THE H^p SPACES OF A CLASS OF FUNCTION ALGEBRAS

BY

P. R. AHERN and DONALD SARASON

University of California, Los Angeles, (1) *and University of California, Berkeley,* (2) *California, U.S.A.*

§ 1. Introduction

This paper is a study of a class of uniform algebras and of the associated Hardy spaces of generalized analytic functions. It is a natural continuation of a number of similar studies which have appeared in recent years; see Bochner [7], Helson and Lowdenslager [15], Hoffman [17], Hoffman and Rossi [18], König [19], Lumer [20], [21], Srinivasan [29], Srinivasan and Wang [30], and Wermer [35]. All of these previous studies are based on premises that force the generalized analytic functions to behave, roughly speaking, like analytic functions in a simply connected domain. In the algebras to be investigated here, the condition of simple connectivity is replaced by one of finite connectivity.

We begin by stating our basic hypotheses. The notations about to be introduced will be retained throughout. Let X be a compact Hausdorff space and A a uniform algebra on X, that is, a uniformly closed subalgebra of $C(X)$ that contains the constants and separates points of X. We shall employ the current notations and terminology pertaining to uniform algebras, for which see [5]. In particular, we denote the space of real parts of functions in A by $\mathcal{Re}\, A$, the set of invertible elements of A by A^{-1}, and the set of logarithms of moduli of functions in A^{-1} by $\log |A^{-1}|$.

Let φ be a multiplicative linear functional on A. Our hypotheses pertain only to the functional φ. We denote by $\mathcal{M}(\varphi)$ the set of representing measures for φ and by S the real linear span of the set of all differences between pairs of measures in $\mathcal{M}(\varphi)$. Our basic hypotheses are these:

(I) No non-zero measure in S annihilates $\log |A^{-1}|$;

(II) S has finite dimension σ.

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Conditions (I) and (II) are local versions of, and are implied by, the following global conditions on A:

(I′) The real linear span of $\log |A^{-1}|$ is uniformly dense in $\mathcal{C}_R(X)$ (the space of real continuous functions on X);

(II′) The uniform closure of $\mathcal{Re}\, A$ has finite codimension in $\mathcal{C}_R(X)$.

A uniform algebra satisfying (I′) and (II′) is called a *hypo-Dirichlet algebra*; such algebras were first studied by Wermer [37]. Before anything else we want to mention two concrete examples.

Example 1. Let R be a finite open Riemann surface, that is, a domain on a Riemann surface whose closure is compact and whose boundary X is the union of finitely many non-intersecting analytic Jordan curves, with R lying on one side of X. Let A be the algebra of all functions on X that are restrictions of functions continuous on $R \cup X$ and analytic in R. Then, as Wermer [37] has proved, A is a hypo-Dirichlet algebra.

Example 2. Let X be the boundary of a compact subset Y of the plane whose complement has only finitely many components. Let A be the algebra of all functions on X that can be uniformly approximated by rational functions with poles off Y. Then, by a theorem of Walsh [34], A is a hypo-Dirichlet algebra.

Before describing the contents of the paper in greater detail, it will be convenient to introduce a few notations and to point out a few immediate consequences of our hypotheses. From (I) and (II) it follows that there are σ functions $Z_1, \ldots, Z_\sigma$ in A^{-1} and σ measures $\nu_1, \ldots, \nu_\sigma$ in S such that

$$\int \log |Z_j|\, d\nu_k = \delta_{jk}.$$

We assume that such functions and measures have been chosen once and for all. We denote the σ-tuple of measures $(\nu_1, \ldots, \nu_\sigma)$ by ν. For u a function on X we let $\int u\, d\nu$ stand for the σ-tuple

$$\left(\int u\, d\nu_1, \ldots, \int u\, d\nu_\sigma\right),$$

provided all the relevant integrals exist. The set of all σ-tuples $\int \log |h|\, d\nu$ with h in A^{-1} will be denoted by $\mathcal{L}$; it is obviously a subgroup of E^σ (σ dimensional Euclidean space).

By a theorem of Arens and Singer [3, Theorem 5.2] there is a measure m in $\mathcal{M}(\varphi)$ such that

$$\log |\varphi(h)| = \int \log |h|\, dm$$

for all h in A^{-1}, and our assumption (I) implies that there is only one such measure. The measure m is called the *Arens–Singer measure* for φ. A further result of Arens and Singer [3, Corollary 6.41] states that for any f in A one has the inequality

$$\log |\varphi(f)| \leqslant \int \log |f| \, dm,$$

which is called *Jensen's inequality*.

We now describe briefly the main contents of the paper. In § 2 we prove a useful lemma, special cases of which have appeared in the literature several times before. The lemma is used in § 3 to prove that if μ is any annihilating measure of A, then the absolutely continuous and singular components of μ with respect to m are also annihilating measures of A. This is a version of the classical theorem of F. and M. Riesz on analytic measures; the idea of the proof stems from F. Forelli. An immediate consequence is that the measures in S are absolutely continuous with respect to m. In § 4 the spaces H^p $(1 \leqslant p \leqslant \infty)$ are introduced as the closures of A in $L^p(m)$ (weak-star closure for $p = \infty$), and the lemma of § 2 is employed to obtain information about them. §§ 5 and 6 are devoted to analogues of two classical theorems of Szegö, the theorem on mean-square approximation by polynomials [16, p. 48] and the theorem on the moduli of H^p functions on the unit circle [16, p. 53]. § 7 contains the analogue of a theorem of Beurling on generators of H^p on the unit circle [16, p. 101]. In § 8 we prove the crucial fact that the group $\mathfrak{L}$ is discrete. In § 9 we show that, not only are the measures in S absolutely continuous with respect to m (as is proved in § 3), they are in fact *boundedly* absolutely continuous with respect to m. This result seems to lie fairly deep, and it occupies an important place in the theory. (For the algebras of Example 1 above, it is fairly easy to prove directly that the measures in S are bounded, but we know of no such direct proof for the algebras of Example 2.) In § 10 we obtain more precise forms of Szegö's two theorems. § 11 contains information on the annihilators of A in the spaces $L^p(m)$. In § 12 we show that when $\sigma \neq 0$ the Gleason part containing φ is non-trivial. § 13 pertains to invariant subspaces. We have not been able to prove here as strong a result as we suspect is true. However, the information we do obtain enables us in § 14 to characterize completely the invariant subspaces of the algebras of Example 1. This has already been accomplished independently by Voichick [32], [33], Hasumi [13], and Forelli [private communication]. The methods of these three, although differing technically from one another, all depend on the same device, namely the transplantation of the situation to the unit disk by means of a uniformizer followed by the application of known theorems from the classical H^p theory. The quite different and more direct treatment we give may therefore be of interest. The concluding § 15 contains some remarks on the algebras of Example 2.

In § 13 we offer a conjecture about invariant subspaces. To prove or disprove this conjecture is, we feel, the most important open problem in the theory. A proof would yield improvements in the results of Wermer [37] and O'Neill [24] on the embedding of analytic structures in the maximal ideal space of a hypo-Dirichlet algebra.

We should like to record explicitly our indebtedness to the papers of Helson and Lowdenslager [15], Hoffman [17], and Lumer [20]. It is in these papers that the pattern of the present study is largely laid out.

A few notational conventions: For f in A we shall write $f(m)$ in place of $\int f\,dm$. For $\alpha = (\alpha_1, \ldots, \alpha_\sigma)$ a σ-tuple of real numbers, we define

$$|Z|^\alpha = |Z_1|^{\alpha_1} \ldots |Z_\sigma|^{\alpha_\sigma},$$

$$|Z(m)|^\alpha = |Z_1(m)|^{\alpha_1} \ldots |Z_\sigma(m)|^{\alpha_\sigma},$$

$$\alpha \cdot \log |Z| = \log (|Z|^\alpha),$$

$$\alpha \cdot \log |Z(m)| = \log (|Z(m)|^\alpha).$$

For $\gamma = (\gamma_1, \ldots, \gamma_\sigma)$ a σ-tuple of integers, we let

$$Z^\gamma = Z_1^{\gamma_1} \ldots Z_\sigma^{\gamma_\sigma}.$$

To conclude this introduction we state the following known lemma, which will be used several times below. For its proof, see [4, Satz 3].

LEMMA 1.1. *Let the real continuous function u on X be annihilated by S. Then*

$$\sup_{\substack{g \in A \\ \mathcal{R}e\, g \leqslant u}} \mathcal{R}e\, g(m) = \int u\,dm = \inf_{\substack{g \in A \\ \mathcal{R}e\, g \geqslant u}} \mathcal{R}e\, g(m).$$

§ 2. A preliminary lemma

The lemma we prove in this section, which will play a crucial role, combines ideas of Forelli [9], Hoffman and Wermer [36, Lemma 5], and O'Neill [24, Lemma A]. Forelli was interested in the F. and M. Riesz theorem for certain Dirichlet algebras, Hoffman and Wermer in the Dirichlet algebra case of Theorem 4.1 below, and O'Neill in a theorem about Arens–Singer measures for hypo-Dirichlet algebras. By extracting the basic idea from their proofs, we arrive at

LEMMA 2.1. *Let $\{v_n\}_1^\infty$ be a sequence of non-negative continuous functions on X such that $\int v_n dm \to 0$. Then there exists a subsequence $\{u_n\}$ of $\{v_n\}$ and a sequence $\{f_n\}$ in A^{-1} such that $|f_n| \leqslant e^{-u_n}$ and $f_n \to 1$ almost everywhere with respect to m.*

Proof. Let $\mathfrak{J}$ denote the lattice of points in E^σ with integral coordinates. For β in E^σ we let $|\beta|$ stand for the maximum of the moduli of the components of β. Choose $\{u_n\}$ such that $\int u_n dm < n^{-\sigma-1}$, and such that the sequence of σ-tuples $\{\int u_n d\nu\}$ converges modulo $\mathfrak{J}$ to a point α, with $|\alpha - \int u_n d\nu| < n^{-\sigma-1}(\text{mod } \mathfrak{J})$. By Dirichlet's theorem on Diophantine approximation [12, p. 170], there is a sequence of positive integers $\{q_n\}$, with $q_n \leqslant n^\sigma$, such that each component of $q_n\alpha$ lies within $1/n$ of an integer. Let $w_n = q_n u_n$ and $\beta_n = \int w_n d\nu$. Then $\int w_n dm \to 0$, and there is a sequence $\{\gamma_n\}$ in $\mathfrak{J}$ such that $|\beta_n - \gamma_n| \to 0$. Let $\varepsilon_n = \beta_n - \gamma_n$ and $c = \sum_1^\sigma \|\log|Z_k|\|_\infty$. By Lemma 1.1, there are functions g_n in A such that

$$\mathcal{R}e\, g_n \geqslant w_n - \beta_n \cdot \log|Z| + |\varepsilon_n| c,$$

$$\mathcal{R}e\, g_n(m) \leqslant 1/n + \int w_n dm - \beta_n \cdot \log|Z(m)| + |\varepsilon_n| c.$$

Define $f_n = e^{-g_n} Z^{-\gamma_n}$. Then

$$\log|f_n| = -\mathcal{R}e\, g_n - \gamma_n \cdot \log|Z| \leqslant -w_n + \varepsilon_n \cdot \log|Z| - |\varepsilon_n| c \leqslant -w_n \leqslant -u_n,$$

and so $|f_n| \leqslant e^{-u_n}$. Also

$$\begin{aligned}\int \log|f_n| dm &= -\mathcal{R}e\, g_n(m) - \gamma_n \cdot \log|Z(m)| \\ &\geqslant -1/n - \int w_n dm + \varepsilon_n \cdot \log|Z(m)| - |\varepsilon_n| c \\ &\geqslant -1/n - \int w_n dm - 2|\varepsilon_n| c.\end{aligned}$$

It follows that $\int \log|f_n| dm \to 0$. Because m is an Arens-Singer measure this implies that $|\int f_n dm| \to 1$. Therefore, multiplying each f_n by the appropriate constant of unit modulus, we may assume that $\int f_n dm \to 1$. Because also $|f_n| \leqslant 1$, the latter implies that $f_n \to 1$ in measure modulo m (or, by an easy computation, that $f_n \to 1$ in $L^2(m)$). Hence a suitable subsequence of $\{f_n\}$ will converge to 1 almost everywhere with respect to m. The proof is complete.

§ 3. The F. and M. Riesz theorem

We now apply the method of Forelli [9] to obtain

THEOREM 3.1. *Let μ be an annihilating measure of A with absolutely continuous and singular components μ_a and μ_s (with respect to m). Then μ_a and μ_s annihilate A.*

Proof. Let ε be a positive real number, and choose a compact m-null set K such that $|\mu_s|(X-K)<\varepsilon$. Let $\{u_n\}$ be a sequence of non-negative continuous functions on X such that $u_n\to\infty$ uniformly on K and $\int u_n dm\to 0$. Because of Lemma 2.1, we may by passing to a subsequence assume that there are functions f_n in A^{-1} such that $|f_n|\leqslant e^{-u_n}$ and $f_n\to 1$ almost everywhere modulo m. We then have $f_n\to 0$ uniformly on K. If g is any function in A, then

$$0=\int f_n g\,d\mu=\int f_n g\,d\mu_a+\int_{X-K} f_n g\,d\mu_s+\int_K f_n g\,d\mu_s.$$

For $n\to\infty$ the first term on the right converges to $\int g\,d\mu_a$ and the last term converges to 0. As the middle term never exceeds $\varepsilon\|g\|_\infty$ in absolute value, it follows that $|\int g\,d\mu_a|\leqslant\varepsilon\|g\|_\infty$. As ε is arbitrary we have $\int g\,d\mu_a=0$ for all g in A, and the proof is complete.

