ON THE GREATEST PRIME FACTOR OF A QUADRATIC POLYNOMIAL

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1. Introduction

The theorem that, if P_x be the greatest prime factor of

$$\prod_{n \leqslant x} (n^2+1),$$

then

$$\frac{P_x}{x} \to \infty$$

as $x \to \infty$, for which as for many other interesting results in the theory of numbers we are indebted to Chebyshev, has attracted the interest of several mathematicians. Revealed posthumously as little more than a fragment in one of Chebyshev's manuscripts, the theorem was first published and fully proved in a memoir by Markov in 1895 [6], while later in the same year a generalisation by Ivanov [4] appeared in which the polynomial n^2+1 was replaced by n^2+A for any positive A (an account of both Markov's and Ivanov's work is to be found in Paragraphs 147 and 149 of Landau's *Primzahlen* [5]). In 1921 Nagell [7] improved and further generalised Chebyshev's theorem by shewing that for any $\varepsilon < 1$ and for all sufficiently large x

$$\frac{P_x}{x} > \log^{\varepsilon} x,$$

where P_x is the largest prime factor in the product obtained by replacing n^2+1 by any irreducible integral non-linear polynomial $f(n)$. The final result is due to Erdös [1], who in 1952 improved Nagell's result by shewing that

$$\frac{P_x}{x} > (\log x)^{c_1 \log \log \log x}$$

by a method which he stated could be developed further to yield

$$\frac{P_x}{x} > e^{\log^{c_2} x} \quad (c_2 < 1).$$

Chebyshev's theorem and the generalisations of it that appertain to quadratic polynomials are evidently closely connected with the celebrated problem of whether or not there be an infinitude of primes of the form $n^2 + A$. There is, however, a considerable divergence between the results described above and the inequality

$$P_x > Bx^2,$$

which for the polynomial $n^2 + A$ would be implied by the truth of long standing conjectures about the distribution of the prime values taken by this polynomial. The purpose of the present memoir is to narrow this divergence by shewing how Chebyshev's method can be combined with both the sieve method and a method involving exponential sums in order to obtain a result of the form

$$P_x > Bx^{1+\alpha} \quad (\alpha > 0)$$

for any irreducible polynomial $n^2 + A$, where A may be positive or negative. Since the problem appears to be of some importance, it has been considered to be worthwhile at the cost of some brevity to derive as accurate a result as possible, the value of α actually obtained being

$$\alpha = \frac{1}{10}.$$

Although any irreducible quadratic polynomial may be considered in a similar way, the method is not applicable at least in its present form to polynomials of higher degree.

2. Notation and conventions

The quadratic polynomial will be written as $n^2 - D$ in order that D may be the determinant of a binary quadratic form associated with the polynomial, D not being a perfect square since the polynomial is irreducible. The proof will only be given for the case in which D is negative, since the proof in the other case is similar although rather harder; accordingly it will be assumed throughout until the statement of the final theorem that $D < 0$ (and thus not a perfect square).

The letters d, k, l, m, n, β, λ, and v are positive integers; h, r, α, μ, and ν are integers; s is an integer except where it is the argument in a Dirichlet's series; p is a positive prime number; $e = \exp 1$.

The meaning of x and y, when not occurring as indeterminates in a quadratic form, is as follows: x is a continuous real variable that is to be regarded as tending to infinity, all appropriate inequalities that are true for sufficiently large x being therefore assumed to hold; y is a real number not less than 1. The letter u indicates a real variable, which will be related to x in such a way that x and u tend to infinity simultaneously.

The positive highest common factor and lowest common multiple of r and s are denoted by (r, s) and $[r, s]$, respectively; $d(h)$ is the number of positive divisors of h; $\sigma_\gamma(h)$ is the sum of the γth powers of the positive divisors of h; moduli of congruences may be either positive or negative; $[t]$ is the greatest integer not exceeding t.

The letters ε, η, η_1, and η_2 indicate arbitrarily small positive constants that are not necessarily the same at each occurrence. The equation $f=O(|g|)$ denotes an inequality of the form $|f| \leqslant A|g|$ that is true for all values of the variables consistent with stated conditions, where A is a positive constant that depends at most on ε and D.

The author's paper [3], to which we shall have recourse on several occasions, will be referred to as I.

3. Development of the method

We shew first that it suffices to consider the problem when D is square-free. This is the case of least complexity, since the expression required in the proof for the number of roots (incongruent solutions) of the congruence

$$\nu^2 \equiv D (\bmod l)$$

can be formulated most easily when D has no square factor. Let $D=\Delta\Omega^2$, where Δ is square-free. Then by restricting n to be a multiple of Ω we can reduce the problem to the consideration of a polynomial with constant term Δ in place of D; writing $n=n_1\Omega$, we obtain

$$n^2-D=\Omega^2(n_1^2-\Delta)$$

so that the prime factors of n^2-D for $n \leqslant x$ include the prime factors of $n_1^2-\Delta$ for $n_1 \leqslant x/\Omega$. We therefore assume throughout that D is square-free (and negative) except in the statement of the final result.

The residue class to which D belongs, modulo 4, affects the proof in some minor details. We therefore only give the proof in detail for the simpler case in which $D \equiv 2, 3 \pmod 4$, reserving until the end a brief discussion of the modifications that are necessary for the other relevant case $D \equiv 1 \pmod 4$. Accordingly we make the additional assumption that D is congruent to 2 or 3 (mod 4) until the end of the proof.

In the very beginning the method follows that of Chebyshev and Markov. Let P_x be the greatest prime factor of

$$\prod_{n \leqslant x} (n^2 - D).$$

Then, defining $N_x(l)$ to be the number of positive integers n not exceeding x with the property that $n^2 - D$ is divisible by l, we have

$$\prod_{\substack{p \leqslant P_x \\ p^\alpha \leqslant x^2 - D}} p^{N_x(p^\alpha)} = \prod_{n \leqslant x} (n^2 - D) > ([x]!)^2,$$

where the sinister product is taken over p and positive values of α. Consequently, by Stirling's theorem,

$$\sum_{\substack{p \leqslant P_x \\ p^\alpha \leqslant x^2 - D}} N_x(p^\alpha) \log p > 2x \log x + O(x). \tag{1}$$

Next

$$\sum_{\substack{p \leqslant P_x \\ p^\alpha \leqslant x^2 - D}} N_x(p^\alpha) \log p = \sum_{x < p \leqslant P_x} N_x(p) \log p + \sum_{p \leqslant x} N_x(p) \log p + \sum_{\substack{p^\alpha \leqslant x^2 - D \\ \alpha > 1}} N_x(p^\alpha) \log p$$

$$= \Sigma_A + \Sigma_B + \Sigma_C, \quad \text{say}, \tag{2}$$

the condition $p \leqslant P_x$ being omitted from Σ_C since it is in reality superfluous. The lower bound for Σ_A required for the application of our method can be obtained from (1) and (2) through upper bounds we now derive for Σ_B and Σ_C.

