ON THE BOUNDARY THEORY FOR MARKOV CHAINS. II

BY

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§ 11. Introduction

This paper may be regarded as a new and fairly self-contained one attached to §§ 2–4 of [I].(2) These sections are entitled "Terminology and notation", "The boundary" and "Fundamental theorems" respectively. The rest of [I] is either contained in a more general treatment (§ 5, § 9 and parts of § 6), or may be set aside as special cases under additional hypotheses (parts of § 6, § 7 and § 8). In particular the whole idea of "dual boundary" is dispensed with here, though this is not to say it should be abandoned forever. References to [I] beyond § 4 will be pinpointed.

In sum, the case of a finite number of passable atomic boundary points (briefly: "exits") will be settled here. Namely: all homogeneous Markov chains satisfying Assumptions A and B′ [I; p. 25 and p. 50] will be completely analyzed, with regard to the stochastic behavior of the sample functions as well as the analytical structure of transition probabilities, in fact both at the same time. To be exact, it will also be assumed that:

ASSUMPTION C_1. *All Φ-recurrent states are merged into one absorbing state.*

ASSUMPTION D. *All exits are distinguishable.*

It is important to note the difference between C_1 above and the erstwhile Assumption C [I; p. 47] which would require the absence of any $\prod$-recurrent state and is a serious restriction. On the contrary, conditions C_1 and D may be justly regarded as unessential for the boundary theory; see respectively the discussion at the end of § 15 here and on p. 38 of [I].

A culminating result of the theory has been that of "complete construction", origi-

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(2) References in roman capitals are listed at the end of the paper; references [1] to [14] are to be found at the end of [I].

nated with Feller [9], developed through Neveu [11],(1) and established first by David Williams [VIII] under substantially the same conditions as Theorem 16.1 here, in a somewhat different form and with a totally different, purely analytic method. This result falls into two parts. The first, to be called "decomposition" here, is an analysis of the basic transition matrix (or its Laplace transform) by decomposing it into several components: the exits z, the entrances η, and a connecting matrix M (see § 16 for these symbols). The second, to be called "construction", consists in showing that any such transition matrix can be so put together from similar components chosen rather arbitrarily but subject to certain analytical conditions. Now it should be apparent that the decomposition is in general not unique without further conditions on the choices (such as choosing both the exits and entrances to be "extreme bases" as in Feller's case), and without uniqueness the two parts of the theorem are not really in direct correspondence. Thus, if a process is constructed and then decomposed, the original components used in the construction are not necessarily thereby retrieved. To put it in another way, in an arbitrary construction the various components need not have the meanings attached to those in a meaningful decomposition, although the corresponding (and cognate) parts look quite like each other formally. To see that this question is not an academic one, consider the following problem: from a given process, to construct a new one by stopping it at certain specified exits (see § 18 for a precise formulation and solution). Obviously, this problem cannot be solved by another construction using only the "through" exists, because the corresponding entrances can no longer be chosen arbitrarily. Rather, one must begin with a correct decomposition of the original process, and then shut off the properly identified entrances.

Such a decomposition will be called "canonical" and it will be derived by the most natural probabilistic considerations.(2) As a matter of fact, the canonical form conceals a more fundamental resolution into elements which are simpler to define and easier to use. These are the probabilities ϱ^a and F^{ab} introduced and studied in § 14. Each ϱ^a is then linked to an entrance law η^a through a measure E^a (Theorem 14.4), the meaning of which is given in § 17. The canonical decomposition itself, in these two stages, is given in Theorem 15.2.

For a fuller understanding of the stochastic as well as analytic structure of the process, however, we must consider a third problem, that of "identification", to be taken up in

(1) It should be pointed out that Neveu's results do not seem to include Feller's since Theorem 4.2.1 of [11] requires, besides "absolute dominance", also e.g. that the cone of entrance laws *relative to* Π (*rather than* Φ in our notation) be of finite dimension. This is a quite different type of assumption from those made by all the other authors.

(2) Observe, *inter alia*, that in the form given here the substochastic case becomes an easy extension of the stochastic one, and that each entrance law is generated by an entrance sequence (excessive measure relative to Φ).

§§ 17–18. Here the major results are given in Theorems 17.1 and 17.3. In contrast to some prior developments, these no longer appear to be "intuitively" obvious", and yet they depend crucially on the set-theoretic properties of "boundary times" given in § 12. One is convinced by the amount of detection needed to identify such simple quantities as ϱ^a and F^{ab} that herein lies indeed the strength of the probabilistic versus the analytico-algebraic method.

In § 18 algebraic transformations between different decompositions are established and an example is given to clarify the question of construction discussed above. In § 19 some consequences of the main results are specified ending with a full description of the sample functions of the process in terms of all the quantities introduced in this paper.

§ 12. Classification of boundary atoms

In this section properties of individual passable atomic boundary points, to be called "boundary atoms" for brevity's sake from now on, will be discussed. No hypothesis beyond Assumption A (p. 25 of [I]) and the existence of such an atom is needed. These atoms will be denoted simply by $a, b, \ldots$ instead of $\infty^a, \infty^b, \ldots$, and the set by $\mathbf{A}$. Correspondingly we shall write $S^a(\omega)$ for $S_{\infty^a}(\omega)$ (p. 38 of [I]) and "$x(t)=a$" for "$t \in S^a(\omega)$" or "x reaches the boundary atom a at time t". Furthermore we shall define $\mathbf{P}^a\{\ldots\}$ by the condition that for every $t_\nu \in \mathbf{T}$, $j_\nu \in \mathbf{I}$, $1 \leqslant \nu \leqslant n$, we have

$$\mathbf{P}^a\{x(t_\nu)=j_\nu,\ 1\leqslant \nu \leqslant n\} = \mathbf{P}\{x^a(t_\nu)=j_\nu \mid \Delta^a\},$$

where the right-hand side is defined in the last paragraph on p. 34 of [I]. This uniquely defines a probability measure on the Borel field $\mathfrak{F}^0$ generated by the Markov chain $\{x_t, t \in \mathbf{T}\}$. It is "the conditional probability when the process starts at a"; note the analogy with the usual $\mathbf{P}_i\{\ldots\}$ for $i \in \mathbf{I}$ (p. 24 of [I]). Similarly for conditional expectation $\mathbf{E}^a\{\ldots\}$.

"For a.e. ω" will mean for every ω except a set N in $\mathfrak{F}^0$ such that $\mathbf{P}_i(N)=0$ for each i in $\mathbf{I}$. It will then follow that we have also $\mathbf{P}^a(N)=0$ for each a in $\mathbf{A}$.

Let us define, for $s \geqslant 0$:

$$\alpha_s(\omega) = \inf\{t : t > s : x(t) \in \mathbf{A}\}, \tag{12.1}$$

where the inf is taken to be $+\infty$ if the set is empty; and

$$\alpha(\omega) = \alpha_0(\omega).$$

Thus α_s is the "first time after s when the boundary is reached"; it is an optional random variable; and α is the τ on p. 25 of [I], the letter τ (with subscripts) being reserved for a general time random variable in this paper. Next, we define

$$\alpha^a(\omega) = \inf\{t: t > 0,\, x(t) = a\};$$

$$K_i^a(t) = \mathbf{P}_i\{\alpha^a \leqslant t\}; \tag{12.2}$$

$$K^{ab}(t) = \mathbf{P}^a\{\alpha^b \leqslant t\}.$$

Thus K^b and K^{ab} are the first entrance time distributions into b, starting at i and a respectively. Extending a familiar notation in [1], we write $i \rightsquigarrow a$ iff $K_i^a(\infty) > 0$; $a \rightsquigarrow b$ iff $K^{ab}(\infty) > 0$;(1) and $a \rightsquigarrow i$ iff $i \in \mathbf{I}^a$, i.e., iff $\xi_i^a(t) \not\equiv 0$ (see pp. 34–5 of [I] for notation). According to § 10 of [I], $\xi_i^a(\cdot)$ is either identically zero or never zero; similar properties hold for K_i^a and K^{ab} but the full strength of this result will not be needed. It is now possible to define the relation $\leftrightsquigarrow$, called "communication", between any two elements of $\mathbf{I} \cup \mathbf{A}$ in the obvious way and deduce the usual classification (see [1; § II.10]). We shall give only the few propositions that will be needed later.

Definition 12.1. The boundary atom a is called *recurrent* if

$$\mathbf{P}^a\{S^a(\omega) \text{ is an unbounded set }\} = 1;$$

otherwise it is called *nonrecurrent*.

THEOREM 12.1. *If a is recurrent, then for every $\delta > 0$:*

$$\mathbf{P}^a\{S^a(\omega) \cap (\delta, \infty) \neq \emptyset\} = 1.$$

Conversely if there exists a $\delta > 0$ for which (12.2) *is true, then a is recurrent.*

Proof. Clearly Definition 12.1 implies

$$\mathbf{P}^a\{\forall \delta > 0 : S^a(\omega) \cap (\delta, \infty) \neq \emptyset\} = 1$$

which implies the first assertion. To prove the converse, let $\tau_0(\omega) \equiv 0$ and for $n \geqslant 0$ define

$$\tau_{n+1}(\omega) = \inf\{t: t > \tau_n(\omega) + \delta,\, x(t) = a\}.$$

By the strong Markov property [1; Theorem II.9.3] the fields $\mathfrak{F}_{\tau_n+\delta}$ and $\mathfrak{F}'_{\tau_n+\delta}$ are independent conditioned on $x(\tau_n + \delta)$ which is in $\mathbf{I}$ with probability one. Also τ_{n+1} is measurable $\mathfrak{F}'_{\tau_n+\delta}$ and is the first entrance time into a in the post-$(\tau_n + \delta)$ process. Hence we have by the Strong Markov property: (2)

(1) Note that this is not the same definition as on p. 43 of [I].

(2) This will be used so often that we cannot mention it every time, but it must be remembered that we are invoking here the form for boundary entrances as given in [I; Theorem 4.4], rather than the usual form as given in (1; Theorem II.9.3). We will indicate this by using the capital S for the former and the small s for the latter.

$$\mathbf{P}\{\tau_{n+1} < \infty \mid \tau_1, \ldots, \tau_n\} = \mathbf{P}\{\tau_{n+1} < \infty \mid x(\tau_n)\} = \mathbf{P}^a\{S^a(\omega) \cap (\delta, \infty) \neq \emptyset\} = 1.$$

Consequently all τ_n are finite, $\tau_n \uparrow \infty$ and S^a is unbounded with probability one.

THEOREM 12.2. *If a is recurrent, then $\mathbf{I}^a$ is a $\prod$-recurrent class and for every $i \in \mathbf{I}^a$ we have $K_i^a(\infty) = 1$. Conversely, if there exists a $\prod$-recurrent state i in $\mathbf{I}^a$ such that $i \rightsquigarrow a$ then a is recurrent.*

Proof. Let $i \in \mathbf{I}^a$, then $\xi_i^a(t)$ is positive from a certain t on (in fact for all $t > 0$ by § 10 of [I]). Hence there exists $\delta > 0$ such that

$$\int_0^\delta \xi_i^a(t)\, dt > 0. \tag{12.3}$$

Using the sequence $\{\tau_n\}$ defined above, we have

$$\int_0^\infty \xi_i^a(t)\, dt \geq \sum_{n=1}^\infty \mathbf{E}\{\mu[S_i \cap (\tau_n, \tau_{n+1})]\} \geq \sum_{n=1}^\infty \int_0^\delta \xi_i^a(t)\, dt = +\infty.$$

Furthermore by Theorem 12.1, we have

$$0 = \mathbf{P}^a\{S^a \cap (\delta, \infty) = \emptyset\} \geq \xi_i^a(\delta)\,[1 - K_i^a(\infty)].$$

But (12.3) implies that $\xi_i^a(\delta) > 0$ by the first sentence of the proof, hence $K_i^a(\infty) = 1$. Since $p_{ii}(t) \geq \int_0^t \xi_i^a(t-s)\, dK_i^a(s)$ it follows that

$$\int_0^\infty p_{ii}(t)\, dt \geq K_i^a(\infty) \int_0^\infty \xi_i^a(t)\, dt = +\infty;$$

hence i is recurrent [1; Theorem II.10.4]. For each j in $\mathbf{I}^a$, we have $i \rightsquigarrow a \rightsquigarrow j$; hence $\mathbf{I}^a$ is one $\prod$-recurrent class.

Conversely, let $i \in \mathbf{I}^a$ and i be $\prod$-recurrent. Then

$$\mathbf{P}^a\{S_i \text{ is an unbounded set}\} = 1. \tag{12.4}$$

If $K_i^a(\infty) > 0$, there exist $\delta > 0$ such that $K_i^a(\delta) > 0$. Define a new sequence $\{\tau_n'\}$ as before but with "a" replaced by "i". It follows from (12.4) that all τ_n' are finite and $\tau_n' \uparrow \infty$ with probability one. We have

$$\mathbf{P}^a\{S^a \cap (\tau_n', \tau_{n+1}') \neq \emptyset\} \geq K_i^a(\delta)$$

and the events Λ_n in the $\{\cdots\}$ above are independent by the strongest Markov property [1; Theorem II.9.5] applied to the τ's. Hence by the Borel–Cantelli lemma, infinitely many of the Λ's occur and so S^a is unbounded with probability one.

COROLLARY. *If a is nonrecurrent, and $i \in \mathbf{I}^a$, then either i is $\prod$-nonrecurrent or $i \not\rightsquigarrow a$ (i.e., the negation of $i \rightsquigarrow a$). Conversely, if there exists an i in $\mathbf{I}^a$ such that either $i \not\rightsquigarrow a$ or i is $\prod$-nonrecurrent, then a is nonrecurrent.*

However, $\mathbf{I}^a$ may contain more than one distinct class. Let us also observe that if i is $\prod$-recurrent and $i \not\rightsquigarrow \mathbf{A}$ (i.e., $\forall a \in A : i \not\rightsquigarrow a$), then i must be Φ-recurrent. For if $i \not\rightsquigarrow A$, then $\forall j \in \mathbf{I}$ we have $p_{ij}(\cdot) \equiv f_{ij}(\cdot)$ and consequently $\prod$-recurrence of i implies its Φ-recurrence. Conversely, if i is Φ-recurrent then it is certainly $\prod$-recurrent and $i \not\rightsquigarrow \mathbf{A}$ by Theorem 3.2 of [I].

