THEOREM 4.3.3. *Let $P_1, P_2\subset G$ be two arbitrary perfect subsets of the compact metrizable group G, then we can find two subsets $E_1\subset P_1$ and $E_2\subset P_2$ such that the algebra $A(E_1+E_2)$ is topologically isomorphic to $V(D_\infty)$.*

§ 4. The embedding of V in A and the problem of spectral synthesis

We start with some trivial remarks, and we preserve all the notations introduced up to now.

Let $E\subset G$ be a compact subset of spectral synthesis of the compact group G, and let $E_1\subset E\subset G$ be a compact subset of E then the following two assertions (α) and (β) below are trivially equivalent

(α) E_1 is a subset of spectral synthesis of G, i.e. a set of spectral synthesis of the algebra $A(G)$.

(β) E_1 identified to a subset of the spectrum $A(E)$, which is E, is a set of spectral synthesis of the algebra $A(E)$.

We shall also need the following trivial consequence of the definition of a $\mathcal{K}$-set.

Let $K\subset G$ be a compact Kronecker (resp. K_p) subset of the compact group G, and let $\varkappa:K\to H$ be a continuous mapping where H is a compact group isomorphic to $\mathbf{T}^\Omega$ (resp. $[\mathbf{Z}(p)]^\Omega$) with Ω some cardinal number. Then for any $W\subset H$ nhd. *of 0_H the zero element of H, there exists $h=h_{W,\varkappa}:G\to H$ a continuous homomorphism such that*

$$\varkappa(k)-h(k)\in W \quad \forall k\in K$$

(p is of course some natural prime).

Let now G be some compact group and let $\{m_j\in\mathbf{Z}\}_{j=1}^n$ be n mutually prime non zero integers ($(m_1, m_2, \ldots, m_n)=1$) and let us denote by

$$\mu: G^n=G\times G\times\ldots\times G\to G;\ \mu(g_1,g_2,\ldots,g_n)=\sum_{j=1}^n m_j g_j \tag{4.4.1}$$

a group homomorphism. Let us also denote by $i_p:G\to G^n$ ($p=1, 2, \ldots, n$) the canonical injection of G in G^n that identifies G to the pth component of the product. We shall now prove the:

LEMMA 4.4.1. *The mapping μ splits, i.e. there exists $L\subset G^n$ a compact subgroup such that,*

$$G^n=L\oplus\operatorname{Ker}\mu;\ L\cong G;\ \operatorname{Ker}\mu\cong G^{n-1}=G\times\ldots\times G.$$

Proof. Let $M = (m_{ij} \in \mathbf{Z})_{i,j=1}^{n}$ be a square, integer entries, matrix such that $\det(M) = 1$, and $m_{1,j} = m_j$ $(j = 1, 2, \ldots, n)$ but arbitrary otherwise, by our hypothesis it is easy to verify that such a matrix exists, and set:

$$\mu_i : G^n \to G;\ \ \mu_i(g_1, g_2, \ldots, g_n) = \sum_{j=1}^{n} m_{i,j} g_j; \quad i = 1, 2, \ldots, n$$

and also

$$\mu^* : G^n \to G^n;\ \mu^*(g^*) = (\mu_1(g^*), \ldots, \mu_n(g^*)) \in G^n;\ \forall g^* \in G^n.$$

Then by our hypothesis on μ, μ^* is an automorphism of G^n, and also $\mu_1 = \pi_1 \circ \mu^*$, where π_1 is the projection of G^n on its first component group; since $\mu_1 = \mu$ our Lemma follows.

Let us denote then by $\mu_L = \mu|_L$ which is then an isomorphism from L to G. Note that L, and therefore μ_L also, is not uniquely determined by the Lemma.

Let us now suppose that $G = \mathbf{T}^{\Omega}$ (resp. $[\mathbf{Z}(p)]^{\Omega}$ with p some natural prime) where Ω is some cardinal number, and let $K_1, K_2, \ldots, K_n$ be disjoint compact subsets such that $K^* = K_1 \cup K_2 \cup \ldots \cup K_n$ is a Kronecker set (resp. a K_p set) of G. And let us suppose that we choose the $\{m_j \in \mathbf{Z}\}_{j=1}^{n}$ such that $m_j = \pm 1$ $(j = 1, 2, \ldots, n)$ but arbitrary otherwise, and let us define μ by (4.4.1); and let us apply our Lemma and let us fix a decomposition $G^n = L \oplus \operatorname{Ker} \mu$ and a μ_L as in Lemma 4.4.1. We see then that $\mu_K = \mu|_K$ (μ restricted to the set K) where $K = K_1 \times K_2 \times \ldots \times K_n \subset G^n$ is $(1-1)$ and has an inverse [cf. ch. 4, § 1, (iv)]:

$$\mu_K^{-1} : \tilde{K} = \left\{ \sum_{j=1}^{n} m_j k_j;\quad k_j \in K_j \quad j = 1, 2, \ldots, n \right\} \to K.$$

Let now $f \in I^{A(G)}(\tilde{K})$ then $f \circ \mu = \check{\mu}(f) \in I^{A(G^n)}(K)$, and since K, being a cartesian product of sets of spectral synthesis, is a set of spectral synthesis [Th. 1.5.1; [12]]. It follows that for any $\varepsilon > 0$ there exists $f_\varepsilon \in I_0^{A(G^n)}(K)$ with

$$\|f_\varepsilon - f \circ \mu\|_{A(G^n)} \leqslant \varepsilon.$$

Let now:

$$\varkappa : K^* \to \operatorname{Ker} \mu;\ \text{defined by } \varkappa(k_j) = l \circ i_j(m_j k_j)\ \ \forall k_j \in K_j\ (j = 1, 2, \ldots, n),$$

where l denotes the projection of G^n on the direct summand $\operatorname{Ker} \mu$ defined by the decomposition $G^n = L \oplus \operatorname{Ker} \mu$. Since then $\operatorname{Ker} \mu \simeq G^{n-1}$, applying our remark at the beginning of this paragraph we see that for W any nhd. of zero of the group $\operatorname{Ker} \mu$ we can find $h = h_W : G \to \operatorname{Ker} \mu$ a continuous homomorphism such that $\varkappa(k) - h(k) \in W$ $\forall k \in K^*$; let us then denote by

$$h_W^* : G \to G^n = L \oplus \operatorname{Ker} \mu;\ \ h_W^*(g) = (\mu_L^{-1}(g),\ h_W(g))$$

We have then trivially $\mu \circ h_W^* = I\partial G$ for all W and also if W' is any nhd. of K we can choose W such that $h_W^*(\tilde{K}) \subset W'$; from these facts we deduce that

$$\|f - f_\varepsilon \circ h_W^*\|_{A(G)} \leqslant \|f \circ \mu - f_\varepsilon\|_{A(G^n)} \leqslant \varepsilon \quad \forall W$$

and that for some W $f_\varepsilon \circ h_W^* \in I_0^{A(G)}(\tilde{K})$, and ε being arbitrary, we see that we have, in fact, proved that $\tilde{K}$ is a set of spectral synthesis of G. We can thus state:

THEOREM 4.4.1. *Let G be an arbitrary compact abelian group, let $\{\varepsilon_j = \pm 1\}_{j=1}^n$ be a choice of ± 1, and $K_1, K_2, \ldots, K_n$ compact disjoint subsets of G such that $K^* = K_1 \cup K_2 \cup \ldots \cup K_n$ is a $\mathcal{K}$-set of G. Then the set $\tilde{K} = \sum_{j=1}^n \varepsilon_j K_j \subset G$ is a set of spectral synthesis of G.*

Proof. The case G arbitrary and K^* a Kronecker set can be deduced from what we have already said by embedding G topologically and algebraically in $\mathbf{T}^\Omega$ for Ω some appropriate cardinal. The case K^* a K_p set for p some prime also follows from what we have done upon observing that $\overline{Gp(K^*)} \cong [\mathbf{Z}(p)]^{\Omega'}$ where Ω' is again some cardinal number [6].

From Theorems 4.4.1 and 4.2.3 and the remarks at the end of Chapter 4, § 2 as well as by the introductory remark of this paragraph about the equivalence of (α) and (β) we deduce

THEOREM 4.4.2. *For any infinite compact abelian group G group there exists $E \subset G$ a compact subset such that $A(E) \cong V(D_\infty) = \mathbf{C}(D_\infty) \hat{\otimes} \mathbf{C}(D_\infty)$ and in such a way that $E_1 \subset E$ a closed subset of E is a set of spectral synthesis of G if and only if $E_1' \subset D_\infty \times D_\infty$ the set that corresponds to E_1 in the above identification is a set of spectral synthesis of $V(D_\infty)$.*

5. Some metric lemmas for a group algebra

As it was pointed out in the introduction, the material of this chapter is technical and out of line with the rest of the paper; it is only inserted here to introduce some classical notations and to clarify a few isolated points.

§ 1. Metric properties of the group algebras and preliminary results

(A) For the calculations and formulas that follow we shall use the letters C (resp. $C_{\alpha,\beta\ldots}$) for an absolute constant (resp. a constant depending only on the parameters $\alpha, \beta, \ldots$) and these C and $C_{\alpha,\beta\ldots}$ will not be the same in all the formulae they appear. We shall preserve this notation till the end of the paper.

We start now with some standard definitions and notations. Let M be a metric space whose metric we denote by d, and let $f \in \mathbf{C}(M)$; we set then

$$\omega_f(\delta) = \sup_{d(x,x') \leqslant \delta} |f(x) - f(x')|, \quad \delta \geqslant 0.$$

We then denote by $\Lambda_\alpha(M)$ (resp. $\lambda_\alpha(M)$) for α any real number $0<\alpha\leqslant 1$ the space of those functions $f\in \mathbf{C}(M)$ for which

$$\omega_f(\delta)=O(\delta^\alpha) \quad (\text{resp. } \omega_f(\delta)=o(\delta^\alpha))$$

λ_α is then always a subspace of Λ_α. For all $f\in\Lambda_\alpha(M)$ we denote by

$$\|f\|_\alpha=\inf\{k;\ \omega_f(\delta)<k\delta^\alpha \quad \forall\delta\geqslant 0\}$$

and by $\|f\|_{\Lambda_\alpha}=\|f\|_\infty+\|f\|_\alpha$. It is then trivial and well known that Λ_α with $\|\ \|_{\Lambda_\alpha}$ is a Banach algebra and that λ_α is a closed subalgebra.

We shall consider $\Lambda_\alpha(\mathbf{T})$ for the natural Euclidean metric of $\mathbf{T}$, then the group translation is a continuous operation on $\lambda_\alpha(\mathbf{T})$ for any $0<\alpha<1$ i.e. the mapping $t\to f_t(x)=f(x+t)$ for any fixed $f\in\lambda_\alpha$ is continuous from $\mathbf{T}$ to λ_α. Thus the principle of regularization by convolution applies to λ_α.

In the case $M=\mathbf{R}$ then $F\in\Lambda_1(\mathbf{R})$ if and only if

$$F(x)=\int_0^x f(t)\,dt+F(0); \quad f(t)\in L^\infty(\mathbf{R})$$

and then $\|F\|_1=\|f\|_\infty$ and $F'(t)=f(t)$ a.e. for the Lebesgue measure in t.

(B) Observe now that for $0<\alpha<\beta\leqslant 1$ and M again a general metric space we always have

$$\lambda_\beta\subset\Lambda_\beta\subset\lambda_\alpha\subset\Lambda_\alpha.$$

Let now $0\leqslant\alpha<\gamma<\beta\leqslant 1$ be real numbers, and let us adopt the convention $\Lambda_0(M)=\mathbf{C}(M)$ and $\|f\|_0=2\|f\|_\infty$ for all $f\in\Lambda_0$, and let us suppose that $f\in\Lambda_\beta$ and therefore also $f\in\Lambda_\alpha\cap\Lambda_\gamma$.

Let $x, x'\in M$ be two points of the space and let $\delta>0$ be an arbitrary real number, we have then

$$d(x,x')\geqslant\delta\Rightarrow|f(x)-f(x')|\leqslant\|f\|_\alpha\delta^{\alpha-\gamma}(d(x,x'))^\gamma$$

(observe that the convention $\|f\|_0=2\|f\|_\infty$ is designed to make the above inequality work with $\alpha=0$), also

$$d(x,x')\leqslant\delta\Rightarrow|f(x)-f(x')|\leqslant\|f\|_\beta\delta^{\beta-\gamma}(d(x,x'))^\gamma.$$

Thus we obtain that for any $\delta>0$

$$\|f\|_\gamma\leqslant\|f\|_\alpha\delta^{\alpha-\gamma}+\|f\|_\beta\,\delta^{\beta-\gamma}.$$

Thus for $\delta^{\beta-\alpha}=\|f\|_\alpha/\|f\|_\beta$ we obtain the interpolation

$$\|f\|_\gamma\leqslant 2\,\|f\|_\alpha^{(\beta-\gamma)/(\beta-\alpha)}\cdot\|f\|_\beta^{(\gamma-\alpha)/(\beta-\alpha)}$$

provided that $\|f\|_\alpha\cdot\|f\|_\beta\neq 0$. [7]

(C) Let us now denote by $A_\alpha(\mathbf{T}) = A(\mathbf{T}) \cap \lambda_\alpha(\mathbf{T})$ for all $0<\alpha<1$ and for any $f \in A_\alpha(\mathbf{T})$ let us denote by $\|f\|_{A_\alpha} = \|f\|_A + \|f\|_\alpha$; $A_\alpha(\mathbf{T})$ becomes then a Banach algebra with that norm. Also let us for any $\alpha>0$ denote by $\tilde{A}_\alpha(\mathbf{T})$ the algebra of functions $f(t) \in \mathbf{C}(\mathbf{T})$ such that

$$f(t) = \sum_{-\infty}^{+\infty} \alpha_\nu e^{i\nu t} \quad \text{with} \quad \|f\|_{\tilde{A}_\alpha} = \sum_{\nu=-\infty}^{+\infty} |\alpha_\nu| (1+|\nu|)^\alpha < +\infty.$$

$\| \; \|_{\tilde{A}_\alpha}$ is then a norm and $\tilde{A}_\alpha(\mathbf{T})$ is also a Banach algebra with that norm, and we have for $0<\alpha<1$ the dense topological inclusions $\mathcal{D}(\mathbf{T}) \subsetneq \tilde{A}_\alpha(\mathbf{T}) \subsetneq A_\alpha(\mathbf{T})$. These inclusions allow us to identify

$$(A_\alpha(\mathbf{T}))' = PM_\alpha(\mathbf{T}) \quad \text{and} \quad (\tilde{A}_\alpha(\mathbf{T}))' = \widetilde{PM}_\alpha(\mathbf{T})$$

the dual spaces with spaces of distributions; the dual norms on these spaces are then denoted by $\| \; \|_{PM_\alpha}$ and $\| \; \|_{\widetilde{PM}_\alpha}$ respectively.

Relative to the above definitions we recall the following well-known theorem [7]: If $f \in \Lambda_\alpha(\mathbf{T})$ for some $0<\alpha\leqslant 1$ and if f is of bounded variation then $f \in A(\mathbf{T})$ and

$$\|f\|_A \leqslant C_\alpha[\|f\|_\alpha^{1/2} V(f)^{1/2} + \|f\|_\infty]$$

where $V(f)$ denotes the total variation of f on $[0, 2\pi]$.

