A NON-STANDARD INTEGRAL EQUATION WITH APPLICATIONS TO QUASICONFORMAL MAPPINGS

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1. Statement of results

A) Let C denote an oriented closed Jordan curve in the extended complex plane (Riemann sphere). We denote by $D_1 = D_1(C)$ and $D_2 = D_2(C)$ the domains interior and exterior to C, respectively, and we denote by $\lambda_j(z)|dz| = \lambda_{D_j(C)}(z)|dz|$ the Poincaré metric in D_j, $j=1, 2$. We denote by q an integer, $q \geqslant 2$.

In this paper we investigate the integral equation

$$\varphi(z) = \iint_{D_1} (\zeta - z)^{-2q} \lambda_1(\zeta)^{2-2q} \overline{\psi(\zeta)}\, d\xi\, d\eta, \quad z \in D_2, \tag{1.1}$$

where the given function φ and the unknown function ψ are assumed to be holomorphic in D_2 and D_1, respectively. We will give conditions under which the equation is uniquely solvable, and some applications of these conditions.

We write equation (1.1) in the abbreviated form

$$\varphi = \mathfrak{L}_C^{(q)} \psi, \tag{1.2}$$

and denote by $-C$ the curve C with the orientation reversed. Thus

$$\varphi = \mathfrak{L}_{-C}^{(q)} \psi$$

is an abbreviation for the formally transposed equation

$$\varphi(z) = \iint_{D_2} (\zeta - z)^{-2q} \lambda_2(\zeta)^{2-2q} \overline{\psi(\zeta)}\, d\xi\, d\eta, \quad z \in D_1.$$

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B) For every simply connected domain D with at least two boundary points let $\lambda(z)|dz| = \lambda_D(z)|dz|$ denote the Poincaré metric. Let G be a discrete group of conformal self-mappings $z \rightarrowtail (z)$ of D; we include the case of the trivial group $G = 1 = \{\mathrm{id}\}$. For every integer $q \geqslant 2$ let $A_q(D, G)$ denote the complex Banach space of integrable automorphic forms of weight $(-2q)$, that is the space of holomorphic functions $\varphi(z)$, $z \in D$, satisfying the conditions

$$\varphi(\gamma(z))\gamma'(z)^q = \varphi(z) \text{ for } \gamma \in G, \tag{1.3}$$

$$\varphi(z) = O(|z|^{-2q}), \quad z \to \infty \text{ if } \infty \in D, \tag{1.4}$$

and

$$\|\varphi\|_{A_q(D,G)} = \iint_{D/G} \lambda_D(z)^{2-q} |\varphi(z)| \, dx \, dy < \infty.$$

Also, let $B_q(D, G)$ denote the complex Banach space of bounded automorphic forms of weight $(-2q)$, that is the space of holomorphic function $\varphi(z)$, $z \in D$, satisfying (1.3), (1.4) and

$$\|\varphi\|_{B_q(D)} = \sup \lambda_D(z)^{-q} |\varphi(z)| < \infty.$$

We set $A_q(D, 1) = A_q(D)$, $B_q(D, 1) = B_q(D)$. Clearly $B_q(D, G) \subset B_q(D)$.

For $\varphi \in A_q(D, G)$ and $\psi \in B_q(D, G)$ we set

$$(\varphi, \psi)_{q, D/G} = \iint_{D/G} \lambda(z)^{2-2q} \varphi(z) \overline{\psi(z)} \, dx \, dy.$$

Concerning these spaces see Bers [4].

THEOREM 1. *The mappings*

$$\mathfrak{L}_C^{(q)} : B_q(D_1) \to B_q(D_2), \tag{1.5}$$

$$\mathfrak{L}_C^{(q)} : A_q(D_1) \to A_q(D_2) \tag{1.6}$$

are anti-linear continuous and injective, and

$$(\mathfrak{L}_C^{(q)} \varphi, \psi)_{q, D_2} = \overline{(\varphi, \mathfrak{L}_{-C}^{(q)} \psi)_{q, D_1}} \tag{1.7}$$

for $\varphi \in A_q(D_1)$, $\psi \in B_q(D_2)$.

C) A group G of Möbius transformations $z \rightarrowtail (az+b)/(cz+d)$ is called a *quasi-Fuchsian* group with fixed curve C if every $\gamma \in G$ maps D_1 onto D_1 and D_2 onto D_2.

THEOREM 2. *Let G be a quasi-Fuchsian group with fixed curve C. Then*

$$\mathfrak{L}_C^{(q)} B_q(D_1, G) \subset B_q(D_2, G), \tag{1.8}$$

and there exists a continuous anti-linear injection

$$\hat{\mathfrak{L}}_C^{(q)} : A_q(D_1, G) \to A_q(D_2, G) \tag{1.9}$$

such that

$$(\hat{\mathfrak{L}}_C^{(q)} \varphi, \psi)_{q, D_2/G} = \overline{(\varphi, \mathfrak{L}_{-C}^{(q)} \psi)}_{q, D_1/G} \tag{1.10}$$

for $\varphi \in A_q(D_1, G)$, $\psi \in B_q(D_2, G)$. (*If* $G=1$, *then* $\hat{\mathfrak{L}} = \mathfrak{L}$.)

D) A Jordan curve C will be called a *quasi-circle* if it is the image of a circle under a quasiconformal automorphism of the Riemann sphere.

THEOREM 3. *Let C be a quasi-circle and G a quasi-Fuchsian group with fixed curve C. Then the mappings*

$$\mathfrak{L}_C^{(q)} : B_q(D_1, G) \to B_q(D_2, G),$$

and

$$\hat{\mathfrak{L}}_C^{(q)} : A_q(D_1, G) \to A_q(D_2, G)$$

are surjective (*and therefore bijective topological isomorphisms*). The theorem applies, in particular, to $G=1$.

E) Let U and L denote the upper and lower half-planes, respectively. A quasiconformal self-mapping $w: U \to U$ is known to be continuous also on $\mathbf{R} \cup \{\infty\}$. We call w normalized if it leaves $0, 1, \infty$ fixed. Two normalized quasiconformal self-mappings $w_1: U \to U$ and $w_2: U \to U$ will be called equivalent if $w_1(x) = w_2(x)$ for all $x \in \mathbf{R}$.

By a Fuchsian group we will mean a discrete group of conformal self-mappings of U (and hence also of L). A normalized quasiconformal self-mapping w of U is called compatible with a Fuchsian group Γ if, for every $\gamma \in \Gamma$, the mapping $w \circ \gamma \circ w^{-1}: U \to U$ is a Möbius transformation. In this case $\Gamma_1 = w\Gamma w^{-1}$ is a Fuchsian group, and the mapping

$$\gamma \rightarrowtail \chi(\gamma) = w \circ \gamma \circ w^{-1}$$

is an isomorphism of Γ onto Γ_1 called the quasiconformal deformation induced by w.

Let Γ, w and χ be as above. If $x_0 \in \mathbf{R}$ is an attracting fixed point of $\gamma \in \Gamma$, then $\xi_0 = w(x_0)$ is an attracting fixed point of $\chi(\gamma)$. Using this observation one concludes easily that χ is determined by the equivalence class of w. Also, if Γ is of the first kind, that is if the fixed points of elements of Γ are dense in $\mathbf{R}$, then the equivalence class of w is determined by χ.

As an application of Theorems 1–3 we will establish

THEOREM 4. *To every normalized quasiconformal self-mapping $w: U \to U$ compatible with the Fuchsian group Γ there belong canonical topological isomorphisms of $A_q(U, \Gamma)$ onto $A_q(U, w\Gamma w^{-1})$ and of $B_q(U, \Gamma)$ onto $B_q(U, w\Gamma w^{-1})$, $q = 2, 3, \ldots$. These isomorphisms depend only on the equivalence class of w.*

In particular, let Γ be a finitely generated Fuchsian group of the first kind. Then $A_q(U, \Gamma) = B_q(U, \Gamma)$, its finite dimension can be computed from q, g, ν_∞ and ν_n, $n = 2, 3, \ldots$,

where g is the genus of the Riemann surface U/Γ, ν_∞ is the number of nonconjugate maximal parabolic subgroups of Γ, and ν_n the number of nonconjugate maximal cyclic subgroups of order n of Γ. Furthermore, two finitely generated Fuchsian groups of the first kind are quasiconformal deformations of each other if and only if they have the same numbers g, ν_j, $j=\infty, 2, 3, \ldots$. In this case, therefore, one part of Theorem 4 is well-known. But the existence of canonical isomorphisms seems to be new even in this classical case.