Definition 12.2. The boundary atom a is called *sticky* iff

$$\mathbf{P}^a\{\forall \delta > 0 : S^a \cap (0, \delta) \neq \emptyset\} = 1; \tag{12.5}$$

otherwise it is called *nonsticky* (it will follow after Theorem 12.5 that the probability above is then equal to 0).

We begin with a simple observation valid for every a.

THEOREM 12.3. *For each a and a.e. ω, $S^a(\omega)$ is a countable set.*

Proof. The definition of "reaching the boundary" (pp. 28–29 of [I]) entails that if $t \in S^a(\omega)$, then there exists $\delta > 0$ such that $(t-\delta, t) \notin S^a(\omega)$. Hence every point in $S^a(\omega)$ is isolated on the left, and the theorem follows from a well-known property of the real line.

THEOREM 12.4. *If a is sticky, then for a.e. ω, $S^a(\omega)$ is dense in itself.*

Remark. In view of the preceding proof this means: for each t in $S^a(\omega)$ and $\delta > 0$, we have $(t, t+\delta) \cap S^a(\omega) \neq \emptyset$. For $t = 0$ this reduces to (12.5).

Proof. For each real r and for a.e. ω for which $\alpha_r(\omega) < \infty$, we have by (12.5) and the Strong Markov property:

$$\forall \delta > 0 : \quad S^a(\omega) \cap (\alpha_r(\omega), \alpha_r(\omega) + \delta) \neq \emptyset.$$

Hence for a.e. ω this is even true for all $\alpha_r(\omega)$ with rational values of r, simultaneously. Since every point in $S^a(\omega)$ is such an $\alpha_r(\omega)$ by definition, Theorem 12.4 is proved.

COROLLARY. *If a is sticky and*

$$\gamma(t, \omega) = \sup[S^a(\omega) \cap (0, t)],$$

then $\gamma(t, \omega) \in \overline{S^a(\omega)} - S^a(\omega)$ where $\overline{S^a}$ is the Euclidean closure of S^a.

Proof. By definition, $\gamma(t, \omega) \in \overline{S^a(\omega)}$; by the Remark above $\gamma(t, \omega) \notin S^a(\omega)$.

The next result is a zero-or-one law for the notion of stickiness. It may be remarked that the so-called zero-or-one law in the theory of Hunt processes is trivially false for Markov chains in general.

THEOREM 12.5. *If a is nonsticky, then for a.e. ω, $S^a(\omega)$ does not have a finite point of accumulation. In particular, the probability in* (12.5) *is equal to zero.*

Proof. Since (12.5) does not hold, there exist $\delta>0$ and $\varepsilon>0$ such that

$$\mathbf{P}^a\{S^a\cap(0,\delta)\neq\emptyset\}=1-\varepsilon. \tag{12.6}$$

Suppose the theorem false and let Λ_0 be a set of positive probability such that if $\omega\in\Lambda_0$ then $S^a(\omega)$ has a finite point of accumulation. Then there exist $t\geqslant 0$, and a subset Λ_1 of Λ_0 with positive probability such that if $\omega\in\Lambda_1$ then $S^a(\omega)$ has a point of accumulation in $(t, t+\delta)$. This implies that $S^a(\omega)\cap(t,t+\delta)$ is an infinite set. Now for each $m\geqslant 1$, let

$$t<\tau_{m1}(\omega)<\tau_{m2}(\omega)<\ldots<\tau_{mN}(\omega)<t+\delta$$

be all the successive distinct members of the set $\{\alpha_{n/m}(\omega),\ n\geqslant 0\}$ defined in (12.1), where $N=N(m,\omega)$ is a nonnegative integer. For each ω in Λ_1, we have $\lim_{m\to\infty}N(m,\omega)=+\infty$. Hence given N_0, there exist m_0 and $\Lambda_2\subset\Lambda_1$, with $2P(\Lambda_2)\geqslant P(\Lambda_1)$, such that

$$\forall\omega\in\Lambda_2:\quad N(m_0,\omega)>N_0.$$

Applying the Strong Markov property to $\tau_{m_01},\ldots,\tau_{m_0N_0}$, we obtain by (12.6):

$$\mathbf{P}\{\tau_{m_0,n+1}-\tau_{m_0n}<\delta\,|\,\tau_{m_01}<\ldots<\tau_{m_0n}<\infty\}\leqslant P^a\{S^a\cap(0,\delta)\neq\emptyset\}=1-\varepsilon.$$

It follows that

$$\mathbf{P}(\Lambda_1)\leqslant 2\mathbf{P}(\Lambda_2)\leqslant 2(1-\varepsilon)^{N_0}.$$

Since N_0 is arbitrary, $\mathbf{P}(\Lambda_1)=0$. This is a contradiction that proves the theorem.

§ 13. Exit and entrance sequences and laws

This short section contains several more-or-less known propositions in the forms to be needed later.

Given the countable index set $\mathbf{I}$, let $\mathcal{M}(\mathbf{I})$ be the space of measures on $\mathbf{I}$, namely al sequences of nonnegative finite real numbers index by $\mathbf{I}$.

Definition 13.1. Given a standard substochastic transition matrix function $\Psi(\cdot)$ on $\mathbf{I}\times\mathbf{I}$, an entrance (exit) sequence e relative to Ψ is an element of $\mathcal{M}(\mathbf{I})$ satisfying

$$\forall t\geqslant 0:\quad e\geqslant e\Psi(t)\quad [e\geqslant\Psi(t)e].$$

Here if $e=\{e_i\}$ and $(\Psi(t))=(\psi_{ij}(t))$, $e\Psi(t)$ is the element of $\mathcal{M}(\mathbf{I})$ whose j-component $[e\Psi(t)]_j$ is $\sum_i e_i\psi_{ij}(t)$ $[\Psi(t)e$ is the element whose i-component is $\sum_j \psi_{ij}(t)e_j]$; and the inequality is taken component-wise. We shall use this type of "vector notation" when confusion is unlikely; but since both subscripts and superscripts will appear as possible components of vectors we shall revert to an explicit notation whenever in doubt.

Since $\Psi(\cdot)$ is standard, we have $\lim_{t\downarrow 0}\psi_{ii}(t)=1$ for every i, from which it follows easily that

$$e=\lim_{t\downarrow 0} e\Psi(t) \quad [e=\lim_{t\downarrow 0}\Psi(t)\,e],$$

so that e is "excessive" in Hunt's usage. It is easy to prove that relative to the minimal solution $\Phi(\cdot)$ (see p. 23 of [I]), e is an entrance (exit) sequence if and only if

$$eQ\leqslant 0 \quad [Qe\leqslant 0],$$

where Q is the initial derivative matrix.

We shall state the next two theorems for the entrance case only since the exit case is entirely similar.

Definition 13.2. An *entrance law* relative to Ψ is a one-parameter family $\eta(\cdot)=\{\eta(s),\, s\geqslant 0\}$ of elements of $\mathcal{M}(\mathbf{I})$ satisfying the functional equation:

$$\forall s>0,\, t\geqslant 0: \quad \eta(s)\Psi(t)=\eta(s+t). \tag{13.1}$$

By [3; Lemma 1], for each j in $\mathbf{I}$, $\eta_j(\cdot)$ is continuous in $[0,\infty)$.

THEOREM 13.1. *If e and Ψ are as in Definition 13.1, there exists an entrance law $\eta(\cdot)$ relative to Ψ such that*

$$e-e\Psi(t)=\int_0^t \eta(s)\,ds. \tag{13.2}$$

Remark. We shall say that the entrance sequence e generates the entrance law $\eta(\cdot)$.

Proof. This is nothing but a general form of Theorem 6.2 of [I] proved in the same way, but a sketch will be given. Let

$$H(t)\overset{\text{def}}{=}e-e\Psi(t). \tag{13.3}$$

It is clear that $H(\cdot)\nearrow$ and the semigroup property $\Psi(s)\Psi(t)=\Psi(s+t)$ implies

$$H(s+t)-H(t)=H(s)\Psi(t). \tag{13.4}$$

By a basic lemma [3; Lemma 2], H has a continuous derivative η so that

$$H(t)=\int_0^t \eta(s)\,ds \tag{13.5}$$

which is (13.2); moreover the equation (13.4) may be differentiated with respect to s, yielding (13.1).

THEOREM 13.2. *Let $\Psi(\cdot)$ be as before and let $\{e(s),\ s>0\}$ be a one-parameter family of elements of $\mathfrak{M}(\mathbf{I})$ satisfying*

$$e(s)\Psi(t)\leqslant e(s+t), \tag{13.6}$$

$$e\overset{\text{def}}{=}\int_0^\infty e(s)\,ds<\infty. \tag{13.7}$$

Then e is an entrance sequence relative to Ψ, and

$$\lim_{t\uparrow\infty} e\Psi(t)=0, \tag{13.8}$$

or equivalently

$$e=\int_0^\infty \eta(s)\,ds, \tag{13.9}$$

where η is the entrance law generated by e according to Theorem 13.1.

Proof. Integrating (13.6) over s in $(0,\infty)$ we obtain

$$e\Psi(t)\leqslant\int_0^\infty e(s+t)\,ds=e-\int_0^t e(s)\,ds\leqslant e.$$

Hence e is an entrance sequence as asserted, and also (13.8) is true by (13.7). The equivalence of (13.8) and (13.9) is obvious from (13.2).

It is instructive to compare the results above with the standard potential theory argument, according to which we should write

$$\int_0^\infty [e-e\Psi(t)]\,\Psi(u)\,du=\int_0^\infty H(t)\,\Psi(u)\,du=\int_0^\infty [H(t+u)-H(u)]\,du=\int_0^t [H(\infty)-H(u)]\,du.$$

Therefore in particular

$$\lim_{t\downarrow 0}\int_0^\infty \frac{1}{t}[H(t+u)-H(u)]\,du=H(\infty)-H(0)=e.$$

Under our conditions it is permitted to interchange the limit and integration above which then becomes (13.9). However, if the same interchange is made a little earlier in

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^\infty H(t)\,\Psi(u)\,du,$$

the result is
$$\int_0^\infty \eta(0)\,\Psi(u)\,du$$

which is in general strictly less than e.

THEOREM 13.3. *Let*
$$z \overset{\text{def}}{=} \lim_{t\uparrow\infty} [I-\Psi(t)]\,1; \tag{13.10}$$

then z is an exit sequence relative to Ψ. If e is any exit sequence, then
$$e \leqslant z \tag{13.11}$$

if and only if
$$e \leqslant 1 \quad \text{and} \quad \lim_{t\uparrow\infty} \Psi(t)\,e = 0. \tag{13.12}$$

Remark. By the analogue of Theorem 13.2, the second condition in (13.12) is equivalent to that $e = \int_0^\infty e(s)\,ds$ where $e(\cdot)$ is the exit law generated by e.

Proof. It is clear that $z \leqslant 1$ and $\lim_{t\uparrow\infty}\Psi(t)z = 0$, hence (13.11) implies (13.12). Conversely if (13.12) holds, then
$$[I-\Psi(t)]e \leqslant [I-\Psi(t)]\,1$$
and letting $t \uparrow \infty$ we obtain (3.11). Q.e.d.

When Ψ is the minimal solution Φ, then z is $L(\infty)$. From the probabilistic point of view there is an obvious choice of exit sequences relative to Φ. They are the $L^a(\infty)$, $a \in \mathbf{A}$ studied in § 4 of [I]. It is the entrance sequences that have to be discovered and this will be done in Theorem 14.3 below.

§ 14. The basic quantities

From here on Assumptions A and B′ will be in force throughout the paper. In this section we introduce the new notions which make the present approach possible. The underlying idea is simple enough: to study the succession of boundary atoms in a sample function, viewing these as "banners" (superscripts from **A**) under which the ordinary states (subscripts from **I**) line up. If all boundary atoms are nonsticky, this is easily carried out, has essentially been done in § 5 of [I] and will be reviewed in § 19. For a sticky atom the important thing is to concentrate on the *change* of banners so that beginning at one of them the sample function is followed through until a new one appears, if ever. Now it turns out that each portion between change of banners "possesses finite potentials" (Theorem 14.2) so that it can be sufficiently well isolated and analyzed before the portions are pieced together. Special attention must be paid to the case when the banner does not change and when it changes suddenly. The first can be treated separately as "traps"; the second produces delicate effects which will be stressed at the appropriate places.

Definition 14.1. For each a in $\mathbf{A}$ let us define a new optional random variable as follows:

$$\beta^a(\omega) = \inf_{b \neq a} \alpha^b(\omega) = \inf\{t: t > 0, x(t) \in \mathbf{A} - \{a\}\},$$

where α^b is defined in (12.2). Thus starting at a, β^a is the "first time for change (of banners)".

Definition 14.2. For each $a \in \mathbf{A}$, $b \in \mathbf{A}$, $a \neq b$ and $t \geqslant 0$:

$$\varrho_j^a(t) \stackrel{\text{def}}{=} \mathbf{P}^a\{\beta^a > t; x(t) = j\},$$

$$F^{ab}(t) \stackrel{\text{def}}{=} \mathbf{P}^a\{\alpha^b = \beta^a \leqslant t\}.$$

Thus $\varrho_j^a(t)$ is the probability, starting at a, that no change of banner has occurred up to time t and that at this time state j appears under the initial banner a; $F^{ab}(t)$ is the probability that a change of banner has occurred before or on time t and that the change is to b (regardless what banner is flying at t).

We have the obvious relations, if $t > 0$:

$$\varrho_*^a(t) \stackrel{\text{def}}{=} \sum_i \varrho_i^a(t) = \mathbf{P}^a\{\beta^a > t\}, \quad \sum_{b \neq a} F^{ab}(t) = \mathbf{P}^a\{\beta^a \leqslant t\}; \tag{14.1}$$

$$\varrho_*^a(t) + \sum_{b \neq a} F^{ab}(t) = 1. \tag{14.2}$$

Note that $\varrho_j^a(0) = \varrho_j^a(0+)$ but $\varrho_*^a(0) \leqslant \varrho_*^a(0+)$ in general since $\sum_j \mathbf{P}^a\{x(0) = j\} \leqslant 1$. The limits of (14.2) at either end of $(0, \infty)$ are important:

$$\sum_{b \neq a} F^{ab}(0) = 1 - \lim_{t \downarrow 0} \uparrow \varrho_*^a(t) = 1 - \varrho_*^a(0+); \tag{14.3}$$

$$\sum_{b \neq a} F^{ab}(\infty) = 1 - \lim_{t \uparrow \infty} \downarrow \varrho_*^a(t) = 1 - \varrho_*^a(\infty). \tag{14.4}$$

Definition 14.3. The boundary atom is called *ephemeral* iff $\varrho_*^a(0+) = 0$; it is called a *trap* iff $\varrho_*^a(\infty) = 1$.

