FOUNDATIONS OF THE THEORY OF DIRICHLET SERIES

BY

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1. A modern reader, familiar with the methods of functional analysis, is struck with the conviction that the classical theory of Dirichlet series [3, 5, 6] must have content expressible in more congenial language. Harald Bohr recognized the analogy between harmonic series and the Fourier series of functions on the circle; later his theory of almost-periodic functions was shown to be part of a theory of Fourier series on compact abelian groups that embraced the classical case of the circle group as well. Various generalizations treat the spaces L^p, and it is fairly clear by now how much of Fourier series can be developed in the more general setting. Nevertheless another part of the theory of Dirichlet series, to which Bohr himself contributed a great deal, does not seem to be harmonic analysis. This is the part depending on the *Dirichlet condition*

$$\lambda_1 < \lambda_2 < \ldots; \quad \lambda_n \to \infty \tag{1}$$

imposed on the exponents of a harmonic series. The purpose of this paper is to suggest that part of this side of the theory can be approached from the point of view of Banach algebra. I shall set forth the ideas as simply as possible, without striving for the greatest possible generality, even though this means losing some results that could be got with sufficient care.

2. A *Dirichlet series* is a formal series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{2}$$

involving a variable $s = \sigma + it$, in which the exponents are assumed to satisfy (1). If the series converges anywhere in the complex plane, then it converges precisely in a half-plane $\sigma > \sigma_0$ (where σ_0 may be $-\infty$), perhaps with the addition of some or all the boundary

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points. The sum of the series is an analytic function $\varphi(s)$, which may to be sure have an analytic continuation beyond the half-plane of convergence. These statements are true as well for various summability methods. One of the broad problems of the subject is to describe relations between function-theoretic behavior of $\varphi(s)$ and summability properties of the series.

If we fix σ in (2), setting $\varphi_\sigma(t)=\varphi(\sigma+it)$, we have a formal expansion in the real variable t:

$$\varphi_\sigma(t) \sim \sum_1^\infty (a_n e^{-\lambda_n \sigma})\, e^{-i\lambda_n t}. \tag{3}$$

In harmonic analysis φ_σ becomes a function on a compactification of the line, whose Fourier series is essentially (3). Now, on the contrary, we are going to put φ_σ into a space B of functions defined on the line that is dual to a space A contained in $L^1(-\infty, \infty)$. The structure of A will determine everything. Before discussing summability we have to introduce these spaces A.

A is a linear dense subset of L^1, a Banach algebra under convolution, in its own norm that satisfies $\|f\| \geqslant \|f\|_1$ (the first norm refers to A). It is assumed to have the following four properties.

(a) *To any real λ_0 and positive ε there is an f in A whose Fourier transform*

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx \tag{4}$$

is not 0 *at λ_0, but vanishes outside the interval $(\lambda_0-\varepsilon, \lambda_0+\varepsilon)$.*

(b) *The ideal A_0 of all f such that $\hat{f}$ is compactly supported is dense in A.*

(c) *Each f in A belongs to the closure I_f of the ideal consisting of all $f \ast g$, g in A.*

(d) *The dual B of A is a space of functions φ with*

$$\varphi(f) = \int_{-\infty}^{\infty} \varphi(-x)\, f(x)\, dx = \varphi \ast f(0), \tag{5}$$

where the integral is assumed to exist absolutely. (Then φ is defined uniquely up to null-sets.)

Algebras satisfying similar hypotheses have been studied, with different objectives, by Wermer [7], extending earlier work of Beurling [1].

Property (a) will be the crucial assumption, and the restriction it puts on A limits the power of any summability theory for Dirichlet series. (b) is technically convenient in connection with the hypothesis that A is dense in $L^1(-\infty, \infty)$. (c) says that A has approximate identities, in the weakest possible sense; we shall require more later. Finally (d) enables us to identify sums of certain Dirichlet series with functions in B, and thus furnishes the link between Dirichlet series and Banach algebra.

These hypotheses are made once for all without further mention. As need arises new assumptions will be made.

3. Let φ belong to the space B dual to an algebra A. We say that *the Dirichlet series*

$$\sum_1^\infty a_n e^{i\lambda_n x} \tag{6}$$

is summable (A) *to* φ *if*

$$\varphi(f) = \sum a_n \hat{f}(\lambda_n) \tag{7}$$

for every f *in* A_0. *In that case we write*

$$\varphi(x) \overset{A}{\sim} \sum_1^\infty a_n e^{i\lambda_n x}, \tag{8}$$

and say that φ *has Dirichlet series* (6) *relative to* A.

The sum in (7) is finite for every f in A_0, so the right side is defined no matter what the series (6). The requirement is that this should be the restriction to A_0 of a continuous functional on A. Since A_0 is dense in A there cannot be more than one such functional; that is, two different φ in B cannot have the same Dirichlet series relative to A. Furthermore a given function φ cannot possess more than one Dirichlet series relative to A, because the difference of two series would be another Dirichlet series defining the null functional on A_0, whose coefficients would all vanish by (a).

A trigonometric polynomial is its own Dirichlet series, relative to any algebra A.