(D) Let now $0<\varepsilon<\pi/2$ and let us denote by $\sum_\varepsilon$ a continuous complex function on the real line periodic with period 2π and such that:

$$\textstyle\sum_\varepsilon(x) = 1 - e^{ix} \quad -\varepsilon \leqslant x \leqslant \varepsilon; \; \sum_\varepsilon(x) \text{ is linear on the interval } [\varepsilon, 2\pi-\varepsilon].$$

The conditions above completely determine $\sum_\varepsilon$. We can compute easily the Fourier series of $\sum_\varepsilon$ and we see at once that for all α $(0<\alpha<1)$ we have:

$$\textstyle\sum_\varepsilon \in \tilde{A}_\alpha(\mathbf{T}); \quad \|\sum_\varepsilon\|_{\tilde{A}_\alpha} \leqslant C_\alpha \varepsilon^{1-\alpha}; \quad 0<\alpha<1.$$

Let now m, n be two integers; $K \subset \mathbf{T}$ a compact subset and $S \in \widetilde{PM}_\alpha(\mathbf{T})$ such that $\operatorname{supp} S \subset K$; let further ε, α be two positive numbers such that $0<\varepsilon<\pi/2$ $0<\alpha<1$ satisfying

$$\sup_{t\in K} |1 - e^{i(m-n)t}| = \sup_{t \in K} |e^{imt} - e^{int}| \leqslant \varepsilon/2. \tag{5.1.1}$$

It then follows that if we denote by

$$\zeta(t) = e^{imt} \textstyle\sum_\varepsilon ((n-m)t),$$

$$\xi(t) = e^{imt} - e^{int} - \zeta(t) = e^{imt}[1 - e^{i(n-m)t} - \textstyle\sum_\varepsilon ((n-m)t)]$$

then:
$$\zeta(t) \in \tilde{A}_\alpha; \quad \|\zeta\|_{\tilde{A}_\alpha} \leqslant C_\alpha (|m|+|n|)^\alpha \|\textstyle\sum_\varepsilon\|_{\tilde{A}_\alpha} \leqslant C_\alpha l^\alpha \varepsilon^{1-\alpha};$$

with $l=\max\{|m|,|n|\}$, also $\xi(t)$ is zero on some neighbourhood of K therefore $\langle \xi, S\rangle = 0$, thus

$$\langle e^{imt}, S\rangle - \langle e^{int}, S\rangle = \langle \zeta, S\rangle$$

and
$$|\langle e^{imt}, S\rangle - \langle e^{int}, S\rangle| \leqslant C_\alpha l^\alpha \varepsilon^{1-\alpha} \|S\|_{\widetilde{PM}_\alpha}. \tag{5.1.2}$$

If in the above considerations, we make the stronger hypothesis $S \in PM(K)$ (i.e. S is an ordinary pseudomeasure supported by K) and if we observe that $\|\zeta\|_A = \|\sum_\varepsilon\|_A \leqslant \|\sum_\varepsilon\|_{\tilde{A}_\alpha}$ we see that (5.1.2) can be sharpened to

$$|\langle e^{imt} - e^{int}, S\rangle| \leqslant C_\alpha \varepsilon^{1-\alpha} \|S\|_{PM}. \tag{5.1.3}$$

Let us now consider $A_\alpha(K) = A_\alpha(\mathbf{T})/I(K) \subsetneq \mathrm{C}(K)$ which is a Banach algebra of functions on K, its norm being simply the quotient norm. Then we can deduce from (5.1.2) and (5.1.3) that if $K, m, n, \varepsilon, \alpha$ satisfy (5.1.1) then we have

$$\|e^{imt}|_K - e^{int}|_K\|_{\tilde{A}_\alpha} \leqslant C_\alpha l_\alpha \varepsilon^{1-\alpha}; \quad \|e^{imt}|_K - e^{int}|_K\|_A \leqslant C_\alpha \varepsilon^{1-\alpha}, \tag{5.1.4}$$

where again $l=\max\{|m|,|n|\}$.

6. The dual of a tensor algebra and the V-Sidon sets

§ 1. Definitions and trivial remarks

Let $\mathcal{K} = \{K_j\}_{j=1}^n$ be compact spaces and let $V(\mathcal{K})$ be the tensor algebra over these spaces. We shall denote by $BM(\mathcal{K}) = (V(\mathcal{K}))'$ the dual space of $V(\mathcal{K})$ and we shall denote by $\| \; \|_{BM}$ the dual norm on this space, also following the standard terminology [8] we shall call $BM(\mathcal{K})$ the space of bimeasures on $K = K_1 \times K_2 \times \ldots \times K_n$.

Since the algebra $V(\mathcal{K})$ is a regular algebra we can define for every $S \in BM(\mathcal{K})$ supp S the support of S as the smallest closed subset of K outside which S reduces to zero.

In the particular case where $K_j = \mathbf{T}^{m_j}$ $m_j \geqslant 1$ $(j=1,2,\ldots n)$ are finite dimensional torus we can identify $BM(\mathcal{K})$ to a space of distributions on $\mathbf{T}^M$, $M = m_1 + m_2 + \ldots + m_n$ $BM(\mathcal{K}) \subset \mathcal{D}'(\mathbf{T}^M)$.

In general it is always true that $M(K) \subsetneq BM(\mathcal{K})$ that is that we can identify the space of Radon measures of K to a space of bimeasures on K; and that for every $\mu \in M(K)$ we have $\|\mu\|_{BM} \leqslant \|\mu\|_M$.

Let now $E \subset K$ be a compact subset. We then denote $V(E) = V(\mathcal{K})/I^{V(\mathcal{K})}(E)$ which is a Banach algebra and can be identified to an algebra of complex functions on E, namely the restrictions on E of the functions of $V(\mathcal{K}) \subsetneq \mathrm{C}(K)$, so that we have a dense norm decreasing inclusion $V(E) \subsetneq \mathrm{C}(E)$.

We then say that $E \subset K$, a closed subset, is a V-Helson set for the algebra $V(\mathcal{K})$, or simply a V-Helson set if $V(E) = \mathrm{C}(E)$ as algebras of functions. A V-Helson set that is countable we call a V-Sidon set. It is then clear that $E \subset K$ is a V-Helson set if and only if there exists $C > 0$ such that

$$\|f\|_{V(E)} \leqslant C\|f\|_{\mathrm{C}(E)} \quad \forall f \in V(E).$$

Further, since every $\mu \in M(E)$ can be identified to $\dot{\mu} \in (V(E))'$ an element of the dual of $V(E)$, and $\|\dot{\mu}\|_{(V(E))'} = \|\mu\|_{BM}$, we see using the duality theory of Banach spaces that $E \subset K$ a closed subset is a V-Helson set if and only if there exists $C^* > 0$ such that

$$\|\mu\|_M \leqslant C^* \|\mu\|_{BM} \quad \forall \mu \in M(E).$$

It is immediate that if we identify topologically $V(\mathcal{K})$ to an algebra $A(E)$ for some compact subset $E \subset G$ of a compact abelian group G as in Ch. 4, § 2, 3; then $BM(\mathcal{K})$ is identified topologically to $(A(E))'$ i.e. to the space of pseudomeasures of the group G whose support lies in E, and which are synthetizable in E; so that if in addition (Ch. 4, § 4) E is a set of spectral synthesis of the group G then $BM(\mathcal{K})$ is in fact identified to

$$PM(E) = \{S \in PM(G);\ \operatorname{supp} S \subset E\}\ [9].$$

If further the identification between $V(\mathcal{K})$ and $A(E)$ is isometric then the identification between $BM(\mathcal{K})$ and the space of pseudomeasures is isometric (for the norm of pseudomeasures $\|S\|_{PM} = \|\hat{S}\|_\infty$).

It is also clear that in an identification of $V(\mathcal{K})$ with $A(E)$, $Q \subset K$ a subset of K is V-Helson if and only if its corresponding subset $\tilde{Q} \subset E$ is a Helson set of the group (Ch. 4, § 1).

§ 2. Norm of the embedding $V(\mathcal{K}) \underset{\rightarrow}{\subseteq} \mathrm{C}(K)$ for finite spaces

Let us suppose here that $\mathcal{K} = \{K_j\}_{j=1}^n$ are n finite spaces, let us denote as usual $K = K_1 \times K_2 \times \ldots \times K_n$, and let us enumerate once and for all each $K_j = \{k_j^1, k_j^2, \ldots, k_j^{p_j}\}$ ($p_j = \operatorname{Card} K_j$) and so obtain a coordinate system for

$$K = \{(k_1^{r_1}, k_2^{r_2}, \ldots, k_n^{r_n});\quad 1 \leqslant r_j \leqslant p_j \quad j = 1, 2, \ldots, n\}.$$

For any $\mu \in M(K)$ let us then denote by

$$\mu_{\mathbf{r}} = \mu(\{k_1^{r_1}, k_2^{r_2}, \ldots, k_n^{r_n}\});\quad \mathbf{r} = (r_1, r_2, \ldots, r_n) \quad 1 \leqslant r_j \leqslant p_j, \quad j = 1, 2, \ldots, n$$

and also denote once and for all

$$R=\{\mathbf{r}=(r_1 r_2, \ldots, r_n); \quad 1 \leqslant r_j \leqslant p_j, j=1,2, \ldots, n\}.$$

Then the tensor $(\mu_{\mathbf{r}})_{\mathbf{r}\in R}$ completely determines μ. Relative to these tensors we shall adopt the summation convention for upper and lower repeated indices; so that for $u_j=(u_j^1, u_j^2, \ldots, u_j^{p_j})$ $(u_j^i \in \mathbf{C};\ 1 \leqslant i \leqslant p_j,\ j=1,2,\ldots,n)$ n vectors we shall write

$$\mu_{r_1, r_2, \ldots, r_n} u_1^{r_1} u_2^{r_2} \ldots u_n^{r_n}=\sum_{\mathbf{r}\in R} \mu_{r_1, r_2, \ldots, r_n} u_1^{r_1} u_2^{r_2} \ldots u_n^{r_n}.$$

We see then that by the definition of the dual norm on $BM(\mathcal{X})$ we have

$$\|\mu\|_{BM}=\sup_{\substack{\|u_j\|\leqslant 1\\ 1\leqslant j\leqslant n}} |\mu_{r_1, r_2, \ldots, r_n} u_1^{r_1} u_2^{r_2} \ldots u_n^{r_n}|, \tag{6.2.1}$$

where the norms of the u's are defined by

$$\|u_j\|=\sup_{1\leqslant r\leqslant p_j} |u_j^r| \quad j=1,2,\ldots,n.$$

From (6.2.1) we deduce, taking the sup first w.r.t. u_n and then w.r.t. $u_1, u_2, \ldots, u_{n-1}$ that:

$$\|\mu\|_{BM}=\sup_{\substack{\|u_j\|\leqslant 1\\ 1\leqslant j\leqslant n-1}} \left(\sum_{r=1}^{p_n} |\mu_{r_1, r_2, \ldots, r_{n-1}, r} u_1^{r_1} u_2^{r_2} \ldots u_{n-1}^{r_{n-1}}|\right) \tag{6.2.2}$$

(summation convention for the r_α's), from (6.2.2) we deduce also

$$\|\mu\|_{BM}\leqslant 2^{n-1} \sup_{S} \left(\sum_{r=1}^{p_n} |\mu_{r_1, r_2, \ldots, r_{n-1}, r} u_1^{r_1} u_2^{r_2} \ldots u_{n-1}^{r_{n-1}}|\right), \tag{6.2.3}$$

where $\qquad S=\{\|u_j\|\leqslant 1, u_j^{r_j} \text{ is real};\ 1\leqslant r_j\leqslant p_j, j=1,2,\ldots,n-1\}.$

We shall now carry out a probabilistic estimation. We suppose that $\mu=M_{\mathbf{R}}(K)$ is a real measure ($\mu_{\mathbf{r}}$; $\mathbf{r}\in R$ are all real) and we consider

$$\{X_j^r;\ 1\leqslant r\leqslant p_j \quad 1\leqslant j\leqslant n\}$$

a double entry family of independent, normalized normal random variables i.e. all equidistributed with some random variable $X\in\mathfrak{N}(0,1)$ $(EX=0 \quad \sigma^2X=1)$ [10].

For any $a>0$ positive number we denote then by:

$$X_j^r(a)=\begin{cases} X_j^r & \text{if } |X_j^r|\leqslant a\\ 0 & \text{if } |X_j^r|>a\end{cases}; \quad X(a)=\begin{cases} X & \text{if } |X|\leqslant a\\ 0 & \text{if } |X|>a\end{cases}$$

the truncated variables for $(1\leqslant r\leqslant p_j;\ j=1,2,\ldots,n)$. Let us also denote by

$$X_j = (X_j^1, X_j^2, \dots, X_j^{p_j}) \quad (j = 1, 2, \dots, n)$$

random vectors. Now since the random vectors X_j $(j = 1, 2, \dots, n)$ are independent they induce an identification of Ω the underlying probability space with

$$\Omega = \Omega_1 \otimes \Omega_2 \otimes \dots \otimes \Omega_n,$$

where Ω_j is the probability space on which X_j is defined $(j = 1, 2, \dots, n)$. Let us also denote by $\omega_j \in \Omega_j$ the generic point of the space, and let us adopt the usual notation

$$E_{q_1, q_2, \dots, q_s} Z = E_{X_{q_1}, X_{q_2}, \dots, X_{q_s}} Z = \int Z(\omega_1, \omega_2, \dots, \omega_n)\, dP(\omega_{q_1}) \dots dP(\omega_{q_s})$$

with $1 \leqslant q_1 < q_2 < \dots q_s \leqslant n$, the conditional expectation of the random variable Z w.r.t. the variables $\{X_{q'_1}, X_{q'_2}, \dots, X_{q'_\sigma}\}$ where $\{q'_1, q'_2, \dots, q'_\sigma\} = \complement_{[1, 2, \dots, n]} \{q_1, q_2, \dots, q_s\}$. We still preserve the summation convention for upper and lower indices and we denote:

$$\nu(a) = \mu_{r_1, r_2, \dots, r_n} X_1^{r_1}(a)\, X_2^{r_2}(a) \dots X_n^{r_n}(a);$$

we proceed to obtain an estimate of $E\,|\nu(a)|$. Let us denote for fixed r $(1 \leqslant r \leqslant p_n)$

$$\nu_r(a) = \mu_{r_1, r_2, \dots, r_{n-1}, r} X_1^{r_1}(a)\, X_2^{r_2}(a) \dots X_{n-1}^{r_{n-1}}(a).$$

We have then:

$$\begin{aligned} E\,|\nu(a)| &= E_{1, 2, \dots, n-1} E_n |\nu_r(a)\, X_n^r(a)| \\ &\geqslant E_{1, 2, \dots, n-1} (E_n |\nu_r(a)\, X_n^r| - E_n |\nu_r(a)\, (X_n^r - X_n^r(a))|); \end{aligned} \tag{6.2.4}$$

and since $\nu_r(a) X_n^r$ for $\omega_1, \omega_2, \dots, \omega_{n-1}$ fixed is a normal variable of Law $\mathfrak{N}(0, \sum_{r=0}^{p_n} |\nu_r(a)|^2)$, using the formula $E\,|X_b| = \sqrt{(2/\pi)}\, b$ valid for every random variable $X_b \in \mathfrak{N}(0, b^2)$, we see that:

$$E_n |\nu_r(a)\, X_n^r| = \sqrt{\frac{2}{\pi}} \left(\sum_{r=1}^{p_n} |\nu_r(a)|^2 \right)^{\frac{1}{2}} \tag{6.2.5}$$

we have also:

$$\begin{aligned} [E_n |\nu_r(a)\, (X_n^r - X_n^r(a))|]^2 &\leqslant E_n |\nu_r(a)\, (X_n^r - X_n^r(a))|^2 \\ &= \sum_{r=1}^{p_n} |\nu_r(a)|^2\, \sigma^2(X_n^r - X_n^r(a)) = \sigma^2(X - X(a)) \sum_{r=1}^{p_n} |\nu_r(a)|^2. \end{aligned} \tag{6.2.6}$$