It is clear how we can eliminate each ephemeral boundary atom by splitting it into others, but this will not be necessary.

THEOREM 14.1. *If a is sticky and distinguishable from any other boundary atom, then*

$$\sum_{b \neq a} F^{ab}(0) = 0. \tag{14.5}$$

If b is nonsticky, then

$$\sum_{a \neq b} F^{ab}(0) = 0. \tag{14.6}$$

Proof. It follows from Definition 14.2 that

$$F^{ab}(0) = \mathbf{P}\{\forall \delta > 0 : S^b \cap (0; \delta) \neq \emptyset\}. \tag{14.7}$$

If a is sticky then by definition, 0 is an accumulation point of $S^a(\omega)$ for almost every ω. Hence by Theorem 4.6 of [I], for almost no ω can 0 be an accumulation point of any $S^b(\omega)$ where b is distinguishable from a. This means $F^{ab}(0) = 0$ by (14.7), and the first assertion of the theorem follows. Next suppose $F^{ab}(0) > 0$; choose i with $L_i^a(\infty) > 0$, then by the Strong Markov property,

$$\mathbf{P}_i\{\forall \delta > 0 : S^b \cap (\alpha^a, \alpha^a + \delta) \neq \emptyset\} \geqslant L_i^a(\infty) F^{ab}(0) > 0.$$

Thus S^b has a finite accumulation point α^a with positive probability and so b must be sticky by Theorem 12.5. The second assertion of the theorem follows. Q.e.d.

It is clear from the meaning of ϱ^a that (see p. 40 of [I] for ζ^a):

$$\zeta^a(t) \leqslant \varrho^a(t) \leqslant \xi^a(t).$$

Now let us put

$$p_{ij}^a(t) \stackrel{\text{def}}{=} f_{ij}(t) + \int_0^t l_i^a(u)\, \varrho_i^a(t-u)\, du. \tag{14.8}$$

By the Strong Markov property, this is the probability of transition when the boundary set $\mathbf{A} - \{a\}$ is taboo; namely when the process is stopped at all boundary atoms except a. Let us call this stopped process, completed as usual by a new absorbing state θ, the a-process (see [1; p. 244 ff.]). It is clear that

$$\prod{}^a \stackrel{\text{def}}{=} (p_{ij}^a(\cdot)), \quad (i, j) \in \mathbf{I}^a \times \mathbf{I}^a,$$

is a substochastic transition matrix (function) whose stochastic completion is the transition matrix of the a-process and that

$$\Phi \leqslant \prod{}^a \leqslant \prod. \tag{14.9}$$

Moreover, if the process is initially at a it is an open Markov chain whose absolute distribution is $\{\varrho^a(t), t > 0\}$. This fact is expressed by the following functional equation

$$\varrho^a(s) \prod{}^a(t) = \varrho^a(s+t); \tag{14.10}$$

which is to be compared with

$$\zeta^a(s)\, \Phi(t) = \zeta^a(s+t), \qquad \xi^a(s) \prod(t) = \xi^a(s+t).$$

Thus we have interposed, for each a, a new process between the minimal Φ and the maximal $\prod$. Successive interposition will lead from Φ to $\prod$, but that will not be necessary.

The continuity of each $\varrho_j^a(\cdot)$ follows from (14.10) by [3; Lemma 1]. Furthermore each $\varrho_j^a(\cdot)$ has the property of being either identically zero or never zero in $(0,\infty)$, by [I; § 10].

If a is a trap, then clearly $\varrho^a(\cdot)\equiv\xi^a(\cdot)$. If a is not a trap, the following result is essential (see p. 29 of [I] for Z).

THEOREM 14.2. *If a is not a trap, and $i\in\mathbf{I}-Z$, then*

$$r_i^a \overset{\text{def}}{=} \int_0^\infty \varrho_i^a(t)\,dt < \infty. \tag{14.11}$$

Proof. If $i\in\mathbf{I}-Z$, then $i\rightsquigarrow\mathbf{A}$. Since a is not a trap, $a\rightsquigarrow\mathbf{A}-\{a\}$ so that in any case $i\rightsquigarrow\mathbf{A}-\{a\}$. Hence there exists $b\in\mathbf{A}-\{a\}$, and $h>0$ with $L_i^b(h)>0$. Now

$$\varrho_i^a(nh)\,L_i^b(h)\leqslant\mathbf{P}^a\{nh<\beta^a\leqslant(n+1)\,h\}$$

and so
$$\sum_{n=1}^\infty \varrho_i^a(nh)\leqslant[L_i^b(h)]^{-1}<\infty.$$

Since
$$\varrho_i^a(t)\,f_{ii}(h)\leqslant\varrho_i^a(t+h)$$

and $\varrho_i^a(\cdot)$ is continuous, we have

$$\max_{nh\leqslant t\leqslant(n+1)h}\varrho_i^a(t)\leqslant[\min_{0\leqslant t\leqslant h}f_{ii}(t)]^{-1}\,\varrho_i^a(nh+h).$$

Consequently
$$r_i^a=\int_0^\infty\varrho_i^a(t)\,dt\leqslant[\min_{0\leqslant t\leqslant h}f_{ii}(t)]^{-1}\sum_{n=0}^\infty\varrho_i^a(nh+h)<\infty$$

and Theorem 14.2 is proved.

The next step is quite similar to the handling of ξ in the one-exit nonrecurrent case given in § 9 of [I]. The present extension to the general situation is made possible by the interposition of ϱ which behaves as "nonrecurrent" in the sense of the preceding theorem. Instead of Assumption C of [I; p. 47] the following assumption will be made from now on.

ASSUMPTION C_0. *There are no Φ-recurrent states.*

This will be slightly liberalized towards the end of § 15 to include the case of substochastic $\prod$, but it is not entirely dispensable without complicating the later results.

THEOREM 14.3. *For each boundary atom a, there exists an entrance sequence e^a relative to Φ having the property that*

$$\lim_{t\uparrow\infty}e^a\Phi(t)=0.$$

Equivalently, if $\eta^a(\cdot)$ denotes the entrance law generated by e^a (see Theorem 13.1), we have

$$e^a = \int_0^\infty \eta^a(s)\, ds. \tag{14.12}$$

Proof. Three cases will be considered though the first two may be combined.

Case 1. a is not a trap. Then by (14.9) and (14.10),

$$\varrho^a(s)\,\Phi(t) \leqslant \varrho^a(s) \prod{}^a(t) = \varrho^a(s+t)$$

and (14.11) is true. Hence we may take Ψ to be Φ, $e(s)$ to be $\varrho^a(s)$, e to be r^a and $\eta(s)$ to be $\eta^a(s)$ in Theorem 13.2 to conclude (14.12).

Case 2. a is a nonrecurrent trap. Then $\varrho^a(\cdot) \equiv \xi^a(\cdot)$ and by the Corollary to Theorem 12.2, each state i in $\mathbf{I}^a$ is either $\prod$-nonrecurrent or $i \not\rightsquigarrow a$. Since a is a trap this means $i \not\rightsquigarrow \mathbf{A}$, and so by the discussion following the corollary just cited, each i in $\mathbf{I}^a$ which is $\prod$-recurrent must in fact be Φ-recurrent. This has been excluded by Assumption C_0. Hence each i in $\mathbf{I}^a$ is $\prod$-nonrecurrent and so

$$r^a = \int_0^\infty \varrho^a(s)\, ds = \int_0^\infty \xi^a(s)\, ds < \infty$$

(see [I; Theorem 6.1], the r^a here being the g^a there). The rest is the same as in Case 1.

Thus in both Case 1 and Case 2 the e^a of the theorem is just the r^a of (14.11).

Case 3. a is a recurrent trap. Then $\mathbf{I}^a$ is a $\prod$-recurrent class. By a well-known theorem [1; Theorem II.13.5], there exists $\{e(s),\ s>0\}$ such that

$$\int_0^\infty e(s)\, ds < \infty, \quad e(s) \prod(t) = e(s+t);$$

and consequently $$e(s)\,\Phi(t) \leqslant e(s+t).$$

In fact since all states are stable we may choose any state 0 in $\mathbf{I}^a$ and set

$$e_j(s) = {}_0p_{0j}(s), \quad \int_0^\infty e_j(s)\, ds = {}_0p^*_{0j}$$

[1; p. 201]. Hence Theorem 13.2 is applicable with $\Psi = \Phi$ and this choice of $e(s)$, so that e^a is the sequence with

$$e^a_j = {}_0p^*_{0j}, \quad j \in \mathbf{I}^a.$$

Remark. It would be interesting to know whether the case of a recurrent trap indeed requires special handling as described above; and if so where is this covered up in the algebraic treatment of other authors.

The next result Theorem 14.4 is one of the two keys to the canonical decomposition. We need an analytical lemma which is essentially known (see [I; § 9] and Neveu [10]); [1] for a purely analytical proof and general discussion see [II].

LEMMA. *Let σ be finite, nonnegative, nonincreasing in $(0,\infty)$ with $\sigma(0+)\leqslant+\infty$ and $\int_0^1\sigma(s)\,ds<\infty$. Let $\hat\sigma(\lambda)$ be its Laplace transform:*

$$\hat\sigma(\lambda)=\int_0^\infty e^{-\lambda t}\sigma(t)\,dt,\quad 0<\lambda<\infty.$$

Given also two constants $\delta\geqslant 0$, $p\geqslant 0$. Then there exists a nonnegative measure $E(\cdot)$ on $[0,\infty)$ such that for $0<\lambda<\infty$:

$$[\delta+p\lambda+\lambda\hat\sigma(\lambda)]\,\hat E(\lambda)=1, \tag{14.13}$$

where

$$\hat E(\lambda)=\int_{[0,\infty)}e^{-\lambda t}E(dt).$$

Moreover, E is a finite measure unless $\delta=0$, in which case it is infinite but sigma-finite. If σ is absolutely continuous in (t,∞) for every $t>0$, then E is absolutely continuous except for a point mass at 0 if $\sigma(0)<\infty$.

Proof. By a particular case of P. Lévy's representation of infinitely divisible laws and the associated processes (see Lévy [IV]) there exists an infinitely divisible process $\{Y(v),\,v\geqslant 0\}$ such that if $F(v;\cdot)$ denotes the distribution of $Y(v)$, we have

$$\hat F(v;\lambda)\overset{\text{def}}{=}\int_{[0,\infty)}e^{-\lambda t}F(v;dt)=\mathbf{E}(e^{-\lambda Y(v)})=\exp\left\{-v\left[p\lambda+\int_0^\infty(e^{-\lambda s}-1)\,d\sigma(s)\right]\right\}.$$

Putting

$$\hat u(\lambda)\overset{\text{def}}{=}\lambda\hat\sigma(\lambda)$$

we have

$$e^{-\delta v}\int_{[0,\infty)}e^{-\lambda t}F(v;dt)=e^{-v[\delta+p\lambda+\hat u(\lambda)]}.$$

Integrating this over v in $[0,\infty)$ and setting

$$E(\cdot)\overset{\text{def}}{=}\int_{[0,\infty)}e^{-\delta v}F(v,\cdot)\,dv, \tag{14.14}$$

[1] Neveu's assertion of a sharper result corresponding to Corollary 1 to Theorem 14.7 below is without substantiation.

we obtain
$$\int_{[0,\infty)} e^{-\lambda t} E(dt) = [\delta + p\lambda + \hat{u}(\lambda)]^{-1}, \tag{14.15}$$

proving (14.13). Letting $\lambda \downarrow 0$ in (14.14) we see that $E([0,\infty)) = \delta^{-1}$ since $\lim_{\lambda\downarrow 0}\hat{u}(\lambda) = 0$, proving that E is an infinite measure if and only if $\delta = 0$, in which case E is still sigma-finite by (14.14). Finally, it is well-known from Lévy's theory that if σ is absolutely continuous in (t, ∞) for every $t > 0$, then $F(v; \cdot)$ is absolutely continuous except for a mass at 0 equal to $\hat{F}(v; +\infty) = e^{-v\sigma(0+)}$ when $p = 0$ and $\sigma(0+) < \infty$, and then $E(\{0\}) = [\delta + \sigma(0+)]^{-1}$ by (14.14). The lemma is completely proved.

COROLLARY. *Suppose $p = 0$. For (Lebesgue) almost every $t > 0$, we have*
$$\int_{[0,t]} [\delta + \sigma(t-s)]\, E(ds) = 1; \tag{14.16}$$
and for every $t > 0$ the inequality "$\leqslant$" holds above.

This corollary will be sharpened below; see Corollary 1 to Theorem 14.7.

THEOREM 14.4. *For each a, $\varrho^a(\cdot)$ and the entrance law $\eta^a(\cdot)$ generated by the e^a in Theorem 14.3 are linked by the following formula, for $t \geqslant 0$:*
$$\varrho^a(t) = \int_{[0,t]} \eta^a(t-s)\, E^a(ds), \tag{14.17}$$
where $E^a(\cdot)$ is a probability measure on $[0,\infty)$, unless a is a recurrent trap in which case it is infinite but sigma-finite. Furthermore $E^a(\cdot)$ is absolutely continuous except for a mass at 0 in case a is nonsticky.