The following two corollaries are immediate.

COROLLARY 1. *The measures in S are absolutely continuous with respect to m.*

COROLLARY 2. *If A is hypo-Dirichlet, then the evaluation functionals on A at points of X have unique representing measures.*

We know of no essentially simpler way of proving Corollary 2.

§ 4. The spaces H^p

Henceforth such phrases as "almost everywhere" will refer to the measure m. We denote the space $L^p(m)$ simply by L^p. For $1\leqslant p<\infty$ let H^p be the closure of A in L^p, and let H^∞ be the weak-star closure of A in L^∞. The space H^∞ is an algebra, i.e. it is closed under multiplication. For f in H^p we shall write $f(m)$ in place of $\int f\,dm$.

The first two theorems in this section establish the equalities $H^q\cap L^p=H^p$, $1\leqslant q\leqslant p\leqslant\infty$. The last theorem is a result about exponentiation which will be useful later. For the case of Dirichlet algebras, the following theorem is due to Hoffman and Wermer [36, Lemma 5]. Being in possession of Lemma 2.1, we are able to use their proof.

THEOREM 4.1. *If h is a bounded function in H^1, then there is a sequence $\{h_n\}$ in A such that $\|h_n\|_\infty\leqslant\|h\|_\infty$ and $h_n\to h$ almost everywhere.*

Proof. Assume without loss of generality that $\|h\|_\infty=1$. Let $\{g_n\}$ be a sequence in A converging to h in L^1 and almost everywhere. Define $E(n)=\{x:|g_n(x)|>1\}$. Then

$$\lim_{n\to\infty}\int_{E(n)}(|g_n|-1)\,dm=0,$$

and therefore also $$\lim_{n\to\infty}\int_{E(n)}\log|g_n|\,dm=0.$$

Hence, by Lemma 2.1, we can by passing to a subsequence assume that there are functions f_n in A such that $\log|f_n|\leqslant -\max(\log|g_n|, 0)$ and $f_n\to 1$ almost everywhere. The functions $h_n=f_ng_n$ then form a sequence with the required properties.

COROLLARY. $H^\infty=L^\infty\cap H^p$, $1\leqslant p<\infty$.

THEOREM 4.2. $H^p=H^1\cap L^p$, $1<p<\infty$.

Proof. Let h be a function in $H^1\cap L^p\,(1<p<\infty)$. Take a sequence $\{g_n\}$ in A converging to h in L^1 and almost everywhere. Take also a sequence $\{u_n\}$ in $C(X)$ converging to h in L^p and almost everywhere. Let $E(n)=\{x:|g_n(x)-u_n(x)|\geqslant 1\}$. Then because $\int|g_n-u_n|\,dm\to 0$, we have

$$\lim_{n\to\infty}\int_{E(n)}\log|g_n-u_n|\,dm=0.$$

Hence, by Lemma 2.1, we may by passing to a subsequence suppose that there are functions f_n in A^{-1} such that $\log|f_n|\leqslant -\max(\log|g_n-u_n|, 0)$ and $f_n\to 1$ almost everywhere. We have

$$\|f_ng_n-h\|_p\leqslant\|f_n(g_n-u_n)\|_p+\|f_n(u_n-h)\|_p+\|(f_n-1)h\|_p.$$

As $n\to\infty$, the first term on the right goes to 0 by the bounded convergence theorem, the second term goes to 0 by the choice of the u_n, and the last term goes to 0 by the dominated convergence theorem. Thus $f_ng_n\to h$ in L^p, and we have proved the inclusion $L^p\cap H^1\subset H^p$. As the reverse inclusion is trivial, the proof is complete.

COROLLARY. $H^p=H^q\cap L^p$, $1<p<q<\infty$.

A function f in H^p is called an *outer function* if $\log|f(m)|=\int\log|f|\,dm>-\infty$.

THEOREM 4.3. *If h is in H^1 and $\mathcal{R}e\,h$ is bounded above, then e^h is an outer function in H^∞.*

Proof. Assume without loss of generality that $\mathcal{R}e\,h\leqslant 0$. Take a sequence $\{g_n\}$ in A converging to h in L^1 and almost everywhere. Let $E(n)=\{x:\mathcal{R}e\,g_n(x)>0\}$. Then

$$\lim_{n\to\infty}\int_{E(n)}\mathcal{R}e\,g_n\,dm=0.$$

Therefore, by Lemma 2.1, we may by passing to a subsequence suppose that there are functions f_n in A^{-1} such that $\log|f_n| \leqslant -\max(\mathcal{Re}\, g_n, 0)$ and $f_n \to 1$ almost everywhere. We then have $|f_n e^{g_n}| \leqslant 1$ and $f_n e^{g_n} \to e^h$ almost everywhere, and so e^h is in H^∞. Moreover,

$$\log\left|\int e^h\, dm\right| = \lim_{n\to\infty} \log|f_n(m)\, e^{g_n(m)}| = \lim_{n\to\infty} [\mathcal{Re}\, g_n(m) + \log|f_n(m)|] = \mathcal{Re}\, h(m)$$

because $|f_n(m)| \to 1$. Hence $\log|\int e^h dm| = \int \log|e^h|\, dm$, i.e., e^h is an outer function.

§ 5. Szegö's first theorem

Let A_0 be the kernel of the functional φ. For w a non-negative function in L^1 and p a positive number, we define

$$\Delta_p(w) = \inf_{f \in A_0} \int |1-f|^p w\, dm.$$

It is our purpose in the present section to prove the following result.

THEOREM 5.1. *Let w be a non-negative function in L^1 such that*

$$\log w \text{ is integrable with respect to all the measures in } \mathcal{M}(\varphi), \qquad (\ast)$$

and such that $p^{-1}\int \log w\, dv$ is in $\mathcal{L}$. Then

$$\Delta_p(w) = \exp\left[\int \log w\, dm\right]. \qquad (5.1)$$

The main part of the proof will be broken up into three lemmas. The crucial step is Lemma 5.2. For the remainder of this section we let w stand for a non-negative function in L^1. We first obtain, as an easy consequence of Jensen's inequality, the following estimate.

LEMMA 5.1. $\Delta_p(w) \geqslant \exp[\int \log w\, dm]$.

Proof. Let f be in A_0. By the arithmetic-geometric mean value inequality,

$$\int |1-f|^p w\, dm \geqslant \exp\left[\int \log(|1-f|^p w)\, dm\right] = \exp\left[p\int \log|1-f|\, dm\right] \exp\left[\int \log w\, dm\right].$$

By Jensen's inequality

$$\int \log|1-f|\, dm \geqslant \log|1-f(m)| = 0,$$

and the lemma follows.

LEMMA 5.2. *Assume w satisfies $(\ast)$, is bounded from 0, and $\int \log w\, dv = (0, \ldots, 0)$. Then* (5.1) *holds.*

Proof. Choose a sequence $\{v_n\}$ in $C_R(X)$ such that $v_n \to \log w$ in $L^1(\varrho)$ for all ϱ in $\mathcal{M}(\varphi)$, and such that also $v_n \to \log w$ almost everywhere. Assume also, as we may, that the sequence $\{v_n\}$ is uniformly bounded below. For $n=1, 2, \ldots$ define

$$u_n = v_n - \left(\int v_n d\nu\right) \cdot \log |Z|.$$

Since
$$\lim_{n\to\infty} \int v_n\, d\nu = \int \log w\, d\nu = (0, \ldots, 0),$$

the sequence $\{u_n\}$ is bounded below and converges to $\log w$ in L^1 and almost everywhere. Furthermore $\int u_n d\nu = (0, \ldots, 0)$ for all n. Thus, by Lemma 1.1, there is for each n a function g_n in A such that $\mathcal{R}e\, g_n \geqslant u_n/p$ and

$$\mathcal{R}e\, g_n(m) \leqslant \frac{1}{p} \int u_n\, dm + \frac{1}{n}.$$

Hence we have
$$\lim_{n\to\infty} \mathcal{R}e\, g_n(m) = \lim_{n\to\infty} \frac{1}{p} \int u_n\, dm = \frac{1}{p} \int \log w\, dm. \tag{5.2}$$

For each n define the function f_n in A_0 by

$$f_n = 1 - e^{-g_n + g_n(m)}.$$

Then
$$\Delta_p(w) \leqslant \int |1-f_n|^p w\, dm = e^{p\, \mathrm{Re}\, g_n(m)} \int e^{-p\, \mathrm{Re}\, g_n} w\, dm \leqslant e^{p\, \mathrm{Re}\, g_n(m)} \int e^{-u_n} w\, dm. \tag{5.3}$$

Now the sequence $\{e^{-u_n}\}$ is bounded in absolute value and converges to $1/w$ almost everywhere. Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int e^{-u_n} w\, dm = 1.$$

This together with (5.2) implies that the right side of (5.3) converges to $\exp\, [\int \log w dm]$ as $n \to \infty$. Hence $\Delta_p(w) \leqslant \exp\, [\int \log w dm]$. The reverse inequality is given by the preceding lemma, and so the proof is complete.

LEMMA 5.3. *Formula* (5.1) *holds provided w satisfies* $(*)$ *and* $\int \log w d\nu = (0, \ldots, 0)$.

Proof. Let w be as described. For $n = 1, 2, 3, \ldots$ define

$$\alpha_n = \int \log (w + 1/n)\, d\nu, \quad w_n = |Z|^{-\alpha_n}(w+1/n), \quad s_n = \sup_{x\in X} |Z(x)|^{\alpha_n},$$

(where $|Z(x)|^{\alpha_n}$ stands for the value of the function $|Z|^{\alpha_n}$ at x). By monotone convergence we have $\log (w+1/n) \to \log w$ in L^1, and therefore

$$\lim_{n\to\infty} \alpha_n = \int \log w \, d\nu = (0, \ldots, 0).$$

This implies in particular that $s_n \to 1$. Because $\int \log w_n d\nu = (0, \ldots, 0)$ for all n, it follows by Lemma 5.2 that each w_n satisfies (5.1). Hence for each n there is a function f_n in A_0 such that

$$\int |1-f_n|^p w_n dm \leqslant \exp\left[\int \log w_n dm\right] + 1/n = 1/n + |Z(m)|^{-\alpha_n} \exp\left[\int \log (w+1/n) dm\right]. \quad (5.4)$$

On the other hand,

$$\int |1-f_n|^p w \, dm \leqslant \int |1-f_n|^p (w+1/n) dm \leqslant s_n \int |1-f_n|^p w_n dm. \quad (5.5)$$

Combining (5.4) and (5.5) we obtain

$$\int |1-f_n|^p w \, dm \leqslant \frac{1}{ns_n} + \frac{1}{s_n} |Z(m)|^{-\alpha_n} \exp\left[\int \log (w+1/n) \, dm\right].$$

As $n \to \infty$ the right side of the preceding inequality converges to $\exp[\int \log w \, dm]$. Hence $\Delta_p(w) \leqslant \exp[\int \log w \, dm]$, and the proof is complete.

In order now to complete the proof of Theorem 5.1, assume w satisfies $(\ast)$ and that $p^{-1} \int \log w \, d\nu$ is in $\mathfrak{L}$. Then there is a function h in A^{-1} such that the function $w_1 = |h|^p w$ satisfies $\int \log w_1 d\nu = (0, \ldots, 0)$. Hence (5.1) holds for w_1, and so

$$\Delta_p(w_1) = \exp\left[\int \log (|h|^p w) \, dm\right] = |h(m)|^p \exp\left[\int \log w \, dm\right].$$

This together with the trivial equality $\Delta_p(w_1) = |h(m)|^p \Delta_p(w)$ implies that w satisfies (5.1).

One question that arises is: if w satisfies $(\ast)$ and (5.1), must $p^{-1} \int \log w \, d\nu$ belong to $\mathfrak{L}$? Although we have been unable to answer this, some information will be obtained in § 10.

§ 6. Szegö's second theorem

Our goal in the present section is to prove the following result.

THEOREM 6.1. *Let v be a non-negative function in L^1 satisfying $(\ast)$ such that $\int \log v \, d\nu$ is in $\mathfrak{L}$. Then there is an outer function f in H^1 such that $|f| = v$ almost everywhere.*

Remarks. 1) Outer functions are defined in § 4.

2) The function f is uniquely determined by v to within a multiplicative constant of unit modulus. This follows from results to be obtained in the next section (see the reasoning at the end of the proof of Lemma 7.3).

3) If v is in L^p for a $p>1$ then f is in H^p, by the results of § 4.

The proof of Theorem 6.1 will be accomplished in a series of lemmas. We first need to know that functions in H^p satisfy Jensen's inequality. This is proved by an argument which, although well-known, we have been unable to locate in the literature. We therefore include the details here.

LEMMA 6.1. *If f is in $H^p(1\leqslant p<\infty)$ then*

$$\log|f(m)|\leqslant\int\log|f|\,dm.$$

Proof. Let $\{f_n\}_1^\infty$ be a sequence of functions in A converging to f in L^p. Then each f_n satisfies Jensen's inequality, and so for any $\varepsilon>0$ we have

$$\log|f_n(m)|\leqslant\int\log(|f_n|+\varepsilon)\,dm.$$

As $n\to\infty$ the left side goes to $\log|f(m)|$ and the right side to $\int\log(|f|+\varepsilon)dm$ (because $\log(|f_n|+\varepsilon)\to\log(|f|+\varepsilon)$ in L^1). Thus $\log|f(m)|\leqslant\int\log(|f|+\varepsilon)dm$. The desired inequality is now obtained by letting $\varepsilon\to 0$ monotonically.

The next lemma is purely measure theoretic.