F) Another application of Theorems 1–3 is to quasiconformal extensions of conformal mappings. Nehari [7] observed that if $\varphi(z)$ is holomorphic and univalent in L, then the Schwarzian derivative $\{w, z\}$ belongs to $B_2(L)$ and $\|\{w, z\}\|_{B_2(L)} \leqslant 6$. He also showed that if $\|\{w, z\}\|_{B_2(L)} \leqslant 2$, then w is univalent in L. The following theorem of Ahlfors and G. Weill [2] (also proved in [1]) is a refinement of Nehari's result.

THEOREM 5. *Every $\varphi \in B_2(L)$ with $\|\varphi\|_{B_2(L)} < 2$ is the Schwarzian derivative of a univalent conformal mapping w which is a restriction to L of a quasiconformal automorphism of the Riemann sphere. This w may be chosen so as to satisfy in U the Beltrami equation*

$$w_{\bar{z}} = \mu w_z$$

with

$$\mu(z) = -2y^2\varphi(\bar{z}).$$

Now let C be again a Jordan curve and let $w(z)$, $z \in D_2(C)$, be holomorphic. Nehari's argument shows easily that if w is univalent, then $\{w, z\} \in B_2(D_2)$. Theorem 5 suggests the question: does there exist an $\varepsilon > 0$ such that every $\varphi \in B_2(D_2)$ with $\|\varphi\|_{B_2(D_2)} < \varepsilon$ is the Schwarzian derivative of a univalent function which admits a quasiconformal extension to the whole plane? Ahlfors [1] showed that this is so if C is a quasi-circle. We shall prove a refinement of this result which gives a complete generalization of Theorem 5.

THEOREM 6. *Let C be a quasi-circle. There exists an anti-holomorphic homeomorphism τ of a neighborhood of the origin N_2 in $B_2(D_2)$ onto a neighborhood N_1 of the origin in $B_2(D_1)$ with the following properties.*

(i) $\tau(0) = 0$.

(ii) *if G is a quasi-Fuchsian group with fixed curve C, then*

$$\tau(N_2 \cap B_2(D_2, G)) = N_1 \cap B_2(D_1, G).$$

(iii) *Every $\varphi \in N_2$ is the Schwarzian derivative of a univalent function w which is the restriction to D_2 of a quasiconformal automorphism of the Riemann sphere, and*

(iv) *this automorphism may be chosen so as to satisfy in D_1 the Beltrami equation*

$$w_{\bar{z}} = \mu w_z,$$

where
$$\mu(z) = \lambda_1(z)^{-2}\overline{\psi(z)}, \quad \psi = \tau\varphi.$$

This theorem is crucial for the construction of general Teichmüller spaces to be considered in a subsequent paper (see the announcement in [5]).

2. Preliminaries

A) Let $D \subset \mathbf{C} \cup \{\infty\}$ be a domain and $f: D \to \mathbf{C} \cup \{\infty\}$ a conformal mapping. Let p and q be integers (or half-integers such that $p+q$ is an integer). For every function $\varphi(\zeta)$, $\zeta \in f(D)$, and for $z \in D$ we set

$$(f^*_{p,q}\varphi)(z) = \varphi(f(z))\, f'(z)^p \overline{f'(z)}^q.$$

Clearly
$$(f \circ g)^*_{p,q} = g^*_{p,q} \circ f^*_{p,q}$$

whenever both sides are defined. For the sake of brevity we set

$$f^*_{p,0} = f^*_p.$$

Let D and G have the same meaning as in § 1. For $q = 2, 3, \ldots$, the mappings

$$f^*_q : A_q(f(D), fGf^{-1}) \to A_q(D, G),$$

$$f^*_q : B_q(f(D), fGf^{-1}) \to B_q(D, G)$$

are bijective isometries and

$$(\varphi, \psi)_{q, f(D)/fGf^{-1}} = (f^*_q\varphi, f^*_q\psi)_{q, D/G}.$$

The verification is trivial.

B) A sequence $\{\psi_j\} \subset B_q(D)$ will be said to converge weakly to $\psi \in B_q(D)$ if

$$\lim (\varphi, \psi_j)_{q,D} = (\varphi, \psi)_{q,D} \quad \text{for all} \quad \varphi \in A_q(D). \tag{2.1}$$

We recall (see [4]) that every continuous linear functional l on $A_q(D)$ can be represented, uniquely, as $l(\varphi) = (\varphi, \psi)_{q,D}$ for some $\psi \in B_q(D)$. Hence, by a well-known property of Banach spaces, (2.1) implies that

$$\|\psi_j\|_{B_q(D)} = O(1). \tag{2.2}$$

Also, since for every $z \in D$ the value $\psi(z)$ is a continuous linear functional on $B_q(D)$, (2.1) implies that

$$\lim \psi_j(z) = \psi(z) \quad \text{for all} \quad z \in D. \tag{2.3}$$

Conversely, statements (2.2) and (2.3) imply (2.1) in view of Lebesgue's theorem on dominated convergence.

C) We recall now the definition and some properties of the Poincaré metric. We have that

$$\lambda_U(z) = |z-\bar{z}|^{-1} = 1/2y, \quad z = x+iy,$$

and for every conformal bijection $f: U \to D$

$$f^*_{\frac{1}{2},\frac{1}{2}}\lambda_D = \lambda_U;$$

it follows that

$$g^*_{\frac{1}{2},\frac{1}{2}}\lambda_D = \lambda_D$$

for every conformal self-mapping $g: D \to D$. By Schwarz' lemma we have (cf. Ahlfors [1])

$$\lambda_{D_1}(z) \leqslant \lambda_{D_2}(z) \quad \text{if} \quad z_1 \in D_2 \subset D_1, \tag{2.4}$$

$$\lambda_D(z)\,|z-\partial D| \leqslant 1, \tag{2.5}$$

and, by Koebe's $\frac{1}{4}$-theorem,

$$\lambda_D(z)\,|z-\partial D| \geqslant \tfrac{1}{4} \quad \text{if} \quad \infty \notin D. \tag{2.6}$$

Here ∂D denotes the boundary of D and $|z-\partial D| = \inf|z-\zeta|$, $\zeta \in \partial D$.

D) An element $\varphi \in A_q(D)$ will be said to belong to the subspace $\tilde{A}_q(D)$ if there is a $\psi \in A_q(D_0)$ with $D \cup \partial D \subset D_0$ and $\psi | D = \varphi$. Since $A_q(D)$ is a subspace of $B_q(D)$, so is $\tilde{A}_q(D)$, cf. [4].

LEMMA 1. *Let D be a Jordan domain and $\psi \in B_q(D)$. Then there is a sequence $\{\psi_j\} \subset \tilde{A}_q(D)$ such that* $\lim \psi_j = \psi$ *weakly in $B_q(D)$, and if $\psi \in A_q(D)$, then also* $\lim \psi_j = \psi$ *in $A_q(D)$.*

Proof. If γ is a Möbius transformation, then $\tilde{A}_q(D) = \gamma^*_q(\tilde{A}_q(\gamma(D)))$, as is easily verified. Let Δ be the unit disc. In view of the preceding remark we lose no generality in assuming that $0 \in D \subset \Delta$. Let f denote the conformal mapping of Δ onto D with $f(0) = 0$, $f'(0) > 0$.

Let D_j, $j = 1, 2, 3, \ldots$ be Jordan domains with $D_j \supset \bar{D}_{j+1}$ and $D = \bigcap D_j$. Let f_j be the conformal mapping of Δ onto D_j with $f_j(0) = 0$, $f_j'(0) > 0$. Then, as is well known, $\lim f_j(\zeta) = f(\zeta)$ uniformly on the closure of Δ.

Now let $\psi \in B_q(D)$ be given. For every j we have that $\hat{\psi}_j = (f \circ f_j^{-1})^*_q \psi \in B_q(D_j)$. Hence $\psi_j = \hat{\psi}_j | D$ belongs to $\tilde{A}_q(D)$. We claim that the sequence $\{\psi_j\}$ has the required properties. First, by (2.4), we have that

$$\begin{aligned} \|\psi_j\|_{B_q(D)} &= \sup_{z \in D} \lambda_D(z)^{-q}\,|\psi_j(z)| \\ &\leqslant \sup_{z \in D_j} \lambda_{D_j}(z)^{-q}\,|\hat{\psi}_j(z)| = \|\hat{\psi}_j\|_{B_q(D_j)} = \|\psi\|_{B_q(D)}. \end{aligned}$$

Also, since $\lim (f \circ f_j^{-1})(z) = z$ uniformly on every compact subset of D, we have that $\lim \psi_j(z) = \psi(z)$ uniformly on every compact subset of D. Hence $\lim \psi_j = \psi$ weakly in $B_q(D)$.