A convenient formalism like that of the ordinary Fourier transform can be established for this abstract definition of summability. First we should like to define the convolution $f * \varphi$ of f in A and φ in B. By hypothesis, the integral that ought to define this convolution

$$f * \varphi(x) = \int f(y)\, \varphi(x-y)\, dy \tag{9}$$

exists absolutely at 0, but nothing so far assumed implies that B is invariant under translation. Instead define $f * \varphi$ to be that ψ in B such that

$$\psi(g) = \varphi(f * g) \quad (\text{all } g \text{ in } A). \tag{10}$$

It follows from the definition that $f * (g * \varphi) = (f * g) * \varphi$, and $f * (g * \varphi)(0) = (f * g) * \varphi(0)$, for all f, g in A and φ in B. Furthermore, (9) is really true if φ is, for example, a trigonometric polynomial.

THEOREM 1. *Let f belong to A, φ to B, and suppose* (8) *holds. Then*

$$f*\varphi(x) \overset{A}{\sim} \sum_{1}^{\infty} (a_n \hat{f}(\lambda_n))\, e^{i\lambda_n x}. \tag{11}$$

The relation is obvious by (9) if φ is a trigonometric polynomial, but in general we have to proceed differently. The meaning of (11) is that for any g in A_0 we have

$$f*\varphi(g) = \sum (a_n \hat{f}(\lambda_n))\hat{g}(\lambda_n). \tag{12}$$

By definition the left side of (12) is $\varphi(f*g)$. Now $f*g$ is in A_0 with g, and its Fourier transform is $\hat{f}\cdot\hat{g}$. Therefore (12) is simply (7) written with $f*g$ in place of f, so (12) is true and the theorem is proved.

COROLLARY. *If $\hat{f}(\lambda_n)=0$ for all n, then $f*\varphi=0$.*

For $f*\varphi$ has the null Dirichlet series.

4. For any Dirichlet sequence $\Lambda=\{\lambda_1, \lambda_2, \ldots\}$ denote by B_Λ the linear set of φ in B having Dirichlet series (relative to A) with exponents in Λ. B_Λ certainly contains the trigonometric polynomials with exponents in Λ.

THEOREM 2. *B_Λ is closed in the weak star-topology of B relative to A, and its trigonometric polynomials are dense in that topology.*

The theorem is equivalent to this statement: *a function φ in B belongs to B_Λ if and only if $\varphi(f)=0$ for all f in A such that $\hat{f}=0$ on Λ.* First suppose $\varphi(f)=0$ for all f in A such that $\hat{f}=0$ on Λ. Define $a_n=\varphi(f)$, where $\hat{f}(\lambda_n)=1$ and $\hat{f}=0$ at other points of Λ. There are such functions f by (a); and the value of a_n does not depend on the choice of f by the hypothesis on φ. Next we verify that (7) holds with these numbers a_n, for every f in A_0. Indeed, every f in A_0 is a sum of finitely many functions f_j, where each $\hat{f}_j$ vanishes outside a neighborhood of λ_j that includes no other element of Λ, and a residual function still in A_0 and having a Fourier transform that vanishes on all of Λ. Since (7) is true for each of these summands it holds for f. That is, φ has Dirichlet series (6), so φ is in B_Λ.

In the opposite direction, suppose φ is in B_Λ with Dirichlet series (6). If f is in A and $\hat{f}=0$ on Λ, the Corollary to Theorem 1 asserts that $f*\varphi=0$. Therefore $\varphi(f*g)=0$ for every g in A, and $\varphi(f)=0$ by (c). This completes the proof.

Since approximation in the star topology is universal (in the sense of the theorem) it cannot be interesting. Most of what follows will support the following doctrinal statement:

The general problem of summability is to determine which functions in B_Λ can be approximated in norm by trigonometric polynomials.

To see what is meant let us suppose, informally, that A has an approximate identity $\{K_1, K_2, \dots\}$ of some kind. We assume that $K_n * \varphi$ is defined for φ in B, and the linear operations so obtained are to be uniformly bounded. If φ is a trigonometric polynomial it will be obvious that $K_n * \varphi$ tends to φ in norm, and the same fact will hold automatically for any φ that can be approximated by trigonometric polynomials in norm. If we can describe such φ, and if we phrase the result in terms of the corresponding Dirichlet series, we have a summability theorem.

5. To develop this idea we have to broaden the notion of an approximate identity. Say that an operator K in A is a *convolution operator* if $K(f*g)=(Kf)*g$ for all f, g in A. Then also $(Kf)*g=f*(Kg)$; the Fourier transform of this equation is

$$\widehat{Kf}/\hat{f} = \widehat{Kg}/\hat{g}, \tag{13}$$

at least on the set where $\hat{f}$ and $\hat{g}$ are not 0. By varying f and g we can give a value $\hat{K}(\lambda)$ to the quotient for every real λ. Then $\hat{K}$ is continuous on the line and

$$\widehat{Kf}(\lambda) = \hat{K}(\lambda)\hat{f}(\lambda) \tag{14}$$

for all f in A.

The set of all such operators forms a Banach algebra in the operator norm, denoted by A'. The natural embedding of A in A' is continuous.