Proof. Case 1. a is not a trap or a is a nonrecurrent trap. Integrating (14.10) over s in $(0,\infty)$ and noting that $e^a = r^a$,
$$e_j^a - \int_0^t \varrho_j^a(s)\, ds = \sum_i e_i^a \left\{ f_{ij}(t) + \int_0^t l_i^a(u)\, \varrho_j^a(t-u)\, du \right\}. \tag{14.18}$$

Using the notation $\langle \cdot , \cdot \rangle$ for the inner product of vectors, we put
$$\sigma^{aa}(t) \overset{\text{def}}{=} \langle e^a, l^a(t) \rangle.$$

Recalling also (13.2) with $e = e^a$ and $\Psi = \Phi$ we may rewrite (14.18) in vector notation as
$$\int_0^t \eta^a(s)\, ds = \int_0^t \varrho^a(u)\, [1 + \sigma^{aa}(t-u)]\, du. \tag{14.19}$$

If we put also $$\theta^{aa}(t) \stackrel{\text{def}}{=} \langle \eta^a(s), l^a(t-s)\rangle, \quad 0<s<t, \tag{14.20}$$

which is independent of s (see p. 49 of [I]), we have by (14.12)

$$\sigma^{aa}(t) = \int_0^\infty \langle \eta^a(s), l^a(t)\rangle\, ds = \int_0^\infty \theta^{aa}(s+t)\, dt = \int_t^\infty \theta^{aa}(s)\, ds. \tag{14.21}$$

Since $\varrho^a(\cdot)$ is continuous and positive unless a is ephemeral, it follows from (14.19) that $\sigma^{aa}(\cdot)$ is locally integrable. If a is ephemeral then $e^a=0$ so that $\sigma^{aa}(\cdot)\equiv 0$. Hence the Lemma above is applicable to σ^{aa} with $\delta=1$, $p=0$; the corresponding E will be denoted by E^a. It follows from the proof of the lemma that E^a is a probability measure. Taking Laplace transforms in (14.19) we have

$$\hat{\eta}^a(\lambda) = \hat{\varrho}^a(\lambda)[1+\lambda\hat{\sigma}^{aa}(\lambda)],$$

which can be inverted by (14.15) to yield

$$\hat{\varrho}^a(\lambda) = \hat{E}^a(\lambda)\hat{\eta}^a(\lambda).$$

From the uniqueness theorem for Laplace transforms, and the continuity of the functions $\varrho^a(\cdot)$ and $\eta^a(\cdot)$ in $[0,\infty)$, we conclude (14.17) as asserted.

Case 2. a is a recurrent trap.

In this case we have, recalling the handling of this case in Theorem 14.3:

$$e_j^a = \sum_i e_i^a p_{ij}(t) = \sum_i e_i^a \left\{ f_{ij}(t) + \int_0^t \langle l_i^a(u), \varrho_j^a(t-u)\rangle\, du \right\}.$$

Proceeding as before, we obtain

$$\int_0^t \eta^a(s)\, ds = \int_0^t \varrho^a(u)\, \sigma^{aa}(t-u)\, du \tag{14.22}$$

which differs from (14.19) only in that the "1" there is replaced by "0". The Lemma is applicable as before but with $\delta=0$, $p=0$, and the resulting E^a is now an infinite but sigma-finite measure.

We know that E^a is absolutely continuous except for a mass at 0 when $\sigma^{aa}(0)<\infty$; that this last condition is equivalent to a being nonsticky will be shown in Theorem 14.6 below.

In order to combine the two cases above we introduce the symbol

$$\delta^a = \begin{cases} 0, & \text{if } a \text{ is a recurrent trap;} \\ 1, & \text{otherwise.} \end{cases} \tag{14.23}$$

Then we have for each a,

$$[\delta^a + \lambda\hat{\sigma}^{aa}(\lambda)]\,\hat{E}^a(\lambda) = 1;$$

or for almost every t:
$$\int_0^t [1+\sigma^{aa}(t-s)]\,E^a(ds) = 1. \tag{14.24}$$

COROLLARY. *For each a,*

$$\eta_*^a(\cdot) \overset{\text{def}}{=} \sum_j \eta_j^a(\cdot) = \langle \eta^a(\cdot), 1\rangle$$

is locally integrable.

Proof. We have from (14.19) and (14.22) for every t:

$$\int_0^t \eta_*^a(s)\,ds = \int_0^t \varrho_*^a(s)\,[\delta^a + \sigma^{aa}(t-s)]\,ds \leqslant \int_0^t [1+\sigma^{aa}(t-s)]\,ds < \infty$$

since σ^{aa} is locally integrable.

THEOREM 14.5. *For each $a, \eta_*^a(\cdot)$ is nonincreasing and*

$$c^a \overset{\text{def}}{=} \eta_*^a(\infty) = \langle \eta^a(t), 1 - L(\infty)\rangle \tag{14.25}$$

for every $t>0$. This number is also equal to $\varrho_*^a(\infty)$ unless a is a recurrent trap in which case $\eta_*^a(\infty)=0$, while $\varrho_*^a(\infty)=1$ for any trap.

Proof. The monotonicity is an immediate consequence of the defining property of an entrance law; and the proof of (14.25) is the same as on p. 50 of [I], although the η there need not be the same as here. Both are properties of any entrance law relative to Φ. Next we have from (14.17):

$$\varrho_*^a(t) = \int_{[0,t]} \eta_*^a(t-s)\,E^a(ds).$$

Letting $t \uparrow \infty$ an recalling that E^a is a probability measure unless a is a recurrent trap, we see that the common value in (14.25) is $\varrho_*^a(\infty)$ except in that case. It is clear from (14.4) that $\varrho_*^a(\infty)=1$ for any trap a. On the other hand, if a is a recurrent trap, then for every i in $\mathbf{I}^a$, we have $L_i(\infty)=L_i^a(\infty)=1$ by Theorem 12.3, hence $\eta_*^a(\infty)=0$ by (14.25).

THEOREM 14.6. *The boundary atom a is sticky if and only if*

$$\sigma^{aa}(0+) = +\infty.$$

Proof. For each $\delta > \varepsilon > 0$, we have clearly

$$\mathbf{P}^a\{S^a \cap (\varepsilon, \delta) \neq \emptyset\} = \langle \varrho^a(\varepsilon), L^a(\delta-\varepsilon)\rangle = \int_{[0,\varepsilon]} \langle \eta^a(\varepsilon - s), L^a(\delta-\varepsilon)\rangle\, E^a(ds)$$

by (14.17). Now by (14.20) and (14.21):

$$\langle \eta^a(\varepsilon-s), L^a(\delta-\varepsilon)\rangle = \int_0^{\delta-\varepsilon} \langle \eta^a(\varepsilon-s), l^a(t)\rangle\, dt$$

$$= \int_0^{\delta-\varepsilon} \theta^{aa}(\varepsilon - s + t)\, dt = \sigma^{aa}(\varepsilon - s) - \sigma^{aa}(\delta - s).$$

Hence $$\mathbf{P}^a\{S^a \cap (0,\delta) \neq \emptyset\} = \lim_{\varepsilon\downarrow 0} \int_{[0,\varepsilon]} [\sigma^{aa}(\varepsilon-s) - \sigma^{aa}(\delta-s)]\, E^a(ds). \tag{14.26}$$

If $\sigma^{aa}(0+) < \infty$, it follows from this that

$$\lim_{\delta\downarrow 0} \mathbf{P}^a\{S^a \cap (0,\delta) \neq \emptyset\} = 0$$

and so a is nonsticky by Definition 12.2. If $\sigma^{aa}(0+) = +\infty$, then $E^a(\{0\}) = 0$ by the lemma, it follows from (14.26) and (14.16) that

$$\mathbf{P}^a\{S^a \cap (0,\delta) \neq \emptyset\} = \lim_{\varepsilon\downarrow 0} \int_{[0,\varepsilon]} \sigma^{aa}(\varepsilon - s)\, E^a(ds) = 1.$$

This being true for every $\delta > 0$, we have (12.5) and so a is sticky. Q.e.d.

Generalizing the definitions in (14.20) and (14.21), we put for $a \in \mathbf{A}$, $b \in \mathbf{A}$:

$$\theta^{ab}(t) \overset{\text{def}}{=} \langle \eta^a(s), l^b(t-s)\rangle, \quad 0 < s < t; \qquad \sigma^{ab}(t) \overset{\text{def}}{=} \int_t^\infty \theta^{ab}(s)\, ds. \tag{14.27}$$

It follows by (14.12) that

$$\sigma^{ab}(t) = \int_0^\infty \theta^{ab}(s+t)\, ds = \begin{cases} \displaystyle\int_0^\infty \langle \eta^a(s), l^b(t)\rangle\, dt = \langle e^a, l^b(t)\rangle, \\ \displaystyle\int_0^\infty \langle \eta^a(t), l^b(s)\rangle\, ds = \langle \eta^a(t), L^b(\infty)\rangle. \end{cases} \tag{14.28}$$

Furthermore, for $t > 0$:

$$\eta^a_*(t) = \langle \eta^a(t), 1 - L(\infty) + \sum_{b\in\mathbf{A}} L^b(\infty)\rangle = c^a + \sum_{b\in\mathbf{A}} \sigma^{ab}(t). \tag{14.29}$$

From here on, as a convention in notation, a Lebesgue-Stieltjes integral such as $\int_0^t \dots dE(s)$ shall mean $\int_{[0,t]} \dots E(ds)$ for finite t and $\int_{[0,\infty)} \dots E(ds)$ for $t = +\infty$. Furthermore, $E(0)$ will be written for $E(\{0\})$.

We proceed to derive new basic relations for ϱ^a and F^{ab}, $a \in \mathbf{A}$, $b \in \mathbf{A} - \{a\}$. Note that some results may be vacuously true for an ephemeral a or a trap. We begin with

$$\mathbf{P}_i\{\beta^a \leqslant t, x(\beta^a) = b\} = L_i^b(t) + \int_0^t F^{ab}(t-s)\, dL_i^a(s). \tag{14.30}$$

We observe that the probability on the left side above is equal to

$$\mathbf{P}_i\{\alpha = \beta^a \leqslant t;\, x(\beta^a) = b\} + \mathbf{P}_i\{\alpha < \beta^a \leqslant t;\, x(\beta^a) = b\}.$$

The first term is $L_i^b(t)$ by definition, and the second is by the Strong Markov property equal to

$$\begin{aligned}\mathbf{P}_i\{\alpha < t;\, x(\alpha) = a;\, \beta^a \leqslant t;\, x(\beta^a) = b\} &= \int_0^t \mathbf{P}^a\{\beta^a \leqslant t-u;\, x(\beta^a) = b\}\, d\mathbf{P}_i\{\alpha \leqslant u\} \\ &= \int_0^t F^{ab}(t-u)\, dL_i^a(u).\end{aligned}$$

Hence (14.30) is proved.

Next, we have

$$\mathbf{P}^a\{s < \beta^a \leqslant s+t;\, x(\beta^a) = b\} = \sum_i \mathbf{P}^a\{s < \beta^a;\, x(s) = i\}\, \mathbf{P}_i\{\beta^a \leqslant t;\, x(\beta^a) = b\}.$$

Using Definition 14.2 and (14.30), this is

$$F^{ab}(s+t) - F^{ab}(s) = \langle \varrho^a(s), L^b(t)\rangle + \int_0^t \langle \varrho^a(s), l^a(u)\rangle\, F^{ab}(t-u)\, du. \tag{14.31}$$

Let us introduce the further notation

$$\varrho^{ab}(t) \overset{\text{def}}{=} \langle \varrho^a(t), L^b(\infty)\rangle. \tag{14.32}$$

Since each $\varrho^a(\cdot)$ is continuous and the right side above is dominated by $\langle \xi^a(t), 1\rangle$ which converges uniformly in every finite interval it follows that $\varrho^{ab}(\cdot)$ is continuous.

THEOREM 14.7. *We have, for every $t > 0$:*

$$E^a(t) = 1 - \varrho^{aa}(t); \tag{14.33}$$

$$E^a(t)\, F^{ab}(\infty) = \varrho^{ab}(t) + F^{ab}(t); \tag{14.34}$$

$$F^{ab}(t) = \int_0^t [F^{ab}(\infty) - \sigma^{ab}(t-s)]\, dE^a(s). \tag{14.35}$$

Proof. From (14.17) and (14.32) we have

$$\varrho^{ab}(t)=\int_0^t \sigma^{ab}(t-s)\,dE^a(s). \tag{14.36}$$

In particular, by (14.24), and the uniqueness of Laplace transforms:

$$\varrho^{aa}(t)=\int_0^t \sigma^{aa}(t-s)\,dE^a(s)=1-\delta^a E^a(t)$$

first for (Lebesgue) almost every t but then for every $t>0$ since both extreme terms above are continuous in $(0,\infty)$. Formula (14.33) is thus a sharp form of (14.24). Now letting $t\uparrow\infty$ and then replacing s by t in (14.31), we obtain

$$F^{ab}(\infty)-F^{ab}(t)=\varrho^{ab}(t)+\varrho^{aa}(t)\,F^{ab}(\infty).$$

This is (14.34) on account of (14.33). It is also (14.35) on account of (14.36).

COROLLARY 1. $\forall t>0$: $\int_0^t[\delta^a+\sigma^{aa}(t-s)]\,dE^a(s)=1$.

COROLLARY 2. *$F^{ab}(\cdot)$ is absolutely continuous.*

Equation (14.34) becomes perhaps more interesting if it is divided through by $F^{ab}(\infty)$, supposed to be positive; showing then that the resulting right side does not depend on b and defines the basic measure $E^a(\cdot)$. A similar relation involving an arbitrary j may be recorded as follows:

$$E^a(t)=\frac{\int_0^t \varrho_j^a(s)\,ds+\sum_i \varrho_i^a(t)\int_0^\infty f_{ij}(s)\,ds}{\int_0^\infty \varrho_j^a(s)\,ds}=\frac{\mathbf{E}^a\{\mu[S_j\cap(0,\alpha_t\wedge\beta^a)]\}}{\mathbf{E}^a\{\mu[S_j\cap(0,\beta^a)]\}}.$$

This is proved by integrating the equation

$$\varrho^a(s+t)=\varrho^a(s)\,\Phi(t)+\int_0^t\langle\varrho^a(s),l^a(u)\rangle\,\varrho^a(t-u)\,du$$

over t. It would be interesting to understand the meaning of this "equilibrium property" of $E^a(\cdot)$ with respect to j in $\mathbf{I}$ as well as to b in $\mathbf{A}$.

An interesting consequence of (14.35) is its limit as $t\downarrow 0$:

$$F^{ab}(0)=E^a(0)\,[F^{ab}(\infty)-\sigma^{ab}(0)]. \tag{14.37}$$

THEOREM 14.8. *For any $a\neq b$, we have*

$$\sigma^{ab}(0)\leqslant F^{ab}(\infty)\leqslant 1. \tag{14.38}$$

Proof. Integrating (14.31) over s in $(0,\infty)$, we obtain

$$F^{ab}(\infty)\,t-\int_0^t F^{ab}(s)\,ds=\langle e^a, L^b(t)\rangle+\int_0^t \sigma^{aa}(u)\,F^{ab}(t-u)\,du,$$

or using (14.28) to transform the first term on the right side,

$$F^{ab}(\infty)\,t=\int_0^t \{\sigma^{ab}(u)+[1+\sigma^{aa}(u)]\,F^{ab}(t-u)\}\,du.$$

Dividing through by t and letting $t\downarrow 0$, observing that $\sigma^{ab}(\cdot)$ is nonincreasing we infer (14.38).