Assume now that $\psi \in A_q(D)$. Let $\varepsilon > 0$ be given. Then there is a compact $K \subset D$ such that

$$\iint_{D-K} \lambda_D(z)^{2-q} |\psi(z)| \, dx\, dy < \varepsilon. \tag{2.7}$$

We have that

$$\iint_{D-K} \lambda_D(z)^{2-q} |\psi_j(z)| \, dx\, dy \leqslant \iint_{D-K} \lambda_{D_j}(z)^{2-q} |\hat{\psi}_j(z)| \, dx\, dy = \iint_{\sigma_j} \lambda_D(z)^{2-q} |\psi(z)| \, dx\, dy,$$

where $\sigma_j \subset D$ is defined by $(f_j \circ f^{-1})(\sigma_j) = D - K$. It follows that

$$\iint_{D-K} \lambda_D(z)^{2-q} |\psi_j(z)| \, dx\, dy < 2\varepsilon \quad \text{for large } j. \tag{2.8}$$

On the other hand, by what was said before,

$$\lim \iint_K \lambda_D(z)^{2-q} |\psi_j(z) - \psi(z)| \, dx\, dy = 0. \tag{2.9}$$

Since ε was arbitrary, relations (2.7), (2.8), and (2.9) imply that $\lim \|\psi_j - \psi\|_{A_q(D)} = 0$.

E) From now on let q, C, D_1, D_2, λ_1, λ_2 have the meaning explained in § 1.

LEMMA 2. *For $z \in D_2$ the function*

$${}_q\omega_z(\zeta) = (\zeta - z)^{-2q}, \quad \zeta \in D_1, \tag{2.10}$$

belongs to $\tilde{A}_q(D_1)$, and

$$\|{}_q\omega_z\|_{A_q(D_1)} \leqslant 4^{q-2}(2\pi/q)|z - C|^{-q},$$

where $|z - C| = \inf |z - \xi|, \quad \xi \in C.$

Proof. The first statement is trivial. Noting (2.6) we have

$$\begin{aligned}
\|{}_q\omega_z\|_{A_q(D_1)} &= \iint_{D_1} \lambda_1(\zeta)^{2-q} |\zeta - z|^{-2q} \, d\xi\, d\eta \\
&\leqslant 4^{q-2} \iint_{D_1} |\zeta - C|^{q-2} |\zeta - z|^{-2q} \, d\xi\, d\eta \\
&\leqslant 4^{q-2} \iint_{D_1} |\zeta - z|^{-2-q} \, d\xi\, d\eta \\
&\leqslant 4^{q-2} \iint_{|\zeta| > |z-C|} |\zeta|^{-2-q} \, d\xi\, d\eta = 4^{q-2}(2\pi/q)|z - C|^{-q},
\end{aligned}$$

as asserted.

F) LEMMA 3. *The functions* ${}_q\omega_z$, $z \in D_2$, *span a dense subspace of* $A_q(D_1)$.

Proof. In view of Lemma 1 it will suffice to show that if $\varphi \in \tilde{A}_q(D_1)$ is given, then there is a sequence of elements φ_j in the space spanned by the ${}_q\omega_z$ with

$$\lim \|\varphi - \varphi_j\|_{A_q(D_1)} = 0. \tag{2.11}$$

Let us assume first that $\infty \in D_1 \cup C$. Then there is a smooth Jordan curve C_0 in D_2 and a function Ψ holomorphic on C_0 and in the unbounded component D_0 of its complement, such that Ψ vanishes at ∞ of order at least $2q$ and $\Psi \mid D_1 = \varphi$. We have that

$$\Psi(\zeta) = \frac{d^{2q-1}F(\zeta)}{d\zeta^{2q-1}},$$

where $F(\zeta)$ is holomorphic for $\zeta \in C_0 \cup D_0$ and $F(\infty) = 0$. But then

$$F(\zeta) = \frac{1}{2\pi i} \int_{C_0} \frac{F(z)\,dz}{z-\zeta} \quad \text{for} \quad \zeta \in D_0$$

and hence

$$\Psi(\zeta) = \frac{(2q-1)!}{2\pi i} \int_{C_0} \frac{F(z)\,dz}{(z-\zeta)^{2q}} \quad \text{for} \quad \zeta \in D_0.$$

It follows that there exist functions $\varphi_j(\zeta)$, $j = 1, 2, 3, \ldots$, each of which is a Riemann sum for the integral written above,

$$\varphi_j(\zeta) = \sum_{k=1}^{N_j} c_{jk}(z_{jk} - \zeta)^{-2q}, \quad z_{jk} \in C_0,$$

such that $\lim \varphi_j(\zeta) = \Psi(\zeta)$ uniformly on every closed set $K \subset D_0$. Since $\Psi(\zeta)$ and $\varphi_j(\zeta)$ are $O(|\zeta|^{-2q})$ for $\zeta \to \infty$ and, as one verifies easily from (2.6), $\lambda_1(\zeta)^{2-q} = O(|\zeta|^{q-2})$, $\zeta \to \infty$, it follows that (2.11) holds.

If $\infty \in D_2$ the same conclusion follows by a similar but simpler argument.

3. Solution of the integral equation. Uniqueness

A) For $\sigma \in \mathbf{L}_\infty(D_1)$ and $z \in D_2$ we define

$$(\mathcal{M}_C^{(q)}\sigma)(z) = \iint_{D_1} \frac{\lambda_1(\zeta)^{2-q}\,\sigma(\zeta)}{(\zeta - z)^{2q}}\,d\xi\,d\eta = \iint_{D_1} \lambda_1(\zeta)^{2-q}\,{}_q\omega_z(\zeta)\,\sigma(\zeta)\,d\xi\,d\eta. \tag{3.1}$$

LEMMA 4. $\mathcal{M}_C^{(q)}$ *is a continuous mapping*

$$\mathcal{M}_C^{(q)} : \mathbf{L}_\infty(D_1) \to B_q(D_2),$$

and for every Möbius transformation γ *there is a commutative diagram*

$$\begin{array}{ccc} \mathbf{L}_\infty(\gamma(D_1)) & \xrightarrow{\gamma^*_{-q/2,\,q/2}} & \mathbf{L}_\infty(D_1) \\ \Big\downarrow \mathcal{M}^{(q)}_{\gamma(C)} & & \Big\downarrow \mathcal{M}^{(q)}_{C} \\ B_q(\gamma(D_2)) & \xrightarrow{\gamma^*_q} & B_q(D_2) \end{array} \tag{3.2}$$

Proof. It is obvious that the upper row is a bijective isometry. To establish commutativity note that for every Möbius transformation γ,

$$(\gamma(\zeta)-\gamma(z))^2 = (\zeta-z)^2\gamma'(\zeta)\gamma'(z).$$

Hence, for $\sigma \in \mathbf{L}_\infty(\gamma(D_2))$,

$$\begin{aligned}
(\gamma^*_q \circ \mathcal{M}^{(q)}_{\gamma(C)}\sigma)(z) &= \gamma'(z)^q(\mathcal{M}^{(q)}_{\gamma(C)}\sigma)(\gamma(z)) \\
&= \gamma'(z)^q \iint_{\gamma(D_1)} \frac{\lambda_{\gamma(D_1)}(\zeta)^{2-q}\sigma(\zeta)\,d\xi\,d\eta}{(\zeta-\gamma(z))^{2q}} \\
&= \gamma'(z)^q \iint_{D_1} \frac{\lambda_{\gamma(D_1)}(\gamma(\zeta))^{2-q}\sigma(\gamma(\zeta))\,|\gamma'(\zeta)|^2\,d\xi\,d\eta}{(\gamma(\zeta)-\gamma(z))^{2q}} \\
&= \iint_{D_1} \frac{\gamma'(z)^q\lambda_{D_1}(\zeta)^{2-q}\,|\gamma'(\zeta)|^{q-2}\sigma(\gamma(\zeta))\,|\gamma'(\zeta)|^2\,d\xi\,d\eta}{(\zeta-z)^{2q}\,\gamma'(\zeta)^q\,\gamma'(z)^q} \\
&= \iint_{D_1} \frac{\lambda_1(\zeta)^{2-q}\sigma(\gamma(\zeta))\,\overline{\gamma'(\zeta)}^{q/2}\gamma'(\zeta)^{-q/2}\,d\xi\,d\eta}{(\zeta-z)^{2q}} \\
&= (\mathcal{M}^{(q)}_C \circ \gamma^*_{-q/2,\,q/2}\sigma)(z).
\end{aligned}$$