From now on A' will be more important than A. Instead of making assumptions about A we could have introduced A' at the beginning and put assumptions on A'. We could even dispense with the requirement that A be an algebra, because A' will be an algebra anyway. This approach may be more general than the one being followed, but it is more complicated and has some other disadvantages.

To K in A' corresponds the adjoint operator $\tilde{K}$ in B:

$$(\tilde{K}\varphi)(f) = \varphi(Kf). \tag{15}$$

A simple computation shows that

$$\tilde{K}(f*\varphi) = (Kf)*\varphi = f*(\tilde{K}\varphi). \tag{16}$$

(The computation consists in showing that these three functionals have the same value on each g in A.) From (15) we also obtain this result, which generalizes Theorem 1: *if φ has a Dirichlet series* (8), *then*

$$\tilde{K}\varphi(x) \overset{A}{\sim} \sum (a_n \hat{K}(\lambda_n))\, e^{i\lambda_n x}. \tag{17}$$

An *approximate identity for* A is a sequence $\{K_n\}$ from A' such that $K_n f$ tends to f in norm for every f in A. (We can make a similar definition for a family $\{K_u\}$ depending on a continuous variable.) The norms of operators K_n in an approximate identity necessarily lie under a common bound.

For any φ we have $\tilde{K}_n\varphi(f) = \varphi(K_n f)$, and this converges to $\varphi(f)$ for every f in A. Therefore $\tilde{K}_n\varphi$ tends to φ in the star topology of B. But if φ is a trigonometric polynomial $\tilde{K}_n\varphi$ tends to φ in norm (because the topology of a finite-dimensional space is unique); and the same is true if φ is in the norm-closure of trigonometric polynomials, since the norms of the operators $\tilde{K}_n$ are bounded.

This fact is obvious but should be mentioned: *for any approximate identity* $\{K_n\}$, *the functions* $\hat{K}_n$ *are uniformly bounded and tend uniformly to* 1, *on every finite interval.* It is a little less obvious that *these functions are uniformly bounded over the whole real line.* The proof uses (d), with its requirement of absolute convergence in (5), but the result will not be needed and the proof is omitted.

We have, of course, the following general summability theorem:

If φ *in* B *has Dirichlet series* (8) *and besides belongs to the norm-closure of trigonometric polynomials, and if* $\{K_n\}$ *is an approximate identity for* A, *then the functions represented by the Dirichlet series*

$$\sum_1^\infty a_n \hat{K}_n(\lambda_n)\, e^{i\lambda_n x} \tag{18}$$

converge to φ *in norm.*

This statement, banal in itself, suggests the following questions to be answered in whatever generality is possible: Which functions in B can be approximated by trigonometric polynomials? Does norm approximation imply pointwise convergence? Are there any approximate identities for A? What happens if x is replaced by a complex variable?

6. It is characteristic of the classical theory of Dirichlet series that hypotheses on the sum $\varphi(\sigma+it)$ of a Dirichlet series bear on φ as a function of t uniformly over a certain interval of σ. In our context the analogous assumption about a Dirichlet series would be that a relation (8) implies the existence of functions φ_σ, $\sigma > 0$, such that

$$\varphi_\sigma(x) \overset{A}{\sim} \sum_1^\infty a_n e^{-\lambda_n \sigma} e^{i\lambda_n x}, \tag{19}$$

with some supplemental statement about the norm of these functions in B. Now actually (8) contains information of this kind, in the presence of some additional hypotheses about

A, and in a broad sense it can be said that the Fourier relation (8) is an adequate substitute for the usual kind of function-theoretic hypothesis.

The property that A needs to have is invariance under translation. For any function f, set $T_t f(x) = f(x-t)$ for all real t. To our list of axioms we add the following one.

(e) *If f is in A, $T_t f$ is in A for all real t. $T_t f$ moves continuously in A as t varies.*

The closed graph theorem implies that T_t is a continuous operator for each fixed t. Denote its norm by $\varrho(t)$, and let ω be the smallest even function that increases for positive t and majorizes ϱ:

$$\omega(t) = \sup_{|u| \leqslant |t|} \varrho(u). \tag{20}$$

Form the spaces L^1_ϱ, L^1_ω of measurable functions on the line with norms

$$\int |k(x)|\, \varrho(x)\, dx, \qquad \int |k(x)|\, \omega(x)\, dx. \tag{21}$$

For k in L^1_ϱ (and *a fortiori* for k in L^1_ω) and any f in A, the Bochner integral

$$\int k(t)\, T_t f\, dt \tag{22}$$

exists in A. Clearly the operation K defined by (22) belongs to A' and

$$\|K\| \leqslant \|k\|_\varrho, \tag{23}$$

where the norms refer to A' and L^1_ϱ respectively. In particular we have that $\widehat{Kf} = \hat{k}\hat{f}$.

LEMMA. *ϱ is bounded on each finite interval, and for all t, u we have*

$$\varrho(t) \geqslant 1, \qquad \varrho(t+u) \leqslant \varrho(t)\varrho(u). \tag{24}$$

The same assertions hold for ω.