COROLLARY.
$$\lim_{t\downarrow 0}\,[\eta_*^a(t)-\sigma^{aa}(t)]<\infty. \tag{14.39}$$

Combining Theorems 14.6 and 8 we conclude that the matrix $(\sigma^{ab}(0))$, $(a,b)\in \mathbf{A}\times\mathbf{A}$, has finite elements off the diagonal while a diagonal element is finite or infinite according as it corresponds to a nonsticky or sticky atom. This simple result used to be an obscure point in previous investigations ([7], [10], [VIII]).

§ 15. Canonical decomposition

As explained above the quantities ϱ^a and F^{ab} serve the purpose of separating the banners from each other as long as possible; now it is necessary to link them together. The leading formula, which is the second key to the canonical decomposition, the first being Theorem 14.4, is given below.

THEOREM 15.1. *For each a, we have*

$$\xi_j^a(t)=\varrho_j^a(t)+\sum_{b\neq a}\int_0^t \xi_j^b(t-s)\,dF^{ab}(s),\quad j\in\mathbf{I}^a. \tag{15.1}$$

Remark. This equation need not be valid for an arbitrary j; for example if $j\in\mathbf{I}^b\backslash\mathbf{I}^a$ and $a\rightsquigarrow b$; then the left member is 0 but the right member is positive.

Proof. The meaning of (15.1) is obious: we rewrite it in terms of random variables:

$$\mathbf{P}^a\{x(t)=j\}=\mathbf{P}^a\{\beta^a>t;\,x(t)=j\}+\sum_{b\neq a}\mathbf{P}^a\{\beta^a\leqslant t;\,\beta^a\in\overline{S^b};\,x(t)=j\}.$$

Let us observe at once the notational quirk which necessitates the use of "$\beta^a\in\overline{S^b}$", or more clearly perhaps: "$\forall\delta>0$: $S^b\cap(\beta^a,\beta^a+\delta)\neq\emptyset$" instead of the obvious "$x(\beta^a)=b$" because the latter would have the wrong meaning when $\beta^a=0$. The rigorous proof of the preceding equation is of course, as always, done by invoking the Strong Markov property, but since Theorems 4.3 and 4.4 of [I] were formulated a little too narrowly for the present purpose we shall indicate the minor modifications needed.

1°. The cited theorems were stated for τ^a which is the first time the boundary is reached if it is reached at the atom a, otherwise equal to $+\infty$. However, the same results hold if this is replaced by any optional random variable τ such that $x(\tau)=a$ where $\tau<+\infty$. By definition then there exist $\tau_0<\tau$ and $\tau_n \uparrow \tau$ such that the jump chain $\chi_n = x(\tau_n)$ starting at time $\tau_0\,(=\tau_0(\omega))$ reaches a at time τ. The proofs given for the cited theorems are valid if this $\{\chi_n\}$ is used there instead of the jump chain starting at time 0.

2°. The cited theorem did not deal with the situation where the given $\{x_t\}$ starts at a boundary atom and is an open Markov chain. However, nothing in the proofs, as revised in 1° above, is changed on the set $\{\tau>0\}$ (in which case the chain starts *in effect* at an ordinary state as far as τ is concerned).

3°. Thus we are left with that case $\{\tau=0\}$ under the initial condition say $x(0)=a$. This is settled by the following simple lemma.

LEMMA 1. *For each Λ in $\mathfrak{F}^0$ (the Borel field generated by $\{x_t\}$) we have*

$$\mathbf{P}\{0\in\overline{S^b};\Lambda\}=F^{ab}(0)\,\mathbf{P}^b(\Lambda).$$

Proof. The meaning of this is again clear: on the set $\{0\in\overline{S^b}\}$ the process acts as if it started at b (rather than a). To prove it we define for each positive integer m:

$$\tau_m(\omega)=\inf\{t: t>m^{-1}; x(t,\omega)=b\};$$

then $\tau_m \downarrow 0$ on $\{0\in\overline{S^b}\}$. If $0<\varepsilon<t$. then since $\{0\in\overline{S^b}\}\in\mathfrak{F}_{\tau_m}$ for each m, we have by the Strong Markov property:

$$\mathbf{P}^a\{0\in\overline{S^b};\tau_m\leqslant\varepsilon; x(t)=j\}=\int_0^\varepsilon \xi_j^b(t-s)\,d\mathbf{P}^a\{0\in\overline{S^b};\tau_m\leqslant s\}.$$

Letting $m\uparrow\infty$ we obtain

$$\mathbf{P}^a\{0\in\overline{S^b}; x(t)=j\}=\mathbf{P}^a\{0\in\overline{S^b}\}\,\xi_j^b(t)=F^{ab}(0)\,\xi_j^b(t).$$

This being true for every $t>0$, and $\{0\in\overline{S^b}\}\in\wedge_{m=1}^{\infty}\mathfrak{F}_{m^{-1}}$, the lemma follows.

Theorem 15.1 is now proved by applying the amendments 1°, 2°, 3° to β^a. Thus, the typical term in the sum at the beginning of the proof is further split into

$$\mathbf{P}^a\{\beta^a=0\in\overline{S^b}; x(t)=j\}+\mathbf{P}^a\{0<\beta^a\leqslant t; x(\beta^a)=b; x(t)=j\}$$
$$=F^{ab}(0)\,\xi_j^b(t)+\int_{(0,t]}\xi_j^b(t-s)\,F^{ab}(ds).$$

Q.e.d.

Taking Laplace transforms in (15.1), we obtain

$$\hat{\xi}^a(\lambda) = \hat{\varrho}^a(\lambda) + \sum_{b \neq a} \hat{F}^{ab}(\lambda)\, \hat{\xi}^b(\lambda), \tag{15.2}$$

where
$$\hat{\xi}^a(\lambda) = \int_0^\infty e^{-\lambda t} \xi^a(t)\, dt, \qquad \hat{\varrho}^a(\lambda) = \int_0^\infty e^{-\lambda t} \varrho^a(t)\, dt,$$

$$\hat{F}^{ab}(\lambda) = \int_0^\infty e^{-\lambda t}\, dF^{ab}(t).$$

Putting $\hat{F}^{aa}(\lambda) \equiv 0$ for each $a \in \mathbf{A}$, and introducing the matrix

$$\hat{F}(\lambda) \stackrel{\text{def}}{=} (\hat{F}^{ab}(\lambda)), \quad (a, b) \in \mathbf{A} \times \mathbf{A}$$

as well as the vectors $\hat{\xi}(\lambda)$, etc. where $\hat{\xi}(\lambda) = \{\hat{\xi}^a(\lambda), a \in \mathbf{A}\}$, we may write (15.2) in matrix notation as follows:

$$[I - \hat{F}(\lambda)]\, \hat{\xi}(\lambda) = \hat{\varrho}(\lambda). \tag{15.3}$$

Note that the vectors as well as matrices above are indexed on the finite set $\mathbf{A}$ of superscripts, the subscripts j in $\mathbf{I}^a$ being understood.

The next task is that of solving for $\hat{\xi}$ from (15.3) and that is done by the lemma below. Although it is but a special case of a "recurring theorem" (see Taussky [VI]) in its definitive form, we shall spell out a constructive proof using the theory of Markov chains.

LEMMA 2. *Let $P = (p_{ab})$, $(a, b) \in \mathbf{A} \times \mathbf{A}$, where $\mathbf{A}$ is a finite set, be a substochastic matrix. A necessary and sufficient condition that $I - P$ be invertible is: there does not exist a subset $\mathbf{C}$ of $\mathbf{A}$ such that $P|_{\mathbf{C}}$ (= the restriction of P to $\mathbf{C} \times \mathbf{C}$) is stochastic. The inverse has nonnegative elements when it exists.*

Proof. Consider the stochastic completion $\tilde{P}$ of P by ϑ (see [I; pp. 22–23]). Unless P is stochastic, $\tilde{P}$ is on the enlarged index set $\tilde{\mathbf{A}} = \mathbf{A} \cup \{\vartheta\}$. A discrete parameter Markov chain with minimial state space $\tilde{\mathbf{A}}$ and one-step transition matrix $\tilde{\mathbf{P}}$ will have ϑ as an absorbing state, and any state a such that $a \rightsquigarrow \vartheta$ will be inessential, hence nonrecurrent.

Case 1. $\tilde{P} \neq P$. Suppose the condition of the lemma holds, then every state except ϑ is nonrecurrent. For if there were any recurrent state distinct from ϑ, there would be a recurrent class $\mathbf{C}$ such that $\vartheta \notin \mathbf{C}$, thus $P|_{\mathbf{C}} = \tilde{P}|_{\mathbf{C}}$ would be stochastic, contrary to hypothesis. Hence for any a and b in $\mathbf{A}$ we have, in familiar notation:

$$s_{ab} \overset{\text{def}}{=} \sum_{n=0}^{\infty} \tilde{p}_{ab}^{(n)} < \infty.$$

Since $\tilde{p}_{ab}^{(n)} = p_{ab}^{(n)}$ for every a and b in $\mathbf{A}$, we have

$$s_{ab} = \sum_{n=0}^{\infty} p_{ab}^{(n)} < \infty. \tag{15.4}$$

Let S denote the matrix (s_{ab}), $(a, b) \in \mathbf{A} \times \mathbf{A}$. It is easy to verify that

$$(I-P)S = I = S(I-P), \tag{15.5}$$

the second equation being of course also a consequence of the first. Hence $I-P$ has the inverse $S \geqslant 0$.

Case 2. $\tilde{P} = P$. Then under the condition of the lemma every state in $\mathbf{A}$ is nonrecurrent with respect to P by the same argument as before, and so (15.4) is still true. The rest is the same as before.

The sufficiency of the condition is proved.

Now suppose the condition of the lemma is not fulfilled, namely that there exists a subset $\mathbf{C}$ of $\mathbf{A}$ such that $P|_{\mathbf{C}}$ is stochastic. Then define the vector $w = \{w_a, a \in \mathbf{A}\}$ as follows: $w_a = 1$ or 0 according as $a \in \mathbf{C}$ or $a \in \mathbf{A} - \mathbf{C}$. Clearly we have $w \neq 0$ and $(I-P)w = 0$ so that $I-P$ is not invertible.

Remark. The "sufficiency" part of the lemma and its proof above can be extended to an infinite index set $\mathbf{A}$, but the resulting "inverse" in the sense of (15.5) is not a true inverse operator because the usual multiplication of infinite matrices, even when it is defined, is not necessarily associative. Thus $(I-P)x = y$ is not equivalent to or even implies $x = (I-P)^{-1}y$. An appropriate extension may be needed for the boundary theory with infinitely many atoms.

We are now in position to formulate the theorem on canonical decomposition.

THEOREM 15.2. *If all boundary atoms are distinguishable then for every λ: $0 < \lambda < \infty$, the matrix $I - \hat{F}(\lambda)$ in* (15.3) *has a nonnegative inverse so that we have*

$$\hat{\xi}(\lambda) = [I - \hat{F}(\lambda)]^{-1} \hat{\varrho}(\lambda) = [I - \hat{F}(\lambda)]^{-1} \hat{E}(\lambda)\, \hat{\eta}(\lambda), \tag{15.6}$$

where $\hat{E}(\lambda)$ is the diagonal matrix with entries $\{\hat{E}^a(\lambda), a \in \mathbf{A}\}$,

$$\hat{E}^a(\lambda) = \int_0^\infty e^{-\lambda t}\, dE^a(t), \qquad \hat{\eta}^a(\lambda) = \int_0^\infty e^{-\lambda t}\, \eta^a(t)\, dt,$$

and E^a and η^a are given in Theorem 14.4.

Proof. Fix a λ and apply Lemma 2 with $P=\hat{F}(\lambda)$. Suppose that there exists $\mathbf{C}\subset\mathbf{A}$ such that $\hat{F}(\lambda)|_{\mathbf{C}}$ is stochastic then there exists $\mathbf{C}_0\subset\mathbf{C}$ such that $\hat{F}(\lambda)|_{\mathbf{C}_0}$ is the one-step transition matrix of a discrete parameter Markov chain where state space $\mathbf{C}_0$ is a recurrent class. Since $\hat{F}(\lambda)$ has zero elements on the diagonal, $\mathbf{C}_0$ must contain at least two states, thus

$$\sum_{b\in\mathbf{C}_0-\{a\}}\hat{F}^{ab}(\lambda)=1,\quad a\in\mathbf{C}_0. \tag{15.7}$$

Furthermore, since $\lambda>0$, $\hat{F}^{ab}(\lambda)\leqslant F^{ab}(\infty)$ and $\sum_{b\neq a}F^{ab}(\infty)\leqslant 1$, (15.7) is possible for each a if and only if

$$\hat{F}^{ab}(\lambda)=F^{ab}(0)=F^{ab}(\infty)\quad\text{and}\quad\sum_{b\neq a}F^{ab}(0)=1.$$

It follows then by (14.4) that $\varrho^a(\cdot)\equiv 0$ and consequently (15.1) is reduced to

$$\xi^a_j(t)=\sum_{b\in\mathbf{C}_0-\{a\}}F^{ab}(0)\,\xi^b_j(t),\quad a\in\mathbf{C}_0,$$

for each $t>0$ and $j\in\mathbf{I}^a$. This means the matrix $(F^{ab}(0))$, $(a,b)\in\mathbf{C}_0\times\mathbf{C}_0$, with $F^{aa}(0)=0$ for each a, is the one-step transition matrix of a discrete parameter recurrent Markow chain and that $\{\xi^a_j(t),a\in\mathbf{C}_0\}$ for fixed t and j is a harmonic (regular) function on $\mathbf{C}_0$ relative to this matrix. Such a function must be a constant, namely:

$$\xi^a_j(t)=\xi^b_j(t),\quad j\in\mathbf{I}^a. \tag{15.8}$$

It follows that
$$1=\sum_{j\in\mathbf{I}^a}\xi^a_j(t)=\sum_{j\in\mathbf{I}^a}\xi^b_j(t).$$

Hence $\mathbf{I}^b\subset\mathbf{I}^a$ and so $\mathbf{I}^a=\mathbf{I}^b$ for every a and b in $\mathbf{C}_0$, since a and b are interchangeable above. Thus (15.8) is true for every j in $\mathbf{I}^a=\mathbf{I}^b$ and every t, and so a and b are indistinguishable by definition (see [I; p. 44]), contrary to hypothesis.