LEMMA 6.2. *Assume $1<p<\infty$, and let $\{f_n\}_1^\infty$ be a sequence of functions converging weakly in L^p to the function f. Then $|f|<\limsup|f_n|$ almost everywhere.*

Proof. It will suffice to show that $\int_E|f|dm\leqslant\int_E\limsup|f_n|dm$ for every Borel set E. By weak convergence we have

$$\int_E|f|\,dm=\int_E f\frac{|f|}{f}\,dm=\lim\int_E f_n\frac{|f|}{f}\,dm\leqslant\limsup\int_E|f_n|\,dm.$$

Hence it only remains to show that

$$\limsup\int_E|f_n|\,dm\leqslant\int_E\limsup|f_n|\,dm. \tag{6.1}$$

For this, let ε be a positive real number, and for each positive integer k let

$$E(\varepsilon,k)=\{x\in E:|f_j(x)|\leqslant\varepsilon+\limsup|f_n(x)|\text{ for all }j\geqslant k\}.$$

Then $\bigcup_k E(\varepsilon, k) = E$. For $j \geq k$ we have

$$\int_E |f_j|\, dm = \int_{E(\varepsilon,k)} |f_j|\, dm + \int_{E - E(\varepsilon,k)} |f_j|\, dm$$

$$\leq \varepsilon + \int_E \limsup |f_n|\, dm + \|f_j\|_p [m(E - E(\varepsilon, k))]^{1/q},$$

where $q = p/(p-1)$. But

$$\lim_{k\to\infty} m(E - E(\varepsilon, k)) = 0,$$

and $\sup_j \|f_j\|_p < \infty$ by weak convergence. Inequality (6.1) now follows, and the proof of the lemma is complete.

Although it is superfluous to our needs, we mention that Lemma 6.2 is also true when $p = 1$. To show this, the only modification one must make in the above proof is the use, in place of Hölder's inequality, of the fact that when a sequence $\{f_n\}$ converges weakly in $L^1(m)$, the sequence of measures $\{|f_n|\, dm\}$ is uniformly absolutely continuous with respect to m [8, p. 294].

The next lemma is the crucial step in the proof of our theorem. As with the crucial lemma of the preceding section, its proof depends on an application of Lemma 1.1. For the rest of this section we let v denote a non-negative function in L^1 satisfying $(*)$.

LEMMA 6.3. *Assume v is in L^∞ and satisfies $\int \log v\, d\nu = (0, ..., 0)$. Then the conclusion of Theorem* 6.1 *holds.*

Proof. Take a sequence $\{u_n\}$ in $C_R(X)$ such that $u_n \to \log v$ in $L^1(\varrho)$ for every ϱ in $\mathcal{M}(\varphi)$, and also almost everywhere. Assume also, as we may, that the sequence $\{u_n\}$ is uniformly bounded above. Define

$$u_n' = u_n - \left(\int u_n d\nu\right) \cdot \log|Z|.$$

Because $\int u_n d\nu \to \int \log v\, d\nu = (0, ..., 0)$, the sequence $\{u_n'\}$ is uniformly bounded above and converges to $\log v$ in L^1 and almost everywhere. Also $\int u_n'\, d\nu = (0, ..., 0)$ for every n. Therefore, by Lemma 1.1, there are functions g_n in A such that $\mathcal{Re}\, g_n \leq \log u_n'$ and

$$\lim_{n\to\infty} \mathcal{Re}\, g_n(m) = \lim_{n\to\infty} \int u_n'\, dm = \int \log v\, dm. \tag{6.2}$$

Let $f_n = \exp(g_n)$. Then $|f_n| \leq \exp(u_n')$, and so the sequence $\{f_n\}$ is uniformly bounded. We may therefore assume that it converges weakly in L^2 to some function f. By the preceding lemma,

$$|f| \leqslant \lim\sup |f_n| \leqslant \lim \exp (u_n') = v \tag{6.3}$$

almost everywhere. Therefore, by Jensen's inequality (Lemma 6.1),

$$\log |f(m)| \leqslant \int \log |f|\, dm \leqslant \int \log v\, dm. \tag{6.4}$$

On the other hand,

$$\log |f(m)| = \lim_{n\to\infty} \log |f_n(m)| = \lim_{n\to\infty} \mathcal{R}e\, g_n(m) = \int \log v\, dm$$

by (6.2). This together with (6.3) and (6.4) implies that $\log|f(m)| = \int \log|f|\, dm$ and that $|f| = v$ almost everywhere. The proof is complete.

LEMMA 6.4. *Assume v is in L^2 and $\int \log v\, d\nu = (0, \ldots, 0)$. Then the conclusion of Theorem* 6.1 holds.

Proof. For $n = 1, 2, \ldots$ define

$$v_n = \min (v, n), \quad \alpha_n = \int \log v_n\, d\nu, \quad v_n' = v_n |Z|^{-\alpha_n}.$$

By the monotone convergence theorem we have $\alpha_n \to (0, \ldots, 0)$. Therefore $v_n' \to v$ pointwise, and the sequence $\{v_n'\}$ is bounded in L^2. Furthermore, the v_n' satisfy the hypotheses of the preceding lemma. Therefore there is for each n an outer function f_n in H^2 such that $|f_n| = v_n'$ a.e. The sequence $\{f_n\}$ is bounded in H^2 and so has a subsequence converging weakly in H^2 to a function f. By Lemma 6.2

$$|f| \leqslant \lim\sup |f_n| = \lim v_n' = v$$

almost everywhere. Hence by Jensen's inequality

$$\log |f(m)| \leqslant \int \log |f|\, dm \leqslant \int \log v\, dm.$$

On the other hand, since $\log|f_n| = \log v_n' \to \log v$ in L^1, and since each f_n is an outer function, we have

$$\int \log v\, dm = \lim_{n\to\infty} \int \log |f_n|\, dm = \lim_{n\to\infty} \log |f_n(m)| = \log |f(m)|.$$

It follows that $\log|f(m)| = \int \log|f|\, dm$ and that $|f| = v$ almost everywhere. The proof is complete.

LEMMA 6.5. *The conclusion of Theorem* 6.1 *holds if* $\int \log v\, d\nu = (0, \ldots, 0)$.

Proof. In this case there is by Lemma 6.4 an outer function g in H^2 such that $|g| = v^{\frac{1}{2}}$ almost everywhere. Let $f = g^2$. Then f is in H^1 and $|f| = v$ almost everywhere. Moreover, because $f(m) = g(m)^2$, it is a triviality to verify that f is an outer function.

We can now complete the proof of Theorem 6.1 in a few words. Namely, suppose only of v that $\int \log v\, dv$ belongs to $\mathfrak{L}$, and choose an h in A^{-1} such that $\int \log v\, dv = \int \log |h|\, dv$. Then by Lemma 6.5 there is an outer function g in H^1 such that $|g| = v|h|^{-1}$ almost everywhere, and the function $f = gh$ meets the requirements of the theorem.

For the purpose of proving the discreteness of $\mathfrak{L}$ we need the following (apparent) extension of Theorem 6.1.

LEMMA 6.6. *Assume v is in L^2. Then the conclusion of Theorem 1 is still true even if $\int \log v\, dv$ only belongs to the closure of $\mathfrak{L}$.*

Proof. Under the hypotheses there is a sequence of σ-tuples $\{\alpha_n\}_1^\infty$ and a sequence of outer function $\{f_n\}_1^\infty$ in H^2 such that $\alpha_n \to (0, \dots, 0)$ and

$$|f_n| = v|Z|^{\alpha_n}$$

almost everywhere. The sequence $\{f_n\}$ is then bounded in L^2 and so we may assume that it converges weakly to a function f. That f is an outer function and $|f| = v$ almost everywhere can now be proved by the same argument used in the proof of Lemma 6.4.

If f is an outer function, must $\int \log|f|\, dv$ belong to $\mathfrak{L}$? It is not difficult to show that the answer is affirmative for the algebras on Riemann surfaces cited in the introduction. In § 10 we shall see that this question is equivalent to the one raised at the end of § 5.

§ 7. Generators of H^p

A function f in $H^p (1 \leqslant p < \infty)$ is called a *generator* of H^p if the linear manifold Af is dense in H^p. In the present section we prove the following result.

THEOREM 7.1. *If f is an outer function in H^p and if $|f|$ satisfies $(\ast)$, then f is a generator of H^p.*

As usual, the proof will be chopped up into a series of lemmas. We denote by H_0^p the set of functions f in H^p such that $f(m) = 0$. A non-negative function w in L^1 will be called a *Szegö function for the exponent p* if $\Delta_p(w) = \exp\,[\int \log w\, dm] > 0$.

LEMMA 7.1. *If f is a generator of H^p then f is an outer function and $|f|^p$ is a Szegö function for the exponent p.*

Proof. Assume f is a generator of H^p. Then obviously $f(m) \neq 0$, and so $\log|f|^p$ is in L^1 (by Jensen's inequality). Furthermore $A_0 f$ is dense in H_0^p, and so the L^p-distance of f from $A_0 f$ is not greater than $|f(m)|$, the distance of f from the function $f - f(m)$. It follows that

$$|f(m)|^p \geqslant \inf_{g \in A_0} \int |f - gf|^p \, dm = \Delta_p(|f|^p).$$

But

$$|f(m)|^p \leqslant \exp\left[\int \log|f|^p \, dm\right] \tag{7.1}$$

by Jensen's inequality, and so

$$\Delta_p(|f|^p) \leqslant \exp\left[\int \log|f|^p \, dm\right]. \tag{7.2}$$

Hence, by Lemma 5.1, inequality (7.2) must be an equality, and therefore inequality (7.1) is also an equality. This completes the proof.

LEMMA 7.2. *Assume that $1 < p < \infty$, and let f be an outer function in H^p such that $|f|^p$ is a Szegö function for the exponent p. Then f is a generator of H^p.*

Proof. If g is in H^p then by Hölder's inequality

$$\left(\int |f - g|^p \, dm\right)^{1/p} \geqslant \int |f - g| \, dm \geqslant |f(m)|.$$

Furthermore equality is achieved here by the function $g = f - f(m)$. It follows that the L^p-distance of f from H_0^p is equal to $|f(m)|$. Since L^p is uniformly convex, the function $f - f(m)$ is the *unique* function in H_0^p for which this distance is achieved.

Now let M be the closure in L^p of $A_0 f$. Then by our hypotheses on f we have

$$\inf_{g \in A_0} \int |f - fg|^p \, dm = \exp\left[\int \log|f|^p \, dm\right] = |f(m)|^p.$$

In other words, the L^p-distance of f from M is equal to $|f(m)|$. Since L^p is uniformly convex, this distance is achieved by a unique function in M. This together with the observations of the preceding paragraph and the inclusion $M \subset H_0^p$ enables us to conclude that $f - f(m)$ belongs to M. Hence the constant function $f(m)$ belongs to the L^p-closure of Af. Therefore the L^p-closure of Af, being invariant under multiplication by A, must contain A and consequently is all of H^p, as desired.

COROLLARY. *If f is an outer function in $H^p (1 < p < \infty)$ such that $|f|$ satisfies $(\ast)$ and $\int \log|f| \, d\nu$ belongs to $\mathfrak{L}$, then f is a generator of H^p.*

Proof. This is an immediate consequence of the preceding lemma and Theorem 5.1.

LEMMA 7.3. *Let f be an outer function in H^1 such that $|f|$ satisfies $(\ast)$ and $\int \log|f|\,d\nu = (0, \ldots, 0)$. Then there is an outer function g in H^2 such that $f = g^2$, and f is a generator of H^1.*

Proof. By Theorem 6.1 there is an outer function g in H^2 such that $|g|^2 = |f|$ almost everywhere. Let $f_1 = g^2$. Then f_1 is an outer function in H^1. The function g is a generator of H^2 by the preceding corollary, and it is a triviality to show from this that f_1 is a generator of H^1. Thus there is a sequence $\{h_n\}_1^\infty$ in A such that $h_n f_1 \to 1$ in L^1. If we let $h = f/f_1$, then we have $|h| = 1$ almost everywhere, and the sequence $\{h_n f\}$ converges in L^1 to h. Hence h is in H^1. Because $f = hf_1$ we have $f(m) = h(m) f_1(m)$. But $|f(m)| = |f_1(m)|$ because f and f_1 are outer and $|f| = |f_1|$ almost everywhere. Hence $|h(m)| = 1$, which together with $|h| = 1$ implies that h is a constant. Therefore f is a constant multiple of f_1, and the proof is complete.

Before completing the proof of Theorem 7.1 we prove a factorization theorem. We shall call a function h in H^p an *inner function* if for some σ-tuple α we have $|h| = |Z|^\alpha$. (Thus our notion of inner function depends on the choice of $Z_1, \ldots, Z_\sigma$.)

THEOREM 7.2. *Let f be a function in $H^p (1 \leqslant p < \infty)$ such that $|f|$ satisfies $(\ast)$. Then there are in H^p an outer function g and an inner function h such that $f = gh$ and $\int \log|g|\,d\nu = (0, \ldots, 0)$. The functions g and h are uniquely determined by f to within multiplicative constants of unit modulus.*

Proof. Let $\alpha = \int \log|f|\,d\nu$. By Theorem 6.1 there is an outer function g in H^p such that $|g| = |f|\,|Z|^{-\alpha}$. Let $h = f/g$. By what we have proved above, the function g is a generator of H^p, and so there is a sequence $\{h_n\}_1^\infty$ in A such that $h_n g \to 1$ in L^p. Because h is bounded, it follows that $h_n f = h_n h g \to h$ in L^p. Thus h is in H^p and so is an inner function. This proves the existence of the desired factorization. The uniqueness follows by the reasoning at the end of the proof of Lemma 7.3.