These facts are mentioned in [7], but I will recall the proofs, especially because Wermer's algebras are not quite the same as ours. The fact that ϱ is bounded on finite intervals is a consequence of the uniform boundedness principle in this context. To prove the first inequality for ϱ observe that the adjoint of T_t is the operator T_{-t} in B. Since B contains constant functions, which are left invariant by T_{-t}, this operator (and thus also T_t acting in A) cannot have norm smaller than 1. The second inequality is obvious from the definition of ϱ. Both inequalities are inherited by ω from ϱ.

The lemma implies that L^1_ϱ and L^1_ω are Banach algebras in their norms with the convolution operation. They are dense in L^1, and their norms are larger than the norm of L^1. So these algebras are just of the type we are studying; and following Wermer's ideas we shall derive information about A from our knowledge concerning the associated algebras L^1_ϱ and L^1_ω. The next lemma, of no precise origin, is basic.

LEMMA. *If ϱ is a weight function satisfying* (24), *the space L^1_ϱ has property* (a) *if and only if*

$$\int_{-\infty}^{\infty} \frac{\log \varrho(t)}{1+t^2}\, dt < \infty. \tag{25}$$

If this condition is satisfied, and if ϱ is even and increasing for positive t, then L^1_ϱ has properties (b, c, d) *as well.*

This result not only serves the purposes of our general theory, but also provides examples of algebras A among which, it happens, are those most important in application. The part about (a) is exactly Lemma 4 of [7]. The sufficiency of the condition for (b) is Lemma 5, and the proof in fact covers (c) as well. Finally (d) is obvious from the fact that every linear functional in L^1_ϱ is given by (5), where φ is a measurable function such that

$$\text{ess sup } |\varphi(-x)\varrho(x)^{-1}| \tag{26}$$

is finite (and this quantity is the norm of φ).

Let ϱ be a weight function satisfying (24) and (25). The Poisson formula defines a harmonic function in the lower half-plane with boundary values $\log \varrho(x)$ on the real axis. This harmonic function is evidently non-negative in the half-plane. It is the real part of an analytic function $\log L(z)$; the function L so obtained is analytic in the lower half-plane and satisfies

$$|L(z)| \geq 1, \qquad |L(x)| = \varrho(x). \tag{27}$$

We also have for $\sigma > 0$

$$\log |L(-i\sigma)| = \frac{1}{\pi}\int \frac{\sigma \log \varrho(x)}{\sigma^2 + x^2}\, dx = \frac{1}{\pi}\int \frac{\log \varrho(\sigma x)}{1+x^2}\, dx. \tag{28}$$

The function L and the relation (28) will be useful in the next proof.

THEOREM 3. *Suppose that* (24) *and* (25) *hold. Then L^1_ϱ has functions K_σ, $\sigma > 0$, such that*

$$\hat{K}_\sigma(\lambda) = e^{-\lambda\sigma} \qquad (\textit{all } \lambda \geq 0) \tag{29}$$

and the norm of K_σ in L^1_ϱ is not greater than

$$\exp \frac{1}{\pi}\int \frac{\log \varrho(\sigma x)}{1+x^2}\, dx.$$

We simply define $$K_\sigma(x) = \varkappa \frac{L(x)^{-1}}{\sigma^2 + x^2}; \quad \varkappa = \frac{\sigma L(-i\sigma)}{\pi}. \tag{30}$$

Then we have $$\hat{K}_\sigma(\lambda) = \varkappa \int \frac{L(x)^{-1}}{\sigma^2 + x^2} e^{-i\lambda x}\, dx. \tag{31}$$

The integrand is analytic in the lower half-plane except for a simple pole at $x = -i\sigma$, where it has residue

$$\frac{L(-i\sigma)^{-1} e^{-\lambda\sigma}}{-2i\sigma}. \tag{32}$$

When λ is positive the residue calculus is applicable, leading directly to (29). For the norm of K_σ in L^1_ϱ we have

$$|\varkappa| \int \frac{|L(x)^{-1}|}{\sigma^2 + x^2} \varrho(x)\, dx = |L(-i\sigma)| \frac{1}{\pi} \int \frac{\sigma}{\sigma^2 + x^2}\, dx = |L(-i\sigma)|. \tag{33}$$

This concludes the proof.

Now we are in a position to extend the sum of a Dirichlet series from a line into a half-plane.

COROLLARY. *Let A be an algebra whose translation norm ϱ satisfies* (25). *Let φ be in B and have Dirichlet series* (6), *with $\lambda_1 \geqslant 0$. Then for $\sigma > 0$ there are functions φ_σ in B such that*

$$\varphi_\sigma(x) \overset{A}{\sim} \sum_1^\infty a_n e^{-\lambda_n \sigma} e^{i\lambda_n x}, \tag{34}$$

and the norm of φ_σ does not exceed the norm of φ multiplied by that of K_σ.

The functions K_σ of the Theorem belong to A' and so the Corollary is an immediate consequence of (17) and (23).

