THE EULER CLASS OF GENERALIZED VECTOR BUNDLES

BY

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1. Introduction

Let $\xi = (B, X, \pi)$ denote an oriented vector bundle(1) of dimension n, X being its base space, B its total space and $\pi: B \to X$ the projection. The obstruction to nonzero cross-sections $s: X \to B$ is a distinguished element χ in $H^n(X)$, the n-dimensional singular integral cohomology of X, known as the *Euler class* of ξ. We begin by briefly recollecting how χ may be defined. Let K denote the singular simplicial complex of X, K^* the singular simplicial complex of B, and K^0 the subcomplex of K^* whose $(n-1)$-skeleton lies in $\dot{B}$, the nonzero part of B. One now defines an integral cocycle ε on K^0 in the following manner:(2) Let Δ_n denote the standard n-simplex, $\dot{\Delta}_n$ its boundary, and let $\sigma: \Delta_n \to B$ be a singular n-simplex in K^0. Then $\pi \circ \sigma: \Delta_n \to X$ induces a bundle $\xi' = (B', \Delta_n, \pi')$ over Δ_n, and one may conclude(3) from the fact that Δ_n is contractible that ξ' is equivalent to a product bundle. Consequently there exists a second projection $p: B' \to V_n$, where V_n denotes a standard oriented n-dimensional vector space. Moreover, the map $\sigma: X \to B$ induces a cross-section $s: \Delta_n \to B'$, and since σ maps $\dot{\Delta}_n$ to $\dot{B}$, $p \circ s$ maps $\dot{\Delta}_n$ to $\dot{V}_n$, the punctured vector space. Since $\dot{\Delta}_n$ and $\dot{V}_n$ are homotopically equivalent to the oriented $(n-1)$-sphere, the restriction $p \circ s \,|\, \dot{\Delta}_n$ has a well-defined degree.(4) It is easy to verify that this integer does not depend on the choice of p, and consequently the formula

$$\varepsilon(\sigma) = \text{degree}\,(p \circ s \,|\, \dot{\Delta}_n) \tag{1.1}$$

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(1) For basic facts regarding vector bundles and characteristic classes we refer to J. Milnor [10].

(2) For basic facts regarding singular homology we refer to Eilenberg and Steenrod [7].

(3) Steenrod [12], Theorem 11.6.

(4) Cf. Eilenberg and Steenrod [7], p. 304.

defines ε as an integral n-cochain on K^0. The fact that ε is actually a cocycle may be checked by an elementary calculation. One now observes that the projection $\pi: B \to X$ induces a cochain homomorphism $\pi^*: C^q(K) \to C^q(K^0)$ and consequently a homomorphism

$$\pi^{\#}: H^q(K) \to H^q(K^0),$$

where C^q and H^q denote the integral cochain and cohomology groups, respectively. But $\pi^{\#}$ turns out to be an isomorphism, a fact which requires for its verification a fairly elaborate argument involving a construction due to Eilenberg.(1) The pertinent details will be described in Section 3, where a generalization of this result is to be proved. The Euler class χ of ξ is now defined by the formula $\pi^{\#}(\chi) = [\varepsilon]$, where $[\varepsilon]$ denotes the cohomology class of ε in $H^n(K^0)$. The fact that χ obstructs nonzero cross-sections in ξ is an immediate consequence of the definition. For such a cross-section $s: X \to \dot{B}$ induces a homomorphism $s^{\#}: H^q(K^0) \to H^q(K)$ such that $s^{\#} \circ \pi^{\#} = 1$. But since, by equation (1.1), $\varepsilon(\sigma) = 0$ for all σ lying entirely in $\dot{B}$, it follows that $s^{\#}$ annihilates $[\varepsilon]$. Consequently $\chi = s^{\#} \circ \pi^{\#}(\chi) = 0$, proving that χ obstructs.

Now let ξ be given as before and let J denote a set of local bundle maps of ξ, i.e., bundle maps $u: \xi | U \to \xi$, where U is an open subset of X and $\xi | U$ the restriction of ξ to U. A cross-section $s: X \to B$ will be called *J-invariant* if, for all $u \in J$,

$$\hat{u} \circ (s | U) = s \circ \check{u},$$

where $\check{u}: U \to X$ and $\hat{u}: \pi^{-1}(U) \to B$ denote the maps of the base and total spaces, respectively, associated with u. It is obvious that if J contains only the identity map of ξ, every cross-section will be J-invariant. The question, *does ξ admit a nonzero cross-section*, may consequently be generalized to read: *given (ξ, J), does ξ admit a nonzero J-invariant cross-section.* The present paper will investigate the possibility of extending the function χ to arbitrary pairs (ξ, J) so as to obtain a natural generalization of the classical obstruction theory as outlined above. To give precise sense to this proposal, the notion of a *map* from (ξ, J) to (ξ', J') must be defined in an appropriate manner, i.e., it is requisite that pairs (ξ, J) can be regarded as objects of a category extending the category of oriented vector bundles. To this end we shall employ a general procedure for extending local categories previously introduced by Y. H. Clifton and the author. These category-theoretic considerations, however, will be postponed until Section 5. For the present we shall adopt an entirely non-functorial point of view and restrict attention to a particular pair (ξ, J).

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(1) More precisely, the required construction is analogous to the procedure introduced by Eilenberg in [5], Ch. VI.

It is now evident that a natural concept of *J-invariance* may be defined for singular cochains on X and $\dot{B}$, and it is reasonable to conjecture that these must be of basic significance to the problem at hand. A cochain α in $C^q(K)$ will be called *J-invariant* if, for every singular q-simplex $\sigma:\Delta_q \to X$ and $u \in J$ for which $\check{u} \circ \sigma$ is defined, $\alpha(\check{u} \circ \sigma) = \alpha(\sigma)$. These cochains obviously constitute a subcomplex of $C^*(K)$, which we denote by $C^*(K/J)$. An alternative approach would be to define a quotient K/J of the simplicial complex K as follows: The pairing of σ with $\check{u} \circ \sigma$ for all σ in K and $u \in J$ defines a binary relation on K which in turn generates an equivalence relation $\sim$. Since $\sim$ preserves incidence, K induces a simplicial complex structure on the quotient K/J of K by $\sim$. It is evident that the integral q-cochains on K/J can be identified with the J-invariant cochains in $C^q(K)$, permitting us to use the symbol $C^q(K/J)$ in both senses. In a similar manner one may define a quotient K^0/J of K^0, and the resulting cochains $C^q(K^0/J)$ can be identified with a subset of $C^q(K^0)$, i.e., with the J-invariant cochains of $C^q(K^0)$. It is easy to verify that the cocycle ε actually lies in $C^n(K^0/J)$, a circumstance which accounts for the naturality(1) of χ under bundle maps. Moreover, the projection $\pi: B \to X$ evidently induces a cochain homomorphism $\pi^*: C^q(K/J) \to C^q(K^0/J)$, and consequently homomorphisms $\pi^{\#}: H^q(K/J) \to H^q(K^0/J)$ of the corresponding cohomology groups. It appears, therefore, that we are making some progress towards generalizing the definition of χ as outlined in the first paragraph. If $\pi^{\#}$ turns out to be an isomorphism, an element $\chi \in H^n(K/J)$ can be defined by the formula

$$\pi^{\#}(\chi) = [\varepsilon, K^0/J], \tag{1.2}$$

where $[\varepsilon, K^0/J]$ denotes the cohomology class of ε with respect to the cochain complex $C^q(K^0/J)$. Moreover, since a nonzero J-invariant cross-section $s: X \to \dot{B}$ clearly induces a cochain homomorphism $s^*: C^q(K^0/J) \to C^q(K/J)$, it follows as before that the new χ obstructs in the desired sense.

However, as may be expected, $\pi^{\#}$ is not necessarily an isomorphism, nor does equation (1.2) admit a unique solution for arbitrary J. This deviation from the classical behavior, moreover, is due to the phenomenon of *holonomy*, to be considered in Section 2. For the present it suffices to observe that the holonomy at a point $x \in X$ is defined as a group Φ_x of linear automorphisms of the fiber B_x. When all these groups are trivial, one proves by an easy adaptation of the classical argument that $\pi^{\#}$ is once more an isomorphism, as will be seen in Section 3. The case of nontrivial holonomy, on the other hand, poses considerable difficulty. In the present paper we shall consider this problem only in the lowest nontrivial dimension (i.e., $n=2$), where a rather complete solution has been obtained. Let

(1) Cf. Milnor [10].

us denote by B_x^* the linear subspace of B_x which is pointwise fixed under the transformations of Φ_x, and let B^* denote the point set union of all B_x^*, $x \in X$. Let $\tilde{K}^0$ denote the subcomplex of K^0 whose 0-skeleton lies in B^*. Naturally $\tilde{K}^0$ coincides with K^0 in the case of trivial holonomy. We will now adopt the viewpoint that $\tilde{K}^0$ should take the place of K^0, i.e., that equation (1.2) should be replaced by

$$\pi^{\#}(\chi) = [\varepsilon, \tilde{K}^0/J] \tag{1.3}$$

understood in a corresponding sense. It will be shown in Section 4 that equation (1.3) admits a unique solution, provided that

$$\text{dimension}\, B_x^* > 0 \quad \text{for all} \;\; x \in X. \tag{1.4}$$

Moreover, one can see by simple examples that condition (1.4) is not superfluous. Since a J-invariant cross-section $s: X \to B$ must lie in B^* (as shown in Section 2), one also sees that condition (1.4) is necessary to the existence of nonzero s. The same observation implies (by the usual argument) that a solution χ of equation (1.3) constitutes an obstruction to nonzero J-invariant s. Thus at least in dimension 2 one finds that a moderate amount of holonomy may still be tolerated, and in fact, that χ can tolerate precisely as much holonomy as the cross-sections which it obstructs. The problem becomes more difficult for $n > 2$, and it appears doubtful that the state of affairs will be quite as favorable.

The question of *naturality* will be taken up in Section 6 after the requisite category-theoretic development has been supplied. To obtain a functorial cohomology theory for objects (ξ, J), it will be necessary to replace the *J-invariant cochains* by *J-invariant sections in the corresponding sheaf of singular cochains*. The extended Euler class will take values in the sheaf-cohomology corresponding to $H^n(K/J)$, and naturality (with respect to bundle maps of the extended category) will pose no particular difficulty. It is of interest to note that the extended Euler class of an object (ξ, J) with trivial holonomy is actually induced from an associated classical vector bundle ξ', which is the quotient of ξ under identifications induced by J. When (ξ, J) has nontrivial holonomy, on the other hand, it will be shown that its extended Euler class (when it exists) is not induced in this manner (because the quotient ξ' does not exist as a vector bundle), and what is more, cannot be induced from *any* classical bundle.

By way of illustration we shall consider objects (ξ, J) arising from a p-dimensional foliation[1] $\mathfrak{F}$ of a differentiable n-manifold. Such a structure $\mathfrak{F}$, it will be recalled, sends a

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[1] All foliations are assumed to be *oriented*. For the precise definition of $\mathfrak{F}$ and its object (ξ, J), we refer to Section 7.

p-dimensional variety $\mathcal{F}_x$ through every point $x \in X$. The Euler class of (ξ, J) is now an obstruction to $(p+1)$-dimensional foliations $\mathcal{F}^*$ of X with the property that $\mathcal{F}_x \subset \mathcal{F}_x^*$ for all $x \in X$. As previously observed,(1) this generalizes the obstruction problem for direction fields on a differentiable manifold. More precisely, when $\mathcal{F}$ is defined by a projection $p: X \to M$, where M is an $(n-p)$-dimensional differentiable manifold, $\mathcal{F}^*$ is equivalent to a direction field on M. As may be expected, (ξ, J) will now be isomorphic (in the extended category) to the tangent bundle $\tau(M)$.

In an earlier paper on this subject,(2) Y. H. Clifton and the author have defined an extended Euler class for a large set of objects (ξ, J) with practically no condition on the holonomy, but at the cost of using a special homology theory with certain undesirable features. The foremost of these, perhaps, is the fact that the theory is entirely ill-adapted to computation. The present approach is fundamentally simpler and commends itself as more natural to the problem.

It is a pleasure to acknowledge the ample contribution of Y. H. Clifton, who has collaborated with us in the earlier phases of this work. The basic ideas of the present paper stem from that earlier period of joint endeavor.