We have thus proved that the condition of the lemma is fulfilled for $\hat{F}(\lambda)$ for each $\lambda>0$, and therefore $I-\hat{F}(\lambda)$ is invertible. This proves the first equation in (15.6); the second follows from (14.17). Theorem 15.2 is completely proved.

It is instructive to compare this proof with that of Theorem 5.5 of [I] to see where progress has been made.

Before proceeding further we will stop for a minor generalization in order to include the case where $\prod$ is substochastic (see pp. 23–24, 25–26 of [I]).(1) Instead of Assumption C_0 in § 14 we make the following slightly weaker one.

ASSUMPTION C_1. *There is at most one Φ-recurrent state (and this is then necessarily Φ-absorbing).*

(1) It is not clear why the substochastic case causes trouble in Williams [VIII].

This state is to be denoted by θ and not included in $\mathbf{I}$, and $\mathbf{I}_\theta = \mathbf{I} \cup \{\theta\}$ is to be the state space if θ is present under C_1. We have [I; (2.7)]

$$f_{i\theta} \equiv 0, \quad f_{\theta i} \equiv 0, \quad f_{\theta\theta} \equiv 1, \quad L_\theta \equiv 0. \tag{15.9}$$

It follows that θ does not belong to $\mathbf{I} - \bigcup_{a \in \mathbf{A}} \mathbf{I}^a$ provided that $\mathbf{P}\{x(0) = \theta\} = 0$; in other words, with probability one θ does not appear before the boundary is reached. It may belong to some $\mathbf{I}^a$ and not to others. If it belongs to $\mathbf{I}^a$, then a cannot be recurrent. For such an θ, we have by (14.10):

$$\varrho_\theta^a(s+t) = \varrho_\theta^a(s) + \sum_{i \in \mathbf{I}_\theta} \varrho_i^a(s) f_{i\theta}(t) + \int_0^t \left[\sum_{i \in \mathbf{I}_\theta} \varrho_i^a(s)\, l_i^a(u) \right] \varrho_\theta^a(t-u)\, du.$$

It is seen at once that the first sum above is equal to zero, and in the second the summation may be replaced by $i \in \mathbf{I}$, on account of (15.9). Since a is not recurrent $r_i^a = \int_0^\infty \varrho_i^a(t)\, dt < \infty$ for every $i \in \mathbf{I}^a - \{\theta\}$, it follows by integrating the equation above that

$$\int_0^\infty [\varrho_\theta^a(s+t) - \varrho_\theta^a)]\, ds = \int_0^t \sigma^{aa}(u)\, \varrho_\theta^a(t-u)\, du.$$

Since $\varrho_\theta^a(\cdot) \nearrow$, the limit
$$d^a \overset{\text{def}}{=} \lim_{t \uparrow \infty} \varrho_\theta^a(t) \tag{15.10}$$

exists and the equation above reduces to

$$d^a t = \int_0^t \varrho_\theta^a(u)\, du\, [1 + \sigma^{aa}(t-u)]\, du. \tag{15.11}$$

Comparing this with (14.19), we see that we should set

$$\eta_\theta^a(\cdot) \overset{\text{def}}{=} d^a \tag{15.12}$$

in order that (14.19) may be valid for θ as well as the other states in $\mathbf{I}^a$. With this definition (14.17) is valid as follows:

$$\varrho_\theta^a(t) = \int_0^t \eta_\theta^a(t-s)\, E^a(ds) = d^a E^a(t). \tag{15.13}$$

Now (15.1), (15.2) and (15.3) are valid for the subscript θ as well as the others in $\mathbf{I}^a$. Finally (14.29) becomes
$$\sum_{i \in \mathbf{I}_\theta} \eta_i^a(t) = d^a + c^a + \sum_{b \in \mathbf{A}} \sigma^{ab}(t).$$

If a is recurrent then $\theta \notin \mathbf{I}^a$ and we may set $d^a = 0$.

Briefly, the above discussion goes to show that if there is only one Φ-recurrent state, or more generally if all Φ-recurrent states are merged into one absorbing state, it acts like an atom in the recurrent part of the boundary as to be expected from the Martin theory.

Assumption C_1 is justified as follows. A Φ-recurrent state belongs to the set Z of states from which with probability one the boundary will not be reached at all [I; Theorem 3.2], so that after entering such a state the process is controlled by Φ above. Such states form a stochastically closed set which splits into possibly infinitely many disjoint Φ-recurrent classes. Each class may be treated as the θ above as an atom on the recurrent part of the boundary; or in the informal language used before, as a banner trap under which the states of that class line up according to the law of transition Φ. By merging all these classes into one single trap θ in Assumption C_1, we are just stopping the process at the recurrent part of the boundary—not so much because it can be totally ignored but because its behavior from there on is well known and may be separated from the rest of the study in the name of convenience.

§ 16. Construction

We shall lead up to the so-called "construction theorem" by reviewing the components and steps, suitably algebraicized, which enter into the canonical decomposition (15.6).

Let Q be given satisfying Assumption A and let $\Phi(\cdot)$ be the minimal solution associated with it. Put

$$z \stackrel{\text{def}}{=} L(\infty) = \lim_{t\uparrow\infty} [I - \Phi(t)]\, 1 \lim_{\lambda\downarrow 0} [I - \lambda\hat{\Phi}(\lambda)]\, 1; \tag{16.1}$$

z is an exit sequence relative to Φ which is maximal in the sense of Theorem 13.3. Let $\mathbf{A}$ be a finite index set; for each a in $\mathbf{A}$ let z^a be an exit sequence relative to Φ such that

$$z = \sum_{a\in\mathbf{A}} z^a. \tag{16.2}$$

Let
$$z^a(\lambda) \stackrel{\text{def}}{=} [I - \lambda\hat{\Phi}(\lambda)]\, z^a.$$

It follows from Theorem 13.3 that

$$\lim_{\lambda\downarrow 0} \lambda\hat{\Phi}(\lambda)\, z^a = 0. \tag{16.3}$$

For each a in $\mathbf{A}$ let e^a be an entrance sequence relative to Φ. This means in terms of Laplace transforms:

$$\forall\lambda \geqslant 0\colon\ e^a \geqslant e^a \lambda\hat{\Phi}(\lambda).$$

Assume that
$$\lim_{\lambda \downarrow 0} e^a \lambda \hat{\Phi}(\lambda) = 0 \tag{16.4}$$

and define the entrance law η^a as follows:

$$\hat{\eta}^a(\lambda) \overset{\text{def}}{=} k^a e^a [I - \lambda \hat{\Phi}(\lambda)], \tag{16.5}$$

where k^a is a positive constant to be fixed later. We set also

$$c^a k^a c_0^a \overset{\text{def}}{=} \langle \lambda \hat{\eta}^a(\lambda), 1 - z \rangle \tag{16.6}$$

which does not depend on λ; and

$$\hat{\sigma}^{ab}(\lambda) \overset{\text{def}}{=} \langle \hat{\eta}^a(\lambda), z^b \rangle, \qquad \hat{u}^{ab}(\lambda) \overset{\text{def}}{=} \lambda \hat{\sigma}^{ab}(\lambda). \tag{16.7}$$

Now we assume that, for each a and $0 \leqslant \lambda < \infty$:

$$\hat{u}^{aa}(\lambda) < \infty; \tag{16.8}$$

and for $a \neq b$:
$$\hat{u}^{ab}(\infty) \overset{\text{def}}{=} \lim_{\lambda \uparrow \infty} \hat{u}^{ab}(\lambda) < \infty. \tag{16.9}$$

For each pair a and b, $a \neq b$, we choose a constant Ω_0^{ab} such that

$$\hat{u}^{ab}(\infty) \leqslant k^a \Omega_0^{ab} < \infty. \tag{16.10}$$

Let $\Omega_0^{aa} = 0$ for each a, and

$$\mathbf{A}_0 \overset{\text{def}}{=} \{a \in \mathbf{A} : c_0^a + \sum_b \Omega_0^{ab} = 0\}$$

and
$$\delta^a \overset{\text{def}}{=} \begin{cases} 0 & \text{if} \quad a \in \mathbf{A}_0, \\ 1 & \text{if} \quad a \in \mathbf{A} - \mathbf{A}_0. \end{cases}$$

Now fix the constant k^a for each a so that

$$k^a c_0^a + k^a \sum_b \Omega_0^{ab} = \delta^a. \tag{16.11}$$

Write Ω^{ab} for $k^a \Omega_0^{ab}$ and let Ω be the matrix (Ω^{ab}), $(a, b) \in \mathbf{A} \times \mathbf{A}$. Furthermore, put

$$\hat{E}^a(\lambda) \overset{\text{def}}{=} \frac{1}{\delta^a + \hat{u}^{aa}(\lambda)} \tag{16.12}$$

and
$$\hat{\varrho}^a(\lambda) \overset{\text{def}}{=} \hat{E}^a(\lambda)\, \hat{\eta}^a(\lambda); \tag{16.13}$$

$$\hat{F}^{aa}(\lambda) \overset{\text{def}}{=} 0; \qquad \hat{F}^{ab}(\lambda) \overset{\text{def}}{=} \hat{E}^a(\lambda)\, (\Omega^{ab} - \hat{u}^{ab}(\lambda)), \quad a \neq b. \tag{16.14}$$

Let $\hat{F}(\lambda)$ be the matrix $(F^{ab}(\lambda))$. If there exists a subset C of A such that $(\hat{F}(0))|_{\mathbf{C}}$ is stochastic, then we decree that all indices in C be identified. When this has been done $I-\hat{F}(\lambda)$ will be invertible by Lemma 2 of § 15. Let $\hat{D}(\lambda)$ be the diagonal matrix with entries $(\hat{u}^{aa}(\lambda), a\in\mathbf{A})$ and let $\check{I}$ be the diagonal matrix with entries $(\delta^a, a\in\mathbf{A})$. Finally, we put

$$\hat{\xi}(\lambda) \overset{\text{def}}{=} [I-\hat{F}(\lambda)]^{-1}\hat{\varrho}(\lambda) = [I-\hat{F}(\lambda)]^{-1}\hat{E}(\lambda)\hat{\eta}(\lambda)$$

$$= [I-\hat{F}(\lambda)]^{-1}[\check{I}+\hat{D}(\lambda)]^{-1}\hat{\eta}(\lambda) = \hat{M}(\lambda)\hat{\eta}(\lambda), \tag{16.15}$$

where $\hat{M}(\lambda) = (\hat{M}^{ab}(\lambda))$, $(a,b)\in\mathbf{A}\times\mathbf{A}$ is the matrix defined below:

$$\hat{M}(\lambda) \overset{\text{def}}{=} [I-\hat{F}(\lambda)]^{-1}[\check{I}+\hat{D}(\lambda)]^{-1}; \tag{16.16}$$

and
$$\hat{\prod}(\lambda) \overset{\text{def}}{=} \hat{\Phi}(\lambda) + \sum_{a\in\mathbf{A}} \hat{z}^a(\lambda)\hat{\xi}^a(\lambda) = \hat{\Phi}(\lambda) + \sum_{a\in\mathbf{A}}\sum_{b\in\mathbf{A}} \hat{z}^a(\lambda)\hat{M}^{ab}(\lambda)\hat{\eta}^b(\lambda). \tag{16.17}$$

THEOREM 16.1. *The* $\hat{\prod}(\lambda)$ *so constructed is the Laplace transform (resolvent) of a stochastic transition matrix function* $\prod(t)$:

$$\hat{\prod}(\lambda) = \int_0^\infty e^{-\lambda t}\prod(t)\,dt.$$

Conversely, under Assumptions A, B′, C_0 *and* D *every such* $\prod$ *may be constructed in the manner described above.*

Proof. The "converse" part, somewhat vaguely stated here, is just an algebraic restatement of Theorem 15.2 and Theorem 5.1 of [I] with $z^a = L^a(\infty)$.

The proof of the "direct" part will be sketched. It is merely a matter of algebraic verification based on the resolvent equations, and is quite similar to pp. 67–68 of [I]. Dropping the "^" on Laplace transforms, we need to verify (8.26) of [I], i.e.

$$(\lambda-\mu)M(\lambda)\Theta(\lambda,\mu)M(\mu) = M(\mu)-M(\lambda)$$

or equivalently, since, $U(\lambda)$ being the matrix $(u^{ab}(\lambda))$,

$$(\lambda-\mu)\Theta(\lambda,\mu) = \Theta(\mu)-\Theta(\lambda) = U(\lambda)-U(\mu)$$

(see (8.25) of [I] and the equations preceding it), that

$$M(\lambda)[U(\lambda)-U(\mu)]M(\mu) = M(\mu)-M(\lambda). \tag{16.18}$$

Since $M(\lambda)^{-1}$ exists by (16.16), this in turn is equivalent to

$$U(\lambda)-U(\mu)=M(\lambda)^{-1}[M(\mu)-M(\lambda)]M(\mu)^{-1}=M(\lambda)^{-1}-M(\mu)^{-1}. \tag{16.19}$$

Now by simple inspection, one sees that

$$M(\lambda)^{-1}=[\tilde{I}+D(\lambda)][I-F(\lambda)]=\tilde{I}-\Omega+U(\lambda). \tag{16.20}$$

For, on the diagonal, $F(\lambda)$ and Ω are both zero while $D(\lambda)$ is by definition the diagonal part of $U(\lambda)$; off the diagonal, the second equation in (16.20) reduces to

$$[\tilde{I}+D(\lambda)]F(\lambda)=\Omega-U(\lambda)$$

which is just (16.14). Since $\tilde{I}-Q$ does not depend on λ, (16.19) follows from (16.20) and so the resolvent equation for $\prod(\lambda)$ is verified (see the calculation after (8.26) in [I]).