We are now able to complete the proof of Theorem 7.1. Let f be an outer function in H^p such that $|f|$ satisfies $(\ast)$, and let $f = gh$ be the factorization of the preceding theorem, where g is outer with $\int \log|g|\,d\nu = (0, \ldots, 0)$ and h is inner with $|h| = |Z|^\alpha$, $\alpha = \int \log|f|\,d\nu$. Because $f(m) = g(m)h(m)$, it is clear from Jensen's inequality that h must be an outer function. As we already know from Lemma 7.3 that g is a generator of H^p, the function h belongs to the L^p-closure of Af, and so all we need show is that h is a generator of H^p.

By the Dirichlet approximation theorem [12, p. 170], there exist a sequence of positive integers $\{k_n\}_1^\infty$ and a sequence $\{\gamma_n\}_1^\infty$ of σ-tuples with integral coordinates such that

$$\lim_{n\to\infty} |k_n\alpha - \gamma_n| = 0$$

(where $|\beta|$ denotes the supremum norm of the σ-tuple β). Let $\varepsilon_n = k_n\alpha - \gamma_n$, and define

$$h_n = h^{k_n} Z^{-\gamma_n}.$$

The functions h_n are then outer functions in H^p and belong to the L^p-closure of Ah. Because $|h_n| = |Z|^{\varepsilon_n}$ we have $|h_n| \to 1$ uniformly. Because h_n is outer we have

$$|h_n(m)| = \exp\left[\int \log |h_n|\, dm\right] = \exp\left[\int \log |Z|^{\varepsilon_n} dm\right] = |Z(m)|^{\varepsilon_n} \to 1.$$

Thus we may assume that $\lim h_n(m)$ exists, say $\lim h_n(m) = \lambda$. We then have

$$\int |\lambda - h_n|^2\, dm = 1 + \int |h_n|^2 dm - 2\,\mathcal{Re}[\bar{\lambda} h_n(m)] \to 1 + 1 - 2|\lambda|^2 = 0,$$

i.e. $h_n \to \lambda$ in L^2. Therefore a subsequence of $\{h_n\}$ converges to λ almost everywhere. Because the h_n are uniformly bounded in supremum norm, this subsequence converges to λ in L^p, and so the constant function λ belongs to the L^p-closure of Ah. The proof is complete.

From Theorem 7.1 and Lemma 7.1 it follows that if f is an outer function in H^p such that $|f|$ satisfies $(\ast)$, then $|f|^p$ is a Szegö function for the exponent p. Once we have in our possession a more precise form of Szegö's first theorem, we shall be able to prove a converse of this. Namely, we shall show that if v is a non-negative function in L^p such that $\int \log v\, dm > -\infty$ and v^p is a Szegö function for the exponent p, then v is the modulus of an outer function in H^p. The following lemma is a preliminary result in this direction.

LEMMA 7.4. *Assume $1 < p < \infty$. Let the positive function v in L^p be bounded from 0, and assume that v^p is a Szegö function for the exponent p. Then there is an outer function f in H^p such that $|f| = v^{-1}$ almost everywhere.*

Proof. By the uniform convexity of the space $L^p(v^p dm)$, there is a function g in the $L^p(v^p dm)$-closure of A_0 such that

$$\int |1 - g|^p v^p dm = \Delta_p(v^p). \tag{7.3}$$

Because v is bounded from 0 the function g is in H_0^p. Let $f = 1 - g$. By the arithmetic-geometric mean value inequality and Jensen's inequality,

$$\int |f|^p v^p dm \geqslant \exp\left[\int \log |f|^p dm\right] \exp\left[\int \log v^p dm\right]$$

$$\geqslant |f(m)|^p \exp\left[\int \log v^p dm\right] = \Delta_p(v^p).$$

Therefore, by (7.3), both of the preceding inequalities are equalities. That the arithmetic-geometric mean value inequality is an equality implies that $|f|^p v^p$ is constant almost everywhere, and that Jensen's inequality for f is an equality implies that f is outer. Hence some constant multiple of f meets the requirements of the lemma.

§ 8. The discreteness of $\mathcal{L}$

Let $\hat{X}$ denote the space of multiplicative linear functionals on L^∞ with the Gelfand topology. Under the Gelfand representation the algebra L^∞ is transformed into the algebra $C(\hat{X})$, and the subalgebra H^∞ is transformed into a certain uniformly closed subalgebra of $C(\hat{X})$, which we denote by $\hat{H}^\infty$. It is not hard to show that $\hat{H}^\infty$ separates points of $\hat{X}$, and so is actually a uniform algebra on $\hat{X}$; the proof is essentially the same as that of the corresponding fact for H^∞ of the disk [16, p. 174].

The measure m induces a bounded linear functional on L^∞ which is transformed by the Gelfand representation into a bounded linear functional on $C(\hat{X})$. The latter is represented by a probability measure $\hat{m}$ on $\hat{X}$. The measure $\hat{m}$ is multiplicative on $\hat{H}^\infty$ and is a Jensen measure for $\hat{H}^\infty$ (by Lemma 6.1). Therefore $\hat{m}$ is also an Arens–Singer measure for $\hat{H}^\infty$. We denote by $\hat{\varphi}$ the functional on $\hat{H}^\infty$ induced by $\hat{m}$. Let $\hat{S}$ be the real linear span of the set of all Borel measures on $\hat{X}$ of the form $\mu - \hat{m}$ with μ a representing measure for $\hat{\varphi}$.

In the same way as we lifted m to obtain a measure $\hat{m}$ on $\hat{X}$, we can lift each ν_k to obtain a measure $\hat{\nu}_k$ on $\hat{X}$. The measures $\hat{\nu}_k$ belong to $\hat{S}$, and so the dimension of $\hat{S}$ is at least σ. We shall let $\hat{\nu}$ stand for the σ-tuple of measures $(\hat{\nu}_1, \ldots, \hat{\nu}_\sigma)$ and adopt the same notational conventions with respect to $\hat{\nu}$ that we have been using up to now with respect to ν.

After these definitions we are ready to prove the main result of this section.

THEOREM 8.1. *The set $\mathcal{L}$ is discrete.*

Proof. The proof is merely an adaptation of Hoffman's proof that the multiplicative linear functionals on a logmodular algebra have unique representing measures [17, Theorem 4.2].

Assume that $\mathcal{L}$ is not discrete. Then because $\mathcal{L}$ is a subgroup of E^σ, its closure contains a linear manifold B_0 of dimension σ_0, $0 < \sigma_0 \leqslant \sigma$. Let B be the set of all functions $\hat{u}$ in $C_R(\hat{X})$ such that the σ-tuple $\int \hat{u}\, d\hat{\nu}$ belongs to B_0. Then B is a subspace of $C_R(\hat{X})$ of codimension $\sigma - \sigma_0$. We now assert:

$$\text{All representing measures for } \hat{\varphi} \text{ agree on } B. \tag{8.1}$$

Once this has been proved the theorem will follow by contradiction. For (8.1) implies that the dimension of $\hat{S}$ is at most $\sigma - \sigma_0$, while we have seen above that the dimension of $\hat{S}$ is at least σ.

To prove (8.1) let $\hat{u}$ be any function in B. By Lemma 6.6 there are outer functions f and g in H^∞ such that $|f| = e^u$ and $|g| = e^{-u}$. (Here u denotes the function in L^∞ having $\hat{u}$ as its Gelfand representative.) Let μ_1 and μ_2 be any two representing measures for $\hat{\varphi}$. Then

$$\int \exp(\hat{u})\, d\mu_1 \int \exp(-\hat{u})\, d\mu_2 = \int |\hat{f}|\, d\mu_1 \int |\hat{g}|\, d\mu_2 \geq |f(m) g(m)|$$

$$= \exp\left[\int \log|f|\, dm\right] \exp\left[\int \log|g|\, dm\right] = 1.$$

Hence for all real t

$$\int \exp(t\hat{u})\, d\mu_1 \int \exp(t\hat{u})\, d\mu_2 \geq 1. \tag{8.2}$$

Also, the left side of (8.2) equals 1 at $t=0$. Thus the derivative with respect to t of the left side of (8.2) vanishes at $t=0$. But this derivative equals $\int \hat{u}\, d\mu_1 - \int \hat{u}\, d\mu_2$, and so (8.1) is proved.

The argument we just gave shows that all representing measures for $\hat{\varphi}$ agree on the set of functions $\hat{u}$ in $C_R(\hat{X})$ satisfying $\int \hat{u}\, d\hat{\nu} = (0, \ldots, 0)$. It follows that $\hat{S}$ has dimension at most σ. As we know that $\hat{S}$ contains $\hat{\nu}_1, \ldots, \hat{\nu}_\sigma$, we may conclude that $\hat{\nu}_1, \ldots, \hat{\nu}_\sigma$ span $\hat{S}$. The following conclusion is now immediate.

THEOREM 8.2. *The functional $\hat{\varphi}$ on $\hat{H}^\infty$ satisfies the conditions* (I) *and* (II) *originally imposed on φ.*

Let $\hat{\mathfrak{L}}$ denote the set of all σ-tuples $\int \log|h|\, d\nu$ with h an invertible function in H^∞. From Theorems 8.1 and 8.2 it follows that $\hat{\mathfrak{L}}$ is discrete. Obviously $\hat{\mathfrak{L}} \supset \mathfrak{L}$, but we do not know whether the inclusion can ever be strict. This question turns out to be equivalent to the one raised at the end of § 5, and we shall discuss it further in § 10.

In connection with Theorem 8.1 we make the following comment. We can map the group A^{-1} homomorphically onto $\mathfrak{L}$ by sending f onto $\int \log|f|\, d\nu$. As $\exp(A)$ is contained in the kernel of this map, we get a homomorphism of $A^{-1}/\exp(A)$ onto $\mathfrak{L}$. By Theorem 8.1, $\mathfrak{L}$ is a free Abelian group of rank σ, and thus $A^{-1}/\exp(A)$ has a free Abelian group of rank σ as a factor. Now a theorem of Arens and Royden [27] states that $A^{-1}/\exp(A)$ is isomorphic to the first Cech cohomology group with integer coefficients of the maximal ideal space of A. So we see that our assumptions on φ put topological restrictions on the maximal ideal space of A, provided $\sigma > 0$. (This observation was suggested by John Wermer.)

In proving in the next section that the measures in S are bounded with respect to m, we shall need the following information.

LEMMA 8.1. *If f is an outer function in H^1 such that $|f|$ satisfies $(\ast)$, then $\int \log|f|\,d\nu$ is in $\hat{\mathfrak{L}}$.*

Proof. Let $\alpha = \int \log|f|\,d\nu$. By Theorem 7.2 there is a factorization $f = gh$ with g and h in H^1, g outer, and $|h| = |Z|^\alpha$. Since f and g are both outer it is immediate from Jensen's inequality that h must be outer. Therefore h is a generator of H^1 (Theorem 7.1). Because h is bounded and bounded from 0, it follows that $hH^1 = H^1$. Hence there is an h_1 in H^1 such that $hh_1 = 1$. But then h_1 is the inverse of h in H^∞, and so $\int \log|h|\,d\nu = \alpha$ is in $\hat{\mathfrak{L}}$.

The preceding lemma shows that the question raised at the end of § 6 is equivalent to the question of whether $\hat{\mathfrak{L}} = \mathfrak{L}$.

§ 9. The boundedness of the measures in S

Let N be the complex vector space spanned by the functions $d\nu_1/dm, \ldots, d\nu_\sigma/dm$. To prove that the measures in S are bounded with respect to m, we must show that $N \subset L^\infty$. The first step will be to show that $N \subset L^2$. Let N^2 denote the orthogonal complement of $A + \bar{A}$ in L^2. (A bar over a space of functions denotes the space of complex conjugates.)

LEMMA 9.1. $N = N^2$.

Proof. We first show that $N^2 \subset N$. Because N is closed under complex conjugation and closed in L^1, it will be enough to show that any real L^∞ function u annihilating N also annihilates N^2. Without loss of generality we may assume that $\int u\,dm = 0$. By Theorem 6.1, for $0 < r \leqslant 1$ there are outer functions f_r in H^∞ such that $|f_r| = e^{ru}$. Since

$$\log|f_r(m)| = \int \log|f_r|\,dm = r\int \log u\,dm = 0,$$

we may suppose that $f_r(m) = 1$ for all r. Let the functions g_r be defined by $g_r = (f_r - 1)/r$. We have

$$r^2 \int |g_r|^2 dm = \int e^{2ru} dm - 1.$$

Because u is bounded,

$$\int e^{2ru} dm = 1 + 2r\int u\,dm + O(r^2) = 1 + O(r^2),$$

and therefore

$$\int |g_r|^2 dm = O(1),$$

i.e. the family of functions $\{g_r\}$ is bounded in L^2. Hence there is a sequence $\{r_n\}$ converging to 0 such that the sequence $\{g_{r_n}\}$ converges weakly in L^2, say to the function g. Because each g_r is in H_0^2 so is g, which means in particular that $\int \mathcal{R}e\, g\, dm = \int u\, dm$. On the other hand

$$\mathcal{R}e\, g_r \leqslant \frac{|f_r|-1}{r} = \frac{e^{ru}-1}{r},$$

and the right side converges uniformly to u as $r \to 0$. Hence if E is any measurable set, then by weak convergence

$$\int_E \mathcal{R}e\, g\, dm = \lim_{n\to\infty} \int_E \mathcal{R}e\, g_{r_n}\, dm \leqslant \int_E u\, dm.$$

Thus $\mathcal{R}e\, g \leqslant u$ almost everywhere, which together with the equality $\int \mathcal{R}e\, g\, dm = \int u\, dm$ implies that $\mathcal{R}e\, g = u$ almost everywhere. Therefore u is orthogonal to N^2, which is the desired conclusion. The proof that $N^2 \subset N$ is complete.