The estimates for Σ_B and Σ_C are formed by considering an expression for $N_x(l)$. We have

$$N_x(l) = \sum_{\substack{n^2 - D \equiv 0 (\mathrm{mod}\, l) \\ n \leqslant x}} 1 = \sum_{\substack{\nu^2 \equiv D (\mathrm{mod}\, l) \\ 0 < \nu \leqslant l}} \sum_{\substack{n \equiv \nu (\mathrm{mod}\, l) \\ n \leqslant x}} 1 = \sum_{\substack{\nu^2 \equiv D (\mathrm{mod}\, l) \\ 0 < \nu \leqslant l}} \left(\left[\frac{x - \nu}{l} \right] - \left[-\frac{\nu}{l} \right] \right), \tag{3}$$

which is a formula that will be used later to develop another expression for $N_x(l)$. Here it is enough to deduce that

$$N_x(l) = \frac{x\varrho(l)}{l} + O\{\varrho(l)\}, \tag{4}$$

where $\varrho(l)$ is the number of roots of the congruence

$$\nu^2 \equiv D (\mathrm{mod}\, l).$$

Now
$$\varrho(p) = \begin{cases} 2, & \text{if} \quad (D \mid p) = 1, \\ 0, & \text{if} \quad (D \mid p) = -1, \\ 1, & \text{if either} \quad (D \mid p) = 0 \text{ or } p = 2, \end{cases}$$

while
$$\varrho(p^\alpha) = O(1)$$
always. Therefore, by (2) and (4),

$$\Sigma_B = x \sum_{p \leqslant x} \frac{\varrho(p) \log p}{p} + O\Big(\sum_{p \leqslant x} \varrho(p) \log p\Big) = 2x \sum_{\substack{p \leqslant x \\ (D \mid p) = 1}} \frac{\log p}{p} + x \sum_{p \mid 2D} \frac{\log p}{p} + O\Big(\sum_{p \leqslant x} \log p\Big)$$
$$= x \log x + O(x). \qquad (5)$$

Furthermore

$$\Sigma_C = O\Bigg(\sum_{p \leqslant (x^2 - D)^{\frac{1}{2}}} \log p \sum_{2 \leqslant \alpha \leqslant \{\log (x^2 - D)\}/\log p} \Big\{ \frac{x}{p^\alpha} + 1 \Big\} \Bigg) = O\left(x \sum_{p=2}^{\infty} \frac{\log p}{p(p-1)} \right) + O\Big(\log x \sum_{p \leqslant 2x} 1\Big) = O(x). \quad (6)$$

The lower bound
$$\Sigma_A > x \log x + O(x) \qquad (7)$$
is obtained immediately from (1), (2), (5), and (6). On the other hand Σ_A is a particular example of sums of the form
$$T_x(y) = \sum_{x < p \leqslant y} N_x(p) \log p,$$
for which we shall shew that an upper bound can be derived by a complicated procedure involving the use of Selberg's sieve method. The comparison of this upper bound with the lower bound given by (7) will yield the required lower estimate for P_x. It is in this determination of the upper bound and in the subsequent comparison that the more individual aspects of the memoir are presented.

Let us prepare the sum $T_x(y)$ for the application of the sieve method, assuming[1] throughout that y is subject to the restriction $x^{12/11} < y < x^2$. A single application of the sieve method to estimate the complete sum directly does not yield the optimum result, since the sum can be divided into a series of segments for which we can derive individual upper estimates which, while being of a common order of magnitude, are nevertheless of variable precision. Each such segment can be estimated by two different methods, the position of the segment in the sum determining which method is the more favourable. Firstly then, dividing the sum into two parts that correspond to the methods to be applied, we write

$$T_x(y) = \sum_{x < p \leqslant xX} N_x(p) \log p + \sum_{xX < p \leqslant y} N_x(p) \log p = T_x(xX) + T'_x(y), \quad \text{say}, \qquad (8)$$

where $X = x^{1/11}$. The sums $T_x(xX)$ and $T'_x(y)$ are now each to be split up into segments. To consider the first let

[1] This restriction on y is only to apply in the context of $T_x(y)$.

$$V_x(v) = \sum_{v<p\leqslant ev} N_x(p).$$

Then
$$T_x(xX) \leqslant \sum_{0\leqslant\alpha<\log X}\ \sum_{xe^{\alpha}<p\leqslant xe^{\alpha+1}} N_x(p)\log p \leqslant \sum_{0\leqslant\alpha<\log X} \log(xe^{\alpha+1})\, V_x(xe^{\alpha}). \tag{9}$$

To effect the corresponding division of $T'_x(y)$ it is necessary to transform the sum. By the definition of $N_x(p)$ we have

$$T'_x(y) = \sum_{\substack{xX<p\leqslant y\\ pm=n^2-D\\ n\leqslant x}} \log p = \sum_{m>x^2/y\log^6 x} + \sum_{m\leqslant x^2/y\log^6 x} = T''_x(y) + T'''_x(y), \quad \text{say.} \tag{10}$$

The conditions of summation for $T'''_x(y)$ imply that $n<\sqrt{pm}\leqslant x/\log^3 x$. Therefore

$$T'''_x(y) \leqslant 2\log x \sum_{\substack{lm=n^2-D\\ n<x/\log^3 x}} 1 = 2\log x \sum_{n<x/\log^3 x} d(n^2-D) = O\left(\frac{x}{\log x}\right), \tag{11}$$

by Theorem 2 in I. Next, since the conditions of summation in $T''(y)$ imply that $m\leqslant ex/X$, we have

$$T''_x(y) \leqslant \sum_{\substack{x^2/y\log^6 x<m\leqslant ex/X\\ pm=n^2-D\\ n\leqslant x;\ p\geqslant x}} \log\frac{ex^2}{m}.$$

Therefore, letting $W_x(w)$ be defined by

$$W_x(w) = \sum_{\substack{w<m\leqslant ew\\ pm=n^2-D\\ n\leqslant x;\ p\geqslant x}} 1,$$

we deduce that

$$T''_x(y) \leqslant \sum_{0\leqslant\alpha<\log Y} \log(xXe^{\alpha+1})\, W_x\left(\frac{xe^{-\alpha}}{X}\right),$$

where $Y = ey\log^6 x/xX$. Hence, by this, (10), and (11),

$$T'_x(y) \leqslant \sum_{0\leqslant\alpha<\log Y} \log(xXe^{\alpha+1})\, W_x\left(\frac{xe^{-\alpha}}{X}\right) + O\left(\frac{x}{\log x}\right). \tag{12}$$

In accordance with the familiar principle of the sieve method the estimations of $V_x(v)$ and $W_x(w)$ depend on explicit formulae for the associated sums that are obtained by replacing p in the conditions of summation by multiples of a given square-free integer λ. The sum corresponding to $V_x(v)$ is evidently of the form

$$\Upsilon(u;\lambda) = \sum_{u<\lambda k\leqslant eu} N_x(\lambda k),$$

while as will be seen later the sum corresponding to $W_x(w)$ is actually of the same type.

The derivation of the formula for $\Upsilon(u;\lambda)$ in the next sections depends in part on ideas that were developed in I. For convenience it will be assumed throughout that u and λ satisfy the conditions

$$\left.\begin{array}{c} x^{4/5} < u < x^{\frac{4}{3}} \\ \lambda \text{ square-free} \\ \lambda < \min\,(u^{\frac{5}{4}} x^{-1}, x u^{-\frac{3}{4}}), \end{array}\right\} \tag{13}$$

which imply moreover that

$$\lambda < x^{\frac{1}{4}} < u^{\frac{1}{2}}. \tag{14}$$

Here it is useful to note that $u^{\frac{5}{4}}x^{-1} \lesseqgtr x u^{-\frac{3}{4}}$ according as $u \lesseqgtr x$.