2. Some concepts related to holonomy

Let $\xi = (B, X, \pi)$ be an oriented vector bundle as before, and J a set of local bundle maps of ξ containing the identity. For $u \in J$, $\check{u}$ and $\hat{u}$ will always denote the associated maps of the base and total spaces, respectively. Now let $x \in X$ be a point in the domain of $\check{u}$ and let $y = \check{u}(x)$. We will denote by u_x the linear isomorphism of B_x (the fiber over x) onto B_y induced by u. Moreover, the following concept will also be needed: Let

$$\gamma_1 = \{u_i : 0 < i \leqslant p\}$$

be a sequence of maps in J with p even, and let

$$\gamma_2 = \{f_i : 0 \leqslant i \leqslant p\}$$

be a sequence of maps from some given topological space W to X. The pair $\gamma = (\gamma_1, \gamma_2)$ will be called a *J-chain* if, for every positive value of the index i, $f_i = \check{u}_i \circ f_{i-1}$ when i is odd and $f_{i-1} = \check{u}_i \circ f_i$ when i is even. In this case we shall say that γ *connects* f_0 to f_p, and we will let $J(f_0, f_p)$ denote the set of all such J-chains. It is clear that if $\gamma \in J(f, g)$ and $\gamma' \in J(g, h)$, one

(1) Cf. Clifton and Smith [4].

(2) Cf. Clifton and Smith [4]. The present paper takes the place of an expanded version of the theory outlined in [4].

can define a composition $\gamma' \circ \gamma \in J(f, h)$, and one can also define a corresponding $\gamma^* \in J(g, f)$ by indexing the terms of γ_1 and γ_2 in reverse order. With every map $f: W \to X$ we will associate the J-chain γ_f for which $p=0$ (γ_1 is then the empty sequence) and $f_0 = f$. We now define an equivalence relation on $\mathrm{Hom}(W, X)$, the space of maps from W to X, as the set of all pairs (f, g) for which $J(f, g)$ is nonempty. It will be called *J-equivalence*, and one observes that when $W = \Delta_q$, the standard q-simplex, it coincides with the relation $\sim$ on K_q considered in Section 1.

Analogous concepts may be defined with respect to the total space B. Let $\gamma = (\gamma_1, \gamma_2)$ be a J-chain with $\gamma_2 = \{f_i : 0 \leqslant i \leqslant p\}$, and let $\bar{\gamma}_2 = \{\bar{f}_i : 0 \leqslant i \leqslant p\}$ be a corresponding sequence of maps from W to B such that $\pi \circ \bar{f}_i = f_i$ for all i. The J-chain γ is said to *connect* $\bar{f}_0$ to $\bar{f}_p$ provided $\bar{f}_i = \hat{u}_i \circ \bar{f}_{i-1}$ when i is odd and $\bar{f}_{i-1} = \hat{u} \circ \bar{f}_i$ when i is even (where u_i denotes the ith term of γ_1). The set of all such J-chains will be denoted by $J(\bar{f}_0, \bar{f}_p)$, and one now defines *J-equivalence* on $\mathrm{Hom}(W, B)$ by taking $\bar{f} \sim \bar{g}$ to mean that $J(\bar{f}, \bar{g})$ is nonempty. It is apparent that for all pairs $(\bar{f}, \bar{g})$ in $\mathrm{Hom}(W, B)$, $J(\bar{f}, \bar{g}) \subset J(\pi \circ \bar{f}, \pi \circ \bar{g})$, and conversely we shall establish the following result:

LEMMA 2.1. *Given maps $f: W \to X$, $\bar{g}: W \to B$ and $\gamma \in J(f, \pi \circ \bar{g})$, there exists a unique map $\bar{f}: W \to B$ such that $f = \pi \circ \bar{f}$ and $\gamma \in J(\bar{f}, \bar{g})$.*

To prove this it will suffice to consider the case $p=2$. Taking $\gamma_1 = \{u, v\}$ and $\gamma_2 = \{f, h, \pi \circ \bar{g}\}$ one has $h = \check{u} \circ f = \check{v} \circ \pi \circ \bar{g}$. Let $\bar{h} = \hat{u} \circ \bar{g}$, and let $w \in W$, $x = f(w)$ and $y = \check{u}(x)$. If $\bar{f}$ exists, then $u_x[\bar{f}(w)] = \bar{h}(w)$, and consequently

$$\bar{f}(w) = u_x^{-1} \, [\bar{h}(w)], \tag{2.1}$$

which proves uniqueness. One now observes that equation (2.1) defines a function $\bar{f}: W \to B$ such that $\pi \circ \bar{f} = f$ and $\hat{u} \circ \bar{f} = \bar{h}$, and it remains only to show that $\bar{f}$ is continuous. We consider for this purpose a particular point $w_0 \in W$, set $x_0 = f(w_0)$ and $y_0 = \check{u}(x_0)$, and select local product representations of ξ in a neighborhood U of x_0 and U' of y_0, so chosen that $\check{u}(U) \subset U'$. Thus we may regard $\pi^{-1}(U) = U \times V$ and $\pi^{-1}(U') = U' \times V$, where V is a standard vector space. The restriction of $\hat{u}$ to $\pi^{-1}(U)$ now takes the form $\hat{u}(x, z) = (\check{u}(x), \tau(x)z)$, where τ is a continuous function from U to the automorphism group $\mathrm{Gl}(V)$. Likewise there exists a neighborhood U'' of w_0 and continuous functions

$$\bar{h}_1 : U'' \to U', \quad \bar{h}_2 : U'' \to V$$

such that $\bar{h} = (\bar{h}_1, \bar{h}_2)$ on U''. Consequently

$$\bar{f}(w) = (f(w), \tau^{-1}(f(w)) \bar{h}_2(w))$$

on U'', which represents a continuous function.

We shall henceforth employ the notation $\bar{f}=\gamma(f, \bar{g})$ to designate the operation defined by Lemma 2.1. One further observation regarding J-chains will be required in the sequel: If γ is a J-chain as before, a map $h: W' \to W$ will induce a J-chain $\gamma \circ h$ by the formula

$$\gamma \circ h = (\gamma_1, \{f_i \circ h : 0 \leqslant i \leqslant p\}).$$

One way to define the holonomy groups is to consider the case where W contains precisely one point w. Given $x \in X$, let $i_x: W \to X$ denote the map taking w to x. For $x, y \in X$, it is now evident that every $\gamma \in J(i_x, i_y)$ induces a linear isomorphism $\gamma_x: B_x \to B_y$. Moreover, the set $\Phi_x = \{\gamma_x : \gamma \in J(i_x, i_x)\}$ constitutes an automorphism group of B_x: this is the *holonomy group* of (ξ, J) at x. It defines a notion of *J-invariance* for vectors in B_x, i.e., such a vector will be called J-invariant if it is fixed under all transformations of Φ_x. This brings us to the following observation:

LEMMA 2.2. *If $s: X \to B$ is a J-invariant cross-section and $x \in X$, then $s(x)$ is J-invariant.*

For suppose $\gamma = (\gamma_1, \gamma_2)$ is a J-chain in $J(i_x, i_x)$.Then γ_2 defines a sequence $\{x_i : 0 \leqslant i \leqslant p\}$ of points in X such that $x_0 = x_p = x$, and the element u_i in γ_1 determines a linear isomorphism $h_i: B_{i-1} \to B_i$ for $0 < i \leqslant p$, such that $\gamma_x = h_p \circ \ldots \circ h_1$. More precisely, u_i induces h_i or h_i^{-1}. In either case it follows by definition of J-invariance for cross-sections (given in Section 1) that $h_i[s(x_{i-1})] = s(x_i)$, which implies that $s(x)$ is a fixed point of γ_x.

3. The case of trivial holonomy

We suppose in this Section that Φ_x is trivial for all $x \in X$, and our first task will be to establish

THEOREM 3.1. *When (ξ, J) has trivial holonomy, the chain homomorphism*(1)

$$\pi_*: C_q(K^0/J) \to C_q(K/J)$$

induced by the projection constitutes a chain equivalence.

When J contains only the identity map of ξ, the theorem reduces to a familiar result, as previously noted. We recall that the classical proof is accomplished by interposing between K and K^0 the singular simplicial complex K^* of B, and that it involves the following diagram:

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(1) When K' is a simplicial complex, $C_q(K')$ will always denote the corresponding integral chain groups.

$$\begin{tikzcd} C_q(K^0) \arrow[rr, shift left, "\mu"] \arrow[dr, "\pi_*"'] & & C_q(K^*) \arrow[ll, shift left, "\tau"] \arrow[dl, shift left, "\alpha"] \\ & C_q(K) \arrow[ur, shift left, "\beta"] & \end{tikzcd} \tag{3.1}$$

Here μ denotes the chain homomorphism induced by the inclusion $K^0 \to K^*$, and α the chain homomorphism induced by the projection $\pi: B \to X$. To prove that π_* is a chain equivalence, it suffices to show this for α and μ. In the first case this is a simple matter, since the zero cross-section $s^0: X \to B$ induces a chain homomorphism $\beta: C_q(K) \to C_q(K^*)$ such that (α, β) constitutes an equivalence pair (i.e., both composites are chain homotopic to the identity). More precisely, there exists a chain homotopy $D: C_q(K^*) \to C_q(K^*)$ such that

$$\alpha \circ \beta = 1, \tag{3.2}$$

$$\beta \circ \alpha - 1 = \partial D + D\partial,\text{[1]} \tag{3.3}$$

where 1 signifies in each case the appropriate identity. The problem is more difficult for μ and one requires essentially the Eilenberg construction[2] to define a chain homomorphism $\tau: C_q(K^*) \to C_q(K^0)$, together with a chain homotopy $D': C_q(K^*) \to C_q(K^*)$, such that

$$\tau \circ \mu = 1, \tag{3.4}$$

$$\mu \circ \tau - 1 = \partial D' + D'\partial. \tag{3.5}$$

To see how this argument may be carried over to the case of nontrivial J, one observes that a chain homomorphism involving the simplicial complexes K, K^* and K^0 will induce corresponding chain homomorphisms involving the quotients K/J, K^*/J and K^0/J, provided it is *J-invariant* in a rather apparent sense. Let us suppose, for example, that

$$\varphi: C_q(K) \to C_q(K^*)$$

is a chain homomorphism with the property that it carries the elementary chain of a singular simplex into a chain of this kind (a condition which will be satisfied by all our chain homomorphisms). Now *J-invariance* means that for every pair (σ, σ') of J-equivalent

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[1] It will be understood that the term ∂D is absent on dimension 0.

[2] Cf. Eilenberg [5], pp. 439–442.

singular simplexes in K, $\varphi(\sigma)$ and $\varphi(\sigma')$ will likewise be J-equivalent.(1) For a chain homotopy $D: C_q(K^*) \to C_q(K^*)$, *J-invariance* shall mean that for every pair (σ, σ') of J-equivalent singular simplexes in K^*, $D(\sigma)$ and $D(\sigma')$ shall have the form

$$D(\sigma) = \sum_i a_i \sigma_i, \qquad D(\sigma') = \sum_i a_i \sigma_i'$$

with σ_i J-equivalent to σ_i' for all i. This will insure that D shall induce a chain homotopy on the quotient $C_q(K^*/J)$. To establish Theorem 3.1, it will suffice to ascertain J-invariance for the chain homomorphisms α, β, μ and τ, and for the chain homotopies D and D'.

Now this is immediately verified for the natural maps α, β and μ. One can define the chain homotopy D entering into equation 3.3 on the generators $\sigma: \Delta_q \to B$ by the formula $D(\sigma) = SP_\sigma$, where $P_\sigma: \Delta_q \times I \to B$ (I denotes the unit interval) is the singular prism given by $P_\sigma(x,t) = t\sigma(x)$, and S denotes its *basic chain.*(2) If $u \in J$ and $\sigma' = \hat{u} \circ \sigma$ is defined, it now follows by linearity that $P_{\sigma'} = \hat{u} \circ P_\sigma$, from which one may conclude that D is J-invariant in the specified sense. Thus the first half of the argument, involving the pair (α, β), carries over to the quotient complexes without difficulty, and without the assumption of trivial holonomy.

We will now briefly recall the classical construction of τ and D', and see what can be done to insure J-invariance. In the first place it is important to note that if σ is a singular simplex in K^*, $\tau(\sigma)$ will be a singular simplex $\bar{\sigma}$ belonging to K^0, but *basically equivalent* to σ, i.e., $\pi \circ \bar{\sigma} = \pi \circ \sigma$. Given such a chain homomorphism τ, one can define the associated chain homotopy D' by setting $D'(\sigma) = SP_\sigma$, the prism $P_\sigma: \Delta_q \times I \to B$ being given by

$$P_\sigma(x,t) = t\bar{\sigma}(x) + (1-t)\sigma(x).$$

This implies equation (3.5) by the usual calculation. Moreover, if τ is J-invariant, it will follow by linearity that D' is likewise J-invariant. Our problem reduces therefore to the construction of a J-invariant chain homomorphism $\tau: C_q(K^*) \to C_q(K^0)$ which preserves basic equivalence, is J-invariant and satisfies equation (3.4). Since $\tau(\sigma)$ is consequently prescribed for σ belonging to K^0, it remains to define τ on the sets K_q' of q-simplexes in K^* which do not lie in K^0. This is accomplished by induction on the dimension q, starting with $q=0$. Thus the classical construction on dimension zero simply associates with every $\sigma \in K_0'$ a basically equivalent $\bar{\sigma} \in K^0$. To achieve J-invariance, it will obviously be expedient to order the simplexes of K_0', and to define τ_0 by transfinite induction. Thus if $\tau(\sigma)$ has been defined for all $\sigma < \sigma_0$, the definition of $\tau(\sigma_0)$ will distinguish two cases:

(1) We shall avoid notational distinction between a singular simplex σ and its elementary chain.
(2) Cf. Eilenberg [5], p. 423.