Because N^2 is contained in N it is finite dimensional and therefore closed in L^1. Hence to prove the inclusion $N \subset N^2$, and thereby complete the proof of the lemma, it will be enough to show that any real L^∞ function v annihilating N^2 also annihilates N. But if v is as described then it is the real part of a function in H^2, say $v = \mathcal{R}e\, h$. By Theorem 4.3 the functions e^h and e^{-h} are both in H^∞, and so the σ-tuple

$$\int \log |e^h|\, d\nu = \int v\, d\nu$$

is in $\hat{\mathfrak{L}}$. The same reasoning shows that $t \int v\, d\nu$ is in $\hat{\mathfrak{L}}$ for every real t. It therefore follows by the discreteness of $\hat{\mathfrak{L}}$ that $\int v\, d\nu = (0, ..., 0)$, i.e. v annihilates N, as desired. The proof is complete.

LEMMA 9.2. *If the real function u in L^2 is bounded above and in the L^1 closure of $A + \bar{A}$, then u annihilates N.*

Proof. Let u be as described, and assume without loss of generality that $u \leqslant 0$. We first show that there is an outer function h in H^∞ such that $u = \log |h|$. Because u is real and in the L^1 closure of $A + \bar{A}$, it is actually in the L^1 closure of $\mathcal{R}e\, A$. Hence there is a sequence $\{g_n\}$ in A such that $\mathcal{R}e\, g_n \to u$ in L^1 and almost everywhere. Let

$$E(n) = \{x : \mathcal{R}e\, g_n(x) > 0\}.$$

Then

$$\lim_{n\to\infty} \int_{E(n)} \mathcal{R}e\, g_n\, dm = 0.$$

Hence by Lemma 2.1, we can by passing to a subsequence assume that there are functions f_n in A^{-1} such that $\log|f_n| \leqslant -\max(\mathcal{R}e\, g_n, 0)$ and $f_n \to 1$ almost everywhere. Let $h_n = f_n \exp(g_n)$. Then $|h_n| \leqslant 1$ for all n, and so by passing to a further subsequence we may assume that the sequence $\{h_n\}$ converges weakly in L^2, say to the function h. Because $|h_n| \to e^u$ almost everywhere, we have by Lemma 6.2 $\log|h| \leqslant u$ almost everywhere. Thus

$$\int \log|h|\, dm \leqslant \int u\, dm = \lim_{n\to\infty} \mathcal{R}e\, g_n(m) = \lim_{n\to\infty} [\mathcal{R}e\, g_n(m) + \log|f_n(m)|] = \lim_{n\to\infty} \log|h_n(m)|$$

$$= \log|h(m)| \leqslant \int \log|h|\, dm.$$

It follows that $\log|h| = u$ almost everywhere and that $\log|h(m)| = \int \log|h|\, dm$ (i.e. h is outer).

We may now conclude by Lemmas 8.1 and 9.1 that $\int u\,dv$ is in $\hat{\mathcal{L}}$. But the same reasoning shows that $t\int u\,dv$ is in $\hat{\mathcal{L}}$ for every positive real number t, and so it follows by the discreteness of $\hat{\mathcal{L}}$ that $\int u\,dv = (0, \ldots, 0)$, i.e. u annihilates N. The proof is complete.

It is now a simple matter to prove the result we have been aiming for.

THEOREM 9.1. $N \subset L^\infty$.

Proof. We first note that N is spanned by the set of functions of the form $1 - d\mu/dm$ with μ a representing measure for φ. Hence N has a basis consisting of real functions that are bounded above.

Let N^∞ be the annihilator of $A + \bar{A}$ in L^∞, and choose a basis $u_1, \ldots, u_s$ for N^∞. It is obvious that $N^\infty \subset N$. From here on we argue by contradiction, assuming that the last inclusion is proper. Then by the observation of the preceding paragraph, there is a real function u_{s+1} in N which is bounded above and which is not linearly dependent on $u_1, \ldots, u_s$. Because the closure of $A + \bar{A}$ in L^1 has a codimension equal to $\dim(N^\infty) = s$, some non-trivial linear combination of $u_1, \ldots, u_{s+1}$ must lie in this closure. The real and imaginary parts of such a function then belong to both N and the L^1 closure of $A + \bar{A}$. We conclude that N contains a non-null real function u that is bounded above and in the L^1 closure of $A + \bar{A}$. By Lemma 9.1 u belongs to L^2, and so we may apply Lemma 9.2 to conclude that u annihilates N. Hence in particular $\int u^2 dm = 0$, which is a contradiction. The theorem is proved.

The result just proved, of course, enables us to simplify the hypotheses in many of the preceding theorems. We see now that a non-negative function w in L^1 satisfies condition $(\ast)$ if and only if $\int \log w\, dm > -\infty$. This will hold automatically if w is the modulus of an outer function. In particular, therefore, Theorem 7.1 on generators of H^p holds for outer functions without restriction.

§ 10. More precise forms of Szegö's two theorems

For α a σ-tuple of real numbers and $1 \leqslant p < \infty$, let $H^{p,\alpha} = |Z|^\alpha H^p$, and let the linear functional $\varphi_{p,\alpha}$ on $H^{p,\alpha}$ be defined by

$$\varphi_{p,\alpha}(f|Z|^\alpha) = f(m)|Z(m)|^\alpha, \quad f \in H^p.$$

Also let $K(p, \alpha) = \|\varphi_{p,\alpha}\|^p$. By straightforward reasoning one can show that for p fixed, $K(p, \alpha)$ depends continuously on α.

The next result we prove is a strengthened form of Szegö's first theorem.

THEOREM 10.1. *Let w be a non-negative function in L^1 such that $\int \log w dm > -\infty$, and let $\alpha = \int \log w dv$. Then*

$$\Delta_p(w) = K(p, \alpha/p)^{-1} \exp\left[\int \log w dm\right], \quad 1 \leqslant p < \infty. \tag{10.1}$$

Proof. By Theorem 6.1, there is an outer function f in H^p such that $|f|^p = w|Z|^{-\alpha}$. Thus if we let $g = f|Z|^{\alpha/p}$, we have $|g|^p = w$, and so $\Delta_p(w) = \Delta_p(|g|^p)$. Now $\Delta_p(|g|^p)^{1/p}$ is the distance in L^p between g and the subspace of L^p spanned by $A_0 g$. By Theorem 7.1, the linear manifold $A_0 f$ spans H_0^p. Therefore $A_0 g$ spans the kernel of the functional $\varphi_{p,\alpha/p}$, and consequently

$$\Delta_p(|g|^p) = \frac{|\varphi_{p,\alpha/p}(g)|^p}{\|\varphi_{p,\alpha/p}\|^p} = \frac{|f(m)|^p |Z(m)|^\alpha}{K(p, \alpha/p)}.$$

Because f and the Z_k are outer functions,

$$|f(m)|^p|Z(m)|^\alpha = \exp\left[\int \log |f|^p |Z|^\alpha dm\right] = \exp\left[\int \log w dm\right],$$

and (10.1) follows.

COROLLARY. *Let w be a non-negative function in L^1 such that $\int \log w dm = -\infty$. Then $\Delta_p(w) = 0$.*

Proof. For $n = 1, 2, \ldots$ let $\alpha_n = \int \log (w + 1/n) dv$. Then by Theorem 10.1,

$$\Delta_p(w) \leqslant \Delta_p(w + 1/n) = K(p, \alpha_n/p)^{-1} \exp\left[\int \log (w + 1/n)\, dm\right]. \tag{10.2}$$

Now we have noted above that for p fixed, $K(p, \alpha)$ is a continuous function of α. Also, it is easily seen that $K(p, \alpha)$ is constant on each coset of $\mathfrak{L}$. It follows that $K(p, \alpha)$ is bounded from 0. Hence the right side of (10.2) goes to 0 as $n \to \infty$, and the corollary is proved.

We are now able to complete the discussion of the relation between Szegö functions and outer functions. Recall that a non-negative function w in L^1 is called a Szegö function for the exponent p if $\Delta_p(w) = \exp [\int \log w dm] > 0$. It is proved in § 7 that if f is an outer function in H^p then $|f|^p$ is a Szegö function for the exponent p. The following theorem gives the converse of this result.

THEOREM 10.2. *Let $1 \leqslant p < \infty$. Let v be a non-negative function in L^p such that $\int \log v dm > -\infty$ and such that v^p is a Szegö function for the exponent p. Then there is an outer function f in H^p such that $|f| = v$ almost everywhere.*

Proof. Let $\alpha = \int \log v d\nu$. By Theorem 6.1 there is an outer function g in H^p such that $|g| = v|Z|^{-\alpha}$. It follows by the preceding theorem that $|Z|^{\alpha p}$ is a Szegö function for the exponent p. Therefore, by Lemma 7.4, there is an outer function h' in H^p such that $|h'| = |Z|^{-\alpha}$. But then $|h'|^p$ is a Szegö function for the exponent p (Theorem 7.1 and Lemma 7.1), and so, again by Lemma 7.4, there is an outer function h in H^p such that $|h| = |h'|^{-1} = |Z|^{\alpha}$. The function $f = gh$ is then an outer function in H^p and $|f| = v$, as desired.

At the end of § 5 we raised the following question: if w is a Szegö function for the exponent p, must $p^{-1} \int \log w d\nu$ belong to $\mathfrak{L}$? The preceding theorem, together with Lemma 8.1 and the results of § 7, shows that this is equivalent to the question of whether $\hat{\mathfrak{L}} = \mathfrak{L}$. We see that the function $K(p, \alpha)$, which by Lemma 5.1 never exceeds 1, is equal to 1 precisely when α is in $\hat{\mathfrak{L}}$.

We make one additional remark on the problem of the equality of $\mathfrak{L}$ and $\hat{\mathfrak{L}}$. Let us suppose for the sake of convenience that we have chosen $Z_1, \ldots, Z_\sigma$ and $\nu_1, \ldots, \nu_\sigma$ in such a manner that $\mathfrak{L}$ is the set of points in E^σ with integral coordinates; the discreteness of $\mathfrak{L}$ enables us to do this. Suppose it is true that $\hat{\mathfrak{L}} \neq \mathfrak{L}$. Then because $\hat{\mathfrak{L}}$ is discrete and contains $\mathfrak{L}$, the points in $\hat{\mathfrak{L}}$ must have rational coordinates. Hence if α is a point of $\hat{\mathfrak{L}}$ not in $\mathfrak{L}$, then for some positive integer q the point $\gamma = q\alpha$ is in $\mathfrak{L}$. By the reasoning used in proving Lemma 8.1, there is an invertible function h in H^∞ such that $|h| = |Z|^\alpha$. The functions h^q and Z^γ are then outer functions with the same modulus and so one is a constant multiple of the other. Hence for the correct choice of h we have $h^q = Z^\gamma$. The function Z^γ therefore has a q-th root in H^∞, although it has no q-th root in A. Thus if $\hat{\mathfrak{L}} \neq \mathfrak{L}$, then in going from A to H^∞ certain invertible functions acquire roots that they did not originally possess. We know of no intuitive reason either for believing or disbelieving in the possibility of this phenomenon.

The next result relates to Szegö's second theorem.

THEOREM 10.3. *Let v be a non-negative function in L^p $(1 \leqslant p < \infty)$ such that*

$$\int \log v\,dm > -\infty.$$

Then there is a function f in H^p such that $|f| = v$ almost everywhere.

The essential part of the proof is contained in the following lemma.

LEMMA 10.1. *Let α be a σ-tuple in E^σ. Then there is a function h in H^∞ such that $|h| = |Z|^\alpha$ almost everywhere.*

Suppose the lemma has been proved, let v be as in Theorem 10.3, and let $\alpha = \int \log v\,dv$. By Theorem 6.1 there is a function g in H^p such that $|g| = v|Z|^{-\alpha}$. Thus, if h is the function of Lemma 10.1, then $f = gh$ is in H^p and $|f| = v$, as desired. It therefore only remains to prove the lemma.

Our proof of Lemma 10.1 exploits an often used technique; namely, we shall obtain the desired function h as one solution of a dual extremal problem. This is precisely the method used by Tumarkin and Havinson [31] to prove a corresponding result about analytic functions in finitely connected plane Jordan domains. The possibility of applying this method in more abstract settings was suggested by Bishop [6].

Proof of Lemma 10.1. Let α be fixed, and let c denote the supremum of $|f(m)|$ as f varies over the class of functions in A satisfying $|f| \leqslant |Z|^\alpha$. If we let $B = |Z|^{-\alpha}A$ and define the linear functional ψ on B by

$$\psi(|Z|^{-\alpha} f) = f(m) \quad (f \in A),$$

then c is just the norm of ψ (computed with respect to the supremum norm on B). Therefore, by the Hahn-Banach and Riesz representation theorems, the functional ψ is represented by a Borel measure μ' on X of total variation c. Letting $d\mu = |Z|^{-\alpha} d\mu'$, we have $\int |Z|^\alpha d|\mu| = c$ and $\int f\,d\mu = f(m)$ for all f in A (i.e. μ is a complex representing measure for φ). We now assert that the measure μ is mutually absolutely continuous with m. In fact, let μ_a and μ_s be the absolutely continuous and singular components of μ with respect to m. Because the measure $\mu - m$ annihilates A, it follows by the generalized F. and M. Riesz theorem (Theorem 3.1) that the singular component of $\mu - m$, i.e. μ_s, annihilates A. Therefore the measure $|Z|^\alpha d\mu_a$ represents ψ, and we have

$$c \leqslant \int |Z|^\alpha d|\mu_a| \leqslant \int |Z|^\alpha d|\mu_a| + \int |Z|^\alpha d|\mu_s| = c.$$

It follows that $\mu = \mu_a$, i.e. $\mu << m$. We must now show that $d\mu/dm$ cannot vanish on a set of positive m-measure. For this we note that if f is in A_0, then

$$\int |1-f| \left| \frac{d\mu}{dm} \right| dm \geqslant \left| \int (1-f)\, d\mu \right| = 1,$$

and so $\Delta_1(|d\mu/dm|) \geqslant 1 > 0$. But if $d\mu/dm$ vanished on a set of positive m-measure we would have $\int \log|d\mu/dm|\,dm = -\infty$, which would imply by the corollary to Theorem 10.1 that $\Delta_1(|d\mu/dm|) = 0$. This completes the proof of the assertion that μ and m are mutually absolutely continuous.