4. Transformation of $\Upsilon(u;\lambda)$

A formula for $N_x(l)$ can be obtained from (3) by using the methods of Sections 3 and 5 of I. Writing as in I $\psi(t) = [t] - t - \frac{1}{2}$ for any real t, and writing also

$$\Psi_l(x) = \sum_{\substack{\nu^2 \equiv D \pmod{l} \\ 0<\nu\leqslant l}} \psi\left(\frac{x-\nu}{l}\right)$$

and

$$\Phi_l(x) = \sum_{\substack{\nu^2 \equiv D \pmod{l} \\ 0<\nu\leqslant l}} \psi\left(-\frac{\nu}{l}\right),$$

we have

$$N_x(l) = \sum_{\substack{\nu^2 \equiv D \pmod{l} \\ 0<\nu\leqslant l}} \left\{\frac{x}{l} + \psi\left(\frac{x-\nu}{l}\right) - \psi\left(-\frac{\nu}{l}\right)\right\} = \frac{x\varrho(l)}{l} + \Psi_l(x) - \Phi_l(x).$$

Then, since $\Phi_l(x)$ is zero unless $l \mid D$ in which case it is $O(1)$, we deduce that

$$\Upsilon(u;\lambda) = \frac{x}{\lambda} \sum_{u<\lambda k\leqslant eu} \frac{\varrho(\lambda k)}{k} + \sum_{u<\lambda k \leqslant eu} \Psi_{\lambda k}(x) + O(1) = \frac{x}{\lambda}\Sigma_D + \Sigma_E + O(1), \quad \text{say}. \tag{15}$$

The sum Σ_E must itself be transformed by the method of Section 5 in I. Equations (24), (25), and (26) in I give the formula

$$\Psi_l(x) = \Psi_{l,\omega}(x) + O\{\Theta_{l,\omega}(x)\}, \tag{16}$$

in which, writing

$$\varrho(h,l) = \sum_{\substack{\nu^2 \equiv D \pmod{l} \\ 0<\nu\leqslant l}} e^{2\pi i h\nu/l},$$

we have

$$\Psi_{l,\omega}(x) = \frac{1}{\pi} \sum_{1\leqslant h\leqslant\omega} \frac{1}{h}\varrho(h,l)\sin\frac{2\pi h x}{l}, \tag{17}$$

$$\Theta_{l,\omega}(x) = \tfrac{1}{2} C_0(\omega)\,\varrho(l) + \sum_{h=1}^{\infty} C_h(\omega)\,\varrho(h,l)\cos\frac{2\pi hx}{l}, \tag{18}$$

and

$$C_h(\omega) = \begin{cases} O\left(\dfrac{\log\omega}{\omega}\right), & \text{always,} \\ O\left(\dfrac{\omega}{h^2}\right), & \text{if } h>0, \end{cases} \tag{19}$$

where $\omega = \omega(x, u, \lambda) \geqslant 2$. We deduce from (15), (16), (17), and (18) that

$$\begin{aligned} \Sigma_E = {} & \frac{1}{\pi}\sum_{1\leqslant h\leqslant\omega}\frac{1}{h}\sum_{u<\lambda k\leqslant eu}\varrho(h,\lambda k)\sin\frac{2\pi hx}{\lambda k} \\ & + O\left(\tfrac{1}{2}C_0(\omega)\sum_{u<\lambda k\leqslant eu}\varrho(\lambda k) + \sum_{h=1}^{\infty}C_h(\omega)\sum_{u<\lambda k\leqslant eu}\varrho(h,\lambda k)\cos\frac{2\pi hx}{\lambda k}\right). \end{aligned} \tag{20}$$

Next let $\mathsf{P}_\lambda^{\pm}(h, u)$ and $\mathsf{P}_\lambda(y)$ be given by

$$\mathsf{P}_\lambda^{\pm}(h,u) = \mathsf{P}_\lambda^{\pm}(h,u,x) = \sum_{u<\lambda k\leqslant eu}\varrho(h,\lambda k)\,e^{\pm 2\pi i hx/\lambda k}$$

and

$$\mathsf{P}_\lambda(y) = \sum_{0<\lambda k\leqslant y}\varrho(\lambda k),$$

so that, since $\varrho(h, \lambda k) = \varrho(-h, \lambda k)$, we have the properties

$$|\mathsf{P}_\lambda^{+}(h,u)| = |\mathsf{P}_\lambda^{-}(h,u)| = S_\lambda(h,u), \quad \text{say,} \tag{21}$$

and

$$S_\lambda(h,u) \leqslant \mathsf{P}_\lambda(eu).$$

Then, by (19) and (20),

$$\begin{aligned} \Sigma_E &= O\left(\sum_{1\leqslant h\leqslant\omega}\frac{S_\lambda(h,u)}{h}\right) + O\left(\frac{\log\omega\,\mathsf{P}_\lambda(eu)}{\omega}\right) \\ &\quad + O\left(\frac{\log\omega}{\omega}\sum_{1\leqslant h\leqslant\omega}S_\lambda(h,u)\right) + O\left(\omega\sum_{\omega<h\leqslant\omega^2}\frac{S_\lambda(h,u)}{h^2}\right) + O\left(\omega\mathsf{P}_\lambda(eu)\sum_{h>\omega^2}\frac{1}{h^2}\right) \\ &= O\left(\log\omega\sum_{1\leqslant h\leqslant\omega}\frac{S_\lambda(h,u)}{h}\right) + O\left(\omega\sum_{\omega<h\leqslant\omega^2}\frac{S_\lambda(h,u)}{h^2}\right) + O\left(\frac{\log\omega\,\mathsf{P}_\lambda(eu)}{\omega}\right) \\ &= O(\log\omega\,\Sigma_E') + O(\omega\Sigma_E'') + O\left(\frac{\log\omega\,\mathsf{P}_\lambda(eu)}{\omega}\right), \quad \text{say.} \end{aligned} \tag{22}$$

A formula for Σ_D will be obtained in the next section, while the estimate for Σ_E will flow from the upper bound to be derived later for $\mathsf{P}_\lambda^{\pm}(h, u)$.

5. Estimation of Σ_D

We express the Dirichlet series

$$g_\lambda(s)=\sum_{k=1}^{\infty}\frac{\varrho(\lambda k)}{k^s}$$

for general square-free λ in terms of the special series $g(s)$ obtained by taking $\lambda=1$, in order to avail ourselves of results already obtained in Section 4 of I.