(i) σ_0 is J-equivalent to some $\sigma_* < \sigma_0$;

(ii) there does not exist a $\sigma_* \in K_0'$ with these properties.

In the first case there exists a $\gamma \in J(\sigma_0, \sigma_*)$, and we may take $\tau(\sigma_0) = \gamma(\pi \circ \sigma_0, \tau(\sigma_*))$. In the second case we may take $\tau(\sigma_0)$ to be any $\bar{\sigma}_0 \in K_0^0$, basically equivalent to σ_0. By this means the condition of J-equivalence will be propagated, and the definition of τ_0 may be completed by transfinite induction.

Now let us see what happens on the next dimension, i.e., for $q=1$. If $\sigma \in K_1'$, the condition $\partial\tau_1(\sigma) = \tau_0(\partial\sigma)$ becomes

$$\bar{\sigma}^{(i)} = \tau_0(\sigma^{(i)}), \quad 0 \leqslant i \leqslant 1; \tag{3.6}$$

where $\bar{\sigma} = \tau_1(\sigma)$ and the superscript i signifies the ith face of the given singular simplex. The classical construction on dimension one simply associates with each $\sigma \in K_1'$ a basically equivalent $\bar{\sigma} \in K_1^0$ satisfying equation (3.6). To achieve J-invariance, we will again order the simplexes of K_1', and posit the inductive hypothesis that $\tau(\sigma)$ has been defined for all $\sigma < \sigma_1$ so as to satisfy equation (3.6). Again one is confronted with two possibilities, and in the second case one is free to define $\tau(\sigma_1)$ as in the classical construction. Let us suppose then that σ_1 is J-equivalent to some preceding σ_*, and let $\gamma \in J(\sigma_1, \sigma_*)$. To achieve J-invariance, we define

$$\tau(\sigma_1) = \gamma(\pi \circ \sigma_1, \tau(\sigma_*)). \tag{3.7}$$

It is clear, in the first place, that if $\tau(\sigma_*) \in K^0$, $\tau(\sigma_1)$ will likewise lie in K^0. The crucial question is whether the singular simplex $\bar{\sigma}_1 = \tau(\sigma_1)$ satisfies equation (3.6). We note that for each value of the index i, $\gamma \circ e_1^i \in J(\bar{\sigma}_1{}^{(i)}, \tau(\sigma_*{}^{(i)}))$, where e_q^i denotes the standard map from Δ_{q-1} onto the ith face of Δ_q. On the other hand, by J-invariance of τ_0 there exists a J-chain $\gamma_i \in J(\tau(\sigma_*{}^{(i)}), \tau(\sigma_1{}^{(i)}))$, so that

$$\tilde{\gamma}^i = \gamma_i \circ (\gamma \circ e_1^i) \in J(\bar{\sigma}_1^{(i)}, \tau(\sigma_1^{(i)})).$$

It is now evident that equation (3.6) cannot be expected to hold in the general case of nontrivial holonomy. The present hypothesis, however, insures that the automorphism $\tilde{\gamma}_y^i$ of B_y must reduce to the identity, y being a point in the image of Δ_0 under $\pi \circ \sigma_1{}^{(i)}$. The validity of equation (3.6) is therefore assured, and the definition of τ_1 may be completed by transfinite induction.

It will now suffice to observe that the general step of the finite induction (involving the dimension q) may be carried out by an exactly analogous consideration, and this completes the proof of Theorem 3.1.

The result implies that the homomorphism $\pi^{\#}: H^q(K/J) \to H^q(K^0/J)$ induced by the projection is an isomorphism for each $q \geqslant 0$, so that equation 1.2 defines an element $\chi \in H^n(K/J)$. As previously observed, χ is an obstruction to J-invariant cross-sections

$s:X\to\dot{B}$. However, a somewhat stronger result will be of interest. Given a vector bundle $\xi=(B, X, \pi)$, let $\xi_0=(S, X, \pi_0)$ denote the associated sphere bundle. For every $x\in X$, we can identify the fiber S_x in ξ_0 with the set of directions in B_x, where *direction* means a half-line centered at the origin. It is known that the Euler class of ξ is an obstruction to cross-sections $s_0:X\to S$ in ξ_0. In case X is paracompact, this is implied by the fact that ξ admits(1) a Riemannian metric, so that every s_0 induces a cross-section $s:X\to\dot{B}$ of unit vectors. We now note that every local bundle map u of ξ induces a corresponding bundle map u_0 of ξ_0, so that the notion of *J-invariance* for cross-sections $s_0:X\to S$ has an obvious meaning. We will prove the following result:

THEOREM 3.2. *Let (ξ, J) have trivial holonomy and χ denote the corresponding solution of equation* 1.2. *Then χ obstructs J-invariant cross-sections in ξ_0.*

Let $K^{\dagger}$ denote the singular simplicial complex of S and $\psi:C_q(K^{\dagger})\to C_q(K)$ the chain homomorphism induced by the projection $\pi_0:S\to X$. We shall be interested in constructing a chain homomorphism $\theta:C_q(K^{\dagger})\to C_q(K^0)$ which assigns to every $\sigma\in K^{\dagger}$ a *nonzero* $\bar{\sigma}\in K^0$, i.e., a singular simplex $\bar{\sigma}:\Delta_q\to\dot{B}$. Such θ will be called J-invariant if it preserves J-equivalence, where *J-equivalence* for $\sigma\in K^{\dagger}$ is defined in the obvious way. The desired result will be an immediate consequence of the following:(2)

LEMMA 3.1. *There exists a J-invariant chain homomorphism $\theta:C_q(K^{\dagger})\to C_q(K^0)$ such that $\pi_*\circ\theta=\psi$ and $\theta(\sigma)$ is nonzero for all $\sigma\in K^{\dagger}$.*

The construction of θ proceeds in a manner entirely analogous to the construction of τ. The following aspect of the matter is sufficiently different, however, to warrant some consideration: Let us consider the diagram

$$\begin{tikzcd}
 & & B \arrow[d, "\nu"] \\
 & & S \arrow[d, "\pi_0"] \\
\dot{\Delta}_q \arrow[r] \arrow[rruu, "\dot{\theta}"] & \Delta_q \arrow[r] \arrow[ruu, "\theta(\sigma)"] \arrow[ru, "\sigma"'] & X
\end{tikzcd}$$

(1) Cf. Milnor [10], p. 20.

(2) Lemma 3.1 does not require trivial holonomy but may be established on the assumption that the holonomy transformations have no positive eigenvalues different from 1.

where $\nu: B \to S$ denotes the natural map. Given $\sigma: \Delta_q \to S$ and $\dot{\theta}: \dot{\Delta}_q \to B$ (determined by θ_{q-1}), one wishes to extend $\dot{\theta}$ to a nonzero lifting $\theta(\sigma)$ of σ. Again there are two cases to be considered, and when σ is J-equivalent to some $\sigma_* < \sigma, \theta(\sigma)$ will be defined in terms of $\theta(\sigma_*)$ by the usual procedure involving J-chains. Case (ii), on the other hand, requires attention. Here one considers the bundles $\xi' = (B', \Delta_q, \pi')$ and $\xi_0' = (S', \Delta_q, \pi_0')$ induced by $\pi_0 \circ \sigma$. Now $\dot{\theta}$ induces a (local) cross-section $\dot{s}: \dot{\Delta}_q \to B'$, and the problem of defining $\theta(\sigma)$ reduces to the construction of a nonzero cross-section $s: \Delta_q \to B'$ extending $\dot{s}$. Since $\dot{\theta}$ is nonzero by inductive hypothesis, $\dot{s}$ is likewise nonzero, and the desired extension s will exist provided $\dot{s}$ is contractible in the nonzero space $\dot{B}'$. To insure that this will be the case, it will suffice to suppose that the homomorphism θ satisfies the following natural condition: *For all $\sigma \in K_p^\dagger$ and $x \in \Delta_p, \bar{\sigma}(x)$ shall lie on the direction $\sigma(x)$, where $\bar{\sigma} = \theta(\sigma)$.* One observes, in the first place, that if the condition holds for a particular σ, it will then hold for all σ' which are J-equivalent to σ. Supposing that the condition holds on dimension $(q-1)$, one now finds that for every $x \in \dot{\Delta}_q$, $\dot{s}(x)$ lies on the direction of $s_\sigma(x)$, where $s_\sigma: X \to B'$ denotes the cross-section induced by σ. But this implies that $\dot{s}$ admits a nonzero extension s, which induces a map $\theta(\sigma): \Delta_q \to B$ such that our condition is again satisfied. The construction of θ is therefore feasible.

To establish Theorem 3.2, it remains to observe that θ, ψ and π_* induce corresponding chain homomorphisms

$$\begin{tikzcd}
C_q(K^\dagger/J) \arrow[rr, "\theta"] \arrow[dr, shift left, "\psi"] & & C_q(K^0/J) \arrow[dl, "\pi_*"] \\
& C_q(K/J) \arrow[ul, shift left, "\varphi"] &
\end{tikzcd}$$

on the quotient complexes. A cross-section $s_0: X \to S$ would now induce a chain homomorphism $\varphi: C_q(K/J) \to C_q(K^\dagger/J)$ such that $\pi_* \circ \theta \circ \varphi = 1$. On the other hand, since $\theta(\sigma)$ is always nonzero, it follows that the associated cochain homomorphism θ^* annihilates the cocycle ε. Consequently

$$\chi = \varphi^{\#} \circ \theta^{\#} \circ \pi^{\#}(\chi) = \varphi^{\#} \circ \theta^{\#}([\varepsilon, K^0/J]) = 0,$$

where $^{\#}$ signifies in each case the associated homomorphism of the cohomology groups.

4. The case $n=2$

Let (ξ, J) be given, with ξ of dimension 2. As noted in Section 1, it is now expedient to replace the simplicial complex K^0 by the subcomplex $\tilde{K}^0$ whose 0-skeleton lies in B^* (the subspace of J-invariant vectors in B). Clearly this gives rise to a quotient complex $\tilde{K}^0/J$, which may be identified with a subcomplex of K^0/J. The cocycle ε admits therefore a restriction $\tilde{\varepsilon} \in C^2(\tilde{K}^0/J)$, and we will let $[\tilde{\varepsilon}]$ denote its cohomology class in $H^2(\tilde{K}^0/J)$.

THEOREM 4.1. *Let $\pi^{\#}: C^2(K/J) \to C^2(\tilde{K}^0/J)$ denote the homomorphism induced by the projection, and let*

$$\operatorname{dimension} B_x^* > 0 \quad \textit{for all } x \in X. \tag{4.1}$$

Then there exists a unique element $\chi \in H^2(K/J)$ such that $\pi^{\#}(\chi) = [\tilde{\varepsilon}]$.

The proof will naturally involve the subcomplex $\tilde{K}^*$ of K^* with 0-skeleton in B^*, and the chain homomorphisms

$$\mu: C_q(\tilde{K}^0) \to C_q(\tilde{K}^*), \quad \alpha: C_q(\tilde{K}^*) \to C_q(K) \quad \text{and} \quad \beta: C_q(K) \to C_q(\tilde{K}^*)$$

induced by the inclusion $\tilde{K}^0 \to K^*$, the projection $\pi: B \to X$ and the zero cross-section s^0: $X \to B$, respectively. Evidently all three chain homomorphisms are J-invariant, and equation (3.2) is again satisfied.

LEMMA 4.1. *There exists a J-invariant chain homotopy $D: C_q(\tilde{K}^*) \to C_q(\tilde{K}^*)$ satisfying equation* (3.3). *Thus* (α, β) *constitutes a J-invariant equivalence pair.*

The chain homotopy D will be defined precisely as before in terms of prisms

$$P_\sigma: \Delta_q \times I \to B.$$

It should be observed that if $\sigma \in \tilde{K}^*$, every singular simplex σ' belonging to the *standard subdivision*(1) of P_σ will likewise lie in $\tilde{K}^*$. For every vertex of σ' is either a vertex of σ, or must lie on the zero cross-section. In either case the vertex will lie in B^*, and consequently $\sigma' \in \tilde{K}^*$. The shift from K^* to $\tilde{K}^*$ therefore causes no difficulty.