To verify "stochasticity" (the "norm condition"), we write

$$I=M(\lambda)M(\lambda)^{-1}=M(\lambda)[\tilde{I}-\Omega+U(\lambda)];$$

hence by (16.11) and (14.29) (Laplace-transformed):

$$\begin{aligned} 1=I1&=M(\lambda)[\tilde{I}1-\Omega 1+U(\lambda)1]\\ &=M(\lambda)[c+U(\lambda)1]=M(\lambda)\langle\lambda\eta(\lambda),1\rangle=\langle\lambda\xi(\lambda),1\rangle. \end{aligned} \tag{16.21}$$

This is necessary and sufficient for $\prod(\cdot)$ to be stochastic. Theorem 16.1 is completely proved.

COROLLARY. *To construct a substochastic* $\prod(\cdot)$, *choose any* $d_0^a\geqslant 0$ *for each* a *and choose* k^a *so that instead of* (16.11), *we have*

$$k^a(d_0^a+c_0^a+\sum_b\Omega_0^{ab})=\delta^a.$$

Apart from this no change is needed in the procedure. Conversely every substochastic $\prod(\cdot)$ *can be constructed in this manner; and it will be strictly substochastic if and only if* $d^a>0$ *for some* a.

The idea of the above extension to the substochstic case is of course a trivial and familiar one: one first constructs the stochastic completion by adjoining a new index to **I**, and the restriction of the completion to **I** will be the most general substochastic case.

There is another complement, the so-called "extension to the boundary" discussed by previous authors. The idea is to construct a Markov chain x^* on the enlarged state space $\mathbf{I}^*\overset{\text{def}}{=}\mathbf{I}\cup\mathbf{A}$ with correspondingly enlarged transition matrix function $\prod^*(\cdot)$ in such a way that if in this process "the time spent in the boundary set **A** be deleted", the resulting shrunk process will be the given (or constructed as the case may be) x, **I**, $\prod$. This idea goes back to Lévy and has been developed by Neveu and Williams. The algebraic part was already indicated by Feller and in the one-atom case by Reuter. We shall show here how the construction of the latter authors can be extended to our case by a simple modification of Theorem 16.1. This is effected by a more general application of the Lemma of § 14, which is indeed a kind of analytical shadow of Lévy's idea of alloting time to fictitious states. The development of this paper makes it clear that this allotment may be done for each boundary atom separately.

Let k denote a general element of $\mathbf{I}^*$ defined above. For each a in $\mathbf{A}$, choose a number $p^a \geqslant 0$ and introduce the new quantities (dropping "$\hat{\ }$" as before):

$$z_b^a \overset{\text{def}}{=} \delta^{ab},$$

$$\eta_b^a(\lambda) \overset{\text{def}}{=} p^a \delta^{ab}.$$

Thus c^a, σ^{ab} and u^{ab} are not affected for $a \neq b$, but if we denote new quantities by affixing "$*$" to the corresponding old ones, we have

$$u^{*aa}(\lambda) = u^{aa}(\lambda) + p^a \lambda,$$

so that

$$U^*(\lambda) = U(\lambda) + p\lambda I,$$

$$\langle \lambda\eta^*(\lambda), 1\rangle = \langle \lambda\eta(\lambda), 1\rangle + p^a \lambda,$$

$$E^{*a}(\lambda) = \frac{1}{\delta^a + p^a\lambda + u^{aa}(\lambda)},$$

$$M^*(\lambda)^{-1} = M(\lambda)^{-1} + p\lambda I,$$

$$\Theta^*(\lambda, \mu) = \Theta(\lambda, \mu) + pI.$$

It follows from (16.19) that

$$\begin{aligned} U^*(\lambda) - U^*(\mu) &= (\lambda - \mu)\,\Theta^*(\lambda, \mu) = U(\lambda) - U(\mu) + p(\lambda - \mu)\,I \\ &= M(\lambda)^{-1} - M(\mu)^{-1} + p(\lambda - \mu)\,I = M^*(\lambda)^{-1} - M^*(\mu)^{-1}, \end{aligned}$$

which is the new (16.19) needed for the verification of the resolvent equation. The stochasticity is verified exactly as before in (16.21) with the appropriate quantities starred.

It should be pointed out that unless one begins with the canonical decomposition, the extended process constructed above will not yield the correct information on the boundary behavior of the original process. For instance, the transition of the new states in $\mathbf{A}$ among themselves will not be controlled by the jump matrix $(F^{ab}(\infty))$, as it should be, but rather by Ω which may not be the same matrix when the decomposition is not canonical. This leads to the subject matter of the next two sections.

§ 17. Etude approfondie

In this section we begin by giving probabilistic meanings to the quantities, E^a, σ^{ab}, $a \neq b$, which were derived analytically in § 14. Using these we shall be able to analyze in more depth the functions $F^{ab}(\cdot)$ and in particular their limit values $F^{ab}(0)$ and $F^{ab}(\infty)$. These results are also requisite for the problems dealt with in the next section.

A fundamental new random variable, NOT an optional one, will now be introduced.

Definition 17.1. For each boundary atom a, let

$$\gamma^a(\omega) = \sup\{S^a(\omega) \cap (0, \beta^a(\omega))\}.$$

This is the "last exit time from a before switch (changing banners)".

Clearly we have, using the definition in (14.23):

$$\mathbf{P}^a\{\gamma^a < +\infty\} = \delta^a.$$

If a is nonsticky, γ^a is just one of the sequence of times, finite in every finite interval, when the process reaches the boundary at a. If a is sticky, it follows as in the Corollary to Theorem 12.4 that $\gamma^a \in \overline{S^a} - S^a$, so that in particular the process is by definition neither at a nor indeed at any other boundary point. Intuitively, $x(\gamma^a - 0)$ is at an "inaccessible boundary" not definable by means of the jump chain alone. Its behavior is best understood by analogy with the last exit time from an instantaneous state (see [1; Addenda] and [2]). A more comprehensive new boundary theory must surely cover this situation but for our present purposes it is opossible to circumvent this difficulty. The basic nature of γ^a is clearly reflected in the theorem below.

THEOREM 17.1. *We have for each a and $a \neq b$:*

$$\mathbf{P}^a\{\gamma^a \leqslant t\} = E^a(t); \tag{17.1}$$

$$\mathbf{P}^a\{\gamma^a < \beta^a;\ \gamma^a \in ds;\ \beta^a \in dt;\ x(\beta^a) = b\} = E^a(ds)\theta^{ab}(t-s)\,dt. \tag{17.2}$$

Remark. The meaning of the differentials will be explained in the proof. Note also that in (17.2), since $\beta^a > 0$, "$x(\beta^a) = b$" is well-defined, but later when β^a may be zero we must write "$\beta^a \in \overline{S^b}$" instead, as in the proof of Theorem 15.1.

Proof. We begin with the relation, valid for each $t \geqslant 0$:

$$\mathbf{P}^a\{\gamma^a > t\} = \sum_i \mathbf{P}^a\{\beta^a > t;\ x(t) = i;\ \alpha_t < \infty;\ x(\alpha_t) = a\}. \tag{17.3}$$

For we have $\gamma^a \leqslant \beta^a$; and on the set $\{\beta^a > t\}$, if $\alpha_t = +\infty$ then $S^a \cap (t, \infty) = \emptyset$; while if $\alpha_t < \infty$

and $x(\alpha_t) \neq a$, then $\alpha_t = \beta^a$ and $S^a \cap (t, \beta^a) = \emptyset$. Hence either case implies $\gamma^a \leqslant t$. This proves (17.3). Using (14.33), the right member of (17.3) may be written as

$$\langle \varrho^a(t), L^a(\infty)\rangle = \varrho^{aa}(t) = 1 - E^a(t), \tag{17.4}$$

proving (17.1). A similar argument establishes the more specific relation

$$\begin{aligned} \mathbf{P}^a\{\gamma^a < s < t < \beta^a < t';\, x(\beta^a) = b\} &= \langle \varrho^a(s), L^b(t'-s) - L^b(t-s)\rangle \\ &= \int_0^s \langle \eta^a(s-u), L^b(t'-s) - L^b(t-s)\rangle\, dE^a(u), \end{aligned} \tag{17.5}$$

the second equation by (14.17). Since the last-written integrand above is

$$\int_{t-s}^{t'-s} \langle \eta^a(s-u), l^b(v)\rangle\, dv = \int_{t-s}^{t'-s} \theta^{ab}(s-u+v)\, dv = \sigma^{ab}(t'-u) - \sigma^{ab}(t-u)$$

by (14.20) and (14.21), the last member in (17.5) reduces to

$$\int_{[0,s)} E^a(du) \int_{[t,t')} \theta^{ab}(r-u)\, dr.$$

It follows that for $0 \leqslant s < s' < t < t'$,

$$\mathbf{P}^a\{s \leqslant \gamma^a < s' < t \leqslant \beta^a < t';\, x(\beta^a) = b\} = \int_{[s,s')} E^a(du) \int_{[t,t')} \theta^{ab}(r-u)\, dr,$$

which is what is meant by (17.2).

Corollary 1. *We have*

$$\mathbf{P}^a\{\gamma^a < \beta^a \leqslant t;\, x(\beta^a) = b\} = \int_0^t [\sigma^{ab}(0) - \sigma^{ab}(t-s)]\, dE^a(s); \tag{17.6}$$

$$\mathbf{P}^a\{\gamma^a = \beta^a \leqslant t;\, \beta^a \in \overline{S^b}\} = E^a(t)\,[F^{ab}(\infty) - \sigma^{ab}(0)]. \tag{17.7}$$

Proof. It follows from (17.2) and Fubini's theorem on product measure that the left member of (17.6) is equal to

$$\int_{[0,t)} E^a(ds) \int_{[s,t)} \theta^{ab}(t-s)\, ds$$

which is the right member of (17.6). Now we have by Definition 14.1 and (14.35) that

$$\mathbf{P}^a\{\beta^a \leqslant t;\, \beta^a \in \overline{S^b}\} = F^{ab}(t) = \int_0^t [F^{ab}(\infty) - \sigma^{ab}(t-s)]\, dE^a(s). \tag{17.8}$$

Subtracting (17.6) from (17.8) we obtain (17.7). Q.e.d.

It is essential to understand the meaning of the probability in (17.7). How is $\gamma^a = \beta^a$ possible? This happens if and only if

$$\forall \delta > 0: \quad S^a \cap (\gamma^a - \delta, \gamma^a] \neq \emptyset; \quad S^b \cap (\gamma^a, \gamma^a + \delta) \neq \emptyset,$$

(a situation envisaged in the Remark on p. 39 of [I]). If a is nonsticky, then $x(\gamma^a) \in S^a$; if a is sticky, we have already noted that $x(\gamma^a) \in \overline{S^a} - S^a$. In either case the second relation above is possible only if b is sticky, by Theorem 12.5. Needless to say all the assertions above are true with probability one only. We have incidentally discovered an important number, now to be defined.

Definition 17.2. For $a \neq b$, let

$$d^{ab} = F^{ab}(\infty) - \sigma^{ab}(0). \tag{17.9}$$

COROLLARY 2. *If a is not a recurrent trap, then for every $b \neq a$,*

$$d^{ab} = \mathbf{P}^a\{\gamma^a = \beta^a \in \overline{S^b} \mid \gamma^a\}. \tag{17.10}$$

In particular, $d^{ab} = 0$ for every a if b is nonsticky.

Proof. If a is not a recurrent trap, we know from Theorem 14.4 that $E^a(\infty) = 1$. Letting $t \uparrow \infty$ in (17.7), we obtain

$$d^{ab} = \mathbf{P}^a\{\gamma^a = \beta^a < \infty;\ \beta^a \in \overline{S^b}\}.$$

Substituting this back into the right member of (17.7), and comparing with (17.1), we obtain (17.10).

COROLLARY 3. *For every a and b, $a \neq b$, we have*

$$d^{ab} = \lim_{t \downarrow 0} \frac{F^{ab}(t)}{E^a(t)}. \tag{17.11}$$

Proof. Observe first that $E^a(t) > 0$ for $t > 0$, from Corollary 1 to Theorem 14.7. We have by (14.34) and (14.36)

$$F^{ab}(\infty) = \frac{1}{E^a(t)} \left\{ \int_0^t \sigma^{ab}(t-s)\, dE^a(s) + F^{ab}(t) \right\}.$$

Letting $t \downarrow 0$ the corollary follows, since $\sigma^{ab}(t) \nearrow \sigma^{ab}(0)$.

COROLLARY 4. *For every $a \neq b$, and on the set $\{\gamma^a < t\}$, we have*

$$\mathbf{P}^a\{t < \beta^a < \infty;\ x(\beta^a) = b \mid \gamma^a\} = \sigma^{ab}(t - \gamma^a); \tag{17.12}$$

$$\mathbf{P}^a\{\gamma^a<\beta^a<\infty;\, x(\beta^a)=b\,|\,\gamma^a\}=\sigma^{ab}(0); \tag{17.13}$$

$$\mathbf{P}^a\{\beta^a\in\overline{S^b}\,|\,\gamma^a\}=F^{ab}(\infty); \tag{17.14}$$

$$\mathbf{P}^a\{\beta^a=+\infty\,|\,\gamma^a\}=\varrho_*^a(\infty). \tag{17.15}$$

Proof. By (17.2)

$$\mathbf{P}^a\{\gamma^a\leqslant s<t<\beta^a;\, x(\beta^a)=b\}=\int_0^s\sigma^{ab}(t-u)\,dE^a(u).$$

This being true for fixed t and arbitrary $s<t$, we have (17.12). Next, we write for each $t\geqslant 0$:

$$F^{ab}(\infty)=\mathbf{P}^a\{\beta^a\leqslant t;\, \beta^a\in\overline{S^b}\}+\mathbf{P}^a\{\gamma^a\leqslant t<\beta^a;\, x(\beta^a)=b\}+\mathbf{P}^a\{t<\gamma^a;\, x(\beta^a)=b\}. \tag{17.16}$$

The first term on the right side is $F^{ab}(t)$; the second is by (17.2) equal to

$$\int_0^t\sigma^{ab}(t-s)\,E^a(ds);$$

hence the third is equal to

$$F^{ab}(\infty)-F^{ab}(t)-\int_0^t\sigma^{ab}(t-s)\,E^a(ds)=[1-E^a(t)]\,F^{ab}(\infty)$$

by (14.35). This means, by (17.1):

$$\mathbf{P}^a\{\gamma^a>t;\, \beta^a\in\overline{S^b}\}=\mathbf{P}^a\{\gamma^a>t\}\,\mathbf{P}^a\{\beta^a\in\overline{S^b}\}$$

so that there is independence and (17.14) follows. From this (17.15) follows by (14.4). Finally subtracting (17.10) from (17.14) we obtain (17.13) which is a limiting form of (17.12).