Thus combining (6.2.4), (6.2.5) and (6.2.6) we obtain

$$E\,|\nu(a)| \geqslant \left[\sqrt{\frac{2}{\pi}} - \sigma(X - X(a)) \right] E_{1, 2, \dots, n-1} \left\{ \sum_{r=1}^{p_n} |\nu_r(a)|^2 \right\}^{\frac{1}{2}} \tag{6.2.7}$$

and this together with Hölder's inequality gives then

$$E|\nu(a)| \geqslant \left[\sqrt{\frac{2}{\pi}} - \sigma(X - X(a))\right] \frac{1}{\sqrt{p_n}} \sum_{r=1}^{p_n} E|\nu_r(a)| \tag{6.2.8}$$

(observe that ν_r depends only on $\omega_1, \omega_2, \ldots, \omega_{n-1}$ $(r = 1, 2, \ldots, p_n)$). Now repeating the above process and applying (6.2.7) and (6.2.8) for the evaluation of each $E|\nu_r(a)|$ in (6.2.8) and so on $n-1$ times we finally obtain:

$$\begin{aligned} E|\nu(a)| &\geqslant \left[\sqrt{\frac{2}{\pi}} - \sigma(X - X(a))\right]^n \frac{1}{\sqrt{p_2 p_3 \cdots p_n}} \sum_{r_2, r_3, \ldots r_n} \left[\sum_{r_1} |\mu_{\mathbf{r}}|^2\right]^{\frac{1}{2}} \\ &\geqslant \left[\sqrt{\frac{2}{\pi}} - \sigma(X - X(a))\right]^n \frac{1}{\sqrt{q_1}} \frac{1}{\sqrt{p_2 p_3 \cdots p_n}} \left(\sum_{\mathbf{r} \in R} |\mu_{\mathbf{r}}|\right) \\ &\geqslant \left[\sqrt{\frac{2}{\pi}} - \sigma(X - X(a))\right]^n \frac{1}{\sqrt{p_1 p_2 \cdots p_n}} \left(\sum_{\mathbf{r} \in R} |\mu_{\mathbf{r}}|\right), \end{aligned} \tag{6.2.9}$$

where q_1 is defined by:

$$q_1 = \sup_{r_2, r_3, \ldots, r_n} [\mathrm{Card}\, \{r; \mu_{r, r_2, \ldots, r_n} \neq 0\}] \leqslant p_1.$$

The reason why we introduced q_1 is that in practice often $q_1 << p_1$. Now from (6.2.3) and (6.2.9) we deduce that for every $a > 0$ large enough we have:

$$\begin{aligned} \|\mu\|_{BM} &\geqslant \frac{2^{n-1}}{a^{n-1}} \sum_{r=1}^{p_n} E|\nu_r(a)| \geqslant C_n \frac{1}{\sqrt{q_1 p_2 \cdots p_{n-1}}} \left(\sum_{\mathbf{r} \in R} |\mu_{\mathbf{r}}|\right) \\ &\geqslant C_n \frac{\|\mu\|_M}{\sqrt{q_1 p_2 \cdots p_{n-1}}} \geqslant C_n \frac{\|\mu\|_M}{\sqrt{p_1 p_2 \cdots p_{n-1}}}, \end{aligned} \tag{6.2.10}$$

where C_n is a constant depending only on n (cf. Ch. 5, § 1) and where (6.2.9) is actually applied for the evaluation of each $E|\nu_r(a)|$ $(r = 1, 2, \ldots, p_n)$. Observe that when $p_1 = p_2 = \ldots = p_n = p$ (6.2.10) gives;

$$\|\mu\|_M \leqslant C_n q_1^{\frac{1}{2}} p^{(n-2)/2} \|\mu\|_{BM} \leqslant C_n p^{(n-1)/2} \|\mu\|_{BM}. \tag{6.2.11}$$

Inequalities (6.2.10) and (6.9.11) can in some cases be improved if extra information on supp μ is given; let us illustrate the method by supposing that $n = 2$ $\mathcal{K} = \{K_1, K_2\}$ and $m = \mathrm{Card}$ (supp μ). Then for every integer $m_1 = 1, 2, \ldots, p_1$ let us decompose K_2 into two disjoint subsets $J = J_{m_1} \subset K_2$ and $\complement J_{m_1} \subset K_2$ defined by

$$k \in J_{m_1} \Leftrightarrow \mathrm{Card}\, \{h \in K_1;\ \mu(\{h, k\}) \neq 0\} \geqslant m_1.$$

It is then clear that m_1 Card $J_{m_1} \leqslant m$. Let then $\xi_J \in \mathrm{C}(K_2)$ be the characteristic function of J, then $1\otimes\xi_J \in V(K_1\times K_2)$ and $\|1\otimes\xi_J\|_V\leqslant 1$ thus if we decompose $\mu=\mu_1+\mu_2$ by setting $\mu_1=(1\otimes\xi_J)\cdot\mu$ we see that

$$\|\mu_1\|_{BM}\leqslant\|\mu\|_{BM},\quad \|\mu_2\|_{BM}\leqslant 2\,\|\mu\|_{BM}.$$

But we can apply (6.2.10) to μ_2 with $q_1=m_1$, and we can also apply (2.2.10) to μ_1 with $q_1=\text{Card}\,J_{m_1}\leqslant m/m_1$ (reverse the order of K_1, K_2) so that we finally obtain

$$3\,\|\mu\|_{BM}\geqslant\|\mu_1\|_{BM}+\|\mu_2\|_{BM}\geqslant C[(m/m_1)^{-\frac{1}{2}}\,\|\mu_1\|_M+m_1^{-\frac{1}{2}}\,\|\mu_2\|_M]$$

and if we set $m_1=[\sqrt{m}]$ and suppose that $m\geqslant 2$ we obtain

$$\|\mu\|_M\leqslant Cm^{\frac{1}{4}}\,\|\mu\|_{BM}=C\,(\text{Card supp}\,\mu)^{\frac{1}{4}}\,\|\mu\|_{BM}. \tag{6.2.12}$$

All our estimations were carried out for real measures of $M_{\mathbf{R}}(K)$. This was done because we did not wish to introduce complex random variables. But of course from (6.2.10), (6.2.11), and (6.2.12) we can pass to the analogous inequalities valid for arbitrary (complex) measures of $M(K)$ by observing that any $\mu\in M(K)$ can be decomposed $\mu=\mathfrak{R}\mu+i\mathfrak{J}\mu$ with $\mathfrak{R}\mu, \mathfrak{J}\mu\in M_{\mathbf{R}}(K)$ and

$$\frac{1}{\sqrt{2}}\,(\|\mathfrak{R}\mu\|_M+\|\mathfrak{J}\mu\|_M)\leqslant\|\mu\|_M\leqslant\|\mathfrak{R}\mu\|_M+\|\mathfrak{J}\mu\|_M$$

$$C_n(\|\mathfrak{R}\mu\|_{BM}+\|\mathfrak{J}\mu\|_{BM})\leqslant\|\mu\|_{BM}\leqslant\|\mathfrak{R}\mu\|_{BM}+\|\mathfrak{J}\mu\|_{BM}.$$

§ 3. The "best possible" of the estimates in § 2

Let here again $\mathcal{K}=\{K_j\}_{j=1}^n$ be n finite spaces (Card $K_j=p_j<+\infty$ $j=1,2,\dots,n$) and let all the notations of the preceding paragraph be preserved. We shall prove the following converse of (6.2.10).

THEOREM 6.3.1. *For $E\subset K=K_1\times K_2\times\dots\times K_n$ there exists $\mu\in M(E)$ a measure with support in E such that $\mu_{\mathbf{r}}=\pm 1$ or 0 for all $\mathbf{r}\in R$ and such that:*

$$\|\mu\|_{BM}\leqslant C\sqrt{\log n\cdot|E|\cdot P},$$

where C is a numerical constant, $|E|=\text{Card}\,E$, and $P=p_1+p_2+\dots+p_n$.

Proof. Let for $\mathbf{r}=(r_1,r_2,\dots,r_n)$ $\delta_{\mathbf{r}}$ be defined by:

$$\delta_{\mathbf{r}}=\begin{cases}1 & \text{if } (k_1^{r_1},k_2^{r_2},\dots,k_n^{r_n})\in E\\ 0 & \text{otherwise}\end{cases}$$

and let us set:

$$T = T[(t_j^r \in \mathbf{R});\ 1 \leqslant r \leqslant p_j,\ j = 1, 2, \ldots, n] = \sum_{\mathbf{r} \in R} \delta_{\mathbf{r}} \exp\left[2\pi i(t_1^{r_1} + t_2^{r_2} + \ldots + t_n^{r_n})\right]$$

which is a trigonometric polynomial of P variables $[(t_j^r);\ 1 \leqslant r \leqslant p_j\ \ j = 1, 2, \ldots, n]$ and of joined degree n. Using then a well-known theorem [9; XI, no. 6] we see that there exists a choice of ± 1 such that:

$$\Big\| \sum_{\mathbf{r} \in R} \pm \delta_{\mathbf{r}} \exp\left[2\pi i(t_1^{r_1} + t_2^{r_2} + \ldots + t_n^{r_n})\right]\Big\|_\infty \leqslant C\sqrt{\log n \cdot |E| \cdot P}. \tag{6.3.1}$$

To satisfy the conditions of our theorem it suffices then to set $\mu_{\mathbf{r}} = \pm \delta_{\mathbf{r}}$ with the above choice of ± 1, for then the left-hand side of (6.3.1) is equal to $\|\mu\|_{BM}$. Theorem (6.3.1) shows that (6.2.10) is, in some sense, best possible. Indeed let us consider n as fixed in Theorem 6.3.1 and suppose without loss of generality that $p_n = \max\{p_1 p_2, \ldots, p_n\}$ then we obtain from Theorem 6.3.1 with $E = K$ that there exists $\mu \in M_{\mathbf{R}}(K)$ such that

$$\begin{aligned}\|\mu\|_{BM} \leqslant C\sqrt{n \log n}\ \sqrt{p_n |E|} &= C_n p_n \sqrt{p_1 p_2 \ldots p_{n-1}} \\ &= C_n \frac{p_1 p_2 \cdots p_n}{\sqrt{p_1 p_2 \cdots p_{n-1}}} = C_n \frac{\|\mu\|_M}{\sqrt{p_1 p_2 \cdots p_{n-1}}}.\end{aligned} \tag{6.3.2}$$

The proof of Theorem (6.3.1) was based on the existence of a choice of ± 1 that satisfy (6.3.1), and that is in turn established in [9] by a probabilistic method; thus it is of interest to give an explicit construction of a measure satisfying the inequality (6.3.2). This we shall now do in the particular case $n = 2$, $p_1 = p_2 = p$, $E = K = K_1 \times K_2$. Towards that let $\mu \in M(K)$ be such that the square matrix $M = (\mu_{i,j})_{i,j=1}^p$ is unitary and $|\mu_{i,j}| = 1/\sqrt{p}\ i, j = 1, 2, \ldots p = p_1 = p_2$. Such a matrix always exists; e.g. it suffices to set as entries of its columns the values of the p distinct characters of G_p that are mutually orthogonal divided by $\sqrt{p}$, where G_p is a finite group of order p, observe also that when $p = 2^q$ for some q then the above construction can give us a real matrix. For this $\mu \in M(K)$ we then have $\|\mu\|_M = p^{3/2}$, also using (6.2.2) we see that

$$\|\mu\|_{BM} = \sup_{\|\mathbf{z}\| \leqslant 1} \sum_{i=1}^p |(M\mathbf{z})_i|, \tag{6.3.3}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_p)^T$ and $\|\mathbf{z}\| = \sup_{1 \leqslant i \leqslant p} |z_i|$ and where $M\mathbf{z}$ is the matrix product of M with the column vector $\mathbf{z}$ having $(M\mathbf{z})_i$ as ith coordinate. But M being unitary we have for all $\mathbf{z}$ with $\|\mathbf{z}\| \leqslant 1$

$$\sum_{i=1}^p |(M\mathbf{z})_i|^2 = \sum_{i=1}^p |z_i|^2 \leqslant p \tag{6.3.4}$$

thus combining (6.3.3) and (6.3.4) we obtain:

$$\|\mu\|_{BM} \leqslant \sup_{\|\mathbf{z}\|\leqslant 1} \sqrt{p}\left(\sum_{i=1}^{p} |(M\mathbf{z})_i|^2\right)^{\frac{1}{2}} \leqslant p.$$

So finally μ satisfies inequality (6.3.2) for our particular values of the parameters:

$$\|\mu\|_{BM} \leqslant 1/\sqrt{p}\,\|\mu\|_M.$$

§ 4. The V-Sidon sets

Let $\mathcal{K} = \{K_j\}_{j=1}^n$ be arbitrary compact spaces and let $E \subset K = K_1 \times K_2 \times \ldots \times K_n$ be any subset, let also λ be some positive number, then we shall say that E is an S_λ subset of K if for every choice of finite subsets $F_j \subset K_j$ of the same number of elements

$$\text{Card } F_j = m \ (j = 1, 2, \ldots n) \quad \text{we have} \quad \text{Card}\,(E \cap F_1 \times F_2 \times \ldots \times F_n) \leqslant \lambda m.$$

Let us now denote by $\Phi = \Phi(\mathcal{K})$ the free abelian group generated by the disjoint union $K_1 \cup K_2 \cup \ldots \cup K_n$. One way to realize concretely Φ is to identify K_j with $\tilde{K}_j \subset G$ $(j = 1, 2, \ldots n)$, where G is some fixed compact group, so that $\tilde{K}_p \cap \tilde{K}_q = \emptyset$ $p \neq q$ and so that $\tilde{K}_1 \cup \tilde{K}_2 \cup \ldots \cup \tilde{K}_n = K^*$ is a Kronecker set and thus independent; $\Phi(\mathcal{K})$ is then isomorphic to $Gp(K^*) \subset G$.

We can now identify $K = K_1 \times K_2 \times \ldots \times K_n$ canonically to a subset of Φ by identifying $k = (k_1, k_2, \ldots k_n)$ with the point $k_1 + k_2 + \ldots + k_n \in \Phi$. When we identify then Φ to $Gp(K^*)$ as above $K \subset \Phi$ is identified to $\tilde{K} = \tilde{K}_1 + \tilde{K}_2 + \ldots + \tilde{K}_n \subset G$ in the way already explained in Ch. 4, § 2. We shall say that $X \subset K$ a subset of K is free if in the above identification of $K \subset \Phi$ X becomes an independent subset of Φ. Let us now suppose that $E \subset K$ is a compact V-Helson subset then using Theorem 6.3.1 we see that it must be an S_λ subset of K for some positive number λ. Indeed suppose that, then for $\alpha > 0$ arbitrarily large we can find $F_j \subset K_j$ finite subsets such that Card $F_j = m \geqslant 1$ $(j = 1, 2, \ldots n)$ and

$$(E' = E \cap F_1 \times F_2 \times \ldots \times F_n) \geqslant \alpha m.$$

But then by Theorem 6.3.1 we can construct $\mu \in M_{\mathbf{R}}(E')$ such that $\|\mu\|_M = \text{Card } E'$ and

$$\|\mu\|_{BM} \leqslant C\sqrt{m\,|E'|} \quad \text{so that} \quad \|\mu\|_{BM}/\|\mu\|_M \leqslant C\alpha^{-\frac{1}{2}}$$

and α being arbitrarily large this contradicts the fact that E is V-Helson.

Let us now suppose that E is a countable set and that there exists $l \geqslant 1$ a positive integer such that every $F \subset E$ finite subset can be decomposed

$$F=\bigcup_{j=1}^{l} F_j;\quad F_j \text{ free } j=1,2,\dots,l$$

as the finite union of l free subsets, we shall call such a set a locally l-free set. We shall now show that for every $l\geqslant 1$ every locally l-free countable compact set of K is a V-Sidon set.