LEMMA 4.2. *There exists a chain homomorphism $\tau: C_q(\tilde{K}^*) \to C_q(\tilde{K}^0)$ and chain homotopy $D': C_q(\tilde{K}^*) \to C_q(\tilde{K}^*)$ satisfying equations* (3.4) *and* (3.5), *such that τ and D' are J-invariant on dimensions* 0 *and* 1.

Again we observe that once τ has been constructed, D' may be defined by the usual formula. Let us therefore recall the construction of τ in Section 3. It is apparent that the only modification required on dimension 0 is to choose $\tau(\sigma_0)$ to lie in B^*, which is possible on account of condition (4.1). We note that case (i) will cause no difficulty, for when $\tau(\sigma_0)$ is defined by the formula

(1) Cf. Eilenberg [5], p. 423.

$$\tau(\sigma_0) = \gamma(\pi \circ \sigma_0, \tau(\sigma_*)), \quad \gamma \in J(\sigma_0, \sigma_*);$$

the fact that $\tau(\sigma_*)$ lies in B^* implies the same for $\tau(\sigma_0)$. In dimension 1, on the other hand, the construction of Section 3 is applicable precisely as it stands. That $\tau(\sigma_1)$ defined by equation (3.7) will satisfy equation (3.6) is now assured, not by the assumption of trivial holonomy, but by the fact that the 0-skeleton of $\tau(\sigma_*)$ lies in the J-invariant part of B. For dimensions $q > 1$ the construction may proceed precisely as in the classical case, since J-invariance is no longer required.(1)

Let us next consider the chain homomorphism $\varphi = \tau \circ \beta$ and its dual $\varphi^* : C_q(\tilde{K}^0) \to C^q(K)$. As usual we regard $C^q(\tilde{K}^0/J)$ as a subcomplex of $C^q(\tilde{K}^0)$, i.e., the subcomplex of J-invariant cochains in $\tilde{K}^0$. Thus $\varphi^*(\bar{\varepsilon})$ is defined and represents a cocycle in $C^2(K)$, which we denote by λ.

LEMMA 4.3. *λ is J-invariant.*

To establish this, we consider two J-equivalent singular 2-simplexes σ_1 and σ_2 in K. Let $\gamma \in J(\sigma_1, \sigma_2)$, and let $\bar{\sigma}_j = \varphi(\sigma_j)$, $j = 1, 2$. Each of the maps $\sigma_j : \Delta_2 \to X$ induces a bundle ξ_j over Δ_2, and we let $h_j : \xi_j \to \xi$, B^j and s_j denote the associated bundle map, total space of ξ_j and cross-section in ξ_j induced by $\bar{\sigma}_j$, respectively. It is also easy to see that γ induces a bundle map $g : \xi_1 \to \xi_2$ such that $\check{g}$ is the identity of Δ_2. Now $\lambda(\sigma_j) = \varepsilon(\bar{\sigma}_j)$, which is simply the winding number of $s_j | \dot{\Delta}_2$ (as explained in the first paragraph of Section 1). One must consequently show that $s_1 | \dot{\Delta}_2$ and $s_2 | \dot{\Delta}_2$ have the same winding number. To this end we consider the cross-section $s_* = \hat{g} \circ s_1$ in ξ_2, and observe that $s_* | \dot{\Delta}_2$ has the same winding number as $s_1 | \dot{\Delta}_2$. The desired conclusion will now follow if it can be established that the formula

$$s(x,t) = ts_*(x) + (1-t)s_2(x), \quad (x,t) \in \dot{\Delta}_2 \times I, \tag{4.2}$$

defines a homotopy in $\dot{B}^2$, the nonzero part of B^2. It remains therefore to show that the right side of equation (4.2) cannot vanish for $(x, t) \in \dot{\Delta}_2 \times I$. Now if $x \in \Delta_2{}^{(i)}$ and $y = \sigma_2(x)$, the points $s_*(x), s_2(x) \in B^2_x$ correspond under h_2 to the points $\sigma_*(x), \bar{\sigma}_2(x) \in B_y$, where $\sigma_* = \gamma(\pi \circ \sigma_2, \bar{\sigma}_1)$. On the other hand, since β and τ are J-invariant on dimension 1, the same holds for φ, so that $\bar{\sigma}_1{}^{(i)}$ and $\bar{\sigma}_2{}^{(i)}$ are J-equivalent. It now follows that $\sigma_*(x)$ and $\bar{\sigma}_2(x)$ differ by an element $\tilde{\gamma}_y \in \Phi_y$, i.e., that $\tilde{\gamma}_y$ maps $\sigma_*(x)$ to $\bar{\sigma}_2(x)$. Thus $\tilde{\gamma}_y$ is an automorphism of the oriented 2-dimensional vector space B_y, and by condition (4.1), it has 1 for an eigenvalue. But this implies that $\tilde{\gamma}_y$ can have no negative eigenvalues, so that the equation

(1) Needless to say, J-invariance could not be achieved on dimensions $q > 1$ under the present hypothesis.

$$t\sigma_*(x)+(1-t)\bar{\sigma}_2(x)=0$$

has no solution for $t\in I$. Lemma 4.3 is therefore established.

Now $\lambda\in C^2(K/J)$, and we assert that the corresponding cohomology class $\chi\in H^2(K/J)$, maps to $[\tilde{\varepsilon}]$ under $\pi^\#$. Let $\pi^*:C^q(K/J)\to C^q(\tilde{K}^0/J)$ denote again the cochain homomorphism induced by the projection, so that

$$\pi^*(\lambda)=[\tau\circ\beta\circ\alpha\circ\mu]^*(\tilde{\varepsilon}).$$

It follows now by Lemma 4.1 that

$$\pi^*(\lambda)=\tilde{\varepsilon}+\delta\nu,$$

where δ denotes the coboundary operator and $\nu=[\tau\circ D\circ\mu]^*(\tilde{\varepsilon})$. Our assertion will be established provided ν is J-invariant. But this is implied by the fact that μ and D are J-invariant, and

LEMMA 4.4. *$\tau^*(\tilde{\varepsilon})$ is J-invariant.*

The latter follows by the argument employed in the proof of Lemma 4.3.

To establish uniqueness, we suppose that $\pi^\#(\chi')=[\tilde{\varepsilon}]$ for some $\chi'\in H^2(K/J)$, and let λ' denote a representative cocycle of χ'. This means that there exists a cochain $\nu\in C^1(\tilde{K}^0/J)$ such that

$$\pi^*(\lambda')=\tilde{\varepsilon}+\delta\nu. \tag{4.3}$$

Applying the cochain homomorphism φ^* to both sides of equation (4.3), one obtains (by Lemma 4.2)

$$\lambda=\lambda'+\delta\nu', \tag{4.4}$$

where
$$\nu'=[\alpha\circ D'\circ\beta]^*(\lambda')-\varphi^*(\nu).$$

Since β, φ and D' are J-invariant on dimension 1 and α is J-invariant on dimension 2, it follows that $\nu'\in C^1(K/J)$, and this establishes uniqueness.

We turn next to the question of J-invariant cross-sections in the associated sphere bundle ξ_0. The previous approach to this problem is not of much use in the present case for the reason that our construction of the chain homotopy θ is no longer feasible. It will be recalled that θ was required to take each singular simplex $\sigma:\Delta_q\to S$ to $\bar{\sigma}:\Delta_q\to B$ which is a *lifting* of $\bar{\sigma}$, i.e., for all $x\in\Delta_q$, $\bar{\sigma}(x)$ must lie on the direction $\sigma(x)$. Since the present assumption regarding the holonomy groups does not rule out holonomy transformations with positive eigenvalues different from 1, it can happen that a J-invariant direction $\sigma(x)$ contains no nonzero J-invariant vector. It is therefore apparent that the proposed construction could not satisfy the lifting condition even on dimension 0. We will consequently adopt an entirely different approach, based on an idea due to Y. H. Clifton.

Given a J-invariant cross-section s_0 in ξ_0, we propose to construct a cochain $\nu \in C^1(K^0/J)$ such that $\varepsilon = \delta\nu$. It should be observed, moreover, that the construction will not depend on condition (4.1) and will in fact be valid for arbitrary J. Given $\sigma: \Delta_1 \to \dot{B}$, we let $\xi' = (B', \Delta_1, \pi')$ denote the vector bundle induced by $\pi \circ \sigma$ and $s: \Delta_1 \to B'$ the cross-section induced by σ. A product structure on ξ' defines a projection p from $\dot{B}'$ to S^1, the standard 1-sphere, which we may take to be the real line R modulo 1. The given cross-section $s_0: X \to S$ induces now a direction field s_0' in ξ' and consequently a map $f_0: \Delta_1 \to S^1$. Let the product representation of ξ' be so chosen that $f_0(x) = 0 \bmod 1$ for all $x \in \Delta_1$, and let $f: \Delta_1 \to R$ denote a lifting of $p \circ s$. We will define ν by the formula

$$\nu(\sigma) = [f(1)] - [f(0)], \tag{4.5}$$

where $[x]$ denotes the largest integer $n \leqslant x$.(1) It is not difficult to verify, in the first place, that the integer $\nu(\sigma)$ does not depend on the choice of p and f, so that equation (4.3) defines a cochain $\nu \in C^1(K^0)$. To show that ν is J-invariant, let $\bar{\sigma} = \hat{u} \circ \sigma$, where $u \in J$. Now $\bar{\sigma}$ gives rise to an induced bundle $\bar{\xi}'$ and corresponding cross-section $\bar{s}$, and s_0 determines a direction field $\bar{s}_0'$ in $\bar{\xi}'$. Moreover, u induces a bundle map $g: \xi' \to \bar{\xi}'$ such that $\check{g}$ is the identity of Δ_1 and $\bar{s} = \hat{g} \circ s$. Since s_0 is J-invariant, $\hat{g}$ also takes s_0' into $\bar{s}_0'$. Having chosen the product structure on ξ' we may consequently choose the product structure on $\bar{\xi}'$ so that $p \circ s = \bar{p} \circ \bar{s}$ (where $\bar{p}: \dot{\bar{B}}' \to S^1$ denotes the corresponding projection). Thus $\nu(\sigma) = \nu(\bar{\sigma})$, establishing J-invariance. Lastly, the fact that $\varepsilon = \delta\nu$ may be verified by a simple calculation. This proves

THEOREM 4.2 (Clifton). *For* (ξ, J) *with* $n = 2$, $[\varepsilon, K^0/J]$ *obstructs J-invariant cross-sections in* ξ_0.

Now let $\tilde{\nu}$ denote the restriction of ν to $C^1(\tilde{K}^0/J)$, so that $\tilde{\varepsilon} = \delta\tilde{\nu}$. Assuming condition (4.1), let $\varphi = \tau \circ \beta$ as before, giving $\lambda = \delta\varphi^*(\tilde{\nu})$. Since φ is J-invariant on dimension 1, therefore $\varphi^*(\tilde{\nu}) \in C^1(\tilde{K}^0/J)$, which implies $\chi = 0$. We have consequently established

THEOREM 4.3. *The cohomology class χ defined by Theorem* 4.1 *obstructs J-invariant cross-sections in* ξ_0.

5. Extension of local categories

The notion of a *local category* was introduced by H. Cartan and Eilenberg(2) to axiomatize the category-theoretic properties of fiber bundles. It will suffice to recall that a local category $\mathcal{A}$ consists of

(1) It should also be noted that we are representing Δ_1 by the unit interval. Cf. Eilenberg and Steenrod [7], p. 185.

(2) For full definitions and basic theory of local categories, we refer to Eilenberg [6].

(i) a category (which we also denote by $\mathcal{A}$);

(ii) a covariant functor $L:\mathcal{A}\to C$ (where C is the category of topological spaces and continuous maps);

(iii) a function which to every object A in $\mathcal{A}$ and every open subset U of $L(A)$ assigns (1) an object $A\,|\,U$ in A, and (2) a map $i_A\,|\,U$ in A;

these data being subject to some rather natural axioms. These insure, for instance, that given $f:A\to A'$ in $\mathcal{A}$, $U\subset L(A)$ and $U'\subset L(A')$, there will exist g such that the diagram

$$\begin{tikzcd}
A\,|\,U \arrow[r, "g"] \arrow[d, "i_A\,|\,U"'] & A'\,|\,U' \arrow[d, "i_A\,|\,U'"] \\
A \arrow[r, "f"] & A'
\end{tikzcd}$$

is commutative, if and only if $L(f)\,U\subset U'$, in which case g is unique. This map g will be designated by the symbol $U'\,|f|\,U$. It should also be observed that in the context of vector bundles, L corresponds to the functor which (in terms of our previous notation) takes ξ to X and u to $\check{u}$, $A\,|\,U$ corresponds to the restricted bundle $\xi\,|\,U$, and $i_A\,|\,U$ to the inclusion map $\xi\,|\,U\to\xi$. Moreover, the category C of topological spaces may be regarded as a local category in an obvious way, i.e., by taking L to be the identity functor, $A\,|\,U$ to be U, and $i_A\,|\,U$ to be the inclusion map $U\to A$.