We have, by Euler's identity, for $\sigma>1$

$$g_\lambda(s)=\prod_{p\nmid\lambda}\left(\sum_{\alpha=0}^{\infty}\frac{\varrho(p^\alpha)}{p^{\alpha s}}\right)\prod_{p\mid\lambda}\left(\sum_{\alpha=0}^{\infty}\frac{\varrho(p^{\alpha+1})}{p^{\alpha s}}\right).$$

Now, if $p\nmid 2D$, then $\varrho(p)=\varrho(p^\beta)$ for $\beta>1$; also, if $p\mid 2D$, then $\varrho(p)=1$ but $\varrho(p^\beta)=0$ for $\beta>1$, since(1) $D\equiv 2, 3 \pmod 4$. Therefore

$$g_\lambda(s)=\prod_{p\nmid\lambda}\left(\sum_{\alpha=0}^{\infty}\frac{\varrho(p^\alpha)}{p^{\alpha s}}\right)\prod_{\substack{p\mid\lambda\\ p\nmid 2D}}\varrho(p)\left(1-\frac{1}{p^s}\right)^{-1}=\varrho(\lambda)\prod_{p\nmid\lambda}\left(\sum_{\alpha=0}^{\infty}\frac{\varrho(p^\alpha)}{p^{\alpha s}}\right)\prod_{\substack{p\mid\lambda\\ p\nmid 2D}}\left(1-\frac{1}{p^s}\right)^{-1}.$$

Consequently, dividing this equation by the special equation derived from it by setting $\lambda=1$, we obtain

$$\frac{g_\lambda(s)}{g(s)}=\varrho(\lambda)\prod_{p\mid\lambda}\left(\sum_{\alpha=0}^{\infty}\frac{\varrho(p^\alpha)}{p^{\alpha s}}\right)^{-1}\prod_{\substack{p\mid\lambda\\ p\nmid 2D}}\left(1-\frac{1}{p^s}\right)^{-1}.$$

To evaluate this let us consider the case $\varrho(\lambda)>0$. Here the value of $\varrho(p^\alpha)$ for $\alpha>0$ in the first product is 2 unless $p\mid 2D$ in which event its value is 1 for $\alpha=1$ and is 0 for $\alpha>1$. Therefore the first product is

$$\prod_{\substack{p\mid\lambda\\ p\nmid 2D}}\left(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+\dots\right)^{-1}\prod_{\substack{p\mid\lambda\\ p\mid 2D}}\left(1+\frac{1}{p^s}\right)^{-1}=\prod_{p\mid\lambda}\left(1+\frac{1}{p^s}\right)^{-1}\prod_{\substack{p\mid\lambda\\ p\nmid 2D}}\left(1-\frac{1}{p^s}\right).$$

Consequently
$$g_\lambda(s)=\varrho(\lambda)\,g(s)\prod_{p\mid\lambda}\left(1+\frac{1}{p^s}\right)^{-1},\tag{23}$$

which result still clearly holds when $\varrho(\lambda)=0$. In the subsequent application of (23) it will be convenient to write

$$\prod_{p\mid\lambda}\left(1+\frac{1}{p^s}\right)^{-1}=\sum_{l=1}^{\infty}\frac{a_{l,\lambda}}{l^s}.$$

(1) This hypothesis is necessary for the statement to be true for $p=2$.

The next stage is to evaluate $\mathsf{P}_\lambda(y)$. Using (23) we have

$$\mathsf{P}_\lambda(y) = \varrho(\lambda) \sum_{lm \leqslant y/\lambda} a_{l,\lambda}\, \varrho(m) = \varrho(\lambda) \sum_{l \leqslant y/\lambda} a_{l,\lambda} \sum_{m \leqslant y/\lambda l} \varrho(m).$$

The inner sum can be evaluated by both equation (7) and Lemma 2 *in* I, which give when D is square-free and congruent to 2, 3 (mod 4)

$$\sum_{m \leqslant z} \varrho(m) = \frac{zL(1)}{\zeta(2)} + O(z^{\frac{3}{4}}), \tag{24}$$

where
$$L(s) = \sum_{l=1}^{\infty} \left(\frac{D}{l}\right) \frac{1}{l^s}.$$

This yields

$$\mathsf{P}_\lambda(y) = \frac{\varrho(\lambda)\, yL(1)}{\lambda\zeta(2)} \sum_{l \leqslant y/\lambda} \frac{a_{l,\lambda}}{l} + O\left(\frac{\varrho(\lambda)\, y^{\frac{3}{4}}}{\lambda^{\frac{3}{4}}} \sum_{l \leqslant y/\lambda} \frac{|a_{l,\lambda}|}{l^{\frac{3}{4}}}\right)$$

$$= \frac{\varrho(\lambda)\, yL(1)}{\lambda\zeta(2)} \prod_{p|\lambda} \left(1 + \frac{1}{p}\right)^{-1} + O\left(\frac{\varrho(\lambda)\, y}{\lambda} \sum_{l > y/\lambda} \frac{|a_{l,\lambda}|}{l}\right) + O\left(\frac{\varrho(\lambda)\, y^{\frac{3}{4}}}{\lambda^{\frac{3}{4}}} \sum_{l \leqslant y/\lambda} \frac{|a_{l,\lambda}|}{l^{\frac{3}{4}}}\right).$$

In this
$$\sum_{l > y/\lambda} \frac{|a_{l,\lambda}|}{l} < \frac{\lambda^{\frac{1}{4}}}{y^{\frac{1}{4}}} \sum_{l > y/\lambda} \frac{|a_{l,\lambda}|}{l^{\frac{3}{4}}};$$

therefore
$$\mathsf{P}_\lambda(y) = \frac{\varrho(\lambda)\, yL(1)}{\lambda\zeta(2)} \prod_{p|\lambda} \left(1 + \frac{1}{p}\right)^{-1} + O\left(\frac{\varrho(\lambda)\, y^{\frac{3}{4}}}{\lambda^{\frac{3}{4}}} \sum_{l=1}^{\infty} \frac{|a_{l,\lambda}|}{l^{\frac{3}{4}}}\right)$$

$$= \frac{\varrho(\lambda)\, yL(1)}{\lambda\zeta(2)} \prod_{p|\lambda} \left(1 + \frac{1}{p}\right)^{-1} + O\left(\frac{\varrho(\lambda)\, \sigma_{-\frac{3}{4}}(\lambda)\, y^{\frac{3}{4}}}{\lambda}\right). \tag{25}$$

We pass from this to the required formula for Σ_D by partial summation. Substituting (25) in the formula

$$\Sigma_D = \lambda \int_u^{eu} \frac{d\mathsf{P}_\lambda(t)}{t} = {}_u^{eu}\!\left[\frac{\lambda \mathsf{P}_\lambda(t)}{t}\right] + \lambda \int_u^{eu} \frac{\mathsf{P}_\lambda(t)\, dt}{t^2},$$

we obtain after a short calculation

$$\Sigma_D = \varrho(\lambda) \prod_{p|\lambda} \left(1 + \frac{1}{p}\right)^{-1} \frac{L(1)}{\zeta(2)} \int_u^{eu} \frac{dt}{t} + O\left(\frac{\varrho(\lambda)\, \sigma_{-\frac{3}{4}}(\lambda)\, \lambda^{\frac{1}{4}}}{u^{\frac{1}{4}}}\right),$$

since the contribution to the term in square brackets due to the explicit part in (25) vanishes. Consequently
$$\Sigma_D = \varrho(\lambda) \prod_{p|\lambda} \left(1 + \frac{1}{p}\right)^{-1} \frac{L(1)}{\zeta(2)} + O\left(\frac{\lambda^{\frac{1}{4}+\varepsilon}}{u^{\frac{1}{4}}}\right). \tag{26}$$

6. Estimation of $\mathsf{P}_\lambda(h, u)$

In general principle the method follows Section 6 in I although there is a considerable divergence in details owing to our different requirements here. Evidently it is enough to restrict our attention to the sum $\mathsf{P}^+(h, u)$, from which for brevity it is now appropriate to omit the $+$ symbol.