To complete the proof of the lemma it suffices to establish the first statement for the case when $\infty \in C$. Let $\sigma \in \mathbf{L}_\infty(D_1)$. Since the integral defining $\varphi(z) = (\mathcal{M}^{(q)}_C\sigma)(z)$ converges absolutely, φ is holomorphic in D_2. Next, for $\zeta \in D_1$ and $z \in D_2$ we have, by (2.5), (2.6),

$$\lambda_1(\zeta)^{2-q} \leqslant (4\,|\zeta-C|)^{q-2} \leqslant (4\,|\zeta-z|)^{q-2}, \tag{3.3}$$

$$\lambda_2(z)^{-q} \leqslant (4\,|z-C|)^{q} \leqslant (4\,|\zeta-z|)^{q}. \tag{3.4}$$

Thus

$$\begin{aligned}
\lambda_2(z)^{-q}\,|\varphi(z)| &\leqslant \lambda_2(z)^{-q}\iint_{D_1} \frac{\lambda_1(\zeta)^{2-2q}\,|\sigma(\zeta)|\,d\xi\,d\eta}{|\zeta-z|^{2q}} \\
&\leqslant \|\sigma\|_{\mathbf{L}_\infty(D_1)}\lambda_2(z)^{-q}4^{q-2}\iint_{D_1}\frac{d\xi\,d\eta}{|\zeta-z|^{q+2}} \\
&\leqslant \|\sigma\|_{\mathbf{L}_\infty(D_1)}\lambda_2(z)^{-q}4^{q-2}(2\pi/q)\,|z-C|^{-q} \\
&\leqslant 4^{2q-2}(2\pi/q)\,\|\sigma\|_{\mathbf{L}_\infty(D_1)},
\end{aligned}$$

so that

$$\|\varphi\|_{B_q(D_2)}/\|\sigma\|_{\mathbf{L}_\infty(D_1)} \leqslant 4^{2q-2}(2\pi/q).$$

B) LEMMA 5. *The mapping*

$$\mathcal{L}_C^{(q)} : B_q(D_1) \to B_q(D_2)$$

is continuous, and for every Möbius transformation γ there is a commutative diagram

$$\begin{tikzcd} B_q(\gamma(D_1)) \arrow[r, "\gamma_q^*"] \arrow[d, "\mathcal{L}_{\gamma(C)}^{(q)}"'] & B_q(D_1) \arrow[d, "\mathcal{L}_C^{(q)}"] \\ B_q(\gamma(D_2)) \arrow[r, "\gamma_q^*"] & B_q(D_2) \end{tikzcd} \tag{3.5}$$

Proof. For every simply connected domain D with more than one boundary point we define the continuous mapping

$$\mathcal{K}_D^{(q)} : B_q(D) \to \mathbf{L}_\infty(D)$$

by

$$(\mathcal{K}_D^{(q)} \varphi)(z) = \lambda_D(z)^{-q} \overline{\varphi(z)}, \tag{3.6}$$

and verify that for every Möbius transformation γ the diagram

$$\begin{tikzcd} \mathbf{L}_\infty(\gamma(D)) \arrow[r, "\gamma^*_{-q/2,\, q/2}"] & \mathbf{L}_\infty(D) \\ B_q(\gamma(D)) \arrow[r, "\gamma_q^*"] \arrow[u, "\mathcal{K}_{\gamma(D)}^{(q)}"] & B_q(D) \arrow[u, "\mathcal{K}_D^{(q)}"'] \end{tikzcd} \tag{3.7}$$

is commutative. We also verify that

$$\mathcal{L}_C^{(q)} = \mathcal{M}_C^{(q)} \circ \mathcal{K}_{D_1}^{(q)}. \tag{3.8}$$

Thus Lemma 5 follows from Lemma 4.

C) We are now in a position to prove Theorem 1. The mappings (1.5) are continuous by Lemma 5. They are antilinear for $\mathcal{M}$ is clearly linear while $\mathcal{K}$ is anti-linear. Also, we have that for $z \in D_2$, $\psi \in B_q(D_1)$,

$$(\mathcal{L}_C^{(q)} \varphi)(z) = ({}_q\omega_z, \psi)_{q, D_1},$$

where ${}_q\omega_z$ is defined by (2.10). If $\mathcal{L}_C^{(q)}\psi=0$, then, by Lemma 3, $(\varphi,\psi)_{q,D}=0$ for all $\varphi\in B_q(D_1)$. Since $B_q(D_1)$ is canonically isomorphic to the dual of $A_q(D_1)$ we conclude that $\psi=0$. Thus $\mathcal{L}_C^{(q)}$ is injective.

We prove next the continuity of (1.6). In view of Lemma 5 we may assume that $\infty\in C$. Using (3.3), (3.4), we have

$$\begin{aligned}\|\mathcal{L}_C^{(q)}\varphi\|_{A_q(D_2)} &\leqslant \iint_{D_2}\lambda_2(z)^{2-q}\iint_{D_1}\frac{\lambda_1(\zeta)^{2-2q}|\varphi(\zeta)|\,d\xi\,d\eta}{|\zeta-z|^{2q}}\,dx\,dy\\ &\leqslant 4^{q-2}\iint_{D_1}\lambda_1(\zeta)^{2-2q}|\varphi(\zeta)|\iint_{D_2}\frac{dx\,dy}{|\zeta-z|^{q+2}}\,d\xi\,d\eta\\ &\leqslant 4^{q-2}(2\pi/q)\iint_{D_1}\frac{\lambda_1(\zeta)^{2-2q}|\varphi(\zeta)|\,d\xi\,d\eta}{|\zeta-C|^{q}}\\ &\leqslant 4^{2q-2}(2\pi/q)\iint_{D_1}\lambda_1(\zeta)^{2-q}|\varphi(\zeta)|\,d\xi\,d\eta=(2^{4q-1}\pi/q)\,\|\varphi\|_{A_q(D_1)}.\end{aligned}$$

Finally, (1.7) holds because by Fubini's theorem both sides equal to

$$\iiiint_{D_1\times D_2}\frac{\lambda_1(\zeta)^{2-2q}\lambda_2(z)^{2-2q}\overline{\varphi(\zeta)}\,\overline{\psi(z)}}{(\zeta-z)^{2q}}\,d\xi\,d\eta\,dx\,dy.$$

This completes the proof.

D) Proof of Theorem 2. We have, by Lemma 5,

$$\mathcal{L}_C^{(q)}\circ\gamma_q^*=\gamma_q^*\circ\mathcal{L}_C^{(q)}\quad\text{for}\quad\gamma\in G. \tag{3.9}$$

This implies (1.8). The remainder of the proof depends on the theory of Poincaré theta-series (see [4] and Earle's paper [6] where several proofs are simplified).

We recall that for a simply connected domain D with more than one boundary point and a discrete group G of conformal self-mappings of D there exists a linear, continuous surjective mapping

$$\Theta_D^{(q)}: A_q(D)\to A_q(D,G)$$

with

$$(\Theta_D^{(q)}\varphi,\psi)_{q,D/G}=(\varphi,\psi)_{q,D}\quad\text{for}\quad\psi\in B_q(D). \tag{3.10}$$

This mapping is defined by

$$\Theta_D^{(q)}\varphi=\sum_{\gamma\in G}\gamma_q^*\varphi,$$

but we do not use this formula explicitly.

Let $\varphi\in A_q(D_1)$ be such that $\Theta_{D_1}^{(q)}\varphi=0$. Then $\Theta_{D_2}^{(q)}\circ\mathcal{L}_C^{(q)}\varphi=0$. For, by (1.7) and (3.10) we have for every $\psi\in B_q(D_2,G)$

$$(\Theta_{D_2}^{(q)} \circ \mathcal{L}_C^{(q)} \varphi, \psi)_{q, D_2/G} = (\mathcal{L}_C^{(q)} \varphi, \psi)_{q, D_2}$$
$$= \overline{(\varphi, \mathcal{L}_{-C}^{(q)} \psi)_{q, D_1}} = \overline{(\Theta_{D_1}^{(q)} \varphi, \mathcal{L}_{-C}^{(q)} \psi)_{q, D_1/G}} = 0.$$

and it is known that $B_q(D_2, G)$ is canonically anti-isomorphic to the dual of $A_q(D_2, G)$. We conclude that the mapping (1.9) may be defined by setting

$$\hat{\mathcal{L}}_C^{(q)} \circ \Theta_{D_1}^{(q)} \varphi = \Theta_{D_2}^{(q)} \circ \mathcal{L}_C^{(q)} \varphi$$

for $\varphi \in A_q(D_1)$.