In previous papers(1) Clifton and the author have described a process for extending the category C, together with functors of a certain kind. This process remains applicable when C is replaced by an arbitrary local category $\mathcal{A}$, and the basic results carry over in a rather obvious way. A brief summary of the extension theory in its general setting will now be given. It shall be understood in the sequel that the generalized *objects*, *maps*, etc., are defined with respect to a given local category $\mathcal{A}$ and should, strictly speaking, be called $\mathcal{A}$-*objects*, $\mathcal{A}$-*maps*, etc.

Definition 5.1. A *preobject* is a pair $\mathbf{A}=(A,J)$, where A is an object in $\mathcal{A}$ and J a set of maps in $\mathcal{A}$ such that

(i) for every $f\in J$, f is a map from $A\,|\,U_f$ to A, U_f being an open subset of $L(A)$;

(1) Cf. Clifton and Smith [2], [3].

(ii) given $f, g \in J$ and $x \in U_f \cap U_g$, there exist $\bar{f}, \bar{g} \in J$ and an open neighborhood V of x such that

$$\bar{f} \circ (U_{\bar{f}} | f | V) = \bar{g} \circ (U_{\bar{g}} | g | V). \tag{5.1}$$

Condition (ii) will play an essential role. It compensates for the fact that the maps in J are not assumed to have an inverse, and will be referred to as the *extension axiom* for objects.

Definition 5.2. An *object* is a preobject $\mathbf{A} = (A, I)$ such that

(i) the identity map 1_A of A belongs to I;

(ii) if $f: A | U_f \to A$ is a map in $\mathcal{A}$ such that for every $x \in U_f$ there exist $\bar{f}, g, \bar{g} \in I$ and an open neighborhood V satisfying equation (5.1), then $f \in I$.

Condition (ii) (the *closure axiom* for objects) insures, for instance, that I will contain all composites and inverses of its elements, when these exist in $\mathcal{A}$.

Definition 5.3. A preobject (A, J) *generates* an object (A, I) if

(i) $J \subset I$;

(ii) for every object $(A, \bar{I})$ with $J \subset \bar{I}$, $I \subset \bar{I}$.

PROPOSITION 5.1. *Every preobject generates a unique object.*

Definition 5.4. Let $\mathbf{A} = (A, I)$ and $\mathbf{A}' = (A', I')$ be objects. *A* premap $F: \mathbf{A} \to \mathbf{A}'$ is a set of maps in $\mathcal{A}$ such that

(i) for every $f \in F$, f is a map in $\mathcal{A}$ from $A | U_f$ to A', U_f being an open subset of $L(A)$;

(ii) $\{U_f : f \in F\}$ is a covering of $L(A)$;

(iii) given $u \in I$, $f, g \in F$ and $x \in U_u \cap U_g$ such that $L(u)(x) \in U_f$, there exist $u', v' \in I'$ and a neighborhood V of x such that

$$u' \circ (U_{u'} | g | V) = v' \circ (U_{v'} | f | U_f) \circ (U_f | u | V). \tag{5.2}$$

F is a *map* if it is a premap and satisfies

(iv) if $g: A | U_g \to A'$ is a map in $\mathcal{A}$ such that for every $x \in U_g$ there exist $f \in F$, $u \in I$, u', $v' \in I'$ and a neighborhood V of x satisfying equation (5.2), then $g \in F$. Conditions (iii) and (iv) will be referred to as the *extension* and *closure axioms* for maps, respectively. A map F is *generated* by a premap G if $G \subset F$.

PROPOSITION 5.2. *Every premap generates a unique map.*

PROPOSITION 5.3. *Let $F:\mathbf{A}\to\mathbf{A}'$ and $F':\mathbf{A}'\to\mathbf{A}''$ be maps, and let H denote the set of all compositions $f'\circ f$ in $\mathcal{A}$, where $f\in F$ and $f'\in F'$. Then $H:\mathbf{A}\to\mathbf{A}''$ is a premap.*

Definition 5.5. In virtue of Propositions 5.2 and 5.3 we may define the composition $F'\circ F$ to be the map generated by H.

One now observes that if $\mathbf{A}=(A,I)$ is an object, $I:\mathbf{A}\to\mathbf{A}$ will be an identity map. Moreover, if A is an object in $\mathcal{A}$, $(A,\{1_A\})$ will be a preobject, and we will let $\hat{A}$ denote the object generated by $(A,\{1_A\})$. Similarly, if $f:A\to B$ is a map in $\mathcal{A}$, $\{f\}:\hat{A}\to\hat{B}$ will be a premap. The map generated by $\{f\}$ will be denoted by $\hat{f}$.

PROPOSITION 5.4. *The class of objects $\mathbf{A}$, $\mathbf{A}'$, ...; and maps $F:\mathbf{A}\to\mathbf{A}'$, together with the given law of composition, constitutes a category $\mathcal{A}^*$. The correspondence $A\to\hat{A}$, $f\to\hat{f}$ identifies $\mathcal{A}$ with a full subcategory of $\mathcal{A}^*$* (an identification which will henceforth be understood).

The next task is to define a functor $\mathbf{L}:\mathcal{A}\to C$ extending L. This is accomplished in two steps:

PROPOSITION 5.5. *If $\mathbf{A}=(A,I)$ is an object in $\mathcal{A}^*$, then $(L(A), L(I))$ is a C-preobject. Similarly, if F is a map in $\mathcal{A}^*$, then $L(F)$ is a C-premap. A covariant functor $\mathfrak{L}:\mathcal{A}^*\to C^*$ may be defined by taking $\mathfrak{L}(A)$ to be the object generated by $(L(A), L(I))$ and $\mathfrak{L}(F)$ the map generated by $L(F)$. Moreover, $\mathfrak{L}$ extends L.*

PROPOSITION 5.6. *Let $\mathbf{X}=(X,I)$ be an object in C^*, and let Q denote the set of pairs $(x,y)\in X\times X$ such that $u(x)=v(y)$ for some $(u,v)\in I\times I$. Then Q is an equivalence relation on X. Let $\mathfrak{P}(\mathbf{X})$ denote the (topological) quotient of X by Q, and $p:X\to\mathfrak{P}(\mathbf{X})$ the projection. Then $\{p\}:\mathbf{X}\to\mathfrak{P}(\mathbf{X})$ is a C-premap. Let $P(\mathbf{X})$ denote the map (in C^*) generated by $\{p\}$. If $F:\mathbf{X}\to\mathbf{X}'$ is a map in C^*, there exists a unique map $\mathfrak{P}(F)$ in C such that the diagram*

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "F"] \arrow[d, "P(\mathbf{X})"'] & \mathbf{X}' \arrow[d, "P(\mathbf{X}')"] \\
\mathfrak{P}(\mathbf{X}) \arrow[r, "\mathfrak{P}(F)"] & \mathfrak{P}(\mathbf{X}')
\end{tikzcd}$$

commutes. $\mathfrak{P}$ constitutes a covariant functor from C^* to C, extending the identity functor of C.

Definition 5.6. $\mathbf{L}=\mathfrak{P}\circ\mathfrak{L}$, where $\mathfrak{L}$ and $\mathfrak{P}$ are defined by Propositions 5.5. and 5.6.

Lastly, we need to extend the stroke-function:

PROPOSITION 5.7. *Let* $\mathbf{A} = (A, I)$ *be an object in* $\mathcal{A}^*$ *and* U *an open subset of* $\mathbf{L}(\mathcal{A}^*)$. *Let* $p: L(A) \to \mathbf{L}(\mathbf{A})$ *be the natural projection (defined in Proposition* 5.6), *and let*

$$i_{\mathbf{A}} \mid U = \{f \circ (i_{\mathbf{A}} \mid p^{-1}(U)) : f \in I\}.$$

Then
$$\mathbf{A} \mid U = (A \mid p^{-1}(U), i_{\mathbf{A}} \mid U)$$

is an object in $\mathcal{A}^*$ *and* $i_{\mathbf{A}} \mid U$ *a map from* $\mathbf{A} \mid U$ *to* $\mathbf{A}$.

PROPOSITION 5.8. *The category* $\mathcal{A}^*$, *together with the functor* $L: \mathcal{A}^* \to C$ *and the stroke-functions defined by Proposition* 5.7 *constitutes a local category extending* $\mathcal{A}$.

We turn next to the problem of extending functors defined on $\mathcal{A}$. A contravariant functor on $\mathcal{A}$ will admit a canonical extension to $\mathcal{A}^*$ provided it has certain sheaf-like properties. Such a functor S has been called a *sheaf on* $\mathcal{A}$ in [3], and it has been shown that every (set-valued) contravariant functor T on $\mathcal{A}$ generates a sheaf S. To begin with, we must recall(1) that a set-valued *presheaf* on a topological space X may be defined as a contravariant functor $G: c(X) \to E$, where $c(X)$ denotes he category consisting of the open subsets of X and inclusion maps, and E is the category of all sets and functions. If U, V are open subsets of X with $V \subset U$, and $s \in G(U)$, then $G(i_U \mid V)(s)$ may be called the *restriction* of s to V, and will be denoted by $s \mid V$. A *sheaf* on X is now a presheaf G satisfying two conditions:

(i) Let $\{U_i : i \in \mathfrak{J}\}$ be a family of open subsets of X and U their union. If for two elements s, $s' \in G(U)$, $s \mid U_i = s' \mid U_i$ for all $i \in \mathfrak{J}$, then $s = s'$.

(ii) Let $\{U_i : i \in \mathfrak{J}\}$ and U be given as before, and let $s_i \in G(U_i)$ for all $i \in \mathfrak{J}$. If $s_i \mid U_i \cap U_j = s_j \mid U_i = U_j$ for $i, j \in \mathfrak{J}$, then there exists an element $s \in G(U)$ such that $s \mid U_i = s_i$, $i \in \mathfrak{J}$.

Definition 5.7. Let $\mathcal{A}$ be a local category, A an object in $\mathcal{A}$ and $T_A: c(L(A)) \to \mathcal{A}$ the covariant functor defined by

$$T_A(U) = A \mid U, \quad U \subset L(A);$$

$$T_A(i_U \mid V) = i_{A \mid U} \mid V, \quad V \subset U \subset L(A).$$

A presheaf on $\mathcal{A}$ is a contravariant functor $S: \mathcal{A} \to E$, and a *sheaf on* $\mathcal{A}$ is a presheaf S on $\mathcal{A}$ such that, for every object A in $\mathcal{A}$, $S \circ T_A$ is a sheaf on $L(A)$. If S is a sheaf on $\mathcal{A}$, A an object in $\mathcal{A}$, $s \in S(A)$ and $U \subset L(A)$, then $S(i_A \mid U)(s)$ is the *restriction* of s to U (denoted by $s \mid U$).

(1) For basic facts regarding sheaves, we refer to Godement [8].

PROPOSITION 5.9. *Let S be a presheaf on $\mathcal{A}$. For every object A in $\mathcal{A}$, let G_A denote the sheaf on $L(A)$ generated by $S \circ T_A$, let $S^*(A) = G_A \circ L(A)$ and let $h(A): S(A) \to S^*(A)$ denote the natural map.*(1) *Given a map $f: A \to A'$ in $\mathcal{A}$, there exists a unique map $S^*(f): S^*(A') \to S^*(A)$ such that $S^*(f) \circ h(A') = h(A) \circ S(f)$. Moreover, S^* defines a sheaf on $\mathcal{A}$ and h a natural transformation from S to S^*.*

Definition 5.8. S^* is the sheaf on $\mathcal{A}$ *generated* by S.

Definition 5.9. Let S be a sheaf on $\mathcal{A}$ and $\mathbf{A} = (A, I)$ an object in $\mathcal{A}^*$. An element $s \in S(A)$ is *I-invariant* if, for every $f \in I$, $S(f)(s) = s \mid U_f$. The set of I-invariant elements $s \in S(A)$ will be denoted by $\mathbf{S}(\mathbf{A})$.

PROPOSITION 5.10. *Let $\mathcal{A}$ be a local category, S a sheaf on $\mathcal{A}$ and $F: \mathbf{A} \to \mathbf{A}'$ a map in $\mathcal{A}^*$. Given $s' \in \mathbf{S}(\mathbf{A}')$, there exists a unique element $s \in S(A)$ such that $S(f)(s') = s \mid U_f$ for all $f \in F$. Moreover, s lies in $\mathbf{S}(\mathbf{A})$. Let $\mathbf{S}(F)$ denote the map from $\mathbf{S}(\mathbf{A}')$ to $\mathbf{S}(\mathbf{A})$ taking s' to s. This defines a sheaf $\mathbf{S}$ on $\mathcal{A}^*$ extending S.*

Definition 5.10. $\mathbf{S}$ is the *canonical extension* of S to $\mathcal{A}^*$.