Let $ax^2+2bxy+cy^2$ be a binary quadratic form of determinant D. Then, if

$$\lambda k = ar^2+2brs+cs^2$$

be a typical primitive representation of λk by the form, a typical value of $\nu/\lambda k$ in $\varrho(h, \lambda k)$ is given for $s \neq 0$ by

$$\frac{\nu}{\lambda k} = -\frac{\bar{s}}{r} + \frac{br+cs}{r(ar^2+2brs+cs^2)}, \tag{27}$$

where $s\bar{s} \equiv 1 \pmod r$, and is also given for $r \neq 0$ by

$$\frac{\nu}{\lambda k} = \frac{\bar{r}}{s} - \frac{ar+bs}{s(ar^2+2brs+cs^2)}, \tag{28}$$

where $r\bar{r} \equiv 1 \pmod s$. Furthermore

$$\frac{x}{\lambda k} = \frac{x}{ar^2+2brs+cs^2}. \tag{29}$$

Therefore, defining $\theta_{r,s}$ to be the expression for

$$\frac{\nu}{\lambda k} + \frac{x}{\lambda k}$$

in terms of r, s given by (29) and one of (27) and (28), we have

$$\mathsf{P}_\lambda(h, u) = \sum_{a,b,c} \sum_{\substack{u < ar^2+2brs+cs^2 \leq eu \\ ar^2+2brs+cs^2 \equiv 0 \pmod\lambda \\ (r,s)=1;\ (M)}} e^{2\pi i h\theta_{r,s}},$$

where a, b, c in the outer sum indicates summation over a set of representative forms of determinant D, and where (M) indicates that only one representation from each possible set of representations is to be included. Since $u > 0$ and D is negative, the forms will be positive and therefore $a, c > 0$. Also a, b, and c may be regarded as being bounded (choose, for example, the representative forms to be reduced forms). We deduce that

$$\mathsf{P}_\lambda(h, u) = \sum_{a,b,c} \varepsilon_{a,b,c} \sum_{\substack{u < ar^2+2brs+cs^2 \leq eu \\ ar^2+2brs+cs^2 \equiv 0 \pmod\lambda \\ (r,s)=1}} e^{2\pi i h\theta_{r,s}}, \tag{30}$$

where $\varepsilon_{a,b,c}$ is either $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{6}$. The estimation is continued by expressing the inner sum as

$$\sum_{\substack{u<ar^2+2brs+cs^2\leqslant eu \\ ar^2+2brs+cs^2\equiv 0 (\bmod \lambda) \\ (r,s)=1}} e^{2\pi i h\theta_{r,s}} = \sum_{|s|<|r|} + \sum_{|r|<|s|} + \sum_{|r|=|s|=1} = \Sigma_F + \Sigma_G + O(1), \quad \text{say}, \tag{31}$$

and then considering Σ_F and Σ_G.

An estimate for an exponential sum similar to Kloosterman's sum is required. This is obtained by proving successively a number of lemmata.

LEMMA 1. *If $h, r \neq 0$; $0 \leqslant \xi_2 - \xi_1 \leqslant 2|r|$; and h_1 be a given integer, then we have*

$$\sum_{\substack{\xi_1 \leqslant s \leqslant \xi_2 \\ (r,s)=1}} \exp\left(\frac{2\pi i(h_1 s - h\bar{s})}{r}\right) = O[|r|^{\frac{1}{2}+\varepsilon}\{(h,r)\}^{\frac{1}{2}}].$$

This result on an "incomplete sum", which may be inferred by a well known method from Lemma 2 of [2], depends essentially on Weil's estimate for the Kloosterman sum.

LEMMA 2. *The result of Lemma 1 still holds if h_1 be any real number.*

This can be deduced from Lemma 1 by a method due to Estermann. Let $h_2 = [h_1]$ so that $h_1 = h_2 + h_3$ where $0 \leqslant h_3 < 1$. Then, since

$$\exp\left(\frac{2\pi i(h_1 s - h\bar{s})}{r}\right) = \left\{\exp\left(\frac{2\pi i(h_2 s - h\bar{s})}{r}\right)\right\}\left\{\exp\left(\frac{2\pi i h_3 s}{r}\right)\right\},$$

we derive the result by applying partial summation and Lemma 1 with h_2 in place of h_1.

LEMMA 3. *If $h, r \neq 0$ and $0 \leqslant \xi_2 - \xi_1 \leqslant 2|r|$, then*

$$\sum_{\substack{\xi_1 \leqslant s \leqslant \xi_2 \\ s \equiv v (\bmod \lambda) \\ (r,s)=1}} \exp\left(-\frac{2\pi i h\bar{s}}{r}\right) = O[|r|^{\frac{1}{2}+\varepsilon}\{(h,r)\}^{\frac{1}{2}}].$$

The sinister side of the above equation is equal to

$$\frac{1}{\lambda}\sum_{\substack{\xi_1 \leqslant s \leqslant \xi_2 \\ (r,s)=1}} \exp\left(-\frac{2\pi i h\bar{s}}{r}\right)\sum_{\varrho=1}^{\lambda} \exp\left(\frac{2\pi i\varrho(s-v)}{\lambda}\right)$$

$$= \frac{1}{\lambda}\sum_{\varrho=1}^{\lambda} \exp\left(-\frac{2\pi i\varrho v}{\lambda}\right)\sum_{\substack{\xi_1 \leqslant s \leqslant \xi_2 \\ (r,s)=1}} \exp\left(\frac{2\pi i(\varrho r\lambda^{-1}s - h\bar{s})}{r}\right) = O[|r|^{\frac{1}{2}+\varepsilon}\{(h,r)\}^{\frac{1}{2}}],$$

on applying Lemma 2 to the inner sum.

Two other lemmata will also be needed. The first is similar to Lemma 1 in I but is easier to prove.

LEMMA 4. *If $h \neq 0$ and $y \geq 1$, we have*

$$\sum_{l \leq y} \{(h, l)\}^{\frac{1}{2}} = O\{y\sigma_{-\frac{1}{2}}(h)\}.$$

LEMMA 5. *Let (a, b, c) be one of the forms that appear in equation (30). Then, if, for any given r, $\Omega_r(\lambda)$ be the number of roots in v of the congruence*

$$ar^2 + 2brv + cv^2 \equiv 0 \pmod{\lambda},$$

we have

$$\Omega_r(\lambda) = O\{d(\lambda)\}$$

for any square-free number λ.

If $p \nmid c$, then $\Omega_r(p)$ is at most 2, whereas, if $p \mid c$, then $\Omega_r(p)$ is at most p and therefore at most c. Since $\Omega_r(\lambda)$ is a multiplicative function of λ and since c has a bounded number of prime factors, the lemma follows.