4. Quasi-circles

A) Let C be a Jordan curve. By a *quasi-reflection* about C we mean an orientation-reversing automorphism $z \rightarrowtail h(z)$ of the Riemann sphere which leaves every point of C fixed ($h|C = \mathrm{id}$) and is an involution ($h \circ h = \mathrm{id}$). It is clear that h maps the domain D_1 interior to C onto the domain D_2 exterior to C and vice versa.

LEMMA 6. *A Jordan curve C is a quasi-circle, that is the image of $\mathbf{R} \cup \{\infty\}$ under a quasiconformal automorphism $z \rightarrowtail \omega(z)$ of the Riemann sphere, if and only if there is a quasi-reflection h about C such that the mapping $z \rightarrowtail \overline{h(z)}$ is quasiconformal.*

This is almost trivial. Let g be a conformal mapping of U onto D_1. If h is given, set $\omega = g$ in $U \cup \mathbf{R} \cup \{\infty\}$ and $\omega(z) = h(g(\bar{z}))$ for $z \in L$. If ω is given, set $h(z) = \omega(\overline{\omega^{-1}(z)})$.

B) The following is due to Ahlfors [1].

PROPOSITION I. *Let C be a Jordan curve passing through ∞. Then C is a quasi-circle if and only if there exists a quasi-reflection about C satisfying a uniform Lipschitz condition.*

The sufficiency is merely a corollary of Lemma 6. The necessity is a deep result which is basic for what follows.

C) We will also need

LEMMA 7. *Let C be a quasi-circle. Let G be a function continuous in the whole plane. Let H and K be measurable functions such that off C we have*

$$H = \frac{\partial G}{\partial x}, \qquad K = \frac{\partial G}{\partial y} \tag{4.1}$$

in the sense of distributions. Assume also that H and K are square integrable over every bounded set. Then (4.1) *is valid in the whole plane.*

Proof. If $C=\mathbf{R}\cup\{\infty\}$, then the assertion is well known and easily proved. In the general case the assertion follows from the special case and from the behavior of partial derivatives under quasiconformal mappings (see, for instance, Ahlfors–Bers [3]).

LEMMA 8. *Let $z\rightarrowtail g(z)$ be an automorphism of the Riemann sphere and assume that g is conformal off a quasi-circle C. Then g is conformal everywhere (and hence a Möbius transformation).*

This is an immediate corollary of Lemma 7.

5. A reproducing formula

A) The aim of this section is to prove

PROPOSITION II. *Let C be a Jordan curve passing through ∞ and let D_1, D_2 denote the domains interior and exterior to C, respectively. Let $z\rightarrowtail h(z)$ be a quasi-reflection about C which satisfies a uniform Lipschitz condition.*

Then every $\psi\in B_q(D_2)$ satisfies the reproducing formula

$$\psi(z)=-\frac{2q-1}{\pi}\iint_{D_1}\frac{(\zeta-h(\zeta))^{2q-2}(\partial h(\zeta)/\partial\bar{\zeta})\,\psi(h(\zeta))\,d\xi\,d\eta}{(\zeta-z)^{2q}}\quad \textit{for}\quad z\in D_2. \tag{5.1}$$

As an example, let C be $\mathbf{R}\cup\{\infty\}$. Then we may set $h(\zeta)=\bar{\zeta}$ and obtain the known identity

$$\psi(z)=-\frac{2q-1}{\pi}\iint_{\eta>0}\frac{(\zeta-\bar{\zeta})^{2q-2}\psi(\bar{\zeta})\,d\xi\,d\eta}{(\zeta-z)^{2q}} \tag{5.2}$$

valid for $z\in U$.

B) Proof of Proposition II. For $\psi\in B_q(D_2)$ set

$$\nu_\psi(\zeta)=-\frac{2q-1}{\pi}(\zeta-h(\zeta))^{2q-2}\frac{\partial h(\zeta)}{\partial\bar{\zeta}}\psi(h(\zeta)). \tag{5.3}$$

Then (5.1) reads
$$\psi(z)=\iint_{D_1}\frac{\nu_\psi(\zeta)\,d\xi\,d\eta}{(\zeta-z)^{2q}}. \tag{5.4}$$

We will show below that
$$|\nu_\psi(\zeta)|\leqslant c\,\|\psi\|_{B_q(D_2)}\,\lambda_1(\zeta)^{2-q}, \tag{5.5}$$

where c depends only on h and q. This implies, first of all, that the integral in (5.4) converges absolutely. Also, if $\lim\psi_j=\psi$ weakly in $B_q(D_2)$, cf. § 2, then

$$\lim\iint_{D_1}\frac{\nu_{\psi_j}(\zeta)\,d\xi\,d\eta}{(\zeta-z)^{2q}}=\iint_{D_1}\frac{\nu_\psi(\zeta)\,d\xi\,d\eta}{(\zeta-z)^{2q}}.$$

Hence it will suffice to prove (5.4) under the assumption that $\psi\in\tilde{A}_q(D_2)$.

In order to prove (5.5), let c_0 be the Lipschitz constant of h. Then the derivative $\partial h/\partial\zeta = \frac{1}{2}(\partial h/\partial\xi + i(\partial h/\partial\eta))$, taken in the sense of distributions, is a measurable function which satisfies the inequality $|\partial h/\partial\bar{\zeta}| \leqslant c_0$. Also, if $\hat{\zeta} \in C$, we have that

$$|\zeta - h(\zeta)| \leqslant |\zeta - \hat{\zeta}| + |\hat{\zeta} - h(\zeta)| = |\zeta - \hat{\zeta}| + |h(\hat{\zeta}) - h(\zeta)| \leqslant (1 + c_0)|\hat{\zeta} - \zeta|$$

so that
$$|\zeta - h(\zeta)| \leqslant (1 + c_0)|\zeta - C|.$$

On the other hand, since for $\zeta \notin C$ the points ζ and $h(\zeta)$ are separated by C,

$$|\zeta - C| \leqslant |\zeta - h(\zeta)|.$$

By the above inequalities and (2.5), (2.6) we have

$$0 < c_1 \leqslant \lambda_j(\zeta)|\zeta - h(\zeta)| \leqslant 1/c_1, \quad j = 1, 2,$$

where c_1 depends only on c, and hence also

$$0 < c_1 \leqslant \lambda_j(h(\zeta))|\zeta - h(\zeta)| \leqslant 1/c_1, \quad j = 1, 2,$$

where c_1 depends only on c_0. Now,

$$|\psi(h(\zeta))| \leqslant \|\psi\|_{B_q(D_1)} \lambda_2(h(\zeta))^q$$

and (5.5) follows.

C) Now let $\psi \in \tilde{A}_q(D_2)$. The argument used in the proof of Lemma 3 shows that there exists a function $F(z)$, holomorphic in $D_2 \cup C \cup \{\infty\}$ and vanishing at $z = \infty$, such that

$$\psi(z) = F^{(2q-1)}(z) = \frac{d^{2q-1}F}{dz^{2q-1}}.$$

We set
$$G(z) = \begin{cases} \sum_{j=0}^{2q-2} (z - h(z))^j F^{(j)}(h(z))/j! & \text{for} \quad z \in D_1, \\ F(z) & \text{for} \quad z \in D_2 \cup C, \end{cases}$$

Then G is continuous everywhere and holomorphic in D_2. In D_2 the derivative $G'(z)$ is bounded on every bounded set. In D_1 the derivatives $\partial G/\partial\bar{z}$ and $\partial G/\partial z$, taken in the sense of distributions, are measurable functions which are bounded on every bounded set and

$$\frac{\partial G}{\partial \bar{z}} = -\frac{\pi}{(2q-1)!} \nu_\psi.$$

We conclude, by Lemma 7, that G has measurable locally bounded partial derivatives everywhere. Hence we may apply the well known identity

$$G(z) = -\frac{1}{\pi} \iint_{|\zeta| < R} \frac{\partial G(\zeta)}{\partial \bar{\zeta}} \frac{d\xi\, d\eta}{\zeta - z} + \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{G(\zeta)\, d\zeta}{\zeta - z}$$

for $|z| < R$. Let $D_{j,R}$ denote the intersection of D_j with the disc $|z| < R$. The formula above reads

$$G(z) = \frac{1}{(2q-1)!} \iint_{D_{1,R}} \frac{\nu_\psi(\zeta)\, d\xi\, d\eta}{\zeta - z} + \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{G(\zeta)\, d\zeta}{\zeta - z}.$$

For $z \in D_{2,R}$ both sides may be differentiated $(2q-1)$ times. This yields

$$\psi(z) = \iint_{D_{1,R}} \frac{\nu_\psi(\zeta)\, d\xi\, d\eta}{(\zeta - z)^{2q}} + \frac{(2q-1)!}{2\pi i} \int_{|\zeta|=R} \frac{G(\zeta)\, d\zeta}{(\zeta - z)^{2q}}.$$

Since $G(\zeta) = O(|\zeta|^{-1})$ for $\zeta \to \infty$, the identity just written becomes, for $R \to +\infty$, the desired relation (5.4).