This completes our summary of the general extension theory. It needs to be pointed out that the category E may of course be replaced by certain algebraic categories, e.g., by the category dM of graded differential modules. If S is a dM-valued presheaf on $\mathcal{A}$, the generated sheaf S^* on $\mathcal{A}$ and its canonical extension $\mathbf{S}^*$ will likewise take values in dM. The singular integral cochains, for example, define a dM-valued presheaf on C. Let S denote the corresponding sheaf on C and $\mathbf{S}$ its canonical extension to C^* .If $\mathbf{X} = (X, I)$ is an object in C^*, $\mathbf{S}(\mathbf{X})$ will be a graded differential module, and we will let $\mathbf{H}^q(\mathbf{X})$ denote its cohomology groups. If $C^q(X)$ denotes again the group of singular integral q-cochains on X and $h: C^q(X) \to S^q(X)$ denotes the natural map (from sections of the presheaf to sections of the sheaf), it is easily verified that h commutes with the coboundary operator and takes I-invariant cochains to $\mathbf{S}(\mathbf{X})$. It consequently defines a homomorphism $\mathbf{h}: H^q(X, I) \to \mathbf{H}^q(\mathbf{X})$, where $H^q(X, I)$ denotes the cohomology of the I-invariant singular integral cochains on X. The cohomology $\mathbf{H}^q$ is clearly functorial on C^*, and will be called the *extended singular integral cohomology*.

6. The extended Euler class

Before proceeding to examine the particular extended categories with which we shall be concerned, a few remarks regarding the general extension theory may help to clarify the essential idea. An object $\mathbf{A}$ in $\mathcal{A}^*$ is by definition a pair (A, I), where A is an object in

(1) Since $A = T_A(L(A))$, therefore $S(A) = S \circ T_A(L(A))$, so that $h(A)$ is simply the natural map from sections of the presheaf to sections of the sheaf. Cf. Godement [8], pp. 109–112.

the original category $\mathcal{A}$ and I a set of local maps of A. However, instead of regarding $\mathbf{A}$ as simply the classical object A with additional structure (given by I), we propose an essentially opposite viewpoint whereby much of the structure of A itself will be regarded as nongeometric, i.e., as not belonging to $\mathbf{A}$. More precisely, we propose that only those properties of A which in some natural sense are *invariant under isomorphisms of* $\mathcal{A}^*$ shall be regarded as *geometric,* i.e., as belonging to the object $\mathbf{A}$. To formalize this viewpoint, it would of course be necessary to introduce some abstract concept of the *geometric object* of which (A, I) is a *representation*, and this might perhaps lead further to geometric maps and so forth. Needless to say, it will be preferable to avoid introducing further levels of abstraction and notational complexities. All that is really required is to observe that certain aspects of the pair (A, I) are geometric in our sense, while perhaps much of the structure is not. Two examples may serve to illustrate the point: (1) An isomorphism

$$F:(X,I)\rightarrow(X',I')$$

in C^* does not generally induce any point correspondence between the spaces X and X', so that these need not even have the same cardinality. For example, if I contains *all* local maps of X, then (X, I) will be isomorphic in C^* to a classical space consisting of one point. (2) Let $\mathcal{V}$ denote the local category of vector bundles and S_σ the contravariant functor on $\mathcal{V}$ which to every vector bundle ξ assigns the set of cross-section in ξ and to every bundle map $f:\xi\rightarrow\xi'$ the naturally induced map $S_\sigma(f):S_\sigma(\xi')\rightarrow S_\sigma(\xi)$. One verifies that S_σ is a sheaf on $\mathcal{V}$. Let $\mathbf{S}_\sigma$ denote the canonical extension of S_σ to $\mathcal{V}^*$. Then $\mathbf{S}_\sigma(\boldsymbol{\xi})$ is precisely the set of I-invariant cross-sections in ξ, where $\boldsymbol{\xi}=(\xi, I)$ denotes an arbitrary object in $\mathcal{V}^*$. Thus "cross-section" (as a geometric concept in $\mathcal{V}^*$) means *I-invariant cross-section.* Generally speaking, the extended local categories $\mathcal{A}^*$ are designed precisely for the purpose of studying those aspects of a classical structure which express themselves as geometric properties in $\mathcal{A}^*$. To lend further emphasis to this point, we shall henceforth refer to A as the *representative object* of $\mathbf{A}$ and I the set of *representative maps,* $\mathbf{A}=(A,I)$ being an arbitrary object in $\mathcal{A}^*$.

Let us now take a closer look at a general object $\boldsymbol{\xi}$ in the extended category $\mathcal{V}^*$ of vector bundles[1] in order to clarify the geometric aspects of its structure. The object $\boldsymbol{\xi}$ has, in the first place, a *dimension* (i.e., the dimension of ξ), and this is of course invariant under isomorphisms of $\mathcal{V}^*$. The functors $\mathfrak{L}$ and $\mathbf{L}$ (see Proposition 5.5 and Definition 5.6, respectively) provide $\boldsymbol{\xi}$ with a *base object* $\mathfrak{L}(\boldsymbol{\xi})$ belonging to C^* and a *basic space* $\mathbf{L}(\boldsymbol{\xi})$. Both reduce to the orindary base space when $\boldsymbol{\xi}$ lies in $\mathcal{V}$. Now let $T:\mathcal{V}\rightarrow C$ denote the (covariant) functor

[1] Strictly speaking we shall be concerned, in regard to the Euler class, with the extended local category of *oriented* vector bundles. But this distinction need not be made in the earlier part of the discussion.

representing the total space.[1] Replacing L by T in the constructions defining $\mathfrak{L}$ and $\mathbf{L}$, one obtains corresponding functors $\mathfrak{T}: \mathfrak{V}^* \to C^*$ and $\mathbf{T}: \mathfrak{V}^* \to C$, both of which extend T. Moreover, the natural transformation $\pi: T \to L$ representing the projection induces natural transformations $\pi: \mathfrak{T} \to \mathfrak{L}$ and $\boldsymbol{\pi}: \mathbf{T} \to \mathbf{L}$ of the extended functors. To see this more clearly, let us recall that $\mathfrak{L}(\boldsymbol{\xi}) = (L(\xi), \check{I})$, where $\check{I}$ contains the set $L(I)$ and possibly some additional maps brought in by the closure axiom (ξ being the representative bundle of $\boldsymbol{\xi}$ and I its set of representative maps). Similarly, $\mathfrak{T}(\boldsymbol{\xi}) = (T(\xi), \hat{I})$, with $T(I) \subset \hat{I}$. Clearly $\{\pi(\xi)\}: \mathfrak{T}(\boldsymbol{\xi}) \to \mathfrak{L}(\boldsymbol{\xi})$ is a premap, and one defines $\pi(\xi)$ to be the corresponding map in C^*. The spaces $\mathbf{L}(\boldsymbol{\xi})$ and $\mathbf{T}(\boldsymbol{\xi})$ are now defined as quotient spaces of $L(\xi)$ and $T(\xi)$, respectively. More precisely, let two points $x, y \in L(\xi)$ be called *$L(I)$-related* if there exist $f, g \in L(I)$ such that $f(x) = g(y)$.[2] As a consequence of the extension axiom, this constitutes an equivalence relation on $L(\xi)$, and $\mathbf{L}(\boldsymbol{\xi})$ is the corresponding quotient. A *$T(I)$-relation* on $T(\xi)$ is defined in a precisely analogous way, and $\mathbf{T}(\boldsymbol{\xi})$ is the resulting quotient. Since $T(I)$-related points project to $L(I)$-related points under $\pi(\xi)$, the latter induces a map $\boldsymbol{\pi}(\boldsymbol{\xi})$ of the quotients.

Let us now write $\xi = (B, X, \pi)$ as before, and abbreviate $\boldsymbol{\pi}(\boldsymbol{\xi})$, $\mathbf{T}(\boldsymbol{\xi})$, and $\mathbf{L}(\boldsymbol{\xi})$ by $\bar{\pi}$, $\bar{B}$ and $\bar{X}$, respectively. Let $\hat{p}: B \to \bar{B}$ and $\check{p}: X \to \bar{X}$ denote the natural projections. It is easily verified that two points on a given fiber B_x in ξ will be $T(I)$-related if and only if they correspond under a transformation in Φ_x (the holonomy group at x). But this shows that B_x will induce a vector space structure on $\hat{p}(B_x)$ precisely when Φ_x is trivial. Since $\hat{p}(B_x) = \bar{\pi}^{-1}(\check{p}(x))$, one sees that ξ will induce a bundle structure on $\bar{\xi} = (\bar{B}, \bar{X}, \bar{\pi})$ if and only if (ξ, I) has trivial holonomy, in which case $p = (\hat{p}, \check{p})$ will evidently be a bundle map from ξ to $\bar{\xi}$, and this will generate a map $P: \boldsymbol{\xi} \to \bar{\xi}$ in $\mathfrak{V}^*$. We may summarize this observation in the form of

THEOREM 6.1. *An object $\boldsymbol{\xi}$ in $\mathfrak{V}^*$ admits a natural quotient $\bar{\xi}$ in $\mathfrak{V}$ if an donly if $\boldsymbol{\xi}$ has trivial holonomy.*

It is of interest to note that the *only if* part of Theorem 6.1. follows from a much sharper result regarding the behavior of holonomy with respect to maps in $\mathfrak{V}^*$, viz.

THEOREM 6.2. *Let $F: \boldsymbol{\xi} \to \boldsymbol{\xi}'$ be a map in $\mathfrak{V}^*$ with $\boldsymbol{\xi} = (\xi, I)$ and $\boldsymbol{\xi}' = (\xi', I')$, and let*

$$(x, x') \in L(\xi) \times L(\xi')$$

correspond under $L(f)$, where $f \in F$. The map f induces then an isomorphism $\varphi_f: \Phi_x \to \Psi_f$, where Ψ_f is a subgroup of the holonomy group $\Phi'_{x'}$. Moreover, the conjugacy class of Ψ_f in $\Phi'_{x'}$ is independent of f.

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(1) In terms of our previous notation, T takes ξ to B and u to $\hat{u}$.

(2) The same relation is defined if $L(I)$ is replaced by $\check{I}$.

To prove this result, it will be convenient to let $|I|,|F|$ and $|I'|$ denote the sets of fiber maps induced by the bundle maps of I, F and I', respectively. We assert now that every $\gamma_x \in \Phi_x$ can be written in the form $\gamma_x = h_2^{-1} \circ h_1$, where $h_1, h_2 \in |I|$. For since $|I|$ contains the identities, one may assume

$$\gamma_x = h_p^{-1} \circ \ldots \circ h_3 \circ h_2^{-1} \circ h_1. \tag{6.1}$$

But by the extension axiom there exist $\bar{h}_2, \bar{h}_3 \in |I|$ such that $\bar{h}_2 \circ h_2 = \bar{h}_3 \circ h_3$, so that $h_3 \circ h_2^{-1} = \bar{h}_3^{-1} \circ \bar{h}_2$. Consequently the number of factors on the right side of equation 6.1 may be reduced by 2, and proceeding in this manner one can achieve the case $p=2$.

Given $f \in F$ such that $\check{f}(x) = x'$ and $\gamma_x \in \Phi_x$, we may define $\varphi_f(\gamma_x) = f_x \circ \gamma_x \circ f_x^{-1}$. To show that $\varphi_f(\gamma_x) \in \Phi'_{x'}$, we choose $g \in |F|$ such that $g \circ h_1$ is defined, which is possible by condition (ii) of Definition 5.4. By the extension axiom for maps there exist $k_i, l_i \in |I'|$ such that

$$k_i \circ f_x = l_i \circ g \circ h_i, \quad i = 1, 2.$$

But this implies

$$\varphi_f(\gamma_x) = k_2^{-1} \circ l_2 \circ l_1^{-1} \circ k_1,$$

so that $\varphi_f(\gamma_x) \in \Phi'_{x'}$. It is now obvious that $\varphi_f : \Phi_x \to \Phi'_{x'}$ is a monomorphism, and we take Ψ_f to be its image. Now suppose $\bar{f} \in F$ is a second map such that $L(\bar{f})$ takes x to x', and let $\varphi_{\bar{f}}(\gamma_x) = \bar{f}_x \circ \gamma_x \circ \bar{f}_x^{-1}$. By the extension axiom for maps, there exist $s, \bar{s} \in |I'|$ such that $s \circ f_x = \bar{s} \circ \bar{f}_x$. Consequently

$$\varphi_{\bar{f}}(\gamma_x) = (\bar{s}^{-1} \circ s) \circ \varphi_f(\gamma_x) \circ (\bar{s}^{-1} \circ s)^{-1}$$

proving that Ψ_f and $\Psi_{\bar{f}}$ are conjugate by $(\bar{s}^{-1} \circ s) \in \Phi'_{x'}$.