Let $\{f_n\}_1^\infty$ be a sequence of functions in A such that $|f_n| \leqslant |Z|^\alpha$ for all n and $f_n(m) = \int f_n d\mu \to c$. Because bounded sets in L^∞ are weak-star relatively compact, this sequence has a weak-star cluster point h, and clearly $|h| \leqslant |Z|^\alpha$ almost everywhere modulo m. Moreover, $\int h d\mu = c$ (because $\mu << m$). If we write the last equality in the form

$$\int h|Z|^{-\alpha}|Z|^\alpha d\mu = c$$

and recall that $\int |Z|^\alpha d|\mu| = c$, we see that we must have $|h|\,|Z|^{-\alpha} = 1$ almost everywhere modulo μ, and therefore also modulo m. The proof of Lemma 10.1 is complete.

Tumarkin and Havinson [31] have shown that in the case of a finitely connected plane Jordan domain, the extremizing function h is continuous to the boundary and has less zeros in the domain than the latter's degree of connectivity. One suspects that in the present abstract setting, the corresponding property of h should be that it generates an invariant subspace of H^2 whose codimension in H^2 does not exceed σ. However, we do not know how to prove this.

§ 11. Annihilators

In the present section we obtain information on the annihilators of A in the spaces L^p. We need first some facts about invariant subspaces. A (closed) subspace of L^p will be called *invariant* if it is invariant under multiplication by the functions in A.

LEMMA 11.1. *Let f be a function in L^p $(1 \leqslant p < \infty)$ such that $\int \log|f|\,dm > -\infty$, let $\alpha = \int \log|f|\,dv$, and let M be the invariant subspace of L^p generated by f. Then there is a function v in L^∞, with $|v| = |Z|^\alpha$ almost everywhere, such that $M = vH^p$. The function v is uniquely determined by f to within a multiplicative constant of unit modulus.*

Proof. By Theorem 6.1 there is an outer function f_1 in H^p such that $|f_1| = |f|\,|Z|^{-\alpha}$. Letting $v = f/f_1$, we have $|v| = |Z|^\alpha$. If M_1 is the invariant subspace generated by f_1, then we obviously have $M = vM_1$. By Theorem 7.1, $M_1 = H^p$, and so $M = vH^p$, as desired.

To prove the uniqueness of v, suppose w has the same properties. Then $w^{-1}vH^p = H^p$, and so $w^{-1}v$ is a generator of H^p. Therefore $w^{-1}v$ is an outer function. This and the fact that $|w^{-1}v| = 1$ almost everywhere imply that $\int w^{-1}v\,dm = 1$, which is possible only if $w^{-1}v$ = constant almost everywhere. The proof is complete.

LEMMA 11.2. *Let the subspace M of L^p $(1 \leqslant p < \infty)$ be invariant under A. Then $M \cap L^\infty$ is L^p-dense in M.*

Proof. Let f be any function in M and let M' be the smallest invariant subspace of L^p containing the function $|f| + 1$. It follows from Lemma 11.1 that $M' \cap L^\infty$ is L^p-dense in M'. Let $g = f/(|f| + 1)$. Since multiplication by g is a bounded operator on L^p, the linear manifold gM' is contained in the invariant subspace of A generated by the function $g(|f| + 1) = f$. Hence $gM' \subset M$, and it is clear that $(gM') \cap L^\infty$ is L^p-dense in gM'. Therefore f is in the L^p-closure of $M \cap L^\infty$, and the lemma is proved.

THEOREM 11.1. (i) *The annihilator of A_0 in L^p is $H^p + N$, $1 \leqslant p \leqslant \infty$.*

(ii) *The annihilator of $A + N$ in L^p is H_0^p, $1 \leqslant p \leqslant \infty$.*

Proof. The case $p = 2$ is an immediate consequence of Lemma 9.1. The cases $1 \leqslant p < 2$ follow from the case $p = 2$ via Lemma 11.2. The cases $2 < p \leqslant \infty$ follow from the cases $1 \leqslant p < 2$ by a simple duality argument.

We also have information on the real annihilating measures of A, namely, that the only real annihilating measures of A that are absolutely continuous with respect to m are the measures in S. This is the content of our next theorem. First a lemma is needed.

LEMMA 11.3. *The only real functions in H^1 are the constants.*

Proof. Let f be a real function in H^1 such that $f(m) = 0$. Then by Jensen's inequality, for any real number t,

$$\int \log |1 + tf|\, dm \geqslant 0.$$

This implies by [17, Lemma 6.6] that $f = 0$ almost everywhere.

THEOREM 11.2. *If the function w in L^1 annihilates $A + \bar{A}$, then w belongs to N.*

Proof. We may obviously assume without loss of generality that w is real. By Theorem 11.1 there are a function h in H_0^1 and real functions u and v in N such that $w = h + u + iv$. We then have $\mathfrak{Im}\, h = -v$, and so by Theorem 4.3 the functions e^{ih} and e^{-ih} are in H^∞. Hence the σ-tuple $\int \log |e^{ih}|\, d\nu = \int v\, d\nu$ is in $\hat{\mathfrak{C}}$. The same reasoning shows that $t \int v\, d\nu$ is in $\hat{\mathfrak{C}}$ for all real t, and so $\int v d\nu = (0, ..., 0)$ because $\hat{\mathfrak{C}}$ is discrete. This means that v annihilates N, and therefore $\int v^2 dm = 0$, i.e. $v = 0$. Hence we have $w = h + u$, and so h is a real function in H_0^1. Consequently $h = 0$ by Lemma 11.3, and $w = u$, as desired.

§ 12. Gleason parts

Two multiplicative linear functionals on a uniform algebra are said to *lie in the same part* if their difference has norm less than 2. This notion was introduced by Gleason in [10], where he observed that the relation of lying in the same part is an equivalence relation. Thus the maximal ideal space of a uniform algebra decomposes into disjoint parts, and one can show that these parts are the largest sets on which one can hope to impose an analytic structure [10].

B. V. O'Neill has shown that in a hypo-Dirichlet algebra, the Arens–Singer measures of two functionals in the same part are mutually boundedly absolutely continuous [24, Lemma A]. His proof can with only minor modifications be carried through under the hypotheses of the present paper. For the sake of completeness we shall present the details. First a lemma about general uniform algebras it needed. Its proof can be found (essentially) in Hoffman's logmodular paper [17, Lemma 7.5].

LEMMA 12.1. *If ψ_1 and ψ_2 are multiplicative linear functionals on a uniform algebra, then the following are equivalent.*

(i) *ψ_1 and ψ_2 are in different parts.*

(ii) *There is a sequence of functions $\{f_n\}$ in the algebra, with $|f_n| \leqslant 1$ for all n, such that $\psi_1(f_n) \to 0$ and $|\psi_2(f_n)| \to 1$.*

THEOREM 12.1 (O'Neill). *Let φ_1 be a multiplicative linear functional on A lying in the same part as φ, and let m_1 be an Arens–Singer measure for φ_1. Then m_1 is boundedly absolutely continuous with respect to m.*

Proof. Assume the conclusion of the theorem is false. Then there is a sequence $\{u_n\}$ of non-negative functions in $\mathcal{C}(X)$ such that $\int u_n dm \to 0$ and $\int u_n dm_1 \to \infty$. By Lemma 2.1, we can by passing to a subsequence suppose that there are functions f_n in A^{-1} such that $|f_n| \leqslant \exp(-u_n)$ and $f_n \to 1$ almost everywhere modulo m. We then have

$$\varphi(f_n) = \int f_n dm \to 1, \quad |\varphi_1(f_n)| = \exp\left[\int \log|f_n| dm_1\right] \leqslant \exp\left[-\int u_n dm_1\right] \to 0.$$

As also $|f_n| \leqslant 1$ for all n, it follows by Lemma 12.1 that φ and φ_1 lie in different parts. This contradiction proves the theorem.

Let φ_1 and m_1 be as in the preceding theorem, and let S_1 be the real linear span of the set of measures of the form $\mu_1 - m_1$ with μ_1 a representing measure for φ_1. If μ_1 is any representing measure for φ_1, then the negative component of $\mu_1 - m_1$ is boundedly absolutely continuous with respect to m_1 and therefore also with respect to m, so that $m + \delta(\mu_1 - m_1)$ is a representing measure for φ whenever δ is a sufficiently small positive real number. We may conclude that S_1 is contained in S. Thus φ_1 satisfies the conditions that we originally placed on φ, and we can interchange the roles of φ and φ_1 in the above reasoning. In particular, m is boundedly absolutely continuous with respect to m_1, and $S_1 = S$.

We assume explicitly for the remainder of this section that $\sigma > 0$. We shall show that then the part containing φ is non-trivial, i.e. that it contains a functional other than φ. For this purpose we introduce a class of kernel functions in the space H^2. If α is any σ-tuple of real numbers, we can obtain a new inner product on H^2 by replacing the measure m by the measure $|Z|^{2\alpha}dm$. Let B_α denote the kernel function with respect to this new inner product for the functional on H^2 induced by m. In other words, then, B_α is the unique function in H^2 such that

$$f(m) = \int f \bar{B}_\alpha |Z|^{2\alpha} dm$$

for all f in H^2. The object of the next few lemmas is to show that there is an α such that B_α is *not* an outer function. Once this has been done the non-triviality of the part containing φ will follow without great difficulty. For any α, let $c_\alpha = \| |Z|^\alpha \|_\infty$.

LEMMA 12.2. *The map $\alpha \to B_\alpha$ is continuous in the L^2 norm.*

Proof. We prove first the following subsidiary assertions.

(i) *$\int |B_\alpha|^2 dm$ stays bounded as α varies over any bounded set.*

(ii) *The map $\alpha \to B_\alpha(m)$ is continuous.*

Proof of (i). Because

$$\int |B_\alpha|^2 \, dm \leq c_{-2\alpha} \int |B_\alpha|^2 \, |Z|^{2\alpha} \, dm = c_{-2\alpha} B_\alpha(m) \leq c_{-2\alpha} \int |B_\alpha| dm,$$

it will be enough to show that $\int |B_\alpha| dm$ stays bounded as α varies over any bounded set. But

$$\frac{1}{c_{-\alpha}} \int |B_\alpha| \, dm \leq \int |B_\alpha| \, |Z|^\alpha \, dm \leq \left(\int |B_\alpha|^2 \, |Z|^{2\alpha} \, dm \right)^{\frac{1}{2}} = (B_\alpha(m))^{\frac{1}{2}} \leq \left(\int |B_\alpha| \, dm \right)^{\frac{1}{2}}.$$

Therefore $\int |B_\alpha| dm \leq c_{-2\alpha}$, and the desired conclusion follows.

Proof of (ii). We have

$$B_\alpha(m) = \int B_\alpha \bar{B}_\beta |Z|^{2\beta} dm, \qquad B_\beta(m) = \overline{B_\beta(m)} = \int B_\alpha \bar{B}_\beta |Z|^{2\alpha} dm.$$

Therefore

$$|B_\alpha(m) - B_\beta(m)| \leqslant \int |B_\alpha B_\beta| \left| |Z|^{2\alpha} - |Z|^{2\beta} \right| dm$$

$$\leqslant \left(\int |B_\alpha|^2 dm\right)^{\frac{1}{2}} \left(\int |B_\beta|^2 dm\right)^{\frac{1}{2}} \max \left| |Z|^{2\alpha} - |Z|^{2\beta} \right|,$$

and the desired continuity follows by (i).

We can now complete the proof of Lemma 12.2. We have

$$\frac{1}{c_{-2\alpha}} \int |B_\alpha - B_\beta|^2 dm \leqslant \int |B_\alpha - B_\beta|^2 |Z|^{2\alpha} dm = \int |B_\alpha|^2 |Z|^{2\alpha} dm - \int B_\alpha \bar{B}_\beta |Z|^{2\alpha} dm$$

$$- \int \bar{B}_\alpha B_\beta |Z|^{2\alpha} dm + \int |B_\beta|^2 |Z|^{2\beta} dm + \int |B_\beta|^2 (|Z|^{2\alpha} - |Z|^{2\beta}) dm$$

$$\leqslant B_\alpha(m) - B_\beta(m) + \left(\max \left| |Z|^{2\alpha} - |Z|^{2\beta} \right| \right) \int |B_\beta|^2 dm,$$

and the desired continuity follows by (i) and (ii).