6. Solution of the integral equation. Existence

A) In this section we prove Theorem 3. Let C be a quasi-circle. Without loss of generality (cf. Lemma 5) we assume that it passes through ∞. In view of Proposition I we may apply Proposition II to the curve C. Let $\psi \in B_q(D_2)$ be given, and let ν_ψ be defined by (5.3). We note (5.5) which shows that setting

$$\sigma = \lambda_1^{q-2} \nu_\psi,$$

we have $\sigma \in \mathbf{L}_\infty(D_1)$. The reproducing formula (5.4) may be written as

$$(\mathfrak{M}_C^{(q)} \sigma)(z) = \psi(z) \quad \text{or} \quad l({}_q\omega_z) = \psi(z),$$

where ${}_q\omega_z$ is defined by (2.10) and $l: A_q(D_1) \to \mathbf{C}$ is the continuous linear functional defined by

$$l(\hat{\varphi}) = \iint_{D_1} \lambda_1(\zeta)^{2-q} \hat{\varphi}(\zeta)\, \sigma(\zeta)\, d\xi\, d\eta, \quad \hat{\varphi} \in A_q(D_2).$$

But in view of the anti-isomorphism between $B_q(D_1)$ and the dual of $A_q(D_1)$, cf. [4], there is a $\varphi \in B_q(D_1)$ with

$$l(\hat{\varphi}) = (\hat{\varphi}, \varphi)_{q,D_1}.$$

In particular,

$$({}_q\omega_z, \varphi)_{q,D_1} = \psi(z),$$

which means that $\mathfrak{L}_C^{(q)} \varphi = \psi$. Thus $\mathfrak{L}_C^{(q)}: B_q(D_1) \to B_q(D_2)$ is surjective.

B) Now let G be a quasi-Fuchsian group with fixed curve C. If $\psi \in B_q(D_2, G) \subset B_q(D_2)$, then there is a $\varphi \in B_q(D_1)$ with $\mathfrak{L}_C^{(q)} \varphi = \psi$. By Theorem 1, it is unique. It follows from (3.9) that $\varphi \in B_q(D_1, G)$. Hence $\mathfrak{L}_C^{(q)}: B_q(D_1, G) \to B_q(D_2, G)$ is surjective.

Assume, finally, that $\hat{\mathfrak{L}}_C^{(q)}: A_q(D_1, G) \to A_q(D_2, G)$ is not surjective. Then there is a $\varphi_0 \in A_q(D_2, G)$ and a $\psi_0 \in B_q(D_2, G)$ such that

$$(\varphi_0, \psi_0)_{q, D_2/G} \neq 0 \tag{6.1}$$

and $$(\hat{\mathfrak{L}}_C^{(q)} \varphi, \psi_0)_{q, D_2/G} = 0 \quad \text{for all} \quad \varphi \in A_q(D_1, G).$$

The latter condition means, by Theorem 2, that

$$(\varphi, \mathfrak{L}_{-C}^{(q)} \psi_0)_{q, D_1/G} = 0 \quad \text{for all} \quad \varphi \in A_q(D_1, G)$$

and this implies that $\mathfrak{L}_{-C}^{(q)} \psi_0 = 0$. But then $\psi_0 = 0$ by Theorem 1, so that (6.1) is impossible.

7. Isomorphisms between spaces of automorphism forms

A) This section contains the proof of Theorem 4.

A normalized quasiconformal self-mapping $w: U \to U$ is completely determined by its Beltrami coefficient $\mu(z) = w_{\bar{z}}/w_z$ (cf. Ahlfors–Bers [3]). We write $w = w_\mu$. The function μ is a bounded measurable function defined in U, with $|\mu(z)| \leqslant k < 1$. We continue it to L be setting $\mu(z) = 0$ for $z \in L$, and we denote by w^μ the unique topological self-mapping of $\mathbf{C}$ which leaves 0, 1 fixed and satisfies the Beltrami equation

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}. \tag{7.1}$$

We shall establish later

PROPOSITION III. *Two normalized quasiconformal self-mappings, w_μ and w_ν, of U coincide on $\mathbf{R}$ if and only if the functions w^μ and w^ν coincide on $L \cup \mathbf{R}$.*

B) Let Γ be a Fuchsian group. If $w_\mu: U \to U$ is compatible with Γ (cf. § 1), a direct computation shows that

$$\gamma^*_{-1,1} \mu = \mu \quad \text{for all} \quad \gamma \in \Gamma. \tag{7.2}$$

Conversely, if (7.2) holds, one computes at once that for $\gamma \in G$ the function $w \circ \gamma$ satisfies (7.1) so that $w \circ \gamma = \hat{\gamma} \circ w$ where γ is a Möbius transformation. This means that w_μ is compatible with Γ. We assume that this is so and set $\Gamma_1 = w_\mu \Gamma w_\mu^{-1}$.

Now relation (7.2) shows, in the same way, that for every $\gamma \in \Gamma$ the mapping

$$w^\mu \circ \gamma \circ (w^\mu)^{-1}$$

is a Möbius transformation. Therefore the group $G = w^\mu \Gamma (w^\mu)^{-1}$ is a group of Möbius transformations. It is, as one sees at once, a quasi-Fuchsian group with fixed curve $C = w^\mu(\mathbf{R}) \cup \{\infty\}$. We have that $D_1(C) = w^\mu(U)$ and $D_2(C) = w^\mu(L)$.

Now let $g_\mu: w^\mu(U) \to U$ be the conformal bijection which sends 0, 1, ∞ into 0, 1, ∞, respectively. Then $g_\mu \circ w^\mu: U \to U$ is a normalized self-mapping satisfying (7.1). Hence

$$w_\mu = g_\mu \circ w^\mu | U.$$

It follows from Proposition III that $w^\mu|L$, D_1, D_2, C and g_μ depend only on the equivalence class of w_μ.

C) *Proof of Theorem 4.* With the notations introduced above and setting, for every function $\varphi(z)$, $(\mathbf{j}\varphi)(z)=\overline{\varphi(\bar{z})}$, consider the sequence of mappings

$$B_q(U,\Gamma)\xrightarrow{\mathbf{j}}B_q(L,\Gamma)\xleftarrow{(w^\mu)_q^*}B_q(D_2,G)\xrightarrow{\mathcal{L}_{-C}^{(q)}}B_q(D_1,G)\xleftarrow{(g_\mu)_q^*}B_q(U,\Gamma_1).$$

Now $\mathbf{j}$ and $\mathcal{L}_{-C}^{(q)}$ and surjective topological anti-isomorphisms; this is trivial for $\mathbf{j}$ and follows from Theorem 3 for $\mathcal{L}_{-C}^{(q)}$. On the other hand, $(w^\mu)_q^*$ and $(g_\mu)_q^*$ are bijective topological isomorphisms, cf. § 2. The resulting mapping $B_q(U,\Gamma)\to B_q(U,\Gamma_1)$ has the required properties.

The mapping $A_q(U,\Gamma)\to A_q(U,\Gamma_1)$ is constructed similarly, with $\mathcal{L}_{-C}^{(q)}$ replaced by $\hat{\mathcal{L}}_{-C}^{(q)}$.

D) *Proof of Proposition III.* Let g_μ have the same meaning as above and let $g_\nu: w^\nu(U)\to U$ be the conformal mapping such that

$$w_\nu = g_\nu\circ w^\nu|U.$$

We note that, since $w^\nu(U)$ and $w^\mu(U)$ are Jordan domains, the mapping g_ν and g_μ can be extended, by continuity, to homeomorphisms of $w^\nu(U\cup\mathbf{R}\cup\{\infty\})$ and $w^\mu(U\cup\mathbf{R}\cup\{\infty\})$, respectively.