As immediate consequences one has

COROLLARY 1. *When F is an isomorphism in $\mathfrak{V}^*$, Φ_x and $\Phi'_{x'}$ are isomorphic* (as abstract groups).

COROLLARY 2. *When ξ' lies in $\mathfrak{V}$, ξ must have trivial holonomy.*

Our construction also implies

COROLLARY 3. *The preimage of an I'-invariant vector under a fiber map $f \in |F|$ is I-invariant.*

It is important to note that the fiber map f foes not in general take I-invariant vectors to I'-invariant ones. Maps F in $\mathfrak{V}^*$ for which this always holds will henceforth be referred to as *invariance preserving* maps. One sees that the objects in $\mathfrak{V}^*$, together with all invariance preserving maps, give rise to a subcategory $\mathfrak{V}^{*\dagger}$.

Theorem 6.2 and its corollaries describe the fundamental facts regarding holonomy in $\mathfrak{V}^*$. It is evident that the holonomy groups modulo inner automorphisms constitute geometric invariants of $\boldsymbol{\xi}$ associated with points of the basic space $\mathbf{L}(\boldsymbol{\xi})$.

This may suffice as a preliminary orientation regarding the generalized vector bundles. The *extended Euler class* will be a function $\boldsymbol{\chi}$, defined on a subcategory $\mathcal{E}$ of $\mathfrak{V}^*$, which to every object $\boldsymbol{\xi}$ assigns an element $\boldsymbol{\chi}(\boldsymbol{\xi}) \in \mathbf{H}^n \circ \mathfrak{L}(\boldsymbol{\xi})$, where n denotes the dimension of $\boldsymbol{\xi}$ and $\mathbf{H}^q$ the extended singular integral cohomology. Since no adequate analysis has been given for $n > 2$, it will be appropriate to illustrate the category-theoretic aspects of the problem by considering the case $n = 2$. In accordance with the analysis of Section 4 we will define a subcategory $\mathcal{E}_2$ whose objects are precisely the 2-dimensional objects in $\mathfrak{V}^*$ satisfying condition (4.1). Let $\boldsymbol{\xi} = (\xi, I)$ be such an object, and let $\check{I}$ denote the set of representative maps of $\mathfrak{L}(\boldsymbol{\xi})$. Theorem 4.1 defines an element $\chi \in H^2(L(\xi), L(I))$, where the notation $H^q(X', I')$ signifies the cohomology of I'-invariant singular integral cochains in X'. Since $\check{I}$ contains $L(I)$ and only such additional maps as are brought in by the closure axiom, one sees that cochains invariant under maps of $L(I)$ will be likewise invariant with respect to $\check{I}$. Consequently one has $\chi \in H^2(L(\xi), \check{I})$. The natural projection $\mathbf{h}$ (see end of Section 5) takes χ to an element of $\mathbf{H}^2(\mathfrak{L}(\boldsymbol{\xi}))$, and this will be $\boldsymbol{\chi}(\boldsymbol{\xi})$. Before completing the definition of $\mathcal{E}_2$, it will be of interest to establish

THEOREM 6.3. *Let $\boldsymbol{\xi}, \boldsymbol{\xi}'$ be objects in $\mathcal{E}_2$ and $F: \boldsymbol{\xi} \to \boldsymbol{\xi}'$ an invariance preserving map in $\mathfrak{V}^*$. Then $\mathbf{H}^2 \circ \mathfrak{L}(F)$ maps $\boldsymbol{\chi}(\boldsymbol{\xi}')$ to $\boldsymbol{\chi}(\boldsymbol{\xi})$.*

Let S denote the dM-valued sheaf of singular integral cochains on C and $\dot{\mathrm{S}}$ its canonical extension to C^* (see end of Section 5). If $\boldsymbol{\xi} = (\xi, I)$ and $\xi = (B, X, \pi)$, $\mathrm{S} \circ \mathfrak{L}(\boldsymbol{\xi})$ will be the subcomplex of $L(I)$-invariant elements in $S(X)$. Now let $\tilde{K}^0$ denote again the singular simplicial complex in B whose 0-skeleton lies in B^* and 1-skeleton in $\dot{B}$. The integral cochains on $\tilde{K}^0$ clearly define a dM-valued presheaf over X. In other words, if U is an open subset of X, one takes $G(U)$ to be the graded differential group of integral cochains on the restricted complex $\tilde{K}^0 | U$, and if $i: V \to U$ is an inclusion, $G(i)$ will be the corresponding restriction map. Let $\tilde{S}$ denote the sheaf over X generated by G. A subcomplex $\tilde{\mathrm{S}}$ of $\tilde{S}(X)$ consisting of $T(I)$-invariant elements (analogous to $\mathrm{S} = \mathrm{S} \circ \mathfrak{L}(\boldsymbol{\xi})$) may be defined in an obvious way, and one observes that the dM-homomorphism $\boldsymbol{\pi}^*: S(X) \to \tilde{S}(X)$ induced by the projection $\pi: B \to X$ maps S to $\tilde{\mathrm{S}}$.

Corresponding definitions apply to the second object $\boldsymbol{\xi}'$, and it will be convenient to distinguish quantities relating to $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ by applying the subscripts 1 and 2, respectively. The map $F: \boldsymbol{\xi}_1 \to \boldsymbol{\xi}_2$ induces now a map $\mathrm{S}_F = \mathrm{S} \circ \mathfrak{L}(F): \mathrm{S}_2 \to \mathrm{S}_1$. Moreover, since F is invariance preserving, $f \circ \sigma$ will belong to $\tilde{K}^0_{(2)}$ whenever $\sigma \in \tilde{K}^0_{(1)}$ and $f \in F$. One again finds, therefore, that F induces a map $\tilde{\mathbf{S}}_F: \tilde{\mathbf{S}}_2 \to \tilde{\mathbf{S}}_1$,(1) and this gives a commutative diagram

$$\begin{tikzcd} \tilde{\mathbf{S}}_2 \arrow[r, "\tilde{S}_F"] & \tilde{\mathbf{S}}_1 \\ \mathbf{S}_2 \arrow[u, "\pi_2^*"] \arrow[r, "S_F"] & \mathbf{S}_1 \arrow[u, "\pi_1^*"'] \end{tikzcd}$$

Let $\tilde{\varepsilon}_i$ denote again the cocycle in $C^2(\tilde{K}^0_{(i)})$ previously considered in Section 4, and let $\tilde{\boldsymbol{\varepsilon}}_i$ denote its projection to $\tilde{S}(X)$, for $i=1,2$. It follows by the well-known invariance of ε under bundle maps that $\tilde{\boldsymbol{\varepsilon}}_i \in \tilde{\mathbf{S}}_i$ and $\tilde{\mathbf{S}}_F(\tilde{\boldsymbol{\varepsilon}}_2) = \tilde{\boldsymbol{\varepsilon}}_1$.

At this point we must recall the cochain homomorphisms $\varphi_i^*: C^q(\tilde{K}^0_{(i)}) \to C^q(K_{(i)})$ defined by the constructions in our proof of Theorem 4.1. They induce maps $\boldsymbol{\varphi}_i^*: \tilde{S}_i(X) \to S_i(X)$, and we let $\boldsymbol{\lambda}_i = \boldsymbol{\varphi}_i^*(\tilde{\boldsymbol{\varepsilon}}_i)$, $i=1$, 2. Since the corresponding cochains λ_i are I_i-invariant (by Lemma 4.3), it follows that $\boldsymbol{\lambda}_i \in \mathbf{S}_i$. By commutativity of diagram (6.2) one now has $\boldsymbol{\pi}_1^*(\boldsymbol{\lambda}') = \tilde{\boldsymbol{\varepsilon}}_1$, where $\boldsymbol{\lambda}' = S_F(\boldsymbol{\lambda}_2)$. Moreover, it is readily verified that the cohomology classes of $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}'$ (with respect to the complex $\mathbf{S}_1$) are precisely $\boldsymbol{\chi}(\boldsymbol{\xi}_1)$ and the image of $\boldsymbol{\chi}(\boldsymbol{\xi}_2)$ under $H^2 \circ \mathfrak{C}(F)$, respectively. It therefore remains to show that $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}'$ represent the same cohomology class with respect to $\mathbf{S}_1$.

Since henceforth only one bundle will be involved in the discussion, we may drop the subscript 1 and consider $\tilde{\boldsymbol{\varepsilon}} \in \tilde{S}(X)$; $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in S(X)$,

$$\boldsymbol{\pi}^*: S(X) \to \tilde{S}(X) \text{ and } \boldsymbol{\varphi}^*: \tilde{S}(X) \to S(X),$$

with $\boldsymbol{\lambda} = \boldsymbol{\varphi}^*(\tilde{\boldsymbol{\varepsilon}})$ and $\boldsymbol{\pi}^*(\boldsymbol{\lambda}') = \tilde{\boldsymbol{\varepsilon}}$. We must show that there exists $\boldsymbol{\nu}' \in \mathbf{S}^1$ such that $\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \boldsymbol{\delta}\boldsymbol{\nu}'$, where $\boldsymbol{\delta}$ denotes the coboundary homomorphism in dM. But this problem is entirely analogous to the uniqueness question previously considered in the proof of Theorem 4.1. Consequently let α, β denote the chain homomorphisms defined in Section 4 and let D' denote the chain homotopy of Lemma 4.2. The cochain homotopy $[\alpha \circ D' \circ \beta]^*$ induces now a corresponding homotopy $\mathbf{D}': \mathbf{S}(X) \to \mathbf{S}(X)$, and one obtains

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \boldsymbol{\delta}\mathbf{D}'(\boldsymbol{\lambda}')$$

(1) Cf. Clifton and Smith [3], p. 448.

corresponding to equation (4.4). Since D' is invariant on dimension 1, $\mathbf{D}'$ maps S^2 to S^1, so that $\mathbf{D}'(\boldsymbol{\lambda}')\in S^1$ as was to be proved.

Theorem 6.3 asserts that $\boldsymbol{\chi}$ is natural with respect to invariance preserving maps. Since this condition was essential to our argument, it will be appropriate to define $\mathcal{E}_2$ as the full subcategory of $\mathfrak{V}^{*\dagger}$ determined by the given class of objects.

7. Foliations(1)

In the differentiable case, a *p-dimensional foliation* $\mathfrak{F}$ of an n-dimensional manifold X (with $0<p<u$) is equivalent to an involutive distribution(2) of p-planes on X. The maximal integral varieties of the distribution are called *leaves of* $\mathfrak{F}$, and we will denote by $\mathfrak{F}_x$ the unique leaf of $\mathfrak{F}$ through a given point $x\in X$. Moreover, we shall be concerned only with oriented foliations, i.e., it will be assumed that X and the distribution are both oriented. An open subset U of X will be called a *distinguished neighborhood* with respect to $\mathfrak{F}$ if there exists a diffeomorphism $\varphi: R^p\times R^{n-p}\to U$ (R being the space of real numbers) such that, for every $y_0\in R^{n-p}$, the points $\{\varphi(x,y_0): x\in R^p\}$ lie on one leaf of $\mathfrak{F}$. It is clear that the distinguished neighborhoods cover X. If U is a distinguished neighborhood and λ a leaf of $\mathfrak{F}$ meeting U, then $U\cap\lambda$ consists of one or more connected p-dimensional varieties, sometimes known as *plaques*. Let us employ the term *coplaque* to designate an $(n-p)$-dimensional variety in U which meets each plaque in precisely one point. Thus if φ is a diffeomorphism as above, the set $\{\varphi(x_0,y): y\in R^{n-p}\}$ defines a coplaque in U. A foliation is called *regular* if every point $x\in X$ admits a distinguished neighborhood U such that no two plaques of U lie on the same leaf.

A foliation $\mathfrak{F}$ on X determines a generalized vector bundle (object in $\mathfrak{V}^*$) in the following manner: Annihilating the p-planes of $\mathfrak{F}$ in the tangent bundle of X, one obtains a vector bundle $\xi=(B,X,\pi)$ of dimension $(n-p)$. Moreover, an orientation of $\mathfrak{F}$ will induce an orientation of ξ. One observes now that if U is a distinguished neighborhood in X and V a coplaque in U, the natural projection $p: U\to V$ (taking x to the point in V on the plaque through x) admits a unique lifting to a bundle map $u:\xi\,|\,U\to\xi$. If J denotes the set of all bundle maps u arising in this way, then (ξ, J) will be a preobject, and this generates an object $\boldsymbol{\xi}$ in $\mathfrak{V}^*$.

Regarding the geometric significance of direction fields in $\boldsymbol{\xi}$(3) we have commented in

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(1) The original monograph on this subject is Reeb [11]. A survey of recent results may be found in Haefliger [9].

(2) Cf. Chevalley [1].

(3) I.e., J-invariant direction fields in ξ.