COROLLARY. *If ω satisfies* (25) *in place of ϱ, then φ_σ is bounded as σ decreases to* 0. *If φ is in the closure of the set of trigonometric polynomials, then φ_σ tends to φ in norm.*

The right side of (28) is increased by replacing ϱ with ω. This quantity is finite by hypothesis, and decreases with σ because of the monotonic character of ω. Therefore the norms of K_σ are bounded in L^1_ϱ as σ decreases to 0; this proves the first assertion. If φ is a trigonometric polynomial it is obvious that $\tilde{K}_\sigma \varphi$ converges to φ as σ decreases to 0, and this property remains true in the closure of the trigonometric polynomials.

One can see in the case of simple weight functions that the bound $|L(-i\sigma)|$ given by the theorem is exact for the norm of the functional that assigns to φ the number $\varphi(-i\sigma)$, defined anyway on the class of trigonometric polynomials with negative exponents. Thus the norm of φ_σ must be expected to increase with σ, even though the Dirichlet series converges more readily for large σ.

7. There is an interesting converse question: suppose φ is in B, and the functions $\varphi_\sigma = \tilde{K}_\sigma \varphi$ have Dirichlet series (34) for large values of σ. Does φ itself have a Dirichlet series, and is it (34) with $\sigma = 0$?

We are really interested only in the subspace B_+ consisting of all φ that vanish as linear functionals on all the functions f in A_0 whose transforms vanish on the positive real axis. This is a star-closed subspace of B containing all φ having Dirichlet series in which $\lambda_1 \geqslant 0$. Here is our result concerning the question just raised.

THEOREM 4. *Suppose that A is an algebra whose translation function ϱ satisfies* (25). *Let φ be in B_+ and let* (34) *hold for at least one positive σ. Then* (34) *is true for all $\sigma \geqslant 0$.*

The hypothesis means that the relation

$$\tilde{K}_\sigma \varphi(f) = \sum a_n e^{-\lambda_n \sigma} \hat{f}(\lambda_n) \tag{35}$$

holds for each f in A_0, for a certain positive σ (and hence for all larger σ, although that is not important). Suppose that $\hat{f}(\lambda) = 0$ for all $\lambda > T$. By the hypothesis made on the translation function L^1_ϱ is a regular Banach algebra; therefore we can find M in L^1_ϱ such that $\hat{M}(\lambda)\hat{K}_\sigma(\lambda) = 1$ for all λ in $(0, T)$. Since Mf is in A_0 with f we have

$$\tilde{M}\tilde{K}_\sigma \varphi(f) = \tilde{K}_\sigma \varphi(Mf) = \sum a_n \hat{f}(\lambda_n). \tag{36}$$

On the other hand,
$$\tilde{M}\tilde{K}_\sigma \varphi(f) = \varphi(MK_\sigma f). \tag{37}$$

But the Fourier transforms of f and of $MK_\sigma f$ are equal on the positive axis, and so φ has the same value on these two functions. Hence

$$\varphi(f) = \sum a_n \hat{f}(\lambda_n) \tag{38}$$

for every f in A_0. That is, φ has a Dirichlet series with the expected coefficients, and the truth of (34) for all positive σ follows.

8. At last we can formulate a genuine summability theorem.

THEOREM 5. *Suppose the translation functions ϱ in A satisfies* (25). *Let φ be an element of B with a Dirichlet series. Then φ_σ can be approximated in norm by trigonometric polynomials, for every $\sigma > 0$.*

For example, we can take L^∞ for B: *if φ is a bounded function with spectrum in a Dirichlet sequence, then φ_σ is uniformly almost-periodic, for each positive σ.* This proposition is not hard to prove directly but its statement shows what the theorem means in a simple case.

We choose and fix a positive number σ, and we assume that $\lambda_1 \geqslant 0$. For positive numbers α define

$$J_\alpha(x) = e^{-\alpha\sigma} e^{i\alpha x} K_\sigma(x). \tag{39}$$

The Fourier transform of $J_\alpha(x)$ is $e^{-\lambda\sigma}$ for $\lambda \geqslant \alpha$, and the norm of J_α in L^1_ϱ is $e^{-\alpha\sigma}$ times the norm of K_σ (a fixed number $\varkappa$, since σ will not vary).

Now let φ have Dirichlet series (6). Denote by $q(\alpha)$ the distance from φ_σ to the linear span of the exponentials

$$e^{i\lambda_1 x}, \ldots, e^{i\lambda_n x} \tag{40}$$

where n is the integer such that $\lambda_n \leqslant \alpha < \lambda_{n+1}$. The theorem amounts to the statement that $q(\alpha)$ tends to 0 as α increases without bound. This span is finite-dimensional, and thus star-closed in B. Therefore

$$q(\alpha) = \sup |\varphi_\sigma(f)|, \tag{41}$$

where f ranges over those elements of A having norm at most 1 and vanishing as linear functionals on the exponentials (40): $\hat{f}(\lambda_j) = 0$ $(j = 1, \ldots, n)$. (If the subspace were not star-closed we should have to replace f in (41) by a general element from the conjugate space of B, which would be very inconvenient.)