We direct our attention to Σ_F. The condition $ar^2 + 2brs + cs^2 \leq eu$ implies that

$$|r| \leq Eu^{\frac{1}{2}},$$

where $E = (-ce/D)^{\frac{1}{2}}$; on the other hand the condition $ar^2 + 2brs + cs^2 > u$ in conjunction with $|s| < |r|$ implies that

$$|r| > Gu^{\frac{1}{2}},$$

where $G = (a + 2|b| + c)^{-\frac{1}{2}}$. Hence

$$\Sigma_F = \sum_{Gu^{\frac{1}{2}} < |r| \leq Eu^{\frac{1}{2}}} \sum_{\substack{u < ar^2 + 2brs + cs^2 \leq eu \\ ar^2 + 2brs + cs^2 \equiv 0 \pmod{\lambda} \\ (r, s) = 1;\ |s| < |r|}} e^{2\pi i h\theta_{r,s}} = \sum_{Gu^{\frac{1}{2}} < |r| \leq Eu^{\frac{1}{2}}} \Sigma_{F, r}, \quad \text{say}. \tag{32}$$

Since in the inner sum the inequalities

$$-r < s < r$$

$$\frac{1}{c^2}(cu + Dr^2) < \left(s + \frac{br}{c}\right)^2 \leq \frac{1}{c^2}(ceu + Dr^2)$$

must hold, we see that for given r there are either no values of s or the values of s range through all the integer values in one or two intervals contained in the interval $(-r, r)$. To estimate $\Sigma_{F,r}$ we are thus led to consider the sum

$$\Sigma'_{F,r} = \sum_{\substack{\xi_1(r) \leq s \leq \xi_2(r) \\ ar^2 + 2brs + cs^2 \equiv 0 \pmod{\lambda} \\ (r, s) = 1}} e^{2\pi i h\theta_{r,s}},$$

in which $-r < \xi_1 \leq \xi_2 < r$.

We have

$$\Sigma'_{F,r} = \sum_{\substack{ar^2+2brv+cv^2\equiv 0 \,(\mathrm{mod}\,\lambda)\\ 0<v\leqslant\lambda}} \sum_{\substack{\xi_1(r)\leqslant s\leqslant\xi_2(r)\\ s\equiv v\,(\mathrm{mod}\,\lambda)\\ (r,s)=1}} e^{2\pi i h\theta_{r,s}} = \sum_{\substack{ar^2+2brv+cv^2\equiv 0 \,(\mathrm{mod}\,\lambda)\\ 0<v\leqslant\lambda}} \Sigma'_{F,r,v}, \quad \text{say.} \tag{33}$$

Writing $$\varphi(r,s) = \exp\left(\frac{2\pi i h(br+cs)}{r(ar^2+2brs+cs^2)} + \frac{2\pi i h\,x}{ar^2+2brs+cs^2}\right),$$

we have from (27), (29), and (33)

$$\Sigma'_{F,r,v} = \sum_{\substack{\xi_1(r)\leqslant s\leqslant\xi_2(r)\\ s\equiv v\,(\mathrm{mod}\,\lambda)\\ (r,s)=1}} \exp\left(-\frac{2\pi i h\bar{s}}{r}\right)\varphi(r,s),$$

which when transformed by partial summation becomes

$$\Sigma'_{F,r,v} = \sum_{\xi_1(r)\leqslant\mu\leqslant\xi_2(r)} g(\mu)\{\varphi(r,\mu)-\varphi(r,\mu+1)\} + g([\xi_2(r)])\varphi(r,[\xi_2(r)]+1), \tag{34}$$

where $$g(\mu) = \sum_{\substack{\xi_1(r)\leqslant s\leqslant\mu\\ s\equiv v\,(\mathrm{mod}\,\lambda)\\ (r,s)=1}} \exp\left(-\frac{2\pi i h\bar{s}}{r}\right).$$

In (34) $$\varphi(r,\mu)-\varphi(r,\mu+1) = O\left(\frac{|h|\,x}{|r|^3}\right),$$

since $|\mu|<|r|$ and $ar^2+2br\mu+c\mu^2\geqslant -Dr^2/c$. Therefore, by this and Lemma 3,

$$\begin{aligned}\Sigma'_{F,r,v} &= O\left(\frac{x^{1+\varepsilon}|h|\{(h,r)\}^{\frac12}}{|r|^{\frac52}}\sum_{\xi_1(r)\leqslant\mu\leqslant\xi_2(r)} 1\right) + O(|r|^{\frac12+\varepsilon}\{(h,r)\}^{\frac12})\\ &= O(x^{1+\varepsilon}|r|^{-\frac32}|h|\{(h,r)\}^{\frac12}) + O(|r|^{\frac12+\varepsilon}\{h,r)\}^{\frac12}),\end{aligned}$$

from which we infer through (33) and Lemma 5 the first part of

$$\left.\begin{array}{l}\Sigma'_{F,r}\\ \Sigma_{F,r}\end{array}\right\} = O(x^{1+\varepsilon}|r|^{-\frac32}|h|\{(h,r)\}^{\frac12}) + O(|r|^{\frac12+\varepsilon}\{(h,r)\}^{\frac12}). \tag{35}$$

The second part follows from the first part by the discussion at the end of the previous paragraph.

We can complete the estimation of $\mathsf{P}_\lambda(h,u)$. Firstly, by (32) and (35),

$$\Sigma_F = O\left(x^{1+\varepsilon}|h|\sum_{l>Gu^{\frac12}}\frac{\{(h,l)\}^{\frac12}}{l^{\frac32}}\right) + O(u^{\frac14+\varepsilon}\sum_{l\leqslant Eu^{\frac12}}\{(h,l)\}^{\frac12}),$$

and then, by Lemma 4,

$$\Sigma_F = O(x^{1+\varepsilon} u^{-\frac{1}{4}} |h| \sigma_{-\frac{1}{2}}(h)) + O(u^{\frac{3}{4}+\varepsilon} \sigma_{-\frac{1}{2}}(h)).$$

Since we see in turn through (31) and (30) that Σ_G and $\mathsf{P}_\lambda(h, u)$ satisfy a similar inequality, we have finally that

$$\mathsf{P}_\lambda(h, u) = O(x^{1+\varepsilon} u^{-\frac{1}{4}} |h| \sigma_{-\frac{1}{2}}(h)) + O(u^{\frac{3}{4}+\varepsilon} \sigma_{-\frac{1}{2}}(h)). \tag{36}$$

7. Estimation of Σ_E and $\Upsilon(u; \lambda)$

The estimate for Σ_E follows from (21), (22), and (36). Firstly we have

$$\begin{aligned} \Sigma'_E &= O(x^{1+\varepsilon} u^{-\frac{1}{4}} \sum_{1 \leqslant h \leqslant \omega} \sigma_{-\frac{1}{2}}(h)) + O\left(u^{\frac{3}{4}+\varepsilon} \sum_{1 \leqslant h \leqslant \omega} \frac{\sigma_{-\frac{1}{2}}(h)}{h}\right) \\ &= O(\omega x^{1+\varepsilon} u^{-\frac{1}{4}}) + O(u^{\frac{3}{4}+\varepsilon} \log \omega). \end{aligned} \tag{37}$$

Next

$$\begin{aligned} \Sigma''_E &= O\left(x^{1+\varepsilon} u^{-\frac{1}{4}} \sum_{\omega < h \leqslant \omega^2} \frac{\sigma_{-\frac{1}{2}}(h)}{h}\right) + O\left(u^{\frac{3}{4}+\varepsilon} \sum_{h > \omega} \frac{\sigma_{-\frac{1}{2}}(h)}{h^2}\right) \\ &= O(x^{1+\varepsilon} u^{-\frac{1}{4}} \log \omega) + O(\omega^{-1} u^{\frac{3}{4}+\varepsilon}), \end{aligned} \tag{38}$$

while, by (25),

$$\mathsf{P}_\lambda(eu) = O\left(\frac{d(\lambda)\, u}{\lambda}\right). \tag{39}$$