To prove this we may suppose without loss of generality that each K_j $(j=1,2,\dots n)$ is countable, indeed it suffices to consider the projection of E on each K_j which is countable. With that hypothesis let us realize $V(\mathcal{K})$ in the group algebra $A(\mathbf{T})$; more explicitly let $\tilde{K}_j$ be compact disjoint subsets of $\mathbf{T}$ $\tilde{K}_j$ topologically homeomorphic to K_j $(j=1,2,\dots n)$ and such that $K^*=\tilde{K}_1\cup\tilde{K}_2\cup\dots\cup\tilde{K}_n$ is a Kronecker set of $\mathbf{T}$; as we have already pointed out $\Phi(\mathcal{K})$ is then realized as $Gp(K^*)\subset\mathbf{T}$ and K is then identified to $\tilde{K}=\tilde{K}_1+\tilde{K}_2+\dots+\tilde{K}_n$. E is then identified to $\tilde{E}$ a compact subset of $\tilde{K}\subset\mathbf{T}$ which is locally l-independent i.e. such that every finite subset $\tilde{F}\subset\tilde{E}$ can be written as the finite union of l-independent subsets of the group $\mathbf{T}$; our assertion then follows from the well-known fact that such a subset of $\mathbf{T}$ is a Helson set of $\mathbf{T}$, i.e. $A(\tilde{E})=\mathbf{C}(\tilde{E})$. More explicitly what is well known (cf. [5]) is that for every $l\geqslant 1$ positive integer there exists C_l a constant depending only on l such that for arbitrary countable independent subsets of $\mathbf{T}$ $A_1, A_2, \dots A_l$ and arbitrary $\mu\in M(A)$ $A=A_1\cup A_2\cup\dots\cup A_l$ we have $\|\mu\|_M\leqslant C_l\|\mu\|_{PM}$, and thus that A is a Helson set. To deduce the result assuming only the local property we approximate any $\mu\in M(\tilde{E})$ by some $\mu_F\in M(\tilde{F})$ with $\tilde{F}$ some finite subset of $\tilde{E}$ so that we have $\|\mu_F\|_M\leqslant C_l\|\mu_F\|_{PM}$ and letting μ_F tend to μ (for the $\|\ \|_M$ norm) we obtain the same inequality $\|\mu\|_M\leqslant C_l\|\mu\|_{PM}$. We are now in a position to prove

THEOREM 6.4.1. *Let $E\subset K$ be a countable compact set then the following three conditions on E are equivalent:*

(i) *E is a V-Sidon set.*
(ii) *E is an S_λ subset for some $\lambda>0$.*
(iii) *E is a locally l-free subset for some $l\geqslant 1$.*

Proof. We have already seen that (iii) ⇒ (i) ⇒ (ii). Thus it suffices to show that (ii) ⇒ (iii). This we now do in the following

LEMMA. *Let $\mathcal{K}=\{K_j\}_{j=1}^{n}$ be n finite sets with* Card $K_j=N$ *$j=1,2,\dots n$; and let $E\subset K=K_1\times K_2\times\dots\times K_n$ be an S_λ subset for some integer $\lambda\geqslant 1$. Then E is a locally $n\lambda$-free subset of K.*

Proof of the lemma. The proof is done by induction on N. The induction starts trivially, so let us suppose that the lemma holds when $N\leqslant M-1$ with some $M\geqslant 2$, and let us prove it for $N=M$. Using the definition of an S_λ subset with $F_j=K_j$ $(j=1, 2, \dots n)$ we see that Card $E \leqslant n\lambda$; thus for $j=1, 2, \dots n$ there exists a $k(j) \in K_j$ such that if we denote by:

$$R_j=\{k=(k_1, k_2, \dots, k_n) \in K: k_j=k(j)\} \quad j=1, 2, \dots, n$$

we have:

$$\text{Card}\,(E \cap R_j) \leqslant \lambda \quad j=1, 2, \dots, n. \tag{6.4.1}$$

Let us then denote by $\mathcal{K}'=\{K_j'=\complement_{K_j}\{k(j)\}=\{K_j \setminus k(j)\}_{j=1}^n$ and $K'=K_1' \times K_2' \times \dots \times K_n'$ and identify K' to a subset of K, and let us denote $E'=E \cap K'$. Now Card $K_j'=M-1$ $(j=1, 2, \dots, n)$ thus by the inductive hypothesis the lemma applies to $\mathcal{K}'$ and E' so we can decompose

$$E'=\bigcup_{j=1}^{n\lambda} F_j'; \; F_j' \subset E' \text{ free subsets } j=1, 2, \dots, n\lambda. \tag{6.4.2}$$

Let us now consider $E \backslash E'$, if it is empty $(=\emptyset)$ then (6.4.2) gives us the required decomposition of E and proves the inductive step. So suppose that $E \setminus E'=\emptyset$ and let us enumerate its elements in any fashion whatsoever $E \setminus E'=\{e_1, e_2, \dots, e_m\}$ and observe that by the definition of E' and (6.4.1) we have $m \leqslant n\lambda$; let us also set

$$F_\nu=F_\nu' \cup e_\nu \quad (i \leqslant \nu < m); \; F_\nu=F_\nu' \cup e_m \quad (m \leqslant \nu \leqslant n\lambda).$$

We have then, of course,

$$E=\bigcup_{\nu=1}^{n\lambda} F_\nu; \tag{6.4.3}$$

we claim that (6.4.3) gives us the required decomposition of E, i.e. that F_ν is a free subset of K $(\nu=1, 2, \dots, n\lambda)$. Thus we have to prove that for any $\nu=1, 2, \dots, n\lambda$ we have

$$\sum_{\gamma \in F_\nu} n_\gamma \gamma=0 \text{ in } \Phi(\mathcal{K}), \quad n_\gamma \in \mathbf{Z} \Rightarrow n_\gamma=0 \quad \forall \gamma \in F_\nu. \tag{6.4.4}$$

But for any ν there exists j $(1 \leqslant j \leqslant n)$ and $\gamma_\nu \in F_\nu$ such that

$$\gamma_\nu \in R_j, \quad F_\nu \cap R_j=\gamma_\nu, \quad F_\nu=F_\nu' \cup \gamma_\nu.$$

Thus $\sum_{\gamma \in F_\nu} n_\gamma \gamma=0$ implies that $n_{\gamma_\nu}=0$ and that $\sum_{\gamma \in F_\nu'} n_\gamma \gamma=0$ and F_ν' being free by our hypothesis (6.4.4) follows and the inductive step is proved. This completes the proof of the lemma and of the Theorem 6.4.1.

Theorem 6.4.2 gives a complete characterization of V-Sidon sets. It proves in particular that they are stable by union i.e. the union of two V-Sidon sets is a V-Sidon set.

7. The metric theory of tensor algebras

§ 1. The Bernstein type theorem for tensor algebras

In this paragraph we shall prove the following

THEOREM 7.1.1. *Let* $m \geq 1$ *be a positive integer and* α *a real number such that* $1/2 < \alpha \leq 1$ *then* $\Lambda_\alpha(\mathbf{T}^m \times \mathbf{T}) \subset \mathbf{C}(\mathbf{T}^m) \hat{\otimes} \mathbf{C}(\mathbf{T})$. *Conversely for every* $\beta \in [0, 1/2]$ *there exists* $f \in \Lambda_\beta(\mathbf{T} \times \mathbf{T})$ *such that* $f \notin \mathbf{C}(\mathbf{T}) \hat{\otimes} \mathbf{C}(\mathbf{T})$.

For our theory we shall assign a canonical translation invariant metric Δ on the space of the group D_∞ defined by

$$\Delta(0, \alpha) = \sum_{j=1}^{\infty} \alpha_j/2^j, \ \forall \alpha = (\alpha_1 \alpha_2, \ldots); \quad \alpha_j = 0, 1; \quad j = 1, 2, \ldots .$$

In general now for $M_1, M_2, \ldots, M_k$ finitely many metric spaces with metrics $\delta_j[(M_j, \delta_j)$, $j = 1, 2, \ldots, k]$ one defines on $M = M_1 \times M_2 \times \ldots \times M_k$ the product metric $\delta = \delta_1 \times \delta_2 \times \ldots \times \delta_k$ by:

$$\delta(m^{(1)}, m^{(2)}) = \sum_{j=1}^{k} \delta_j(m_j^{(1)}, m_j^{(2)}); \ m^{(i)} = (m_1^{(i)}, m_2^{(i)} \ldots m_k^{(i)}) \in M, \quad i = 1, 2.$$

We shall then define for any finite $\omega = 1, 2, \ldots$, Δ^ω the product metric of Δ with itself ω times on $(D_\infty)^\omega$. It is with respect to Δ^ω that all the classes $\Lambda_\alpha(D_\infty^\omega)$ will always be considered $(0 \leq \alpha \leq 1)$. From our definitions of $d: \mathrm{D}_\infty \to \mathbf{T}$ in Ch. 3, § 4, and Δ it follows that for every finite $\omega = 1, 2, \ldots, d_\omega : (D_\infty)^\omega \to \mathbf{T}^\omega$ is a mapping that belongs to Λ_1 i.e. if we denote by $|t_1 - t_2|_\omega$ the Euclidean metric on $\mathbf{T}^\omega$ for $t_1, t_2 \in \mathbf{T}^\omega$ we have:

$$|d_\omega(e_1) - d_\omega(e_2)|_\omega \leq C\Delta^\omega(e_1, e_2) \quad e_1, e_2 \in (D_\infty)^\omega, \tag{7.1.1}$$

where C denotes as always an absolute constant.

Notice that in considerations involving Λ_α we cannot identify $(D_\infty)^\omega$ to D_∞ as we did in Ch. 3, § 4, for such an identification does not preserve the metric.

Let us now consider $D_r = [\mathbf{Z}(2)]^r$ for $r = 1, 2, \ldots$ and identify D_r to the space of the r first coordinates of D_∞ so that we have the canonical projection (3.5.4) $D_\infty \underset{\xi_r}{\to} D_r$. This mapping induces then for $\omega, \omega_1, \omega_2$ positive integers and $r \geq 1$:

$$\xi_r^\omega : (D_\infty)^\omega \to (D_r)^\omega; \quad \check{\xi}_r^\omega : \mathbf{C}(D_r^\omega) \to \mathbf{C}(D_\infty^\omega)$$

$$\check{\xi}_r^{\omega_1} \hat{\otimes} \check{\xi}_r^{\omega_2} : \mathbf{C}(D_r^{\omega_1}) \hat{\otimes} \mathbf{C}(D_r^{\omega_2}) \to \mathbf{C}(D_\infty^{\omega_1}) \hat{\otimes} \mathbf{C}(D_\infty^{\omega_2})$$

and $\check{\xi}_r^\omega, \check{\xi}_r^{\omega_1} \hat{\otimes} \check{\xi}_r^{\omega_2}$ are isometric (cf. Ch. 3, § 2). It is also clear that for every $r = 1, 2, \ldots$ and every $x \in D_r^\omega$ the diameter of the set

$$Z_r(x) = \{t \in D_\infty^\omega;\quad \xi_r^\omega(t) = x\} \subset D_\infty^\omega$$

is equal to $\omega 2^{-r}$ and from this it follows that if $f \in \Lambda_\alpha(D_\infty^\omega)$ for some $\alpha \in [0, 1)$ and some positive integer ω, and if we define

$$\pi_r(f) = f_r(x) = 2^{\omega r}\int_{Z_r(x)} f(t)\, dt \in \mathbf{C}(D_r^\omega);\ \forall x \in D_r^\omega \quad r = 1, 2, \ldots \tag{7.1.2}$$

(dt = Haar measure of D_∞^ω) then we have

$$\|\check{\xi}_r^\omega(f_r) - f\|_\infty \leqslant \omega 2^{-\alpha r}\|f\|_\alpha;\ \|f_r\|_\infty \leqslant \|f\|_\infty;\quad r = 1, 2, \ldots. \tag{7.1.3}$$

This is just a consequence of the Lipschitz character of f. A repeated application of (7.1.3) shows then that for any $0 < \alpha \leqslant 1$ and any $\omega = 1, 2, \ldots$ and any $f \in \Lambda_\alpha(D_\infty^\omega)$ there exists a sequence $f_j \in \mathbf{C}(D_j^\omega)\ j = 1, 2, \ldots$ such that:

$$f = \sum_{j=1}^\infty \check{\xi}_j^\omega(f_j);\quad \|f_j\|_\infty \underset{j\to\infty}{=} O(2^{-\alpha' j})\quad \forall \alpha' < \alpha. \tag{7.1.4}$$

Let now $\omega_1 \geqslant \omega_2 \geqslant \ldots \omega_s \geqslant 1$ be positive integers and set $\omega_1 + \omega_2 + \ldots + \omega_s = \omega$ and let us consider for $r = 1, 2, \ldots \infty$ a positive integer of ∞, the algebra

$$V_r = \mathbf{C}(D_r^{\omega_1}) \hat{\otimes} \mathbf{C}(D_r^{\omega_2}) \hat{\otimes} \ldots \hat{\otimes}\, \mathbf{C}(D_r^{\omega_s}) \underrightarrow{\supseteq} \mathbf{C}(D_r^\omega) \tag{7.1.5}$$

Then for every finite r $V_r = \mathbf{C}(D_r^\omega)$ but for every $f \in V_r$ $\|f\|_{V_r}$ and $\|f\|_\infty$ are not always equal. Using (6.2.10) we see in fact that for every finite r

$$\left.\begin{aligned} \|f\|_{V_r} &= \sup_{\substack{\mu \in M(D_r^\omega)\\ \|\mu\|_{V_r'} \leq 1}} |\langle f, \mu\rangle| \leqslant \sup_{\substack{\mu \in M(D_r^\omega)\\ \|\mu\|_M \leq C_s 2^{r/2(\omega-\omega_1)}}} |\langle f, \mu\rangle| \\ &\leqslant C_s 2^{\frac{1}{2}r(\omega_2+\omega_3+\cdots+\omega_s)}\|f\|_\infty;\quad \forall f \in V_r. \end{aligned}\right\} \tag{7.1.6}$$

Let us now apply (7.1.6) with $s = 2$ and $\omega_2 = 1$ to (7.1.4) ($\omega = \omega_1 + 1$); we see then that for every $f \in \Lambda_\alpha(D_\infty^\omega)$ with $\alpha > 1/2$ and every $\alpha' < \alpha$ we have:

$$\left.\begin{aligned} &f = \sum_{j=1}^\infty \check{\xi}_j^\omega(f_j);\\ &f_j \in \mathbf{C}(D_j^{\omega_1}) \hat{\otimes} \mathbf{C}(D_j) = V_j;\quad \|f\|_{V_j} = O(2^{(1/2-\alpha')j}) \quad \text{as} \quad j \to \infty \end{aligned}\right\} \tag{7.1.7}$$

and since for every $f_j \in V_j$ $j=1,2,\dots$, we have $\check{\xi}_j^{\omega}(f_j)=\check{\xi}_j^{\omega_1}\hat{\otimes}\check{\xi}_j^1(f_j)$ we see that for each f_j of (7.1.7)

$$\check{\xi}_j^{\omega}(f_j)\in \mathbf{C}(D_\infty^{\omega_1})\hat{\otimes}\mathbf{C}(D_\infty)=V_\infty;\quad \|\check{\xi}_j^{\omega}(f_j)\|_{V_\infty}=O(2^{(1/2-\alpha')j})\quad \text{as}\quad j\to\infty.$$

So the series $\sum_{j=1}^{\infty}\check{\xi}_j^{\omega}(f_j)$ converges in V_∞ as soon as $\alpha'>1/2$ and thus $f\in V_\infty$.