Now let f be one of the functions envisaged in (41), and take $\alpha = \lambda_n$. Then $K_\sigma f$ and $J_\alpha f$ have Fourier transforms agreeing at every λ_j: they are both zero at $\lambda_1, \ldots, \lambda_n$, and they are both $e^{-\lambda\sigma}\hat{f}(\lambda)$ beyond that point. By Theorem 2,

$$\varphi_\sigma(f) = \varphi(K_\sigma f) = \varphi(J_\alpha f). \tag{42}$$

Hence
$$|\varphi_\sigma(f)| \leqslant \|J_\alpha\| \, \|\varphi\| = \varkappa e^{-\alpha\sigma} \|\varphi\|. \tag{43}$$

By taking the supremum on f we see that $q(\alpha)$ tends to 0, as was to be proved.

9. In all this development no mention has been made of the continuity of functions φ in B. Even the extensions φ_σ of a function φ with a Dirichlet series can presumably be discontinuous. Now finally by specializing still further we shall find the classical situation, in which sums of Dirichlet series are continuous analytic functions in a region of the complex plane. The new axiom we need is this.

(f) *There is an approximate identity for A consisting of functions K_n belonging to A, with norms bounded in A.*

Under hypothesis (e) we can always find approximate identities for A in A', and these functions have bounded norms in A'. We are strengthening this fact by requiring the functions to lie in A, and to have their norms in A bounded (and not merely their norms as elements of A').

THEOREM 6. *Let A satisfy hypotheses* (a, b, c, d, e, f). *Denote the closure of the trigonometric polynomials in B by B_0. Then*

(i) *each φ in B_0 is continuous;*

(ii) *if φ_n in B_0 tends to φ in norm, $\varphi_n(-x)/\varrho(x)$ tends uniformly to $\varphi(-x)/\varrho(x)$;*

(iii) *$\varphi(x)=o(\omega(x))$ for φ in B_0, at least if ω is unbounded.*

We have postulated the invariance of A under translation, but no restriction has been put on the growth of ϱ. On the other hand, the requirement of an internal approximate identity for A is a severe restriction in another direction.

For the proof, first observe that $f \ast \varphi$ is a continuous function, for any f in A and φ in B, because $T_t f$ moves continuously in A. Indeed we have

$$f \ast \varphi(x) = \varphi(T_{-x} f) = O(\varrho(-x)). \tag{44}$$

This is true for any algebra with translation operators; our problem is to refine (44) by means of (f) for φ in B_0.

For the functions K_n of the hypothesis, $K_n \ast \varphi(0)$ represents a bounded sequence of linear functionals on B. If φ is a trigonometric polynomial the sequence converges to $\varphi(0)$; for φ in B_0 it follows that the sequence converges to some number, which we denote provisionally by $\varphi'(0)$. Similarly $K_n \ast \varphi(x)$ converges to a number $\varphi'(x)$ for any x and φ in B_0, and the function φ' is identical with φ if φ is a trigonometric polynomial. Furthermore we have

$$|\varphi'(x)| \leqslant \varkappa_1 \|\varphi\| \varrho(-x), \tag{45}$$

where $\varkappa_1$ is the upper bound of the numbers $\|K_n\|$. The next step depends on the following result, which is valid in algebras satisfying (a, b, c, d).

LEMMA. *There is a constant $\varkappa_2$ such that*

$$\int |\psi(-x) f(x)| \, dx \leqslant \varkappa_2 \|f\| \, \|\psi\| \tag{46}$$

for all f in A and ψ in B.

Let B' be the space of functions ψ having the following property: the transformation that carries f to $\psi(-x)f(x)$ is a continuous mapping of A into L_1. The requirement of continuity can be omitted; such a transformation is automatically continuous by the closed graph theorem. (The proof is easy if one observes that convergence of a sequence in A carries with it pointwise convergence of a subsequence.) It follows from (d) that B is contained in B'. On the other hand, every element of B' obviously defines a continuous linear functional on A by integration, so B' is no larger than B.

B' is a Banach space in its operator norm. It is obvious for every ψ that

$$\|\psi\|' \geqslant \|\psi\|, \tag{47}$$

where the prime denotes the norm in B'. The closed graph theorem asserts that these norms are equivalent, and that proves the lemma.

The rest of the proof of the Theorem is simple. For each φ' derived from φ in B_0 we have

$$\int |\varphi'(-x)f(x)|\,dx \leqslant \liminf \int |K_n * \varphi(-x)f(x)|\,dx \leqslant \varkappa_1\varkappa_2\|f\|\,\|\varphi\|. \tag{48}$$

The first inequality is Fatou's lemma, and the second one is a case of the lemma just proved. Hence φ' is in B, with norm at most $\varkappa_1\varkappa_2$ times the norm of φ. Now the transformation that carries φ to φ' is defined and continuous in B_0, with range in B. But this transformation is the identity on trigonometric polynomials, and so it is the identity on all of B_0. Thus (45) is a statement about φ itself; the second assertion of the theorem follows immediately. In particular every function in B_0 is the pointwise limit of trigonometric polynomials, uniformly on each bounded interval, so all the functions of B_0 are continuous. Finally, if ω is unbounded (iii) is true for trigonometric polynomials, and the same will be true for their limits. This concludes the proof.