Assume that $w^\nu|L=w^\mu|L$. Then $w^\nu(L)=w^\mu(L)$, $w^\nu|\mathbf{R}=w^\mu|\mathbf{R}$ and $w^\nu(U)=w^\mu(U)$. The last equality shows that $q_\nu=q_\mu$. Hence $w_\nu|\mathbf{R}=q_\nu\circ w^\nu|\mathbf{R}=g_\mu\circ w^\mu|\mathbf{R}=w_\mu|\mathbf{R}$.

Assume next that $w_\nu|\mathbf{R}=w_\mu|\mathbf{R}$. Define

$$h(z)=\begin{cases} g_\nu^{-1}\circ g_\mu(z) & \text{for} \quad z\in w^\mu(U\cup\mathbf{R}\cup\{\infty\}),\\ w^\nu\circ(w^\mu)^{-1}(z) & \text{for} \quad z\in w^\mu(L\cup\mathbf{R}\cup\{\infty\}).\end{cases}$$

This function is well defined and continuous everywhere. Only points on $C=w^\mu(\mathbf{R}\cup\{\infty\})$ must be checked. There

$$(g_\nu^{-1}\circ g_\mu)|C = w^\nu\circ w_\nu^{-1}\circ w_\mu\circ(w^\mu)^{-1}|C = (w^\nu|\mathbf{R})\circ(w_\nu^{-1}|\mathbf{R})\circ(w_\mu|\mathbf{R})\circ(w^\mu)^{-1}|C = w^\nu\circ(w^\mu)^{-1}|C.$$

Also h is an automorphism of the Riemann sphere. It is conformal in $w^\mu(U)$ and in $w^\mu(L)$ and, since C is a quasi-circle, everywhere (by Lemma 8). Since h leaves 0, 1, ∞ fixed, $h(z)=z$. Hence $w^\nu\circ(w^\mu)^{-1}=\mathrm{id}$ in $w^\mu(L)$. This means that $w^\nu|L=w^\mu|L$.

8. Schwarzian derivatives

A) We recall that the Schwarzian derivative $\{w, z\}$ of a meromorphic function function $w(z)$ is defined as

$$\{w, z\} = \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2.$$

It is holomorphic wherever either w is holomorphic and $w' \neq 0$ or w has a pole of order one. Also, $\{w,z\} \equiv 0$ if and only if w is a Möbius transformation, and if z is a meromorphic function of ζ, then

$$\{w, z\} dz^2 = \{w, \zeta\} d\zeta^2 + \{\zeta, z\} dz^2.$$

In particular, if γ is a Möbius transformation, then

$$\{\gamma \circ w, z\} = \{w, z\}, \tag{8.1}$$

$$\{w \circ \gamma, z\} = (\{w, z\} \circ \gamma)\gamma'(z)^2 = \gamma_2^* \{w, z\}. \tag{8.2}$$

B) Lemma 9. *Let D be a simply connected domain with at least two boundary points. Let $w(z)$ be meromorphic and univalent in D. Then $\{w, z\} \in B_2(D)$ and $\|\{w, z\}\|_{B_2(D)} \leqslant 96$.*

If D is a half-plane or a disc this was proved by Nehari [7] with 96 replaced by 6. Nehari's argument also works in the general case.

In view of (8.1) and (8.2) we lose no generality in assuming that w is holomorphic in D and that $\infty \notin D$. Let $z_0 \in D$ and set $\delta = |z_0 - \partial D|$. The function $W(\zeta) = w(z_0 + \delta/\zeta)$ is univalent for $|\zeta| > 1$ and so is the function

$$\frac{w'(z_0)\,\delta}{w(z_0 + \delta/\zeta) - w(z_0)} = \zeta + \text{const.} - \left(\frac{\delta^2}{6}\{w, z\}_{z_0}\right)\frac{1}{\zeta} + \dots .$$

By the "area theorem" $$\frac{\delta^2}{6}|\{w, z\}_{z_0}| \leqslant 1.$$

This means, since $z_0 \in D$ is arbitrary and, by (2.6), $\lambda_D(z_0)\delta \geqslant \frac{1}{4}$, that $\lambda_D(z)^{-2}|\{w, z\}| \leqslant 96$, as asserted.

C) Now let D_1, D_2, C have the same meaning as before and let $\mathbf{L}_\infty(D_1)_1$ denote the open unit ball in the Banach space $\mathbf{L}_\infty(D_1)$. Every element $\mu \in \mathbf{L}_\infty(D_1)$ we consider as defined everywhere in $\mathbf{C}$ with $\mu(z) = 0$ for $z \notin D_1$.

For $\mu \in \mathbf{L}_\infty(D_1)_1$ let $z \rightarrowtail w^\mu(z)$ denote the unique automorphism of $\mathbf{C}$ which leaves $0, 1$ fixed and satisfies the Beltrami equation (7.1). Then $w^\mu(z)$ is conformal and univalent in D_2. For $z \in D_2$ we set

$$\varphi^\mu = \{w^\mu, z\}. \tag{8.3}$$

By Lemma 9 we have that $\varphi^\mu \in B_2(D_2)$. The mapping $\mu \rightarrowtail \varphi^\mu$ will be denoted by $\mathcal{N}_C$.

LEMMA 10. *Let γ be a Möbius transformation. Then the diagram*

$$\begin{tikzcd} \mathbf{L}_\infty(\gamma(D_1))_1 \arrow[r, "\gamma^*_{-1,1}"] \arrow[d, "\mathcal{N}_{\gamma(C)}"'] & \mathbf{L}_\infty(D_1)_1 \arrow[d, "\mathcal{N}_C"] \\ B_2(\gamma(D_1)) \arrow[r, "\gamma^*_2"] & B_2(D_1) \end{tikzcd}$$

is commutative.

Proof. Let $\mu \in \mathbf{L}_\infty(\gamma(D_1))_1$ and set $\nu = \gamma^*_{-1,1}\mu$. One sees at once that $\nu \in \mathbf{L}_\infty(D_1)_1$ and that $w^\mu \circ \gamma$ satisfies the Beltrami equation $w_{\bar z} = \nu w_z$. Hence $w^\nu = \hat\gamma \circ w^\mu \circ \gamma$ where $\hat\gamma$ is another Möbius transformation. Now, using (8.1) and (8.2) we have

$$\mathcal{N}_C \gamma^*_{-1,-1}\mu = \mathcal{N}_C \nu = \{w^\nu, z\} = \{\hat\gamma \circ w^\mu \circ \gamma, z\} = \{w^\mu \circ \gamma, z\} = \gamma_2^* \{w^\mu, z\} = \gamma_2^* \circ \mathcal{N}_{\gamma(C)}\mu$$

as asserted.

D) Let G be a quasi-Fuchsian group with fixed curve C and let $\mathbf{L}_\infty(D_1, G)$ be the closed subspace of $\mathbf{L}_\infty(D_1)$ consisting of those μ for which $\gamma^*_{-1,1}\mu = \mu$ for all $\gamma \in G$. Also, set

$$\mathbf{L}_\infty(D_1, G)_1 = \mathbf{L}_\infty(D_1)_1 \cap \mathbf{L}_\infty(D_1, G).$$

LEMMA 11. *If G is a quasi-Fuchsian group with fixed curve C, then*

$$\mathcal{N}_C \mathbf{L}_\infty(D_1, G)_1 \subset B_2(D_2, G).$$

This follows at once from Lemma 10.