LEMMA 12.3. *$\int \log |B_\alpha| dm > -\infty$ for all α.*

Proof. By Jensen's inequality,

$$\exp\left[\int \log |B_\alpha| dm\right] \geqslant |B_\alpha(m)| = \int |B_\alpha|^2 |Z|^{2\alpha} dm > 0.$$

LEMMA 12.4. *Suppose $\{u_n\}_1^\infty$ is a sequence of non-negative functions converging in L^1 to the function u_0. Suppose further that $\log u_n$ is in L^1, $n = 0, 1, 2, \ldots$, and that $\int \log u_n dm \to \int \log u_0 dm$. Then $\log u_n \to \log u_0$ in L^1.*

Proof. As is easily seen, we may assume without loss of generality that $u_n \to u_0$ almost everywhere. If it happens that $u_n \geqslant 1$ for all n, then $|\log u_0 - \log u_n| \leqslant |u_0 - u_n|$ and the desired conclusion is immediate. If, on the other hand, it happens that $u_n \leqslant 1$ for all n, then we have

$$w_n = \log u_0 - \log u_n \geqslant \log u_0, \tag{12.1}$$

$$\int w_n dm \to 0, \tag{12.2}$$

$$w_n \to 0 \quad \text{almost everywhere.} \tag{12.3}$$

If E_n is the set where $w_n \leqslant 0$, then (12.1) and (12.3) together with the dominated convergence theorem give $\int_{E_n} w_n dm \to 0$. Hence, by (12.2),

$$\int |w_n| dm = \int w_n dm - 2\int_{E_n} w_n dm \to 0,$$

and so in this case also the desired conclusion holds. To handle the general case we simply note that $\log u_n = \log \max (u_n, 1) + \log \min (u_n, 1)$ and apply the two special cases just treated.

LEMMA 12.5. *If the map $\alpha \to \int \log |B_\alpha| dm$ is continuous, then the map $\alpha \to \log |B_\alpha|$ is continuous in the L^1 norm.*

Proof. This follows immediately from the three preceding lemmas.

LEMMA 12.6. *There is an α such that B_α is not an outer function.*

Proof. Assume the lemma is false. Then we have $\int \log |B_\alpha| dm = \log |B_\alpha(m)|$ for all α, and so it follows by Lemmas 12.2 and 12.5 that the map $\alpha \to \int \log |B_\alpha| d\nu$ of E^σ into itself is continuous. The range of this map is therefore a connected subset of E^σ. But also this range is contained in the discrete set $\hat{\mathfrak{L}}$ (Lemma 8.1), and therefore it must consist of a single point. On the other hand, if γ is a σ-tuple with integral coordinates, then a simple computation shows that $B_\gamma = \overline{Z^{-\gamma}(m)} Z^{-\gamma}$. Hence the range of the map $\alpha \to \int \log |B_\alpha| d\nu$ contains all σ-tuples with integral coordinates. This contradiction proves the lemma.

Is the map $\alpha \to \int \log |B_\alpha| dm$ continuous? We suspect that this is so but have been unable to prove it except in the case $\sigma = 1$. As observed in the above proof, the continuity of the map $\alpha \to \int \log |B_\alpha| dm$ implies the continuity of the map $\alpha \to \int \log |B_\alpha| d\nu$. Once the latter is known one can show using topological considerations that the map $\alpha \to \int \log |B_\alpha| d\nu$ is surjective, and from this and the factorization theorem (Theorem 7.2) it follows that for each σ-tuple β there is a function h in H^∞ such that $h = |Z|^\beta$ almost everywhere. Of course, we have already proved this last result (Lemma 10.1). However, the proof just sketched would be interesting, we feel, if the gap in it could be filled.

Before stating the next lemma we mention that the functions B_α are all bounded. This follows from Theorem 9.1, because the measures

$$\frac{|B_\alpha|^2 |Z|^{2\alpha}}{B_\alpha(m)} dm$$

are representing measures for φ.

LEMMA 12.7. *For any σ-tuple α, the invariant subspace of H^2 generated by the function B_α has a codimension in H^2 of at most σ.*

Proof. Let M and K be the closures in L^2 of $B_\alpha A$ and $B_\alpha \bar{A}_0$ respectively. (Thus M is the invariant subspace of H^2 generated by B_α.) Let $L^2(\alpha)$ denote the L^2 space of the measure $|Z|^{2\alpha}dm$. Then L^2 and $L^2(\alpha)$ consist of the same (classes of) functions, and the identity map of either onto the other is bounded. Hence H^2, M and K can all be regarded as subspaces of $L^2(\alpha)$. It is easily seen that K is orthogonal to H^2 in $L^2(\alpha)$, and so we can prove the lemma by showing that the orthogonal complement of $M+K$ in $L^2(\alpha)$ has a dimension of at most σ. But if J is this orthogonal complement, then the subspace $\bar{B}_\alpha |Z|^{2\alpha} J$ is orthogonal to $A+\bar{A}$ in L^2 and so its dimension is at most σ (Lemma 9.1). Because B_α and $|Z|^{2\alpha}$ are non-zero almost everywhere, it follows that the dimension of J is at most σ, as desired.

We are finally able to prove the result we have been aiming for.

THEOREM 12.2. *The part containing φ is non-trivial.*

Proof. By Lemma 12.6 we can choose a σ-tuple α such that B_α is not an outer function. Let M be the invariant subspace of H^2 generated by B_α and let J be the orthogonal complement of M in H^2. The subspace J is non-trivial (Lemma 7.1) and finite dimensional (Lemma 12.7). Let P be the orthogonal projection in L^2 with range J. For each f in H^∞ let the operator T_f on J be defined by

$$T_f h = P(fh), \quad h \in J.$$

It is easy to show that $T_f T_g = T_{fg}$ for all f and g in H^∞. Hence $\{T_f: f \in H^\infty\}$ is a commuting family of operators on a finite dimensional space, and so this family has a common eigenvector, say h_0. We may suppose that $\int |h_0|^2 dm = 1$. For f in H^∞ let $\hat{\psi}(f)$ denote the eigenvalue of T_f corresponding to the eigenvector h_0. Then $\hat{\psi}$ is a multiplicative linear functional on H^∞, and so $\psi = \hat{\psi}|A$ is a multiplicative linear functional on A. If f is in H^∞, then $fh_0 - \hat{\psi}(f)h_0$ is orthogonal to J, and therefore

$$\hat{\psi}(f) = \int f |h_0|^2 dm.$$

Hence $|h_0|^2 dm$ is a representing measure for ψ. This implies that φ and ψ are in the same part, because if two functionals are in different parts then any representing measure for the one is singular with respect to any representing measure for the other [11, Proposition 4]. It remains to show that $\psi \neq \varphi$. Now $\hat{\psi}$ is weak-star continuous. Therefore, if ψ were the same as φ, then $\hat{\psi}$ would equal $\hat{\varphi}$ (the functional on H^∞ induced by m), because A is weak-star dense in H^∞. But the function $B_\alpha h_0$ belongs to M and so is orthogonal to h_0. Hence

$$\hat{\varphi}(B_\alpha) = \int B_\alpha h_0 \bar{h}_0 dm = 0,$$

while, on the other hand, $\hat{\varphi}(B_\alpha) = B_\alpha(m) \neq 0$. The proof is complete.

§ 13. Invariant Subspaces

We shall say that an invariant subspace M of L^p $(1 \leqslant p < \infty)$ is of *type B* if $A_0 M$ is not dense in M. Given a function f in L^p, the invariant subspace of L^p generated by f is of type B if and only if $\Delta_p(|f|^p) \neq 0$. By Lemma 5.1 and the Corollary to Theorem 10.1, this will be true if and only if $\int \log|f| dm > -\infty$. Moreover, when the latter happens the invariant subspace of L^p generated by f is by Lemma 11.1 of the form wH^p where w is a function in L^∞ that agrees in modulus almost everywhere with $|Z|^\alpha$ for some σ-tuple of real numbers α. We shall call such a function w a *rigid function* and such a subspace wH^p a *Beurling subspace*. On the basis of what is known about logmodular algebras and the like (see for example [28]), it seems reasonable to make the following

CONJECTURE. *Every invariant subspace of L^p of type B is a Beurling subspace.*

We have been unable to prove this conjecture in general, but have managed to reduce the problem somewhat. The reduction is described in the present section. It will enable us in the next section to treat the case of finite Riemann surfaces. We have succeeded in proving the conjecture for the case $\sigma = 1$, and this proof is given at the end of the present section.

For the rest of this section we confine our attention to the case $p = 2$. The means of relating invariant subspaces in L^p to invariant subspaces in L^2 is provided by Lemma 11.2. Let G be the Gleason part containing φ. For ψ in G we let A_ψ denote the kernel of ψ and H^2_ψ the closure of A_ψ in H^2. (Thus $A_\varphi = A_0$ and $H^2_\varphi = H^2_0$.) If $\psi \neq \varphi$ then H^2_ψ contains a function f such that $f(m) \neq 0$, and from this it follows that H^2_ψ is of type B. It does not seem immediately evident that H^2_φ is of type B. Of course, this conclusion is implied by the above conjecture, because H^2_φ is of type B with respect to ψ for any ψ in G distinct from φ. What we shall prove is that if the above conjecture holds for the subspaces H^2_ψ $(\psi \in G)$, then it holds in general.

THEOREM 13.1. *If the subspace H^2_ψ is a Beurling subspace for every ψ in G, then every invariant subspace of L^2 of type B is a Beurling subspace.*

The proof of the theorem will be relegated to three lemmas.

LEMMA 13.1. *Let M be an invariant subspace of L^2 of type B and let h be a non-zero function in M which is orthogonal to $A_\varphi M$. Then the invariant subspace of L^2 generated by h has a codimension in M of at most σ.*

Proof. The proof is about the same as that of Lemma 12.7. Assume for convenience that $\int |h|^2 dm = 1$. If f is any function in A_φ, then f and fh are orthogonal, and therefore $\int f|h|^2 dm = 0$. This implies that $|h|^2 dm$ is a representing measure for φ. Hence h is bounded and $\int \log|h|\,dm > -\infty$ (the latter by the corollary to Theorem 10.1). Let J be the orthogonal complement in M of the invariant subspace generated by h. Then because $h\bar{A}_\varphi$ is orthogonal to M, the subspace J is orthogonal to $hA + h\bar{A}_\varphi$. Therefore $\bar{h}J$ is orthogonal to $A + \bar{A}_\varphi$, and so the dimension of $\bar{h}J$ is at most σ (Lemma 9.1). Since h is non-zero almost everywhere, it follows that the dimension of J is at most σ, as desired.

LEMMA 13.2. *Assume that H^2_φ is a Beurling subspace, and let w be a rigid function such that $H^2_\varphi = wH^2$. Let M be any invariant subspace of L^2 of type B. Then wM has codimension one in M.*

Proof. We first note that wM equals the L^2 closure of $A_\varphi M$. In fact, that $A_\varphi M$ is contained in wM is trivial, and that wM is contained in the L^2 closure of $A_\varphi M$ follows because w is in the weak-star closure of A_φ. Now suppose the lemma is false, i.e. that the orthogonal complement of wM in M has codimension greater than one. Choose a non-zero vector h_1 in this orthogonal complement, and let M_1 be the invariant subspace of L^2 generated by h_1. Let J be the orthogonal complement of M_1 in M, let P be the orthogonal projection in L^2 with range J, and let the operator T on J be defined by $Tf = P(wf)$. The adjoint of T is then given by $T^*f = P(\bar{w}f)$. Now M_1 is a Beurling subspace, and so the orthogonal complement of wM_1 in M_1 is spanned by h_1. Therefore any non-zero vector in M orthogonal to both wM and h_1 must be in J. We have assumed that there is such a vector, and consequently the operator T^* has 0 as an eigenvalue. By Lemma 13.1, J is finite dimensional, and thus T has 0 as an eigenvalue. It follows that there is a non-zero vector h_2 in J such that wh_2 is in M_1. Let M_2 be the subspace spanned by M_1 and h_2. Because multiplication by w is an isomorphism on L^2, the subspace wM_1 has codimension one in wM_2. But this is absurd because obviously $wM_1 = wM_2$. This contradiction proves the lemma.

By Lemma 13.1, if M is an invariant subspace of L^2 of type B, then there is a function in M that generates an invariant subspace whose codimension in M is finite. Hence the proof of Theorem 13.1 will be complete once we have proved the next lemma.

LEMMA 13.3. *Assume that H^2_ψ is a Beurling subspace for every ψ in G. Let M be an* *invariant subspace of L^2 of type B, and let h_1 be a function in M that generates an invariant subspace having finite codimension $\delta > 0$ in M. Then there is a function in M that generates an invariant subspace of codimension $\delta - 1$ in M.*

Proof. Let M_1 be the invariant subspace of L^2 generated by h_1 and let J be the orthogonal complement of M_1 in M. Let P be the orthogonal projection in L^2 with range J, and for f in A let the operator T_f on J be defined by $T_f h = P(fh)$. As in the proof of Theorem 12.2, the family of operators $\{Tf: f \in A\}$ is commutative, and so this family has a common eigenvector h_2. For f in A let $\psi(f)$ denote the eigenvalue of T_f for the eigenvector h_2. Then just as before, ψ is a multiplicative linear functional on A lying in the part G (see the proof of Theorem 12.2). Let w be a rigid function such that $H^2_\psi = wH^2$. Also let M_2 be the subspace spanned by M_1 and h_2. If f is in A_ψ then fh_2 is orthogonal to J and therefore in M_1. It follows that M_2 is an invariant subspace and that wM_2 is contained in M_1. But wM_2 has codimension one in M_2 by Lemma 13.2, and therefore $wM_2 = M_1$. Hence M_2 is the invariant subspace generated by h_1/w, and the proof is complete.

Although Theorem 13.1 reduces the above stated invariant subspace conjecture to a special case, it is in this special case where much of the interest resides because it is connected with the problem of imposing an analytic structure on the part G. This problem has been studied by Wermer [37] and O'Neill [24], and their results could be improved and simplified if it were known that the subspaces H^2_ψ are Beurling subspaces.