THEOREM 7. *Let A satisfy* (a, b, c, d, e, f) *and* (25). *If φ is in B and has a Dirichlet series, then $\varphi_\sigma(-x)$ is an analytic function of $s=\sigma+ix$ in the half-plane $\sigma>0$.*

The new hypothesis (25) is needed to give meaning to φ_σ, and to guarantee the conclusion of Theorem 5. Approximation by trigonometric polynomials in norm gives uniform approximation with respect to the weight function $\varrho(-x)^{-1}$, and the rest of the proof is easy.

10. It lies beyond my interest and competence to describe all the classical results that can be derived from these theorems, and I certainly do not wish to claim that everything can be. I do wish to make the point that functional analysis, or more accurately algebraic analysis, has something to say about the subject of Dirichlet series. For this purpose I shall discuss two families of algebras of recognized importance, one giving rise to summation by typical means, and the other to a theorem of F. Carlson.

First take for A the algebra L^1_ϱ with $\varrho(x)=(1+|x|)^\alpha$, where α is a fixed positive number. The translation operator T_t has norm $(1+|t|)^\alpha$, because

$$\sup_x \frac{(1+|x-t|)^\alpha}{(1+|x|)^\alpha}=(1+|t|)^\alpha.$$

Good approximate identities can be found in A. Indeed an approximate identity in L^1 is likely to be one in A if only its functions belong to A, because the process of pushing mass towards the origin has the effect of reducing norm in A.

To a Dirichlet series (6) are attached [5] two types of *typical means:*

$$\sum a_n\left(1-\frac{\lambda_n}{T}\right)^\varkappa e^{i\lambda_n x}, \qquad \sum a_n(1-e^{\lambda_n-T})^\varkappa e^{i\lambda_n x} \tag{49}$$

where the sums range over n such that $\lambda_n<T$, and $\varkappa$ is a positive number called the *order* of summation. (Ordinary convergence corresponds to $\varkappa=0$, but this case is exceptional.) We assume that $\lambda_1\geqslant 0$.

The relation
$$\int_0^\infty \lambda^\varkappa e^{-\lambda s}d\lambda=s^{-\varkappa-1}\int_0^\infty \lambda^\varkappa e^{-\lambda}\,d\lambda \tag{50}$$

is obvious if s is real and positive, and it follows immediately for complex s with positive real part. That is, aside from a constant factor, the function vanishing for negative λ and equal to $\lambda^\varkappa e^{-\lambda\sigma}$ for positive λ is the transform of $(\sigma-ix)^{-\varkappa-1}$; this function is in A provided $\varkappa>\alpha$. Convolute this function with an element of A whose transform is $e^{\lambda\sigma}$ on the interval $(0, 1)$; the result is a function L in A such that $\hat{L}(\lambda)$ vanishes for negative λ, and equals $\lambda^\varkappa$ for $0\leqslant\lambda\leqslant 1$.

By linear change of variable we obtain a function K in A such that $\hat{K}(\lambda)$ is $(1-\lambda)^\varkappa$ for $0\leqslant\lambda\leqslant 1$, and vanishes for $\lambda\geqslant 1$. Define $K_T(x)$ to be $TK(Tx)$ for positive T, so that $\hat{K}_T(\lambda)=K(\lambda/T)$. These functions are bounded in A as T tends to infinity, and they have the effect of an approximate identity on trigonometric polynomials with positive frequencies. By Theorem 5, if φ has a Dirichlet series (6) with $\lambda_1\geqslant 0$ then

$$K_T * \varphi_\sigma(x)=\sum a_n\left(1-\frac{\lambda_n}{T}\right)^\varkappa e^{-\lambda_n\sigma}e^{i\lambda_n x} \quad (\lambda_n<T) \tag{51}$$

converges in norm to φ_σ, for every positive σ. Thus we obtain the well-known result [5, Theorem 41] on the summability of Dirichlet series by typical means of the first kind in the region where φ_σ is uniformly $O(|x|^\alpha)$, $\alpha<\varkappa$. (Under a supplementary hypothesis we can take $\alpha=\varkappa$ [5, Theorem 44], but the improvement is only apparent.)

Evidently any function K in A with $\hat{K}(0)=1$ and $\hat{K}$ compactly supported leads to a summability method in exactly the same way. In constructing the function K above we used the fact that A has local inverses; this eliminated some of the explicit calculations that usually have to be made, and the device would be even more useful if the kernel were more complicated.

The means of the second kind are not obtained by dilation of a single function K in A, and are more complicated to treat. For any positive T we can find a function in A with transform equal to $(1-e^{\lambda-T})^{\varkappa}$ for $0\leqslant\lambda\leqslant T$, and vanishing for $\lambda\geqslant T$; but it is easy to see that the family cannot be bounded in A as T tends to infinity. Instead we choose and fix a positive number δ, and seek functions K_T such that

$$\hat{K}_T(\lambda)=\begin{matrix}(1-e^{\lambda-T})^{\varkappa}\,e^{-\lambda\delta} & (0\leqslant\lambda\leqslant T)\\ 0 & (\lambda\geqslant T).\end{matrix} \tag{52}$$

Provided δ can be taken as close to 0 as we please, these functions will establish summability in the same half-plane as we should get with $\delta=0$.