E) The proof of Theorem 6 depends on the following

PROPOSITION IV. *The mapping $\mathcal{N}_C : \mathbf{L}_\infty(D_1)_1 \to B_2(D_2)$ is holomorphic. Its derivative at $\mu = 0$ is the linear mapping $(-6/\pi)\mathcal{M}_C^{(2)}$, that is the mapping*

$$\mu(\zeta) \rightarrowtail -\frac{6}{\pi} \iint_{D_1} \frac{\mu(\zeta)\, d\xi\, d\gamma}{(\zeta - z)^4}.$$

Proof. In proving the holomorphic character of the mapping $\mathcal{N}_C$ we assume that D_2 is bounded. In view of Lemma 10 this involves no loss of generality. We choose a fixed but arbitrary $\mu \in \mathbf{L}_\infty(D_1)_1$ and numbers k_0 and k such that $|\mu| \leqslant k_0 < k < 1$. Let ν denote an arbitrary element of $\mathbf{L}_\infty(D_1)$ with $|\nu| < k - k_0$, and let t denote a complex variable restricted by the condition $|t| < 1$. We will prove that $\varphi^{\mu + t\nu} = \mathcal{N}_C(\mu + t\nu)$ satisfies the inequalities

$$\|\varphi^{\mu+t\nu}-\varphi^{\mu}\|_{B_2(D_2)}\leqslant c'|t| \quad \text{for} \quad |t|<\tfrac{1}{2}, \tag{8.4}$$

$$\left\|\frac{1}{t_1}(\varphi^{\mu+t_1\nu}-\varphi^{\mu})-\frac{1}{t_2}(\varphi^{\mu+t_2\nu}-\varphi^{\mu})\right\|_{B_2(D_2)}\leqslant c''(|t_1|+|t_2|) \quad \text{for} \quad 0<|t_1|,|t_2|<\tfrac{1}{4}, \tag{8.5}$$

where the constants c', c'' depend only on k_0 and k. Inequality (8.4) shows that the mapping $\mathcal{N}_C$ is continuous, while (8.5) shows the existence of the limit, in $B_2(D_2)$,

$$\lim_{\substack{t\to 0\\ t\in\mathbf{C}}}\frac{1}{t}(\varphi^{\mu+t\nu}-\varphi^{\mu})$$

for every $\mu\in\mathbf{L}_\infty(D_1)_1$, $\nu\in\mathbf{L}_\infty(D_1)$.

Let z_0 be a fixed but arbitrary point in D_2 and set

$$r=|z_0-C|, \quad \sigma(z)=\mu(z_0+rz), \quad \tau(z)=\nu(z_0+rz). \tag{8.6}$$

Then one verifies immediately that

$$w^{\sigma+t\tau}(z)=\frac{w^{\mu+t\nu}(z_0+rz)-w^{\mu+t\nu}(z_0)}{w^{\mu+t\nu}(z_0+r)-w^{\mu+t\nu}(z_0)},$$

and therefore

$$\varphi^{\sigma+t\tau}(0)=r^2\varphi^{\mu+t\nu}(z_0). \tag{8.7}$$

Since $\sigma(z)$, $\tau(z)$ vanish for $|z|<1$, the function $w^{\sigma+t\tau}(z)$ is holomorphic for $|z|<1$, and so is the function $g(z)=z/w^{\sigma+t\tau}(z)$ whose value at $z=0$ is

$$g(0)=1\Big/\left(\frac{dw^{\sigma+t\tau}(z)}{dz}\right)_{z=0}.$$

Hence

$$\frac{d^jw^{\sigma+t\tau}(z)}{dz^j}=\frac{j!}{2\pi i}\int_{|\zeta|=1}\frac{w^{\sigma+t\tau}(\zeta)\,d\zeta}{(\zeta-z)^{j+1}}$$

and similarly for g.

We will denote by c_j constants depending only on k_0 and k. The results of [3] imply that for every fixed z the numbers $w^{\sigma+t\tau}(z)$ and $g(z)=z/w^{\sigma+t\tau}(z)$ depend holomorphically on t and that

$$|w^{\sigma+t\tau}(z)|<c_1, \quad |g(z)|<c_2 \quad \text{for} \quad |z|=1.$$

This implies that

$$\left|\frac{d^jw^{\sigma+t\tau}(z)}{dz^j}\right|_{z=0}<c_3, \quad j=1,2,3,$$

$$\left|\frac{dw^{\sigma+t\tau}(z)}{dz}\right|_{z=0}>c_4>0,$$

and hence $|\varphi^{\sigma+t\tau}(0)|<c_5$ and

$$|\varphi^{\sigma+t\tau}(0)-\varphi^{\sigma}(0)|<c_6|t| \quad \text{for} \quad |t|<\tfrac{1}{2},$$

$$\left|\frac{1}{t_1}(\varphi^{\sigma+t_1\tau}(0)-\varphi^{\sigma}(0))-\frac{1}{t_2}(\varphi^{\sigma+t_2\tau}(0)-\varphi^{\sigma}(0))\right|<c_7(|t_1|+|t_2|) \quad \text{for} \quad 0<|t_1|,|t_2|<\tfrac{1}{4}.$$

Noting (8.7), the fact that $z_0\in D_2$ was arbitrary and the inequality (cf. (2.6)) $\lambda_1(z_0)^{-2}\leqslant 16\,r^2$, we obtain (8.4) and (8.5).

F) In proving that the derivative of $\mathcal{N}_C$ at the origin is $(-6/\pi)\mathcal{M}_C^{(2)}$ we assume that D_1 is bounded. This involves no loss of generality in view of Lemmas 4 and 10. For every $\mu\in\mathbf{L}_\infty(D_1)$ we have

$$\left(\frac{\partial w^{t\mu}(z)}{\partial t}\right)_{t=0}=-\frac{z(z-1)}{\pi}\iint_{D_1}\frac{\mu(\zeta)\,d\xi\,d\eta}{\zeta(\zeta-1)(\zeta-z)}. \tag{8.8}$$

This follows easily from the formulas for w^μ, with μ of compact support, contained in [3]. (As a matter of fact, (8.8) holds in general, even if $D_1=\mathbf{C}$, but we do not use this here.) Also $w^{t\mu}(z)$ is, for every fixed $\mu\in\mathbf{L}_\infty(D_1)$, a holomorphic function of both t (for $|t|$ small) and $z\in D_2$. So therefore is $\varphi^{t\mu}(z)$. For the sake of brevity we set $w^{t\mu}=w$, $\varphi^{t\mu}=\varphi$, and denote differentiation with respect to z by a prime, that with respect to t be a dot. Since differentiation with respect to t and z commute, we have

$$\dot\varphi=\{w,z\}^{\cdot}=\frac{(w')^3\dot w'''-\dot w'(w')^2w'''-3\dot w''(w')^2w''+6\dot w'w'w''}{(w')^4}.$$

For $t=0$, however, we have $w\equiv z$ hence $w'\equiv 1$, $w''=w'''=0$ and $\dot\varphi=\dot w'''$. This means, in view of (8.8), that

$$\begin{aligned}\left(\frac{\partial\varphi^{t\mu}(z)}{\partial t}\right)_{t=0}&=\frac{\partial^3}{\partial z^3}\left(\frac{\partial w^{t\mu}(z)}{\partial t}\right)_{t=0}\\&=-\frac{1}{\pi}\frac{\partial^3}{\partial z^3}z(z-1)\iint_{D_1}\frac{\mu(\zeta)\,d\xi\,d\mu}{\zeta(\zeta-1)(\zeta-z)}\\&=-\frac{6}{\pi}\iint_{D_1}\frac{\mu(\zeta)\,d\xi\,d\eta}{(\zeta-z)^4}.\end{aligned}$$

Since we already know that the limit $\lim_{t\to 0}((1/t)\mathcal{N}_C(t\mu))$ exists in $B_2(D_2)$, it follows from the above relation that this limit is $(-6/\pi)\mathcal{M}_C^{(2)}$.

9. Quasiconformal extension

We are now in a position to prove Theorem 6. Let C be a quasi-circle and G a quasi-Fuchsian group with fixed curve C. Consider the mapping $\mathcal{K}_{D_1}^{(2)}$ defined by (3.6). It follows from the commutativity of the diagram (3.7) that

$$\mathcal{K}_{D_1}^{(2)}:B_2(D_1,G)\to\mathbf{L}_\infty(D_1,G);$$

it is clear that this mapping is anti-linear and maps the open ball β about the origin in $B_2(D_1, G)$ of radius 1 into $\mathbf{L}_\infty(D_1, G)_1$. The mapping

$$Q = \mathcal{N}_C \circ \mathcal{K}_{D_1}^{(2)} : \beta \to B_2(D_2, G)$$

is therefore well defined (cf. Lemma 11). By Proposition IV, Q is an anti-holomorphic mapping, $Q(0) = 0$, and the derivative of Q at the origin is the anti-linear mapping

$$(-6/\pi)\, \mathcal{M}_C^{(2)} \circ \mathcal{K}_{D_1}^{(2)} = (-6/\pi)\, \mathcal{L}_C^{(2)},$$

cf. (3.8)). But by Theorem 3

$$(-6/\pi)\, \mathcal{L}_C^{(2)} : B_2(D_1, G) \to B_2(D_2, G)$$

is bijective. Hence, by the implicit function theorem in Banach space, there are neighborhoods of the origin $N_1 \subset B_2(D_1, G)$ and $N_2 \subset B_2(D_2, G)$ such that $Q : N_1 \to N_2$ is a bijection with an anti-holomorphic inverse. This is the assertion of Theorem 6.

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