Therefore, by (22), (37), (38), and (39),

$$\Sigma_E = O(\omega \log \omega \cdot x^{1+\varepsilon} u^{-\frac{1}{4}}) + O(u^{\frac{3}{4}+\varepsilon} \log^2 \omega) + O\left(\frac{u^{1+\varepsilon} \log \omega}{\omega \lambda}\right).$$

Let ω be chosen so that $\omega x u^{-\frac{1}{4}} = 4u\omega^{-1}\lambda^{-1}$ with the consequence that

$$\omega = 2 u^{\frac{5}{8}} x^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} > 2$$

by (13). Substituting this value of ω we obtain finally

$$\Sigma_E = O(x^{\frac{1}{2}+\varepsilon} u^{\frac{3}{8}} \lambda^{-\frac{1}{2}}) + O(u^{\frac{3}{4}+\varepsilon}) = O(x^{\frac{1}{2}+\varepsilon} u^{\frac{3}{8}} \lambda^{-\frac{1}{2}}), \tag{40}$$

since $u^{\frac{3}{4}} < x^{\frac{1}{2}} u^{\frac{3}{8}} \lambda^{-\frac{1}{2}}$ by (13).

The estimate for $\Upsilon(u; \lambda)$ is now immediate. Collecting together the results of (15), (26), and (40) we obtain

$$\Upsilon(u; \lambda) = \frac{x \varrho(\lambda)}{\lambda} \prod_{p \mid \lambda} \left(1 + \frac{1}{p}\right)^{-1} \frac{L(1)}{\zeta(2)} + O(x^{\frac{1}{2}+\varepsilon} u^{\frac{3}{8}} \lambda^{-\frac{1}{2}}), \tag{41}$$

since (13) implies that the error term due to (26) may be absorbed into that of due to (40).

8. Application of the sieve method

A suitable formula for $\Upsilon(u; \lambda)$ having been obtained, Selberg's sieve method [8] can be applied to the estimation of $V_x(v)$ and $W_x(w)$.

Defining, as is customary in Selberg's method, $\varrho_d = \varrho_d(z)$ by

$$\varrho_d = \begin{cases} 1, \text{ if } d = 1, \\ \text{arbitrary real number, if } d \text{ be square-free}^{(1)} \text{ and } 1 < d \leqslant z, \\ 0, \text{ otherwise,} \end{cases}$$

we introduce the non-negative function

$$\left(\sum_{d|n} \varrho_d\right)^2,$$

which is equal to 1 when n is a prime number exceeding z.

In the estimation of $V_x(v)$ it will be assumed that $x < v < x^{\frac{4}{3}}$ and $z < x^{\frac{1}{2}} v^{-\frac{3}{8}}$ so that in particular v satisfies the first condition imposed on u by (13). Then, because $z < v$,

$$V_x(v) \leqslant \sum_{v < l \leqslant ev} N_x(l) \left(\sum_{d|l} \varrho_d\right)^2 = \sum_{d_1, d_2 \leqslant z} \varrho_{d_1} \varrho_{d_2} \sum_{\substack{v < l \leqslant ev \\ l \equiv 0 \,(\mathrm{mod}\, [d_1, d_2])}} N_x(l) = \sum_{d_1, d_2 \leqslant z} \varrho_{d_1} \varrho_{d_2} \Upsilon(v; [d_1, d_2]).$$

Therefore, since $[d_1, d_2]$ does not exceed $xv^{-\frac{3}{4}}$ and may be regarded as being square-free, we have

$$V_x(v) \leqslant \frac{xL(1)}{\zeta(2)} \sum_{d_1, d_2 \leqslant z} \frac{\varrho_{d_1} \varrho_{d_2}}{f([d_1, d_2])} + O\left(x^{\frac{1}{2}+\varepsilon} v^{\frac{3}{8}} \sum_{d_1, d_2 \leqslant z} \frac{|\varrho_{d_1}| |\varrho_d|}{[d_1, d_2]^{\frac{1}{2}}}\right), \tag{42}$$

by (41), where $f(n)$ is the multiplicative function defined by

$$\frac{1}{f(n)} = \frac{\varrho(n)}{n} \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1}.$$

The first term in (42) is now minimized in the usual way. Setting

$$f_1(m) = f(m) \prod_{p|m} \left(1 - \frac{1}{f(p)}\right), \quad (>0)$$

we find that

$$V_x(v) \leqslant \frac{xL(1)}{\zeta(2) \sum_{l \leqslant z} \frac{\mu^2(l)}{f_1(l)}} + O\left(x^{\frac{1}{2}+\varepsilon} v^{\frac{3}{8}} \sum_{d_1, d_2 \leqslant z} \frac{|\varrho_{d_1}| |\varrho_{d_2}|}{[d_1, d_2]^{\frac{1}{2}}}\right), \tag{43}$$

(1) Although the condition that d_1 be square-free is usually omitted (often incorrectly) at this stage of Selberg's sieve method, the condition is virtually implicit in the method. In a case such as this, where the function $f(n)$ is not totally multiplicative, it is essential that the condition be imposed explicitly.

where $$|\varrho_d| \leqslant \prod_{p|d}\left(1-\frac{1}{f(p)}\right)^{-1} = O\{(\log\log 10\,d)^2\} = O(x^\varepsilon).$$

To evaluate the first term in the dexter side of (43) let l_1 indicate, generally, either 1 or any number composed entirely of prime factors that divide the square-free number l', that is to say $p|l_1$ implies $p|l'$. If $p\nmid 2D$ and $\varrho(p)>0$, then

$$f_1(p) = \frac{p}{2}\left(1+\frac{1}{p}\right) - 1 = \frac{p}{2}\left(1-\frac{1}{p}\right)$$

so that $$\frac{1}{f_1(p)} = \frac{2}{p}\left(1-\frac{1}{p}\right)^{-1};$$

also, if $p|2D$, then $$\frac{1}{f_1(p)} = \frac{1}{p}.$$

Therefore, as $f_1(m)$ is a multiplicative function, we have for any square-free number l'

$$\frac{1}{f_1(l')} = \frac{\varrho(l')}{l'}\prod_{\substack{p|l'\\ p\nmid 2D}}\left(1-\frac{1}{p}\right)^{-1} = \sum_{l_1=1}^{\infty}\frac{\varrho(l'l_1)}{l'l_1},$$

the identity being trivial if $\varrho(l')=0$. We deduce that as $y\to\infty$

$$\sum_{l\leqslant y}\frac{\mu^2(l)}{f_1(l)} = \sum_{l'\leqslant y}\frac{1}{f_1(l')} = \sum_{l'\leqslant y}\sum_{l_1=1}^{\infty}\frac{\varrho(l'l_1)}{l'l_1} \geqslant \sum_{m\leqslant y}\frac{\varrho(m)}{m} \geqslant (1-\eta_1)\frac{L(1)\log y}{\zeta(2)}, \tag{44}$$

by (24) and partial summation.