First of all there is an element L of A' such that

$$\hat{L}(\lambda)=\begin{matrix}(1-e^{\lambda-T})^{\varkappa} & (\lambda\leqslant T)\\ 0 & (T\leqslant\lambda\leqslant T+1).\end{matrix} \tag{53}$$

Indeed, $1-\hat{L}$ can easily be completed to be the transform of a function in A. Furthermore the norm of L does not depend on T.

There is a function M in A whose transform satisfies

$$\hat{M}(\lambda)=\begin{matrix}e^{-\lambda\delta} & (0\leqslant\lambda\leqslant T)\\ 0 & (T+1\leqslant\lambda).\end{matrix} \tag{54}$$

For a single T we merely subtract from the exponential a function that is equal to $e^{-\lambda\delta}$ for $\lambda\geqslant T+1$, vanishes for $\lambda\leqslant T$, and is infinitely differentiable. The same correction function works for larger T, after translation and multiplication by constants that become smaller. Thus the norm of M is bounded by a number independent of T.

Now $\hat{L}\hat{M}$ has the properties required in (52) of $\hat{K}_T$; this function is the transform of a function in A whose norm is bounded by a number independent of T.

The summability theorem for means of second type is the same as the theorem for means of first type, and the rest of the proof is evident.

The argument just given can be applied to any kernels that arise by translation from the transform of a single element of A'. The introduction of the exponential factor overcomes the difficulty inherent in the fact that such a transform must have the same limit at one end of the real axis as at the other end.

In classical summability theory very little is said about the sum of a Dirichlet series outside the region in which it has polynomial growth, which is to say the region in which the typical means of various orders are effective. The stronger methods that have been defined (for example one investigated by Hardy [6, p. 39]) are rightly considered esoteric. The reason is that the classical literature does not recognize the barrier (25), which determines whether any compactly supported kernel can be effective for all Dirichlet series subject to a given growth condition. Thus no compactly supported kernel can define a summation method (by the processes we have been studying) for functions allowed to grow exponentially. On the other hand, there are interesting function spaces within the barrier (25) that are outside the reach of the typical means of finite order. For example, we can form L^1_ϱ with $\varrho(x) = \exp|x|^{\frac{1}{2}}$. In this space the translation norm is once again the same function ϱ. Since ϱ satisfies (25), the space contains functions whose transforms vanish outside finite intervals. We can easily write down summation methods of first and of second kind for Dirichlet series of corresponding growth.

11. Functional analysis has created a great array of Banach spaces on the line, for their own interest and for application to differential equations, harmonic analysis, and other subjects. Some of these norms give algebras of the type we have been studying. Here is an example rather different from the weight spaces of the last section. It is one space in a family defined by Beurling [2]; although our results can be applied to many of his algebras, I shall only describe one in which we find a striking classical result.

Really it is only necessary to describe B, taking the properties (a, b, c, d) for A from Beurling's paper. B is the set of functions φ on the line that are locally square-summable, with finite norm defined by

$$\sup_{r>0}\left(\frac{1}{2r+1}\int_{-r}^{r}|\varphi(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{55}$$

Now A is invariant under translation; from (55) it is easy to show that the norm of translations is given by $\varrho(t) = (2|t|+1)^{\frac{1}{2}}$. Thus Theorem 5 is applicable: for φ in B with a Dirichlet series, and $\sigma > 0$, φ_σ belongs to B_0.

If φ is a trigonometric polynomial with coefficients a_n, then obviously

$$\lim_{r\to\infty}\frac{1}{2r+1}\int_{-r}^{r}|\varphi(x)|^2\,dx = \sum|a_n|^2 \leqslant \|\varphi\|^2. \tag{56}$$

It follows that every function in B_0 has a harmonic expansion whose coefficients are the limits of the coefficients of approximating polynomials, and that (56) holds. Therefore, if φ is in B with Dirichlet series (6), we have for $\sigma > 0$

$$\lim_{r\to\infty} \frac{1}{2r+1} \int_{-r}^{r} |\varphi(\sigma+ix)|^2\, dx = \sum |a_n|^2 e^{-2\lambda_n\sigma} < \infty. \tag{57}$$

The existence of this limit is part of the assertion.

This result is a well-known theorem of F. Carlson [4]. His hypothesis was naturally phrased differently. He asserts that (57) holds for a Dirichlet series as far to the left in σ as $\varphi(\sigma+ix)$ is uniformly of polynomial growth, and has finite supremum (rather than limit) in r on the left side.

Carlson's proof is the origin of this paper. He shows clearly that (57) is the consequence of approximation in the norm of a linear space. The linear thread is recognizable all through the function-theoretic detail. This achievement in 1922 gives its author the right to a distinguished place in the history of linear spaces.

12. There is satisfaction in having found out, from my own point of view, what this part of the theory of Dirichlet series is about, but I would like to claim a more substantial point of novelty for the exposition now concluded. In Fourier series people stopped long ago trying to sum series to functions, and began instead to use functions to generate series. This change in point of view set a new direction that the subject has followed ever since. The classical subject of Dirichlet series, on the contrary, has never escaped from its dependence on contour integration, and could not make that shift. My optimistic hope is that I have found a way round the difficulty so that Dirichlet series can follow power series to the center of algebraic analysis.

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