We are now in a position to prove the first part of our theorem (7.1.1). Indeed let $f\in\Lambda_\alpha(\mathbf{T}^{m+1})$ for some α $(1/2<\alpha\leqslant 1)$ (Λ_α is taken for the usual $|\ |_{m+1}$ euclidean metric of course). Then if we denote by $\tilde{f}=f\circ(d_m\times d)\in\mathbf{C}(D_\infty^m\times D_\infty)$ we see that $\tilde{f}\in\Lambda_\alpha(D_\infty^{m+1})$ (7.1.1) so from our considerations above with $\omega_1=m$ it follows that $\tilde{f}\in\mathbf{C}(D_\infty^m)\hat{\otimes}\mathbf{C}(D_\infty)$; and this together with Theorem 3.5.2 implies that $f\in\mathbf{C}(\mathbf{T}^m)\hat{\otimes}\mathbf{C}(\mathbf{T})$ and proves the first part of our theorem. To prove the second part of the theorem it suffices to show that for every $\beta\in[0,1/2]$ there exists $f\in\Lambda_\beta(\mathbf{I}^2)$ such that $f\notin\mathbf{C}(\mathbf{I})\hat{\otimes}\mathbf{C}(\mathbf{I})=V(\mathbf{I}^2)$ where $\mathbf{I}=[0,1]$ is the unit interval with the euclidean metric, for such an interval can be embedded as a closed subset of $\mathbf{T}$ and the existence of the above f and (Ch. 3, § 3, Case 1) imply then our result. Towards that it suffices to show that for any $A\geqslant 0$ positive number arbitrarily large there exists $f\in\Lambda_1(\mathbf{I}^2)$ such that $\|f\|_{\Lambda_\beta}\leqslant 1$ and $\|f\|_{V(\mathbf{I}^2)}\geqslant A$, for if $\Lambda_\beta(\mathbf{I}^2)\subset V(\mathbf{I}^2)$ for some β, then by the closed graph theorem the canonical injection must be continuous.

To do this we consider for $n\geqslant 1$ a positive integer the finite set $I_n=[0;\,1/n;\,2/n;\dots n-1/n;\,1]\subset\mathbf{I}$ and $I_n\times I_n\subset\mathbf{I}\times\mathbf{I}$ and denote by $V_n=\mathbf{C}(I_n)\hat{\otimes}\mathbf{C}(I_n)$ and define $\mu\in BM(I_n^2)=M(I_n^2)$ by:

$$|\mu(\{x\})|=1\ \ \forall x\in I_n^2;\ \ \|\mu\|_{BM}\leqslant Cn^{\frac{3}{2}}\quad (C=\text{abs. constant}) \tag{7.1.8}$$

(cf. Ch. 6, §3). Let us also define $f\in\mathbf{C}(\mathbf{I}^2)$ by:

$$f(x)\,\mu(\{x\})=1\ \ \forall x\in I_n\times I_n$$

and such that for every $p,q=0,1,2,\dots,n-1$ the function $f(x)$ coincides with a linear function (of two variables) when x lies in the triangle $T_{p,q}^+$ and it coincides with another linear function when $x\in T_{p,q}^-$. The two triangles $T_{p,q}^{\pm}\subset\mathbf{I}\times\mathbf{I}$ are defined by their vertices:

$$T_{pq}^+=\left\{\left(\frac{p}{n},\frac{q}{n}\right);\ \left(\frac{p+1}{n},\frac{q}{n}\right);\ \left(\frac{p+1}{n},\frac{q+1}{n}\right)\right\};\ T_{pq}^-=\left\{\left(\frac{p}{n},\frac{q}{n}\right);\ \left(\frac{p}{n},\frac{q+1}{n}\right);\ \left(\frac{p+1}{n},\frac{q+1}{n}\right)\right\}.$$

It is then clear that $\|f\|_\infty\leqslant 1$, $\|f\|_{\Lambda_1}\leqslant Cn$ therefore by Ch. 5, § 1(B), we deduce that for any $\beta\in[0,1]$ we have $\|f\|_{\Lambda_\beta}\leqslant Cn^\beta$; it is also clear that

$$f_n=f|_{I_n\times I_n}\in V_n;\quad \|f_n\|_{V_n}\leqslant\|f\|_{V(\mathbf{I}^2)}$$

but also by (7.1.8) we have:

$$\|f_n\|_{V_n} \geq \frac{C}{n^{\frac{3}{2}}} |\langle f_n, \mu\rangle| = C\sqrt{n}$$

and putting all this together we see that if we define $\tilde{f} = f/n^{\beta}$ ($\beta \in [0, 1]$) we have:

$$\|\tilde{f}\|_{\Lambda_\beta} \leq C, \quad \|\tilde{f}\|_{V(\mathbf{I}^2)} \geq Cn^{1/2-\beta}$$

and so $\tilde{f}$ gives us the required function as soon as n is large enough provided that $\beta < 1/2$. The critical case $\beta = 1/2$ we shall settle in the next chapter using harmonic analysis.

§ 2. The Beurling–Pollard type of theorems for tensor algebras

In this paragraph we preserve all the notations of the preceding paragraph. In particular let us fix once and for all $\omega_1 \geq \omega_2 \geq \dots \omega_s$ positive integers and let us denote by V_r (r a positive integer or ∞) the algebra defined in (7.1.5), and let $\omega = \omega_1 + \omega_2 + \dots + \omega_s$; let also $\pi_r: \mathbf{C}(D_\infty^\omega) \to \mathbf{C}(D_r^\omega)$ ($r \geq 1$) be the linear mapping defined in (7.1.2), and let us denote by $\dot{\pi}_r = \pi_r|_{V\infty}: V_\infty \to V_r$ the restricted mapping (observe $V_\infty \subset \mathbf{C}(D_\infty^\omega)$). It is then easy to verify that $\|\dot{\pi}_r\| \leq 1$ ($r \geq 1$) (for the V-norms).

Let us observe now that $\check{\xi}_r^\omega \circ \dot{\pi}_r$ tends to the identity of V_∞ as $r \to \infty$ in the strong operator topology ($\check{\xi}_r^\omega$ can be considered as an isometric mapping $V_r \to V_\infty$) i.e. $\check{\xi}_r^\omega \circ \dot{\pi}_r(f) \xrightarrow[r\to\infty]{} f$ in V_∞ ($\forall f \in V_\infty$). Let us observe also that if we denote by $\eta_r: BM_\infty = (V_\infty)' \to BM_r = (V_r)'$ the dual mapping of $\check{\xi}_r^\omega$ we have $\|\eta_r\| \leq 1$ and supp $(\eta_r S) \subset \xi_r^\omega$ (supp S) $\forall S \in BM_\infty$ ($r = 1, 2, \dots$). Let now $E \subset D_\infty^\omega$ be a closed set $f \in I^{V_\infty}(E) \subset V_\infty$ be such that

$$|f(x)| \leq C[\Delta^\omega(x, E)]^\varrho = C[\inf_{e\in E} \Delta^\omega(x, e)]^\varrho; \;\; \forall x \in D_\infty^\omega, \;\; \text{some} \;\; \varrho \geq 0. \tag{7.2.1}$$

Let also $S \in BM_\infty(E)$ i.e. $S \in BM_\infty$ and such that supp $S \subset E$, and let us consider $a = \langle f, S\rangle$. We have then

$$\langle \check{\xi}_r^\omega \circ \dot{\pi}_r(f), S\rangle = a_r \xrightarrow[r\to\infty]{} a \tag{7.2.2}$$

and also $a_r = \langle \dot{\pi}_r(f), \eta_r S\rangle$ with supp $(\eta_r S) \subset \xi_r^\omega(E)$, so by (7.2.1) we see that

$$\sup_{x \in \operatorname{Supp}(\eta_r S)} |\dot{\pi}_r f(x)| \leq C\, 2^{-r\varrho}. \tag{7.2.3}$$

Also using (6.2.10) we obtain that

$$\|\eta_r S\|_M \leq C\, 2^{\frac{1}{2}r(\omega_2+\omega_3+\dots\omega_s)} \|\eta_r S\|_{BM_r} \leq C\, 2^{\frac{1}{2}r(\omega_2+\dots+\omega_s)} \|S\|_{BM_\infty}. \tag{7.2.4}$$

So finally combining (7.2.2), (7.2.3), and (7.2.4) we obtain that:

$$|a_r| \leqslant C\, 2^{r[\frac{1}{2}(\omega_2+\omega_3+\dots+\omega_s)-\varrho]}\, \|S\|$$

and that if $\frac{1}{2}(\omega_2+\omega_3+\dots+\omega_s)<\varrho$ we have $a_r \xrightarrow[r\to\infty]{} 0=a$. Now since S above was chosen arbitrary in $BM_\infty(E)$ and since the orthogonal in V_∞ of $BM_\infty(E)$ is $J^{V_\infty}(E)$ what we have shown can be stated in the following

THEOREM 7.2.1. *If* $\omega_2+\omega_3+\dots+\omega_s<2\varrho$ *and if* $f\in I^{V_\infty}(E)$ *satisfies* (7.2.1) *then* $f\in J^{V_\infty}(E)$.

Observe that often if we know the shape of E we can use sharper estimates [(6.2.10), (6.2.12)] in (7.2.4) and improve Theorem 7.2.1 in an obvious way. We shall illustrate that idea at the end of this paragraph.

Let us now for $\omega_1\geqslant\omega_2\geqslant\dots\geqslant\omega_s$ and $\omega=\omega_1+\omega_2+\dots+\omega_s$ positive integers as above consider

$$V_T=\mathbf{C}(\mathbf{T}^{\omega_1})\hat{\otimes}\mathbf{C}(\mathbf{T}^{\omega_2})\hat{\otimes}\dots\hat{\otimes}\mathbf{C}(\mathbf{T}^{\omega_s})\subseteq\mathbf{C}(\mathbf{T}^{\omega})$$

and let $E\subset\mathbf{T}^\omega$ be a closed set and $f\in I^{V_T}(E)\subset V_T$ be such that:

$$|f(t)|\leqslant C|t-E|_\omega^\varrho=C[\sup_{e\in E}|t-e|_\omega]^\varrho;\ \forall t\in\mathbf{T}^\omega,\ \text{ some }\ \varrho\geqslant 0 \tag{7.2.5}$$

($|t_1-t_2|_\omega$ is as in 7.1.1). Let us set then $\tilde{f}=\check{d}_\omega(f)=f\circ d_\omega\in V_\infty$ and $\tilde{E}=d_\omega^{-1}(E)$, using (7.1.1) we see then that $|\tilde{f}(t)|\leqslant C[\Delta^\omega(x,\tilde{E})]^\varrho$ $(\forall x\in D_\infty^\omega)$, thus from Theorem 7.2.1 it follows that $\tilde{f}\in J^{V_\infty}(\tilde{E})$ as soon as $\omega_2+\omega_3+\dots+\omega_s<2\varrho$. From this, using Lemma 1.4.1 and Ch. 3, § 4, we see that $f\in J^{V_T}(E)$. So we have proved

THEOREM 7.2.2. *If* $\omega_2+\omega_3+\dots+\omega_s<2\varrho$ *and if* $f\in I^{V_T}(E)$ *satisfies* (7.2.5) *then* $f\in J^{V_T}(E)$.

Let us finish this paragraph by illustrating how particular information on the shape of E allows us to improve Theorem 7.2.2. Towards that let us suppose that $\omega_1=\omega_2=\dots=\omega_s=1$ and that E is the surface of a small sphere in $\mathbf{T}^s$ i.e. that E is the image of the surface of the sphere $S_s^\alpha=\{\mathbf{x}\in\mathbf{R}^s;\ |x|^2=x_1^2+x_2^2+\dots+x_s^2=\alpha\}$ (for some $0<\alpha<\pi$) by the exponential mapping $e(\mathbf{x})=(e^{ix_1},\ e^{ix_2},\dots,e^{ix_s})\in\mathbf{T}^s$ $\forall\mathbf{x}\in\mathbf{R}^s$. We then assert that the conclusion of the Theorem 7.2.2 holds for any $\varrho>(s-2)/2$.

Indeed preserving the notations of the proof of the theorem let us consider $\tilde{E}=d_s^{-1}(E)$, then for any $S\in BM_\infty(\tilde{E})$ we can apply (6.2.10) with $q_1=C$ an absolute constant (to do that we have to decompose the surface of the sphere into zones using a bounded partition of unity, and apply (6.2.10) separately to each zone choosing each time the axis x_1 appropriately) and obtain $\|\eta_r S\|_M \leqslant 2^{\frac{1}{2}r(s-2)}\|S\|_{BM_\infty}$ $(r \geqslant 1)$ instead of the coarser (7.2.4). So as soon as $\varrho > (s-2)/2$ and $f \in I^{V_T}(E)$ and $|f(t)| \leqslant C|t-E|^{\varrho}_{\omega}$ $\forall t \in \mathbf{T}^s$ we have $\langle \check{\pi}_r \check{f}, \eta_r S\rangle \xrightarrow[r\to\infty]{} 0$ and $\langle \check{f}, S\rangle = 0$. From there we finish the proof as above and obtain $f \in J^{V_T}(E)$.

The interest of the above example will be seen later when we compare it with L. Schwartz's example of failure of spectral synthesis in $A(\mathbf{R}^s)$ [11].

8. $A(G)$ as a subalgebra of $V(G)$

We shall denote throughout in this chapter by G a compact abelian group, by $V(G) = \mathbf{C}(G)\widehat{\otimes}\mathbf{C}(G)$ and as it is customary by $A(G) = \mathfrak{J}L^1(\hat{G})$, $\hat{G}$ being the dual group, also using J we shall always identify $V(G)$ to a dense subalgebra of $\mathbf{C}(G\times G)$ (cf. Ch. 2, § 1 and § 2). The main fact from harmonic analysis which we shall use is that if $f_1, f_2 \in L^2(G)$ then $f_1 * f_2 \in A(G)$ and $\|f_1 * f_2\|_A \leqslant \|f_1\|_{L^2}\|f_2\|_{L^2}$ which is an immediate consequence of Plancherel's theorem; the space $L^2(G)$ is of course taken with respect to the normalized Haar measure of G which we shall simply denote throughout as dx $(x \in G)$.

As it was pointed out in the introduction the formulation of the results of § 1 below is not the one I originally gave in [12]; I follow C. S. Herz in introducing the mappings M and P [13] and in arguing directly on the algebra rather than on the dual space. This brings out the ideas much more neatly.

Also we should like to point out that the results of this chapter differ essentially from what was presented up to now in the fact that here the global group structure of G on which the tensor algebra is considered is essentially used. When in Ch. 3, § 2, we considered tensor algebras over group spaces we only used the local regularization that is provided by convolving with an approximating identity, i.e., we used the group structure in a far less fundamental way.