The error term in (43) is

$$O\left(x^{\frac{1}{2}+\varepsilon}v^{\frac{3}{8}}\sum_{d_1,\,d_2\leqslant z}\frac{1}{[d_1,d_2]^{\frac{1}{2}}}\right) = O\left(x^{\frac{1}{2}+\varepsilon}v^{\frac{3}{8}}\sum_{d\leqslant z}\sum_{\substack{d_1',\,d_2'\leqslant z/d\\ (d_1',\,d_2')=1}}\frac{1}{d^{\frac{1}{2}}d_1'^{\frac{1}{2}}d_2'^{\frac{1}{2}}}\right)$$

$$= O\left(x^{\frac{1}{2}+\varepsilon}v^{\frac{3}{8}}\sum_{d\leqslant z}\frac{1}{d^{\frac{1}{2}}}\frac{z}{d}\right) = O(x^{\frac{1}{2}+\varepsilon}v^{\frac{3}{8}}z). \tag{45}$$

Choosing z to be $x^{\frac{1}{2}-\eta}v^{-\frac{3}{8}}$ (consistent with earlier conditions), we deduce the required estimate

$$V_x(v) < \frac{(1+\eta_2)\,x}{\log(x^{\frac{1}{2}}v^{-\frac{3}{8}})} \tag{46}$$

from (43), (44) and (45).

The estimation of $W_x(w)$ being very similar to that of $V_x(v)$ after the first stage, we suppress all but the earlier details. Here we assume $x^{4/5}<w<x$ and $z<x^{2/7}w^{-3/14}$. Then, since [1] $z<x$,

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[1] This relates to the condition $p\geqslant x$ in the definition of $W_x(w)$.

$$W_x(w) \leqslant \sum_{\substack{w<m\leqslant ew \\ lm=n^2-D \\ n\leqslant x}} \Big(\sum_{d|l} \varrho_d\Big)^2 = \sum_{d_1,\, d_2\leqslant z} \varrho_{d_1}\varrho_{d_2} \sum_{\substack{w<m\leqslant ew \\ m[d_1,\, d_2]\varrho=n^2-D \\ n\leqslant x}} 1$$

$$= \sum_{d_1,\, d_2\leqslant z} \varrho_{d_1}\varrho_{d_2}\, \Upsilon([d_1, d_2]\, w;\, [d_1, d_2]).$$

The conditions on w and z shew that in the above sum $u=[d_1, d_2]w$ and $\lambda=[d_1, d_2]$ satisfy (13). Therefore, by (41), we obtain as an analogue of (42)

$$W_x(w) \leqslant \frac{xL(1)}{\zeta(2)} \sum_{d_1,\, d_2\leqslant z} \frac{\varrho_{d_1}\varrho_{d_2}}{f([d_1, d_2])} + O\left(x^{\frac{1}{2}+\varepsilon} w^{\frac{3}{8}} \sum_{d_1,\, d_2\leqslant z} \frac{|\varrho_{d_1}|\,|\varrho_{d_2}|}{[d_1, d_2]^{\frac{1}{8}}}\right)$$

$$= \frac{xL(1)}{\zeta(2)} \sum_{d_1,\, d_2\leqslant z} \frac{\varrho_{d_1}\varrho_{d_2}}{f([d_1, d_2])} + O(x^{\frac{1}{2}+\varepsilon} w^{\frac{3}{8}} z^{7/4}) \leqslant \frac{(1+\eta_2)\,x}{\log\,(x^{2/7} w^{-3/14})}, \tag{47}$$

on choosing z to be $x^{2/7-\eta} w^{-3/14}$.

9. The greatest prime divisor

We arrive at our final result by using the estimates for $V_x(v)$ and $W_x(w)$ to obtain an upper bound for $T_x(y)$ for $y=x^{11/10}$. To this end let $\gamma=\log x$.

First it is necessary to estimate $T_x(xX)$. Since in (9) $xe^\alpha < x^{\frac{4}{3}}$, we have by (9) and (46)

$$T_x(xX) \leqslant x(1+\eta_2) \sum_{0\leqslant\alpha<\log X} \frac{\alpha+\gamma+1}{\frac{1}{8}\gamma - \frac{3}{8}\alpha} \leqslant 8\,x(1+\eta_2) \int_0^{\log X} \frac{(\gamma+t)\,dt}{\gamma-3t} + O(x)$$

$$= 8\,x(1+\eta_2) \Big[-\tfrac{1}{3}t - \frac{4\gamma \log\,(\gamma-3t)}{9} \Big]_0^{\log X} + O(x)$$

$$= 8\,x(1+\eta_2)\,(-\tfrac{1}{33} + \tfrac{4}{9}\log \tfrac{11}{8}) \log x + O(x)$$

$$= (\cdot 89010\ldots)\,(1+\eta_2)\,x \log x + O(x) \leqslant \cdot 8902\, x \log x. \tag{48}$$

Next, since $xX^{-1} e^{-\alpha} > x^{4/5}$ in (12), equation (47) implies that

$$T'_x(y) \leqslant x(1+\eta_2) \sum_{0\leqslant\alpha<\log Y} \frac{\alpha+\frac{12}{11}\gamma+1}{\frac{1}{11}\gamma+\frac{3}{14}\alpha} + O\left(\frac{x}{\log x}\right)$$

for $y=x^{11/10}$. Therefore

$$T'_x(y) \leqslant 14\,x(1+\eta_2) \int_0^{\log Y} \frac{(12\gamma+11\,t)\,dt}{14\gamma+33\,t} + O(x)$$

$$= 14\,x(1+\eta_2) \Big[\tfrac{1}{3}t + \frac{2\gamma \log\,(14\gamma+33\,t)}{9} \Big]_0^{\log Y} + O(x)$$

$$= 14\,x(1+\eta_2)\,(\tfrac{1}{330} + \tfrac{2}{9}\log \tfrac{143}{140}) \log x + O(x \log\log x)$$

$$= (\cdot 10808\ldots)\,(1+\eta_2)\,x \log x + O(x \log\log x)$$

$$\leqslant \cdot 1081\, x \log x. \tag{49}$$

Combining (48) and (49) we thus have from (8) that

$$T_x(x^{11/10}) \leqslant \cdot 9983x \log x.$$

If this be compared with (7), we obtain at once that

$$P_x > x^{11/10} \tag{50}$$

for the case $D \equiv 2, 3 \pmod 4$.

A few changes in detail are necessary when $D \equiv 1 \pmod 4$ because $\varrho(2^\alpha)$ is no longer always zero for $\alpha > 1$. The first variations appear in Section 5, where new formulae must replace both (23) and (24) with incidental minor modifications in the remainder of the section. No changes are necessary in Sections 6 and 7 except in the main term of the final formula (41). The argument of Section 8, although affected by the above modifications and in particular by the consequent change in the form of the functions $f(p)$ and $f_1(p)$, leads to the same final formulae (46) and (47) and thence to (50).

We have thus arrived at the following theorem for the case in which D is negative, the proof for D positive being similar in principle.

THEOREM. *If D be not a perfect square, then for all sufficiently large values of x the greatest prime factor of*

$$\prod_{\substack{n \leqslant x \\ n^2 > D}} (n^2 - D)$$

exceeds $x^{11/10}$.

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