Section 1. It should also be noted that the basic space $\mathbf{L}(\boldsymbol{\xi})$ is precisely the set of leaves, and by Theorem 6.1, this carries a natural bundle structure $\bar{\xi}$ if and only if $\boldsymbol{\xi}$ has trivial holonomy. When $\mathfrak{F}$ is regular (which implies trivial holonomy), the natural projection $P:\boldsymbol{\xi}\to\bar{\xi}$ will be an isomorphism in $\mathfrak{V}^*$. For if $\mathfrak{F}$ is regular, every point of $\mathbf{L}(\boldsymbol{\xi})$ has a neighborhood $\bar{U}$ which can be lifted to a coplaque V in X by a homeomorphism $s:\bar{U}\to V$, and moreover, s admits a unique lifting to a bundle map $f:\bar{\xi}\,|\,\bar{U}\to\xi$. It requires some construction to verify that the set F of all such bundle maps f satisfies the extension axiom (as a premap from $\bar{\xi}$ to $\boldsymbol{\xi}$). F therefore gnerates a map $P':\bar{\xi}\to\boldsymbol{\xi}$, and this will be an inverse of P. The fact that $P'\circ P=I$ (the identity of $\boldsymbol{\xi}$) depends of course on the circumstance that I contains maps $u:\xi\,|\,U\to\xi$ with the property that $\check{u}$ projects U onto a coplaque V.(1) Conversely, analogous considerations indicate that $\mathfrak{F}$ will be regular whenever P is an isomorphism. For arbitrary $\mathfrak{F}$, the object $\boldsymbol{\xi}$ represents therefore a natural generalization of the classical vector bundle $\bar{\xi}$ associated with a regular foliation.

A simple example may now serve to illustrate the geometric significance of the extended Euler class in the context of foliation theory. Take X to be an oriented Euclidean space of dimension 3, let (x, y, z) denote Cartesian coordinates on X and let (r, θ, z) denote cylindrical coordinates, defined by

$$x=r\cos\theta, \quad y=r\sin\theta.$$

We will let $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $(\partial/\partial r, \partial/\partial\theta, \partial/\partial z)$ denote the natural frame fields associated with the two coordinate systems, respectively, bearing in mind that the tangent vectors $\partial/\partial r$ and $\partial/\partial\theta$ are defined only for $r>0$. The formula

$$\mathbf{v}=\cos r\ \partial/\partial z+\sin r\ \partial/\partial\theta$$

defines now a vector field $\mathbf{v}$ on X, and this determines a 1-dimensional oriented foliation $\mathfrak{F}$. The leaves $\mathfrak{F}_p$ (as a function of $p\in X$) are easily described: When p lies on the z-axis, $\mathfrak{F}_p$ coincides with the z-axis. Otherwise let $p=(\bar{r},\bar{\theta},\bar{z})$. For $0<\bar{r}<\pi/2$, $\mathfrak{F}_p$ will be a helix about the z-axis; for $\bar{r}=\pi/2$, a circle given by $r=\bar{r}$, $z=\bar{z}$; and so forth. Actually, only the region $0\leqslant r<\varrho$ for some $\varrho>\pi/2$ will be of interest. It should be observed that the behavior of the helical leaves near the cylinder $r=\pi/2$ causes $\mathfrak{F}$ to be nonregular.

We assert that the generalized vector bundle $\boldsymbol{\xi}$ associated with $\mathfrak{F}$ belongs to the subcategory $\mathcal{E}_2$. Since $\boldsymbol{\xi}$ clearly has dimension 2 (the codimension of $\mathfrak{F}$), it remains to

(1) These maps u are of course not invertible. When the representative maps of an object (in an extended category) are all invertible, its identity reduces to a pseudo-group on the representative object. In general, the extension and closure axioms are needed to replace the restrictive condition of a pseudo-group.

examine the holonomy groups Φ_p. Since these must be trivial unless $\mathfrak{F}_p$ is a circular leaf,(1) we must look at points on the cylinder $r=\pi/2$, e.g., the point p with Cartesian coordinates $(\pi/2,\ 0,\ 0)$. We recall that Φ_p operates on the fiber B_p over p in the representative bundle $\xi=(B,X,\pi)$ of $\boldsymbol{\xi}$. Since $\mathbf{v}(p)=(\partial/\partial y)_p$, one may identify B_p with the subspace of X_p spanned by $(\partial/\partial x,\ \partial/\partial z)_p$, where X_p denotes the tangent vector space to X at p. One now observes that for $\delta>0$ sufficiently small,

$$V=\{(x,0,z):(x,z)\in R^2 \text{ and } |x-\pi/2|<\delta\}$$

constitutes a coplaque through p. Clearly every helical leaf of $\mathfrak{F}$ meeting V at a point $(x,0,z)$ reenters again at points of the form $(x,0,z+n\pi^2 \cot x)$, n being an integer, and this implies by an elementary calculation that Φ_p is infinite cyclic with generator given by the matrix

$$\begin{pmatrix} 1 & \pi^2 \\ 0 & 1 \end{pmatrix}$$

relative to the basis $(\partial/\partial x,\ \partial/\partial z)_p$ of B_p. Thus $(\partial/\partial z)_p$ constitutes an I-invariant vector in B_p, I being the identity map of $\boldsymbol{\xi}$. Condition (4.1) is therefore satisfied, and $\boldsymbol{\xi}$ belongs to $\mathcal{E}_2$.

Since the computational aspects of the extended cohomology theory are as yet undeveloped, an actual calculation of the groups $\mathbf{H}^q \circ \mathfrak{L}(\boldsymbol{\xi})$ would not be feasible at this point. Instead, we will show by a direct argument that $\boldsymbol{\chi}(\boldsymbol{\xi})$ does not vanish. Let the singular simplicial complexes K and $\tilde{K}^0$ be defined as before, and let l be a circular leaf of $\mathfrak{F}$ on the cylinder $r=\pi/2$. The leaf l gives rise to a cycle $\mathbf{c}\in C_2(K/I)$ in the following way: Let $\sigma:\Delta_2\to X$ be a singular simplex such that $\sigma|\dot{\Delta}_2$ maps $\dot{\Delta}_2$ onto l with degree $+1$.(2) Let p_i denote the point on l corresponding to the ith vertex of Δ_2, $i=0,1,2$; and let $\sigma':\Delta_2\to X$ denote the singular simplex mapping Δ_2 on p_0. Since each of the segments p_ip_j of l can clearly be mapped to p_0 by $\check{u}$ for some $u\in I$, the chain $c=\sigma-\sigma'$ will correspond to a cycle $\mathbf{c}$ in $C_2(K/I)$. Let χ denote the cohomology class in $H^2(K/I)$ determined by Theorem 4.1. We recall that χ derives from a cocycle $\lambda=\varphi^*(\tilde{\varepsilon})$, where $\varphi: C_q(K)\to C_q(\tilde{K}^0)$ represents a lifting operation applied to the elements $\sigma\in K$. It is important to note that if this lifting operation has been carried out on a subcomplex K' of K, it can be extended to K. The period $\chi(\mathbf{c})$ is now given by $\lambda(c)$, and moreover, $\lambda(c)=\varepsilon(\tilde{c})$ for an admissible lifting $\tilde{c}$ of c. A lifting of σ' to $\tilde{K}^0$ is defined by taking

$$\sigma'(w)=(\partial/\partial z)_{p_0}, \quad w\in\Delta_2;$$

(1) It is well known for general foliations that the holonomy groups are trivial on simply-connected leaves. Cf. Reeb [1].

(2) This simply means that if p traverses $\dot{\Delta}_2$ once in a positive sense, $\sigma(p)$ will traverse l in the same way.

and one can certainly find a lifting $\tilde{\sigma}$ of σ such that

$$\tilde{\sigma}(w) = (\partial/\partial z)_{\sigma(w)}, \quad w \in \dot{\Delta}_2.$$

This gives an admissible lifting $\tilde{c} = \tilde{\sigma} - \tilde{\sigma}'$ of c, and since $\varepsilon(\tilde{\sigma}')$ is obviously zero, one has $\chi(c) = \varepsilon(\tilde{\sigma})$. Now suppose $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthogonal frame field on X with $\mathbf{e}_3 = \mathbf{v}$. This implies that $(\partial/\partial z)_p$ must lie in the plane spanned by $(\mathbf{e}_1, \mathbf{e}_2)$ for p on l, and one may verify that $\varepsilon(\tilde{\sigma})$ is precisely the winding number of $(\partial/\partial z)_p$ with respect to $(\mathbf{e}_1, \mathbf{e}_2)$ as p traverses l in a positive sense. It is not difficult, of course, to find a frame field with the desired property. Since the vector field

$$\mathbf{e} = -\sin r\, \partial/\partial z + \cos r\, \partial/\partial \theta,$$

defined for $r > 0$, is orthogonal to $\mathbf{v}$ and $\partial/\partial r$, our problem is to express $\mathbf{e}_1$ and $\mathbf{e}_2$ as linear combinations of $\mathbf{e}$ and $\partial/\partial r$ in such a way that $(\mathbf{e}_1, \mathbf{e}_2)$ may be continuously extended to the z-axis. Setting

$$\mathbf{e}_1 = \cos\theta\, \partial/\partial r - \sin\theta\, \mathbf{e} \tag{7.1}$$

$$\mathbf{e}_2 = \sin\theta\, \partial/\partial r + \cos\theta\, \mathbf{e}$$

will be admissible, for clearly $\mathbf{e}_1 \to \partial/\partial x$ and $\mathbf{e}_2 \to \partial/\partial y$ as $r \to 0$. But $\mathbf{e} = -(\partial/\partial z)$ on l, so that equations (7.1) give $\varepsilon(\tilde{\sigma}) = -1$.

This proves that χ does not vanish. The fact that the extended Euler class $\boldsymbol{\chi}$ must also be nonzero may now be established by a standard sheaf-theoretic consideration. The idea of the argument is this:(1) The cohomology class $\boldsymbol{\chi}$ derives from a section $\boldsymbol{\lambda}$ in the sheaf of cochains over X corresponding to the cocyle λ (see above). The vanishing of $\boldsymbol{\chi}$ would imply that $\boldsymbol{\lambda}$ cobounds a section $\boldsymbol{\alpha}$ which is I-invariant in the natural sense. But the canonical map from cochains to sections is always surjective when the underlying space is paracompact,(2) so that $\boldsymbol{\alpha}$ derives from a cochain α. There now exists an open covering $\mathcal{U}$ of X such that, with respect to singular simplexes subordinate to $\mathcal{U}$, α is I-invariant in the usual sense and $\lambda = \delta\alpha$. For a sufficiently fine barycentric subdivision c^* of the chain c (defined above), one has therefore $\lambda(c^*) = \delta\alpha(c^*)$. Since c^* projects to a cycle in $C_2(K/I)$, this implies $\lambda(c^*) = 0$. Similarly, by the fact that c also projects to a cycle in $C_2(K/I)$ and λ is I-invariant, one may conclude that $\lambda(c) = \lambda(c^*)$. But this contradicts the result of the preceding calculation, thus establishing that $\boldsymbol{\chi}$ is nonzero.

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(1) For an arbitrary sheaf S on X, the elements of $S(X)$ may be identified with sections in an associated *espace étalé* $p: E \to X$. If S is generated by a presheaf G, a point in $p^{-1}(x)$ is precisely the germ of an element $s \in G(U)$ at x, U being a neighborhood of X. Cf. Godement [8].

(2) Cf. Godement [8], p. 160.

References

[1]. Chevalley, C., *Theory of Lie groups*, I. Princeton University Press, 1946.

[2]. Clifton, Y. H. & Smith, J. W., The category of topological objects. *Proc. Nat. Acad. Sci. U.S.A.*, 47 (1961), 190–195.

[3]. ——, Topological objects and sheaves. *Trans. Amer. Math. Soc.*, 105 (1962), 436–452.

[4]. ——, The Euler class as an obstruction in the theory of foliations. *Proc. Nat. Acad. Sci. U.S.A.*, 50 (1963), 949–954

[5]. Eilenberg, S., Singular homology theory. *Ann. of Math.*, 45 (1944), 407–447.

[6]. ——, *Foundations of fiber bundles*. Lecture notes, University of Chicago, 1957.

[7]. Eilenberg, S. & Steenrod, N., *Foundations of algebraic topology*. Princeton University Press, 1952.

[8]. Godement, R., *Théorie des faisceaux*. Hermann, Paris, 1958.

[9]. Haefliger, A., Variétés feuilletées. *Ann. Scuola Norm. Sup. Pisa*, III, 106 (1962), 367–397.

[10]. Milnor, J., *Lectures on characteristic classes*. Notes by J. Stasheff, Princeton University, 1957.

[11]. Reeb, G., *Sur certaines propriétés topologique des variétés feuilletées*. Hermann, Paris, 1952.

[12]. Steenrod, N., *The topology of fiber bundles*. Princeton University Press, 1951.

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