§ 1. The mappings M and P

Let us define two linear mappings $\bar{M}$ and $\bar{P}$

$$\mathbf{C}(G) \underset{\bar{M}}{\rightarrow} \mathbf{C}(G\times G) \underset{\bar{P}}{\rightarrow} \mathbf{C}(G)$$

by setting

$$\bar{M}f(x,y) = f(x+y);\ \ \bar{P}F(x) = \int_G F(x-z,z)\,dz;\ \ f \in \mathbf{C}(G), F \in \mathbf{C}(G\times G);\ \ x, y \in G.$$

It is then clear that $\bar{P}\circ\bar{M} = I\partial(\mathbf{C}(G))$. Let us also identifying $A(G)$ to a subalgebra of $\mathbf{C}(G)$ and $V(G)$ to a subalgebra of $\mathbf{C}(G\times G)$ define:

$$M = \bar{M}|_{A(G)} : A(G) \to \mathbf{C}(G\times G);\ \ P = \bar{P}|_{V(G)} : V(G) \to \mathbf{C}(G).$$

Let now $\chi \in \hat{G}$ be a character of G which we can identify to $\chi \in A(G)$, we have then $M\chi(x, y) = \chi(x+y) = \chi(x)\,\chi(y)\ (x, y \in G)$ so that $M\chi \in V(G)$ and $\|M\chi\|_V \leqslant 1$. From this it follows by linearity that for all $f \in A(G)$ we have

$$Mf \in V(G);\ \|Mf\|_V \leqslant \|f\|_A. \tag{8.1.1}$$

Similarly let $f, g \in \mathbf{C}(G)$ and let $\psi = f \otimes g \in V(G)$ it is then clear that $P\psi(x) = f * g(x)\ (x \in G)$ and therefore $P\psi \in A(G)$ and $\|P\psi\|_A \leqslant \|f\|_{L^2} \|g\|_{L^2} \leqslant \|f\|_\infty \cdot \|g\|_\infty \leqslant \|\psi\|_V$. From this it follows again by linearity that for any $F \in V(G)$ we have:

$$PF \in A(G),\ \|PF\|_A \leqslant \|F\|_V. \tag{8.1.2}$$

So combining (8.1.1) and (8.1.2) we obtain

$$A(G) \underset{M}{\rightarrow} V(G) \underset{P}{\rightarrow} A(G)$$

two linear mappings such that

$$\|M\| \leqslant 1,\ \|P\| \leqslant 1,\quad P \circ M = I\partial(A(G)). \tag{8.1.3}$$

The norms of M and P are the operator norms for the $\|\ \|_A$ and $\|\ \|_V$ norms of the spaces. From (8.1.3) it follows at once that M is an isometry, and also by the definition of $\overline{M}$ it follows that M is a unitary algebra homomorphism which identifies $A(G)$ to $A^*(G)$ a closed subalgebra of $V(G)$. We shall prove now that $A^*(G)$ is equal to

$$V_\sigma = \{F \in V;\ F(x+g, y-g) = F(x, y) \quad x, y, g \in G\}$$

V_σ is the subalgebra of functions of $V(G)$ that respect the equivalence relation whose classes are the fibers Λ_g parallel to the antidiagonal of $G \times G$:

$$\Lambda_g = \{(x, y) \in G \times G;\ x + y = g\} \quad g \in G$$

Indeed by the very definition of $\overline{M}$ we have $A^* \subset V_\sigma$. Also for any $F \in V_\sigma$ we have

$$PF(x) = \int_G F(x-z, z)\, dz = F(x, 0) \quad \forall x \in G$$

$$M \circ PF(x, y) = F(x+y, 0) = F(x, y) \quad \forall x, y \in G$$

so that $F \in A^*$, and therefore $A^* = V_\sigma$. The identification of $A(G)$ with the closed subalgebra $A^* = V_\sigma \subset V(G)$ will prove a powerful tool for the study of tensor algebras as we shall illustrate in the next paragraph.

§ 2. The subalgebra $A(G)\cong A^*(G)\subseteq V(G)$

We shall prove here that the algebra homomorphism $M:A(G)\to V(G)$ satisfies the conditions of Lemma 1.4.1.

Indeed the mapping $P:V(G)\to A(G)$ is by (8.1.3) an inverse of M. Also for $\tilde{M}$ the transposed mapping of M we have:

$$\tilde{M}:G\times G\to G;\quad \tilde{M}(x,y)=x+y\in G$$

so that for any $F\in V(G)$ we have supp $(PF)\subset\tilde{M}(\text{supp } F)$ which is no other than the condition of Lemma 1.4.1 with an approximating inverse that reduces to the single mapping P, a situation that is in fact much simpler than the one considered there.

We can therefore draw the conclusions of Lemma 1.4.1 and denoting for any subset $E\subset G$

$$E^*=\tilde{M}^{-1}(E)=\{(x,y)\in G\times G;\, x+y\in E\}$$

we have

THEOREM 8.2.1. *For any closed set $E\subset G$ we have*

$$M^{-1}(I^V(E^*))=I^A(E);\quad M^{-1}(J^V(E^*))=J^A(E).$$

In particular if E is not a set of spectral synthesis of the group G and $I^A(E)\neq J^A(E)$ then $I^V(E^*)\neq J^V(E^*)$.

The above theorem has a converse that was first pointed out to me verbally by C. S. Herz, namely

THEOREM 8.2.2. *If $E\subset G$ is a set of spectral synthesis for the algebra $A(G)$ then E^* is a set of spectral synthesis for the algebra $V(G)$.*

We shall give a proof of Theorem 8.2.2 for the sake of completeness, although we shall not actually have the opportunity to use this theorem later.

Proof of Theorem 8.2.2. First observe that for every fixed $F\in V(G)$ if we define $F_g(x,y)=F(x+g,y-g)$ $(x,y,g\in G)$ the mapping $G\to V(G):g\to F_g$ is continuous. Thus we can define the convolution of any $F\in V(G)$ with any $\varphi\in \mathrm{C}(G)$ by setting:

$$F_\varphi=\int_G F_g\cdot\varphi(g)\,dg\in V(G).$$

It is then clear that if $F\in I^V(E^*)$ for some $E\subset G$ then $F_\varphi\in I^V(E^*)$ also, for all $\varphi\in \mathrm{C}(G)$. Also if $\chi\in\hat{G}(\subset \mathrm{C}(G))$ we have for all $x,y,g\in G$:

$$F_\chi(x+g,y-g)=\bar{\chi}(g)\,F_\chi(x,y).$$

In other words we have:

$$F_\chi(x, y) = \bar{\chi}(x)\,\tilde{F}_\chi(x, y);\quad \tilde{F}_\chi \in A^*(G);\quad \forall F \in V(G), \chi \in \hat{G};\ x, y \in G. \tag{8.2.1}$$

So now let us choose $\{\varphi_\beta \in \mathbf{C}(G)\}_{\beta \in B}$ a directed family of functions such that:

$$\varphi_\beta \geqslant 0,\quad \int_G \varphi_\beta(x)\,dx = 1,\quad \int_{G\setminus\Omega} \varphi_\beta(x)\,dy \xrightarrow[\beta \in B]{} 0 \tag{8.2.2}$$

for all Ω open nbd. of $0_G \in G$, and such that each φ_β is a trigonometric polynomial of G (i.e. $\varphi_\beta = \sum_{\chi \in \hat{G}} \gamma_\chi \chi$ a finite sum) such φ_β always exist. It follows then from (8.2.1) that if $F \in I^V(E^*)$ then for every $\beta \in B$ we have:

$$F_\beta = F_{\varphi_\beta} = \sum_{\chi \in \hat{G}} \chi F_{\beta,\chi}\ \text{(finite sum)};\quad F_{\beta,\chi} \in A^*(G) \cap I^V(E^*);\ \forall \beta, \chi.$$

So using the identification of $A^*(G)$ and $A(G)$ and Theorem 8.2.1 we see that for every $\beta \in B$ and $\chi \in \hat{G}$ $F_{\beta,\chi}$ can be identified to an element of $I^A(E) = J^A(E)$ and from this it follows that $F_{\beta,\chi} \in J^V(E^*)$ and therefore $F_\beta \in J^V(E^*)$. From this our theorem follows since by condition (8.2.2) $F_\beta \xrightarrow[\beta \in B]{} F$ in the algebra V.

Let us give here finally one illustration of the homomorphism M by settling the question we left open at the end of Ch. 7, § 1, and completing the proof of Theorem 7.1.1 for the critical case $\beta = 1/2$. Indeed towards that it suffices to consider some

$$f \in \Lambda_{\frac{1}{2}}(\mathbf{T}) \setminus A(\mathbf{T}) \subset \mathbf{C}(\mathbf{T})\ [7]$$

and set $F = \overline{M} f \in \mathbf{C}(\mathbf{T}^2)$ we have then $F \in \Lambda_{\frac{1}{2}}(\mathbf{T}^2)$ but $F \notin \mathbf{C}(\mathbf{T}) \hat{\otimes} \mathbf{C}(\mathbf{T})$.

9. The radial theory

In this chapter we shall base the study of the problem of spectral synthesis and symbolic calculus both for tensor algebras and group algebras on the theory of radial functions of $A(\mathbf{R}^n)$, and we start by recalling the definitions and main results of that theory.

§ 1. The radial functions

Let us for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ $(n \geqslant 1)$ denote $\mathbf{x} = (x_1, x_2, \dots, x_n)$ the coordinates of $\mathbf{x}$, and $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ the scalar product of $\mathbf{x}$ and $\mathbf{y}$ and also $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ the Euclidean norm of $\mathbf{x}$. Let us also denote by SO_n the n dimensional rotation group of $\mathbf{R}^n$ and for any $\mathbf{x} \in \mathbf{R}^n$ and $\sigma \in SO_n$ let $\sigma\mathbf{x}$ be the image of the vector $\mathbf{x}$ after rotation through σ. We shall also denote by $P^n \subset \mathbf{R}^n$ the positive quadrant i.e.

$$P^n = \{\mathbf{x} \in \mathbf{R}^n;\ x_j \geqslant 0 \quad j = 1, 2, \ldots, n\}.$$

Let us now for $n_1, n_2, \ldots, n_r$ positive integers denote

$$R_{n_1, n_2, \ldots, n_r} = \{f \in \mathbf{C}_0(P^r);\ f(|\mathbf{x}^{(1)}|, |\mathbf{x}^{(2)}|, \ldots, |\mathbf{x}^{(r)}|) \in A(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \ldots \times \mathbf{R}^{n_r})$$
$$\mathbf{x}^{(j)} \in \mathbf{R}^{n_j} \quad j = 1, 2, \ldots, r\}$$

an algebra of functions on P^r which we can also identify to a subalgebra of functions of $A(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \ldots \times \mathbf{R}^{n_r})$ the algebra of multiradial functions. We shall norm $R_{n_1, n_2, \ldots, n_r}$ with the norm it inherits as a closed subalgebra of A; it becomes then a Banach algebra and it is easy to verify that it is a regular semisimple *symmetric algebra whose spectrum can be identified to P^r. We shall adopt the abusive notation

$$f(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(r)}) = f(|\mathbf{x}^{(1)}|, |\mathbf{x}^{(2)}|, \ldots, |\mathbf{x}^{(r)}|) \in A(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \ldots \times \mathbf{R}^{n_r});$$
$$\mathbf{x}^{(j)} \in \mathbf{R}^{n_j}\, j = 1, 2, \ldots, r$$

for every $f \in R_{n_1, n_2, \ldots, n_r}$. When $r = 1$ and $n_1 = n$ we obtain $R_n \subset A(\mathbf{R}^n)$ the classical radial algebra that has been studied by many authors [14, 15, 16].

We shall summarize now some of the most important properties of the radial algebras.

Let $f \in R_n \subset A(\mathbf{R}^n)\ (n \geqslant 1)$ then $f = \hat{g}$ for some "radial" $g \in L^1(\mathbf{R}^n)$, more explicitly there exists $g(\varrho) = g_f(\varrho)$ $\varrho \in P^1$ a Borel function of $\varrho \geqslant 1$ such that

$$f(\mathbf{x}) = \int_{\mathbf{R}^n} e^{i\langle \mathbf{x}, \mathbf{y}\rangle} g_f(|\mathbf{y}|)\, d\mathbf{y};\quad \|f\|_A = \int_{\mathbf{R}^n} |g_f(|\mathbf{y}|)|\, d\mathbf{y} \tag{9.1.1}$$

factorizing then the integration of (9.1.1) as an integration along the radius vector and one over the surface of the unit sphere S_n we obtain [16]:

$$\left.\begin{aligned} f(\alpha) &= \frac{(2\pi)^{\frac{1}{2}n}}{\alpha^{\frac{1}{2}(n-2)}} \int_0^\infty g_f(\varrho)\, \varrho^{\frac{1}{2}n} J_{\frac{1}{2}(n-2)}(\alpha\varrho)\, d\varrho;\ \alpha \in P^1 \quad \alpha \neq 0 \\ \|f\|_A &= w_{n-1} \int_0^\infty |g_f(\varrho)|\, \varrho^{n-1}\, d\varrho;\ w_{n-1} = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} = \text{surface area of } S_n, \end{aligned}\right\} \tag{9.1.2}$$

where $J_\lambda(t)$ denotes the Bessel function of the first kind of order λ ($\lambda \in \mathbf{C}$). Relative to these functions we recall the formula [17]:

$$\frac{d^s}{d(t^2)^s}\left(\frac{J_\lambda(t)}{t^\lambda}\right) = \left(\frac{-1}{2}\right)^s \frac{J_{\lambda+s}(t)}{t^{\lambda+s}};\quad s \geqslant 1, \lambda \in \mathbf{C}. \tag{9.1.3}$$

Applying (9.1.3) to (9.1.2) and differentiating under the integral sign we obtain that for all $k \geqslant 1$ and $n = 2k+2$ and $\alpha > 0$ we have

$$\frac{d^k}{d(\alpha^2)^k} f(\alpha) = \frac{(-1)^k 2\pi^{k+1}}{\alpha^{2k}} \int_0^\infty g_r(\varrho)\,\varrho^{n-1} J_{2k}(\alpha\varrho)\,d\varrho; \quad \left|\frac{d^k f(\alpha)}{d(\alpha^2)^k}\right| \leqslant \frac{\Gamma(k+1)}{\alpha^{2k}} \|f\|_A. \tag{9.1.4}$$

Now for any $\sigma = (\sigma_1, \ldots, \sigma_r) \in SO = SO_{n_1} \times \ldots \times SO_{n_r}$ ($n_j \geqslant 1$ $j = 1, 2, \ldots, r$; $r \geqslant 1$) and any $f(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)}) \in A(\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_r}) = A$ ($\mathbf{x}^{(j)} \in \mathbf{R}^{n_j}, j = 1, 2, \ldots, r$) let us define

$$\sigma f(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)}) = f(\sigma_1 \mathbf{x}^{(1)}, \ldots, \sigma_r \mathbf{x}^{(r)})$$

and also for any $\mu \in M(SO)$ let us define:

$$\mu * f(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)}) = \int_{SO} \sigma f(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)})\,d\mu(\sigma).$$

It is clear then that $h_{SO} * f \in R_{n_1, \ldots, n_r}$ ($f \in A$) and also $h_{SO} * f = f$ for all $f \in R_{n_1, \ldots, n_r}$ h_{SO} denotes the normalized Haar measure of SO.

Now as we pointed out in Chapter 1, § 5 we can identify $A = A(\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_r})$ with $A(\mathbf{R}^{n_1}) \hat{\otimes} \ldots \hat{\otimes} A(\mathbf{R}^{n_r})$ so taking into account our previous remarks and the fact that for every $f = f_1 \otimes \ldots \otimes f_r \in A$ ($f_j \in A(\mathbf{R}^{n_j})$, $j = 1, 2, \ldots, r$) we have $h_{SO} * f = (h_{SO_{n_1}} * f_1) \otimes \ldots \otimes (h_{SO_{n_r}} * f_r)$ we obtain a canonical isometric identification

$$R_{n_1, \ldots, n_r} \cong R_{n_1} \hat{\otimes} \ldots \hat{\otimes} R_{n_r}. \tag{9.1.5}$$

An immediate application of (9.1.4) to (9.1.5) with $n_1 = n_2 = \ldots = n_r = 4$ $k = 1$ gives

$$\left|\left(\frac{\partial^r f(\alpha_1, \ldots, \alpha_r)}{\partial(\alpha_1^2), \ldots, \partial(\alpha_r^2)}\right)_{\alpha_1 = \ldots = \alpha_r = 1}\right| \leqslant \|f\|_A; \quad \forall f \in R_{4,4,\ldots,4}. \tag{9.1.6}$$

Let now again S_n be the unit sphere in $\mathbf{R}^n$, and let μ_n be the rotation invariant (uniform) measure on S_n of total mass 1. Then either by direct computation of the Fourier transform [11], or by dualizing (9.1.4) and the analogous result for $n = 2k+1$ (= odd), we see that the distribution $(\partial^s \mu_n)/\partial r^s$, the radial derivative of order s of μ_n, is a pseudomeasure (i.e. has bounded Fourier transform) for $s = 1, 2, \ldots [(n-1)/2] = m =$ integer part of $(n-1)/2$. From this it follows that if $f \in R_n \subset A(\mathbf{R}^n) = A$, $f(1) = 0$, $f'(1) \neq 0$ then [14, 18]

$$f \in I^A(S_n); \quad f^m \notin J^A(S_n), \quad m = \left[\frac{n-1}{2}\right]. \tag{9.1.7}$$

From (9.1.7) it follows in particular that $S_n \subset \mathbf{R}^n$ is not a set of spectral synthesis of the group $\mathbf{R}^n$ ($n \geqslant 3$). From this using the fact that the two groups $\mathbf{R}^n$ and $\mathbf{T}^n$ are locally isomorphic it follows that a small sphere S_n^α in $\mathbf{T}^n$ (cf. end of Chapter 7, § 2) is not a set of spectral synthesis for $n\geqslant 3$. In this context taking into account that $\mathbf{T}^n$ is a real analytic manifold (Lie group) we can rewrite (9.1.7) as follows: For all $n\geqslant 3$ and $\alpha<\pi$ there exists $f\in \mathcal{D}(\mathbf{T}^n)\subset A(\mathbf{T}^n)=A$ which is real analytic in some open nhd. of $S_n^\alpha\subset \mathbf{T}^n$ and such that

$$f\in I^A(S_n^\alpha);\ f^m\notin J^A(S_n^\alpha);\ m=\left[\frac{n-1}{2}\right]. \tag{9.1.8}$$

§ 2. The problem of spectral synthesis for tensor algebras and group algebras

We shall say here that spectral synthesis fails in a commutative regular semisimple Banach algebra R if there exists $E\subset \mathfrak{M}_R$ a closed subset that is not a set of spectral synthesis.