We now prove the conjecture for the case $\sigma = 1$.

THEOREM 13.2. *Assume $\sigma = 1$. Then the subspace H^2_ψ is a Beurling subspace.*

Proof. We shall be fairly sketchy, as the proof resembles several we have already given (see especially Theorem 12.2 and Lemma 13.1). By Lemma 12.6 we can choose a real number α such that the kernel function B_α is not an outer function. Let M be the invariant subspace of L^2 generated by B_α. Then M has codimension one in H^2 (Lemma 12.7). Choose a vector h in H^2 which is orthogonal to M with respect to the inner product induced by the measure $|Z|^{2\alpha} dm$ and which is a unit vector with respect to the same inner product. Let this inner product be denoted by $(\ , \ )_\alpha$. A simple computation shows that the functional ψ on A defined by $\psi(f) = (fh, h)_\alpha$ is multiplicative. As ψ is represented by the measure $|h|^2 |Z|^{2\alpha} dm$, it lies in the part G. Thus h is bounded and non-zero almost everywhere. Let M_1 be the invariant subspace of L^2 generated by h. If the function f in H^2 is orthogonal to M_1 relative to the inner product $(\ , \ )_\alpha$, then $\bar{h}f$ is orthogonal to $A + \bar{A}_\psi$ for the same inner product. It follows that M_1 has codimension one in H^2. But

$$h(m) = \int h\bar{B}_\alpha |Z|^{2\alpha}\, dm = (h, B_\alpha)_\alpha = 0,$$

and therefore M_1 is contained in H^2_φ. Hence $M_1 = H^2_\varphi$, and the proof is complete.

§ 14. Finite Riemann surfaces

Let R be a finite open Riemann surface of degree of connectivity $\sigma+1$. Let $X = \partial R$, and let A be the hypo-Dirichlet algebra on X described in § 1. Choose a point z_0 in R, which shall be fixed for the remainder of the discussion. The harmonic measure m on X evaluated at z_0 is then the Arens-Singer measure for the functional φ on A of evaluation at z_0. The functions in $L^1 = L^1(m)$ have natural harmonic extensions into R, and we shall regard these functions as so extended. The extensions of functions in H^1 are analytic in R.

Our main objective in the present section is to determine all invariant subspaces of L^p, $1 \leqslant p < \infty$. We should first like to point out that in the present case, it is easy to describe explicitly the measures in S. Namely, if Γ is any smooth closed contour in R, we can define a real annihilating measure η of A by setting

$$\int u\, d\eta = \int \frac{\partial u}{\partial n}\, ds, \quad u \in C(X),$$

where ds denotes the arc length differential along Γ and $\partial/\partial n$ denotes differentiation along the positive normal to Γ, both computed in suitable local coordinates. (The function u is regarded as extended harmonically into R.) It is not hard to show that η is boundedly absolutely continuous with respect to m, and so belongs to S, and that further, every measure in S is a linear combination of such measures η. As Γ runs over a homology basis for R the corresponding measures η run over a linear basis for S.

In order to apply the results of § 13 to the present situation, we need the following result from function theory (see [25, Lemma 2.5]).

LEMMA 14.1. *For each point a in R there is a function in A that has a simple zero at a and no other zeros in $R \cup X$.*

This lemma tells us that if ψ is the functional on A of evaluation at some point of R, then the subspace H^2_ψ is a Beurling subspace. Now Arens [2] has proved that the only multiplicative linear functionals on A are the evaluations at points of $R \cup X$. Consequently the part containing φ consists of the evaluations at points of R (Theorem 12.1). The hypotheses of Theorem 13.1 are therefore satisfied, and so we have the

LEMMA 14.2. *Every invariant subspace of L^2 of type B is a Beurling subspace.*

Although it would be a simple matter now to extend this result to general p, the case $p=1$ will suffice.

LEMMA 14.3. *Every invariant subspace of L^1 of type B is a Beurling subspace.*

Proof. Let M be an invariant subspace of L^1 of type B. By Lemma 11.2, $M \cap L^2$ is dense in M. Thus $M \cap L^2$ is an invariant subspace of L^2 of type B, and so by Lemma 14.2 it has the form wH^2 for some rigid function w. Hence $M = wH^1$, as desired.

To go further we need to know that a non-null function in H^1 cannot vanish identically in R. This can be proved as follows. Let $\{R_n\}_1^\infty$ be a sequence of finite Riemann surfaces contained in R, each containing the point z_0, with the closure of R_n contained in R_{n+1} for every n, and with $R = \bigcup R_n$. For each n let X_n denote the boundary of R_n and m_n the harmonic measure on X_n evaluated at z_0. Think of the measures m_n as points in the closed unit ball of the dual of $C(R \cup X)$. Clearly, any cluster point m' of the sequence $\{m_n\}$ for the weak-star topology of $C(R \cup X)^*$ is a positive measure supported by X that satisfies $\int u\,dm' = u(z_0)$ for every u continuous in $R \cup X$ and harmonic in R. By the uniqueness of harmonic measure it follows that no measure other than m can be a weak-star cluster point of $\{m_n\}$. Because the closed unit ball in $C(R \cup X)^*$ is weak-star compact, we may conclude that $m_n \to m$ in the weak-star topology of $C(R \cup X)^*$.

If U is any (complex valued) harmonic function in R, then

$$\int |U|\,dm_n \leq \int |U|\,dm_{n+1}$$

for all n. In fact, the quantity on the right is the value at z_0 of the solution of the Dirichlet problem in R_{n+1} with boundary values $|U|$, and this solution is at least as great as $|U|$ everywhere on X_n. (This well-known reasoning is due to F. Riesz [26].) Hence $\lim \int |U|\,dm_n$ exists for all U harmonic in R; we let $L^1(R)$ denote the family of those U for which this limit is finite. Obviously $L^1(R)$ is a linear space, and we norm it by setting

$$\|U\| = \lim \int |U|\,dm_n.$$

(Although there is no need to do so here, it is a simple matter to show directly that $L^1(R)$ is a Banach space.) If u is a function in $L^1 = L^1(m)$ then $\int |u|\,dm$ is the value at z_0 of the harmonic function in R that at any z takes the value $\int |u|\,dm_z$, (where m_z is the harmonic measure on X evaluated at z), and consequently $\int |u|\,dm_n \leq \int |u|\,dm$. Thus we can define a norm decreasing linear transformation T of L^1 into $L^1(R)$ by setting $(Tu)(z) = \int u\,dm_z$. But because $m_n \to m$ in the weak-star topology of $C(X \cup R)^*$, the transformation T preserves the norms of continuous functions, and therefore it is an isometry. In particular, Tu vanishes identically in R only when u is a null function, which is the result we need.

LEMMA 14.4. *If M_0 is a Beurling subspace of L^1, then every non-trivial invariant subspace of M_0 is a Beurling subspace.*

Proof. It obviously suffices to consider the case where $M_0 = H^1$. Suppose therefore that M is a non-trivial invariant subspace of H^1. If M contains a function that does not vanish at z_0 then it is obviously of type B and so is a Beurling subspace by Lemma 14.3. Hence suppose all functions in M vanish at z_0. By the preceding argument, a non-zero function in M does not vanish identically in R and so has a zero of some finite order at z_0. Let k be the smallest natural number such that there is a function in M with a zero of order k at z_0. By Lemma 14.1 there is a function h in A with a zero of order k at z_0 and no other zeros in $R \cup X$. Thus $h^{-1}M$ is an invariant subspace of H^1 containing a function that does not vanish at z_0. Consequently $h^{-1}M$ is of type B, and therefore a Beurling subspace, and so M itself is a Beurling subspace.

LEMMA 14.5. *H^∞ is a maximal proper weak-star closed subalgebra of L^∞.*

Proof. The proof we give is an adaptation of one due to Srinivasan for the case where R is the unit disk (see [14, p. 27]). Let K be a proper weak-star closed subalgebra of L^∞ containing H^∞, and let M be the annihilator of K in L^1. Then M is a non-trivial invariant subspace and is contained in $H_0^1 + N$ (Theorem 11.1). The invariant subspace $H_0^1 + N$ is contained in the invariant subspace $H^1 + N$, which is obviously of type B. Hence $H_0^1 + N$ and M are by Lemmas 14.3 and 14.4 both Beurling subspaces, and so there is a rigid function w such that $M = w(H_0^1 + N)$. It follows that $K = w^{-1}H^\infty$. Because K is an algebra it contains the function w^{-2}, and so there is an h in H^∞ such that $w^{-2} = w^{-1}h$. Therefore $h = w^{-1}$, i.e. w^{-1} is in H^∞. Consequently $K \subset H^\infty$, and the proof is complete.

LEMMA 14.6. *Every invariant subspace of L^2 has either the form wH^2 with w a rigid function or the form $\chi_E L^2$ with E a measurable subset of X.*

Proof. Let M be an invariant subspace of L^2. If M is of type B then it has the first of the described forms (Lemma 14.2). Assume therefore that M is not of type B. Let h be a function in A that has a simple zero at z_0 and no other zeros in $R \cup X$. If hM were not equal to M then it would be a proper closed subspace of M because h is bounded from 0 on X. But this would clearly imply that M is of type B, contrary to assumption. Hence $hM = M$, and M is invariant under multiplication by h^{-1}. Let K be the algebra of all functions f in L^∞ such that $fM \subset M$. Then K is weak-star closed and, as we have just seen, it contains H^∞ properly. Therefore $K = L^\infty$ by the preceding lemma. But it is a well-known result (which we shall not prove here) that the subspaces of L^2 that are invariant under multiplication by every function in L^∞ are just those of the form $\chi_E L^2$ with E a measurable subset of X. The proof is complete.

THEOREM 14.1. *Every invariant subspace of L^p $(1 \leqslant p < \infty)$ has either the form wH^p with w a rigid function or the form $\chi_E L^p$ with E a measurable subset of X.*

Proof. The case $p=2$ is given by the preceding lemma. The cases $1 \leqslant p < 2$ follow from the case $p=2$ via Lemma 11.2. The cases $2 < p < \infty$ are obtainable from the cases $1 < p < 2$ by a simple duality argument.

A concluding observation: Consider the case where R is a plane domain, so that $\sigma+1$ is the number of boundary components of R. The subspace $H^2 + N$ is invariant and of type B, and therefore there is an inner function w such that $H^2 = w(H^2+N)$. The invariant subspace wH^2 has codimension σ in H^2, and from this it follows, as one can easily convince oneself (or by the reasoning below), that w has precisely σ zeros in R. We shall show that *the zeros of w are the critical points of the Green's function for R with singularity at z_0.*

To prove this, let G be the Green's function for R with singularity at z_0 and let H be a harmonic conjugate of G. Because the boundary X of R is a union of finitely many non-intersecting analytic Jordan curves, the functions G and H extend so as to be harmonic in a neighborhood of X. The multiple valued analytic function $G+iH$ has a single valued derivative Q, and Q is analytic on $R \cup X$ except for a pole of order one at z_0. The zeros of Q are by definition the critical points of G, and it is known that there are precisely σ of them in R and none on X [23, p. 133]. The function $Q_0(z) = (z-z_0)Q(z)$ is thus in A, has σ zeros in R, and is non-zero on X. Now we want to show that Q_0 has the same zeros as w, and for this it will suffice to show that $Q_0 H^2 = wH^2$. From the properties of Q_0 just mentioned it follows that $Q_0 H^2$ has codimension σ in H^2. As this is also true of the subspace wH^2, it will be enough to show that $wH^2 \subset Q_0 H^2$, or equivalently, that $Q_0^{-1} H^2 \subset w^{-1} H^2$. To do this we choose the usual positive orientation on X and apply the well-known relation

$$dm(z) = \frac{-Q}{2\pi i}\,dz \quad (z \in X). \tag{14.1}$$

Suppose f is any function in H^2. Then by Cauchy's theorem

$$\int_X f(z)\,g(z)\,dz = 0$$

for all g in H^2. This together with (14.1) implies that $Q_0^{-1} f$ is orthogonal to $\bar{H}_0^2$, and so $Q_0^{-1} f$ is in $H^2 + N = w^{-1} H^2$. We have proved the desired inclusion $Q_0^{-1} H^2 \subset w^{-1} H^2$.

§ 15. Hypo-Dirichlet algebras in the plane

Let Y be a compact subset of the plane whose complement has only finitely many components, and let A be the algebra of functions on $X = \partial Y$ that can be uniformly approximated by rational functions whose poles lie off Y. As was pointed out in § 1, A is a hypo-Dirichlet algebra. We shall discuss algebras of this kind more fully in a separate paper [1]. We limit ourselves here to two remarks.

The first remark is that the lemmas and theorem of the preceding section apply without change to the algebras of the present section; the proofs carry over *verbatim*. In fact, when transplanted to the present context the discussion of § 14 becomes basically more elementary, as the function theoretic Lemma 14.1 reduces to a triviality, and the theorem of Arens is not needed to identify the maximal ideal space.

Our second remark pertains to one of Mergelyan's approximation theorems. The theorem in question states that the algebra A contains the restriction of every function that is continuous on Y and analytic in the interior of Y (i.e., every such function can be uniformly approximated by rational functions with poles off Y) [22, p. 24]. By exploting the theory of Dirichlet algebras, Glicksberg and Wermer [11] have given a new proof of the special case of this theorem in which Y has a connected complement. What we want to point out is that the theory developed in this paper makes it possible to adopt the proof of Glicksberg and Wermer to the present more general setting, and thus to obtain a new proof of the more general theorem of Mergelyan cited above. As this involves no essentially new ideas, we shall not present any of the details.

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