We have seen in the previous paragraph that spectral synthesis fails in $A(\mathbf{T}^3)$. This together with Theorem 8.2.1 shows that spectral synthesis fails in $V(\mathbf{T}^3)$, from this using Lemma 1.4.1 and Chapter 3, § 4 we obtain

THEOREM 9.2.1. *Spectral synthesis fails in the algebra* $V(D_\infty)$.

Now Theorem 9.2.1 together with Theorem 4.2.3 provides us with a new proof of the following classical theorem which is due to P. Malliavin [19].

THEOREM 9.2.2. *Spectral synthesis fails in $A(G)$ for every infinite compact abelian group G.*

Let us now introduce the following

Definition (P. Malliavin [19]). Let G be any compact abelian group, then we say that $E\subset G$ a closed subset is a set of spectral resolution if every closed subset $E_1\subset E$ is a set of spectral synthesis.

Taking then into account Theorem 9.2.1 and Theorem 4.3.3 we see that we have

THEOREM 9.2.3. *Let G be any compact metrizable abelian group and let $P, Q\subset G$ be two perfect subsets of G. Then the set $P+Q\subset G$ is not a set of spectral resolution.*

It is very easy to suppress the condition of metrizability of G from Theorem 9.2.3 by considering metrizable quotients; we leave this to the reader.

§ 3. The problem of symbolic calculus

Let K be a compact space and let $R\subset \mathbf{C}(K)$ be a symmetric (under complex conjugation), unitary regular Banach algebra of functions on K, whose spectrum can be identified with K, and let further $r \in R$ be a real element (i.e. a real valued function on K). We then set

$$[r]^R = \{\Phi|_I;\ \Phi \in \mathbf{C}(\mathbf{R}),\ \Phi \circ r \in R\} \subset \mathbf{C}(I)$$

$$[R] = \bigcap_{r(K) \subset I} [r]^R \subset \mathbf{C}(I),$$

where I throughout in this paragraph will denote the interval $[-1, 1]$.

Let now $\sigma = \{s_j > 0\}_{j=1}^{\infty}$ ($\sum_{j=1}^{\infty} s_j < +\infty$) be a summable sequence of positive numbers, and let us denote by $\mathcal{D}(I)$ the ∞-differentiable functions on I (including the end points) and let

$$C_\sigma = \{f \in \mathcal{D}(I);\ \sup_{x \in I,\, n \geqslant 1} |s_1 s_2, \ldots, s_n f^{(n)}(x)|^{1/n} < +\infty\}$$

We shall now prove using the theory of radial functions the following

LEMMA 9.3.1. *Let $\varepsilon > 0$ and let $\sigma = \{s_j > 0\}_{j=1}^{\infty}$ be a positive summable sequence ($s = \sum_{j=1}^{\infty} s_j < +\infty$) we can then find $f = f_{\sigma,\varepsilon} \in A(\mathbf{T}^\infty) = A$ a real function such that $[f]^A \subset C_\sigma$ and such that $\|f\|_A \leqslant 1 + \varepsilon$.*

Proof. Towards that let us denote by $\gamma = p \times p \times p \times p : \mathbf{R}^4 \to \mathbf{T}^4$ the canonical projection ($p : \mathbf{R} \to \mathbf{T}$, $p(r) = e^{ir} \in \mathbf{T}$, $\forall r \in \mathbf{R}$) and let us fix once and for all $\varphi, h \in R_4$ two real functions such that:

$$[0; 2] \subset \{\alpha;\ h(\alpha) = 1\} \subset \operatorname{supp} h \subset [0, 3]$$

$$\varphi(0) = \varphi(1) = 0;\ \operatorname{supp} \varphi \subset [0, 2],\ \frac{d\varphi(\alpha)}{d(\alpha^2)} = 1 \quad \text{at} \quad \alpha = 1$$

φ and h can then be identified to functions of $A(\mathbf{R}^4)$ whose support is contained in $\{\mathbf{x} \in \mathbf{R}^4;\ |x_j| \leqslant 3,\ 1 \leqslant j \leqslant 4\}$. Let us then denote by $\theta \in A(\mathbf{T}^4)$ the function that is defined by $\theta \circ \gamma(\mathbf{x}) = \varphi(\mathbf{x})$ $\forall \{\mathbf{x} \in \mathbf{R}^4;\ |x_j| \leqslant \pi,\ 1 \leqslant j \leqslant 4\}$. Let also $\{\tilde{G}_j \cong \mathbf{R}^4\}_{j=1}^{\infty}$ and $\{G_j \cong \mathbf{T}^4\}_{j=1}^{\infty}$ be two infinite sequences of groups isomorphic to $\mathbf{R}^4$ and $\mathbf{T}^4$ respectively, and let $\{\gamma_j\}$, $\{(\varphi_j, h_j)\}$ and $\{\theta_j\}$ the sequences that correspond to γ, (φ, h) on θ respectively in the identification of $\tilde{G}_j$ with $\mathbf{R}^4$ and G_j with $\mathbf{T}^4$. Let us finally denote by $G = \mathbf{T} \times G_1 \times G_2 \times \ldots \cong \mathbf{T}^\infty$ and let us fix ζ an arbitrary positive number and let us set:

$$f(g) = \sin t + \zeta \sum_{j=1}^{\infty} s_j \theta_j(g_j);\ \ g = (t, g_1, g_2, \ldots) \in G.$$

It is then clear that $f \in A(G)$ and $\|f\|_A \leqslant 1 + \zeta s \|\theta\|_A$. Let now $\Phi \in \mathbf{C}(\mathbf{R})$ be such that $F = \Phi \circ f \in A(G)$ and let us also set for every $t \in \mathbf{T}$ and $k \geqslant 1$

$f_{t,k}(g_1, \ldots, g_k) = f(t, g_1, \ldots, g_k, 0_{k+1}, 0_{k+2}, \ldots)$; $F_{t,k}(g_1, \ldots, g_k) = F(t, g_1, \ldots, g_k, 0_{k+1}, 0_{k+2}, \ldots)$; $(g_1, \ldots, g_k) \in G_1 \times \ldots \times G_k$; 0_k is the zero of G_k.

It is then clear that

$$f_{t,k}, F_{t,k} \in A(G_1 \times \ldots \times G_k);\ F_{t,k} = \Phi \circ f_{t,k};\ \|F_{t,k}\|_A \leqslant \|F\|_A. \tag{9.3.1}$$

Let us finally set:

$$\tilde{F}_{t,k} = [F_{t,k} \circ (\gamma_1 \times \ldots \times \gamma_k)]\,(h_1 \otimes \ldots \otimes h_k) \in A(\tilde{G}_1 \times \ldots \times \tilde{G}_k);$$

then since for all $(\tilde{g}_1, \ldots, \tilde{g}_k) \in \operatorname{supp}\,(h_1 \otimes \ldots \otimes h_k) \subset \tilde{G}_1 \times \ldots \times \tilde{G}_k$ we have:

$$[f_{t,k} \circ (\gamma_1 \times \ldots \times \gamma_k)]\,(\tilde{g}_1, \ldots, \tilde{g}_k) = \sin t + \zeta \sum_{j=1}^{k} s_j \varphi_j(\tilde{g}_j)$$

it follows that:

$$\tilde{F}_{t,k} \in R_{4,4,\ldots,4};\ \|\tilde{F}_{t,k}\|_A \leqslant \|F_{t,k}\| \cdot \|h\|_A^k \tag{9.3.2}$$

and that when $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ lies in a small enough nhd. of $\{1\}^k = (1, 1, \ldots, 1)$ in P^k we have

$$\tilde{F}_{t,k}(\alpha_1, \ldots, \alpha_k) = \Phi(\sin t + \zeta \sum_{j=1}^{k} s_j \varphi_j(\alpha_j)). \tag{9.3.3}$$

Thus applying (9.1.6) on (9.3.3) and using (9.3.1) and (9.3.2) we see that

$$|s_1, s_2, \ldots, s_k\, \Phi^{(k)}(\sin t)| \leqslant \|h\|_A^k\, \zeta^{-k}\, \|F\|_A. \tag{9.3.4}$$

So if we set $\zeta = \varepsilon/(s\|\theta\|_A)$ and take into account that t and k in (9.3.4) are arbitrary we obtain

$$\sup_{x \in I;\ k \geqslant 1} |\Phi^{(k)}(x)\, s_1 s_2 \ldots s_k|^{1/k} < +\infty,\ \|f\| \leqslant 1 + \varepsilon$$

in other words f above provides the required function $f_{\sigma,\varepsilon}$ of our lemma.

Now quite generally for any compact group G using the identification of $A(G)$ with $A^*(G) \subset V(G)$ of Chapter 8, § 1 we see that for any $f \in A(G)$ we have

$$[Mf]^{V(G)} = [f]^{A(G)};\ \|Mf\|_V = \|f\|_A. \tag{9.3.5}$$

Finally using Theorem 3.5.4 with $\omega = \infty$ we see that for any $F \in V(\mathbf{T}^\infty)$ if we set $\tilde{F} = \check{d}_\infty \hat{\otimes} \check{d}_\infty(F) \in V(D_\infty)$ we have:

$$[\tilde{F}]^{V(D_\infty)} = [F]^{V(\mathbf{T}^\infty)};\ \|\tilde{F}\|_V = \|F\|_V. \tag{9.3.6}$$

So combining Lemma 9.3.1 and (9.3.5) and (9.3.6) we can state

THEOREM 9.3.1. *For any $\varepsilon > 0$ and any positive summable sequence*

$$\sigma = \{s_j > 0\}_{j=1}^{\infty} \quad (\sum_{j=1}^{\infty} s_j < +\infty)$$

we can find $f = f_{\sigma,\varepsilon} \in V(D_\infty)$ such that $[f]^V \subset \mathbf{C}_\sigma$ and $\|f\|_V \leqslant 1 + \varepsilon$.

As an immediate corollary of Theorem 9.3.1 we shall now obtain an improvement of P. Malliavin's maximal individual symbolic calculus theorem [20]; namely

THEOREM 9.3.2. *Let $\sigma=\{s_j>0\}_{j=1}^{\infty}$ be an arbitrary positive summable sequence ($\sum_{j=1}^{\infty} s_j< +\infty$) and let $\varepsilon>0$ be some positive number, then:*

(i) *For every compact abelian group G that contains a perfect Kronecker set we can find $f=f_{\sigma,\varepsilon}\in A(G)$ such that $[f]^A\subset \mathrm{C}_\sigma$ and $\|f\|_A\leqslant 1+\varepsilon$.*

(ii) *For every compact abelian group G that contains a perfect K_p set (p some natural prime) we can find $f=f_{\sigma,\varepsilon}\in A(G)$ such that $[f]^A\subset \mathrm{C}_\sigma$ and $\|f\|_A\leqslant a_p+\varepsilon$, where a_p is a constant depending only on p such that $1\leqslant a_p\leqslant 4$ for all p.*

Proof. From Theorem 4.2.1 (α) we can find $E\subset G$ a compact subset such that $A(E)$ is isometrically isomorphic to $V(D_\infty)$. Our theorem is then a corollary of Theorem 9.3.1.

(ii) In case (ii) we can find $E\subset G$ a compact subset such that $A(E)$ can be identified topologically to $V(D_\infty)$ with the inequality on the norms $\|\ \|_A\leqslant 4\|\ \|_V$, thus Theorem 9.3.1 implies that we can find $f=f_{\sigma,\varepsilon}\in A(G)$ such that

$$[f]^A\subset \mathrm{C}_\sigma;\ \|f\|_A\leqslant 4+\varepsilon \tag{9.3.7}$$

and this proves our result.

Observe that every infinite compact group falls in one of the two cases (i) or (ii) above (cf. Chapter 4, § 1 (vi)).

Observe also that contrary to what was stated in [12] Theorem 9.3.2 (ii) is best possible in the sense that the bound for $\|f\|_A$ cannot be improved to $1+\varepsilon$.

The exact value of a_p is in fact:

$$a_p=\inf\{\|f\|_A;\ f\in A([\mathbf{Z}(p)]^\infty);\ f([\mathbf{Z}(p)]^\infty)\supset[-1,1]\}$$

and $a_p>1$ for $p\neq 2$. To prove this we may suppose, taking if need be a subgroup of G [cf. Chapter 4, § 1, (v)], that $G=[\mathbf{Z}(p)]^\infty$. Let then $f_1\in A(G)$ be such that $\|f_1\|_A<a_p+\varepsilon$, $f_1(G)\supset[-1,1]$; and with f satisfying (9.3.7) and ζ some positive number let us consider

$$\varphi=\varphi_\zeta(g_1,g)=f_1(g_1)+\zeta f(g)\in A(G\times G);\ (g_1,g)\in G\times G$$

We have then
$$\|\varphi\|_A=\|f_1\|+\zeta\|f\|<a_p+\varepsilon+5\zeta. \tag{9.3.8}$$

Also for every fixed $g_1\in G$ and every $\Phi\in\mathbf{C}(\mathbf{R})$ we have $\psi(g)=\Phi[f_1(g_1)+\zeta f(g)]\in A(G)$ as soon as $\Phi\circ\varphi\in A(G\times G)$; so that if we set $\Phi_{g_1,\zeta}(t)=\Phi[f_1(g_1)+\zeta t], t\in I$ we have $\Phi_{g_1,\zeta}(t)\in[f]^{A(G)}$ and therefore $\Phi_{g_1,\zeta}\in\mathrm{C}_\sigma$; and since this holds for arbitrary g_1 we see from our choice of

f_1 that $\Phi \in \mathcal{C}_\sigma$ and this is valid for arbitrary $\zeta > 0$. So taking into account (9.3.8) and the fact that $G \times G \simeq [\mathbf{Z}(p)]^\infty = G$ we see that φ_ζ gives us the required function as soon as ζ is sufficiently small.

Relative to the classes of ∞-differentiable functions that we considered in Theorems 9.3.1 and 9.3.2 we would like to observe that we essentially obtain all non quasi-analytic classes, in the sense that given $\mathcal{C} \subset \mathcal{D}(I)$ a non quasi-analytic class we can find a summable sequence $\sigma = \{s_j > 0\}_{j=1}^n$ such that $\mathcal{C}_\sigma \subset \mathcal{C}$ [21]. Also if we denote by:

$$\mathcal{A}(I) = \{f \in \mathcal{D}(I);\ f \text{ analytic on } (-1, 1)\}$$

we have $\mathcal{A}(I) \supset \cap_\sigma \mathcal{C}_\sigma$ the intersection being taken over all positive summable sequences σ [22].

Using the above remarks we see that we have proved the following

THEOREM 9.3.3. (i) $[V(D_\infty)] \subset \mathcal{A}(I)$

(ii) $[A(G)] \subset \mathcal{A}(I)$ *for every compact group* G.

Theorem 9.3.3 (ii) above is due to Y. Katznelson [23]. Let us finally introduce the

Definition. A compact set $E \subset G$ of the compact abelian group G is called a set of analyticity if $[A(E)] \subset \mathcal{A}(I)$.

Then we see that we have actually proved the following [cf. Theorems 4.3.3 and 9.3.3 (i)]:

THEOREM 9.3.4. *Let G be a compact metrizable abelian group and let $P, Q \subset G$ be two perfect subsets. Then the set $P + Q \subset G$ is a set of analyticity.*

In fact a theorem more general than Theorems 9.3.3 and 9.3.4 holds. We state it here without proof.

THEOREM 9.3.5. (i) *Let K_1, K_2 be two infinite compact spaces then $[\mathbf{C}(K_1) \hat{\otimes} \mathbf{C}(K_2)] \subset \mathcal{A}(I)$.*

(ii) *Let G be a compact abelian group and $E \subset G$ a closed subset such that for any $N \geq 1$ we can find $X = \{x_1, x_2, \ldots, x_N\} \subset G$, $Y = \{y_1, y_2, \ldots, y_N\} \subset G$ two families of N distinct points each such that $X + Y \subset E$. Then E is a set of analyticity.*

The proof is based on a classical argument [24] of evaluations of exponentials which deliberately we wish to avoid in this paper. The proof of (i) above is to be found in [12]. The proof of (ii) has never been published but it is an easy although slightly technical exercise in harmonic analysis and part (i) of the theorem.

§ 4. The best possible of some Beurling–Pollard constants of Chapter 7, § 2

Let us now for $n \geq 3$ and some sufficiently small positive α denote by $S_n^\alpha = E \subset \mathbf{T}^n$ the sphere of radius α in $\mathbf{T}^n$ [cf. (9.1.8)], and let us choose $f \in A(\mathbf{T}^n) = A$ which is real analytic in some nhd. of $E \subset \mathbf{T}^n$ and such that:

$$f \in I^A(E) \cap \mathcal{D}(\mathbf{T}^n);\ f^m \notin J^A(E) \quad m = \left[\frac{n-1}{2}\right]. \tag{9.4.1}$$

Using then the notations of Chapter 8, § 2 we see that $F = Mf \in V(\mathbf{T}^n) = V$ is analytic in some nhd. of E^* in $\mathbf{T}^{2n}$ and also

$$F \in I^V(E^*) \cap \mathcal{D}(\mathbf{T}^{2n});\quad F^m \notin J^V(E^*);\quad m = \left[\frac{n-1}{2}\right]. \tag{9.4.2}$$

Let us now construct for $r \leq n$ and $\varepsilon > 0$ arbitrary

$$\theta : \mathbf{T}^r \to \mathbf{T}^n;\quad \theta(t) = (\theta_1(t), \ldots, \theta_n(t));\quad t \in \mathbf{T}^r \tag{9.4.3}$$

a continuous mapping such that

$$\theta_j \in \Lambda_{(r/n)-\varepsilon} \quad j = 1, 2, \ldots, n;\ \theta(\mathbf{T}^r) = \mathbf{T}^n. \tag{9.4.4}$$

This can be done using a modification of the classical construction of a Peano curve. We can in fact choose our θ in (9.4.3) so that in addition to (9.4.4) we have

$$\left.\begin{aligned} &\theta_j(t_1, \ldots, t_r) = \sum_{n_1, \ldots, n_r \in \mathbf{Z}} \alpha_{n_1, \ldots, n_r} \exp\left[2\pi i(n_1 t_1 + \ldots + n_r t_r)\right] \in \mathbf{T} \subset \mathbf{C} = \mathbf{R}^2 \\ &\sum |\alpha_{n_1, \ldots, n_r}| (1 + |n_1| + \ldots + |n_r|)^{(r/n)-\varepsilon} < +\infty;\quad j = 1, 2, \ldots, n. \end{aligned}\right\} \tag{9.4.5}$$

To satisfy (9.4.5) is less trivial. We do not give the proof, however, since much sharper results than those obtained from (9.4.5) can be given.

Let now θ satisfy (9.4.3) and F satisfy (9.4.2) and let us consider the induced mapping: $\check{\theta} \hat{\otimes} \check{\theta} : V(\mathbf{T}^n) \to V(\mathbf{T}^r)$. Then denoting by $\tilde{E} = (\theta \times \theta)^{-1}(E^*) \subset \mathbf{T}^{2r}$ we have

$$\chi = \check{\theta} \hat{\otimes} \check{\theta}(F) = F \circ (\theta \times \theta) \in \Lambda_{(r/n)-\varepsilon}(\mathbf{T}^{2r}) \cap V(\mathbf{T}^r) \cap I(\tilde{E}) \tag{9.4.6}$$

and also by Lemma 1.4.1 and Chapter 3, § 2 $\chi^m \notin J^V(\tilde{E})\ (m = [(n-1)/2])$, also (9.4.6) implies that

$$|\chi^m(x)| \leq C\, |x - \tilde{E}|_{2r}^{((mr)/n) - m\varepsilon} \qquad \text{(cf. 7.2.5)}.$$

So letting r be fixed and $n \to \infty$ we have $m/n \to 1/2$ so that, ε being arbitrary, we obtain

THEOREM 9.4.1. *For any $r \geq 1$ and $\alpha < r/2$ we can find $\tilde{E} \subset \mathbf{T}^{2r}$ a closed subset and $h \in V(\mathbf{T}^r) = V$ such that: $h \in I^V(\tilde{E}) \setminus J^V(\tilde{E})$; $|h(x)| \leq C|x - \tilde{E}|_{2r}^\alpha$. In other words Theorem 7.2.2 with $s = 2$ is best possible.*

Let us now suppose that θ actually satisfies (9.4.5). Then since F is analytic in some nhd. of E^* and since $\mathbf{T}^{2n}$ (with the canonical embedding) is analytically embedded in $\mathbf{R}^{4n}$ it follows that χ defined again as before from (9.4.6) coincides with some $\tilde{\chi}\in A(\mathbf{T}^{2r})\subset V(\mathbf{T}^r)$ in some nhd. of $\tilde{E}$. So letting again r be fixed and $n\to\infty$ we obtain

THEOREM 9.4.2. *For $r\geqslant 1$ and $\alpha<r/2=\alpha_0$ we can find $\tilde{E}\subset\mathbf{T}^{2r}$ a closet subset and $f\in A(\mathbf{T}^{2r})$ such that $f\in I^A(\tilde{E})\setminus J^A(\tilde{E})$, $|f(x)|<C|x-\tilde{E}|^{\alpha}_{2r}$.*

Proof. What we actually obtain by the process described above is some function h satisfying the conclusion of Theorem 9.4.1 and being such that $h\in A$ in some nhd. of $\tilde{E}$. From this our theorem follows since $A\underset{\rightarrow}{\subseteq} V$.

Comparing this last theorem with the classical Beurling–Pollard estimations [9], we see that we are very far from the critical constant which is $\alpha_0=r$ and not $\alpha_0=r/2$. Sharp results in that direction have been obtained by J.-P. Kahane [25], [30].

10. The symplectic form and applications to the problem of spectral synthesis

Let $R\subset\mathbf{C}(K)$ be a unitary regular *symmetric Banach algebra of functions on the compact space K whose spectrum can be identified to K, let further $0\neq\mu\in M^+(K)$ be a positive Radon measure fixed once and for all and let also $f\in R\subset\mathbf{C}(K)$ be a real valued function of R.

We now consider for every $u\in\mathbf{R}$ the element of R' (=the dual of R) defined by $F_u=e^{iuf}d\mu$ i.e. the functional $F_u\in R'$ defined by

$$\langle F_u, r\rangle=\int_K e^{iuf(k)}\,r(k)\,d\mu(k),\quad \forall r\in R$$

and we denote by $\|e^{iuf}\|'=\|F_u\|_{R'}$ the norm of F_u as an element of R'. We can state the following theorem which is due to P. Malliavin [19].

THEOREM. *If $f\in R$ a real function is such that for some positive integer $p\geqslant 1$:*

$$\int_{-\infty}^{+\infty}|u|^p\,\|e^{iuf}\|'\,du<+\infty$$

then there exists $a\in R$ a real number such that

$$(f-a)\in I^R(f^{-1}(a));\ (f-a)^p\notin J^R(f^{-1}(a)).$$

We shall now exploit the above theorem to obtain information on the problem of spectral synthesis for tensor algebras.

§ 1. The evaluation of the exponential of the symplectic form

Let us consider for $n \geqslant 1$

$$V_n = \mathbf{C}(\mathbf{I}) \hat{\otimes} \mathbf{C}(\mathbf{I}) \hat{\otimes} \dots \hat{\otimes} \mathbf{C}(\mathbf{I}); \ \mathbf{I} = [0, 1]$$

the product being taken n times, and let us denote by $\mathbf{I}^n = \mathbf{I} \times \mathbf{I} \times \dots \times \mathbf{I}$; for every $x \in \mathbf{I}^n$ let $\mathbf{x} = (x_1, x_2, \dots x_n)$ $(x_j \in \mathbf{I}; j = 1, 2, \dots n)$ be its coordinates, let also $d\mathbf{x} = dx_1 \times \dots \times dx_n$ be the Lebesgue measure volume element on $\mathbf{I}^n$. The spaces $L^2(\mathbf{I})$ and $L^\infty(\mathbf{I})$ which we shall consider in this paragraph are of course taken with respect to the Lebesgue measure of $\mathbf{I}$. Let us finally denote by:

$$\sigma_n = \sigma_n(\mathbf{x}) = \sum_{j=1}^{n-1} x_j x_{j+1} \in V_n \subsetneq \mathbf{C}(\mathbf{I}^n), \ (n \geqslant 2);$$

which for obvious reasons we shall call the symplectic form; we shall also set conventionally $\sigma_1 \equiv 0$.

We now consider $T_n(u) = \|e^{iu\sigma_n}\|' = \|e^{iu\sigma_n}\|_{BM_n}$ with the notations introduced at the introduction of this chapter with $R = V_n$, $K = \mathbf{I}^n$, $R' = (V_n)' = BM_n$, $f = \sigma_n$, $d\mu = d\mathbf{x}$. We shall prove the following

THEOREM 10.1.1. $\qquad T_n(u) \leqslant |u|^{-(n-1)/2} \quad (n \geqslant 1, u \in \mathbf{R}).$

This theorem is an immediate corollary of the following more general

THEOREM 10.1.2. *Let us for $f_1, f_2, \dots, f_{n-1} \in L^\infty(\mathbf{I})$ $(n \geqslant 1)$ and $g \in L^2(\mathbf{I})$ denote by:*

$$L_u(f_1, f_2, \dots, f_{n-1}; g) = \int e^{iu\sigma_n} f_1(x_1) f_2(x_2) \dots f_{n-1}(x_{n-1}) g(x_n) \, d\mathbf{x}.$$

Then for all $n \geqslant 1$ g and f_j $(j = 1, 2, \dots, n-1)$ as above we have:

$$|L_u(f_1, f_2, \dots, f_{n-1}; g)| \leqslant |u|^{-((n-1)/2)} \|f_1\|_\infty \dots \|f_{n-1}\|_\infty \|g\|_{L^2}. \tag{10.1.1}$$

Proof. The proof is done by induction on n; for $n = 1$ the set of f's is vacuous and

$$|L_u(g)| = \left| \int_{\mathbf{I}} e^{iu\sigma_1} g \, dx_1 \right| \leqslant \|g\|_{L^2}$$

so (10.1.1) holds and the induction starts. So let us suppose that (10.1.1) holds with n replaced by $n-1$ and all choices of $f_1, f_2, \dots, f_{n-2}; g$.

Let us now for $f_1, f_2, \dots, f_{n-1}; g$, as in the theorem define $f^*_{n-1} \in L^\infty(\mathbf{R})$ and $g^* \in L^2(\mathbf{R})$ by:

$$f^*_{n-1}(t) = \begin{cases} f_{n-1}(t); & t \in \mathbf{I} \\ 0 & ; t \notin \mathbf{I} \end{cases} \qquad g^*(t) = \begin{cases} g(t); & t \in \mathbf{I} \\ 0 & ; t \notin \mathbf{I}. \end{cases}$$

We have then identifying $\sigma_{n-1} \in V_{n-1}$ to a function on $\mathbf{I}^n$

$$\left.\begin{aligned} L_u(f_1, \ldots, f_{n-1}; g) &= \int_0^1 \ldots \int_0^1 e^{iu\sigma_{n-1}} f_1(x_1) \ldots f_{n-2}(x_{n-2})\, g_u(x_{n-1})\, dx_1 \ldots dx_{n-1}; \\ g_u(t) &= f_{n-1}(t) \int_0^1 e^{iut\tau} g(\tau)\, d\tau = f_{n-1}(t)\, \hat{g}^*(ut), \quad t \in \mathbf{I}; \end{aligned}\right\} \tag{10.1.2}$$

where ^ as usual denotes the Fourier transform. But then Plancherel's theorem gives us

$$\left.\begin{aligned} \|g_u(t)\|^2_{L^2(\mathbf{I})} &= \int_{-\infty}^{+\infty} |f^*_{n-1}(t)\, \hat{g}^*(ut)|^2\, dt \\ &\leqslant \|f^*_{n-1}\|^2_\infty\, |u|^{-1}\, \|g^*\|^2_{L^2(\mathbf{R})} = |u|^{-1}\, \|f_{n-1}\|^2_\infty\, \|g\|^2_{L^2(\mathbf{I})}. \end{aligned}\right\} \tag{10.1.3}$$

So using (10.1.3) and applying the inductive hypothesis on the second member of (10.1.2) we obtain the inductive step and complete the proof of the theorem.

§ 2. A best possible constant in the Beurling–Pollard theory

Let us now use P. Malliavin's theorem of the introduction to this chapter on $R = V_n \underset{\rightarrow}{\subseteq} \mathrm{C}(\mathbf{I}^n)$, $f = \sigma_n$ and $d\mu = d\mathbf{x}$ with the notations of our previous paragraph. We obtain then taking into account Theorem 10.1.1

THEOREM 10.2.1. *For all $n \geqslant 6$ there exists $a_n \in \mathbf{R}$ a real number such that:*

$$\sigma_n - a_n \in I^{V_n}(\Sigma_n); \;\; (\sigma_n - a_n)^{r_n} \notin J^{V_n}(\Sigma_n); \;\; \Sigma_n = \sigma_n^{-1}(a_n),$$

where r_n is the largest integer strictly smaller than $(n-3)/2$.

If now with a_n, Σ_n and r_n as in the theorem we denote by $\varphi_n = (\sigma_n - a_n)^{r_n}$ we see that

$$\varphi_n \in I^{V_n}(\Sigma_n) \setminus J^{V_n}(\Sigma_n); \quad \varphi_n(\mathbf{x}) \leqslant C[d(\mathbf{x}, \Sigma_n)]^{r_n}, \;\; \forall \mathbf{x} \in \mathbf{I}^n; \tag{10.2.1}$$

where d denotes the euclidean distance on $\mathbf{I}^n$. So if we embed $\mathbf{I}$ as a closed arc in $\mathbf{T}$ and take into account Ch. 3, § 3, Case 1, and if we let $n \to \infty$ in (10.2.1) we see that $r_n \sim n/2$ and thus that Theorem 7.2.2 is best possible asymptotically at least as $s \to \infty$ for

$$\omega_2 = \omega_3 = \ldots = \omega_s = 1.$$

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