The next theorem is the completed version of Theorem 5.3 in [I]. The proof there is analytic and leaves one important point unsettled, concering $L^{ab}(0+)$, which is specifically mentioned on p. 42 of [I]. The difficulty is resolved here since the relevant sample function behavior has now been clarified and the new proof, given in perhaps excessive detail here owing to past failure, is probabilistic. We put $F^{aa}(0)=0$ for each a below.

THEOREM 17.2. *For each a and b in* $\mathbf{A}$ *and* $0<t\leqslant+\infty$ *the limits below exist:*

$$L^{ab}(t)\stackrel{\text{def}}{=}\lim_{s\downarrow 0}\langle\xi^a(s),L^b(t-s)\rangle, \tag{17.17}$$

$$\tilde{L}^{ab}(t)\stackrel{\text{def}}{=}\lim_{s\downarrow 0}\langle\varrho^a(s),L^b(t-s)\rangle; \tag{17.18}$$

and we have $$L^{ab}(t)-\tilde{L}^{ab}(t)=F^{ab}(0)=L^{ab}(0+)\geqslant\tilde{L}^{ab}(0+)=0. \tag{17.19}$$

Furthermore, if a is sticky, then

$$L^{ab}(t)\equiv\tilde{L}^{ab}(t)\equiv\delta^{ab}. \tag{17.20}$$

Proof. For convenience' sake, let us state the following lemma which is trivial to prove as soon as it is formulated but useful also in other similar circumstances.

LEMMA. *Let $\{\Lambda_s, s \geqslant 0\}$ be a family of sets in $\mathfrak{F}$, and Λ also in $\mathfrak{F}$. Suppose that there exists a sequence of sets Ω_n in $\mathfrak{F}$ such that $\Omega_n \nearrow \Omega$ and such that*

$$\Lambda_s \cap \Omega_n = \Lambda \cap \Omega_n$$

whenever $s < \varepsilon_n$ where $\varepsilon_n \searrow 0$, then

$$\lim_{s \downarrow 0} \mathbf{P}(\Lambda_s) = \mathbf{P}(\Lambda).$$

To prove Theorem 17.2 we prove first (17.20). If a is sticky, for each $t > 0$ and a.e. ω, there exists an integer n_0 such that:

$$\forall n \geqslant n_0 \colon \ \alpha_{n^{-1}}(\omega) < t, \ \ x(\alpha_{n^{-1}}) = a.$$

Let $n_0(\omega)$ be the least such integer and put $s_0(\omega) = \alpha_{n_0(\omega)^{-1}}$. Then $s_0(\omega)$ is a random variable satisfying for a.e. ω:

(i) $$0 < s_0(\omega) < t,$$

(ii) $$x(s_0(\omega)) = a,$$

(iii) $$\bigcup_{b \neq a} S^b(\omega) \cap (0, s_0(\omega)) = \emptyset.$$

The last as a consequence of Definition 12.2 and Theorem 4.6 of [I].

Let
$$\Omega_n \overset{\text{def}}{=} \{\omega : s_0(\omega) > n^{-1}\},$$

$$\Lambda_s^b \overset{\text{def}}{=} \{s < \beta^a;\ \alpha_s \leqslant t;\ \alpha_s \in \overline{S^b}\}, \quad s \geqslant 0,\ b \in \mathbf{A}$$

We have $\Lambda_s^a \cap \Omega_n = \Omega \cap \Omega_n$ whenever $s < n^{-1}$; hence by the Lemma above

$$\lim_{s \downarrow 0} \mathbf{P}^a(\Lambda_s^a) = \mathbf{P}^a(\Omega) = 1.$$

Now by definition we have

$$\mathbf{P}^a(\Lambda_s^a) = \langle \varrho^a(s), L^a(t-s) \rangle,$$

and consequently we have proved that $\tilde{L}^{aa}(t) \equiv 1$. Since

$$\sum_{b \in \mathbf{A}} \tilde{L}^{ab}(t) \leqslant \lim_{s \downarrow 0} \langle \varrho^a(s), 1 \rangle \leqslant 1,$$

this implies $\tilde{L}^{ab}(t) \equiv 0$ for $b \neq a$. Since $\xi^a(\cdot) \geqslant \varrho^a(\cdot)$, the first equation in (17.20) now follows. We have proved (17.17)–(17.20) for a sticky a.

From now on in this proof let a be nonsticky. Let

$$\Delta^c \overset{\text{def}}{=} \{\omega : 0 \in \overline{S^c(\omega)}\}, \quad \Delta' = \Omega \setminus \bigcup_{c \in \mathbf{A}} \Delta^c.$$

On $\Delta', \beta^a \geqslant \alpha > 0$ a.e. and if $s < \alpha(\omega)$, then $\alpha_s(\omega) = \alpha(\omega)$. Hence if $\Omega_n' = \{\omega : \alpha(\omega) > n^{-1}\}$, we have

$$\Omega_n' \cap \Delta' \cap \Lambda_s^b = \Omega_n' \cap \Delta' \cap \Lambda_0^b$$

whenever $s < n^{-1}$. Hence as before

$$\lim_{s \downarrow 0} \mathbf{P}^a(\Delta' \cap \Lambda_s^b) = \mathbf{P}^a(\Delta' \cap \Lambda_0^b).$$

Since a is nonsticky, under $\mathbf{P}^a$ we have $\alpha^a > 0$ as well as $\beta^a > 0$ on Λ_s^b for $s > 0$, so that

$$\mathbf{P}^a(\Delta' \cap \Lambda_s^b) = \mathbf{P}^a(\Lambda_s^b) = \langle \varrho^a(s), L^b(t-s)\rangle.$$

Together with the preceding relation this proves

$$\tilde{L}^{ab}(t) = \mathbf{P}^a(\Delta' \cap \Lambda_0^b).$$

Next, let $$M_s^b \stackrel{\text{def}}{=} \{\alpha_s \leqslant t, x(\alpha_s) = b\}, \quad s > 0.$$

Then we have by definition

$$\langle \xi^a(s), L^b(t-s)\rangle = \mathbf{P}^a(M_s^b) = \sum_{c \in \mathbf{A}} \mathbf{P}^a(\Delta^c \cap M_s^b) + \mathbf{P}^a(\Delta' \cap M_s^b).$$

Since $\Delta' \cap (M_s^b \backslash \Lambda_s^b) \subset \Delta' \cap \{\beta^a \leqslant s\} \searrow \emptyset$ as $s \downarrow 0$, we have

$$\lim_{s \downarrow 0} \mathbf{P}^a(\Delta' \cap M_s^b) = \lim_{s \downarrow 0} \mathbf{P}^a(\Delta' \cap \Lambda_s^b),$$

hence the last term above converges to $\tilde{L}^{ab}(t)$ as just shown. For each sticky c, we have by Lemma 1 of § 15:

$$\lim_{s \downarrow 0} \mathbf{P}^a(\Delta^c \cap M_s^b) = F^{ac}(0) \lim_{s \downarrow 0} \mathbf{P}^c(M_s^b) = F^{ac}(0) L^{cb}(t) = F^{ac}(0)\, \delta^{cb},$$

the second equation above being an application of (17.17) and the third of (17.20) both with $a = c$. Combining these, we obtain

$$L^{ab}(t) = \sum_{c \in \mathbf{A}} F^{ac}(0)\, \delta^{cb} + \tilde{L}^{ab}(t) = F^{ab}(0) + \tilde{L}^{ab}(t).$$

This proves the first equation in (17.19); to prove the second, we observe that by definition

$$L^{ab}(t) \leqslant \mathbf{P}^a\{S^b \cap (0, t) \neq \emptyset\}$$

and consequently $L^{ab}(0+) \leqslant F^{ab}(0) \leqslant L^{ab}(0+)$. The rest follows. Q.e.d.

The following result is an essential sharpening of the preceding theorem and is also the major step towards the identification problem in the next section. It seems to depend almost precariously on the finer properties of boundary atoms.

THEOREM 17.3. *For each a, we have*

$$\lim_{s \downarrow 0} \frac{1 - \langle \xi^a(s), L^a(\infty)\rangle}{E^a(s)} = \delta^a; \tag{17.21}$$

and if $\delta^a = 1$, for $0 < t \leqslant +\infty$:

$$\lim_{s \downarrow 0} \frac{1 - \langle \xi^a(s), L^a(t)\rangle}{1 - \langle \xi^a(s), L^a(\infty)\rangle} = 1 + \sigma^{aa}(t). \tag{17.22}$$

For each $c \neq a$ and $0 < t \leqslant +\infty$:

$$\lim_{s \downarrow 0} \frac{\langle \xi^a(s), L^c(t)\rangle}{1 - \langle \xi^a(s), L^a(\infty)\rangle} = F^{ac}(\infty) - \sigma^{ac}(t); \tag{17.23}$$

in particular

$$\lim_{s \downarrow 0} \frac{\langle \xi^a(s), L^c(\infty)\rangle}{1 - \langle \xi^a(s), L^a(\infty)\rangle} = F^{ac}(\infty). \tag{17.24}$$

Proof. We have by (15.1):

$$\langle \xi^a(s), L^c(t)\rangle = \langle \varrho^a(s), L^c(t)\rangle + \sum_{b \neq a} \int_0^s \langle \xi^b(s-u), L^c(t)\rangle \, dF^{ab}(u). \tag{17.25}$$

The first term on the right is equal to

$$\int_0^s \langle \eta^a(s-u), L^c(t)\rangle \, dE^a(u) = \int_0^s [\sigma^{ac}(s-u) - \sigma^{ac}(s-u+t)] \, dE^a(a).$$

Recalling (17.11), we have as $s \downarrow 0$:

$$F^{ab}(s) \sim E^a(s) \, d^{ab}. \text{ (}^1\text{)}$$

Substituting the last two equations in (17.25), we have as $s \downarrow 0$:

$$\langle \xi^a(s), L^c(t)\rangle \sim E^a(s) \{\sigma^{ac}(0) - \sigma^{ac}(t) + \sum_{b \neq a} d^{ab} L^{bc}(t)\}.$$

But by Corollary 2 to Theorem 17.1, $d^{ab} = 0$ unless b is sticky, and if so $L^{bc}(t) = \delta^{bc}$ by (17.20); hence

$$\sum_{b \neq a} d^{ab} L^{bc}(t) = \sum_{b \neq a} d^{ab} \delta^{bc} = \begin{cases} d^{ac} & \text{if} \quad a \neq c, \\ 0 & \text{if} \quad a = c. \end{cases}$$

Consequently if $a \neq c$, we have by (17.9):

$$\langle \xi^a(s), L^c(t)\rangle \sim E^a(s) \{\sigma^{ab}(0) - \sigma^{ac}(t) + d^{ac}\} = E^a(s) \{F^{ac}(\infty) - \sigma^{ac}(t)\}. \tag{17.26}$$

(1) $u(s) \sim v(s)$ means $\lim_{s \downarrow 0} u(s)/v(s) = 1$.

Next, putting $a=c$ and $t=+\infty$ in (17.25), we have

$$\langle \xi^a(s), L^a(\infty)\rangle = \int_0^s \sigma^{aa}(s-u)\,dE^a(u) + \sum_{b\neq a}\int_0^s \langle \xi^b(s-u), L^a(\infty)\rangle\, dF^{ab}(u).$$

The first term on the right side is equal to $1-\delta^a E^a(s)$ by Corollary 1 to Theorem 14.7. Hence as $s\downarrow 0$ we have as before:

$$1-\langle \xi^a(s), L^a(\infty)\rangle \sim E^a(s)\,[\delta^a - \sum_{b\neq a} d^{ab} L^{ba}(\infty)] = E^a(s)\,\delta^a.$$

This proves (17.21), and together with (17.26) proves (17.23). If $\delta^a=1$ a similar argument yields

$$1-\langle \xi^a(s), L^a(t)\rangle \sim E^a(s)\,[1+\sigma^{aa}(t)].$$

Hence (17.22) follows from this and (17.21).

Remark. (17.24) may be written as

$$\lim_{\substack{s\downarrow 0\\ s>0}} \frac{\mathbf{P}^a\{x(\alpha_s)=c\}}{1-\mathbf{P}^a\{x(\alpha_s)=a\}} = F^{ac}(\infty),$$

or even more suggestively as

$$\lim_{\substack{s\downarrow 0\\ s>0}} \frac{\mathbf{P}^a\{x(\alpha_s)=b\}}{\mathbf{P}^a\{x(\alpha_s)=c\}} = \frac{F^{ab}(\infty)}{F^{ac}(\infty)}$$

provided $F^{ac}(\infty)>0$. Is either of these relations "intuitively obvious"?

§ 18. Identification

In this section we consider the following problem. Given $\prod=(p_{ij}(\cdot))$, how to "find" the quantities L^a, ξ^a, F^{ab}, ϱ^a, e^a, E^a, η^a, σ^{ab}, introduced earlier in the paper? These have all been defined in terms of the process $\{x_t\}$, but how can they be expressed, or at least, determined by means of $\prod$? The words "expression" and "determination" are of course themselves subject to interpretation but our results below will be stated in specific ways. This problem, to be called that of "identification", has its proper interest, but we shall also use its solution to answer the questions raised in § 11, concerning the tracking down of the canonical decomposition and the construction of stopped processes.

Let Λ be a set in the Borel field $\mathfrak{F}^0$ generated by $\{x_t, 0\leqslant t<\infty\}$. Since the basic probability measure $\mathbf{P}$ on $\mathfrak{F}^0$ is uniquely determined by its values on cylinder sets of the form $\{x(t_n)=j_n,\ 0\leqslant n\leqslant l\}$, $t_n\geqslant 0$, $j_n\in\mathbf{I}$, it is determined by $\prod$ and the initial distribution of the process. In particular, $\mathbf{P}_i\{...\}$ for each $i \in \mathbf{I}$ is determined by $\prod$ alone. However, many interesting sets Λ must in practice be defined in terms of a "nice version" of $\{x_t\}$ (see [1; § II.7]); hence the following simple observation is necessary.

LEMMA. *Let two arbitrary stochastic processes $\{x_t\}$ and $\{\tilde{x}_t\}$ defined on the probability triple $(\Omega, \mathfrak{F}, \mathbf{P})$ be standard modifications of each other, then the Borel fields $\mathfrak{F}^0$ and $\tilde{\mathfrak{F}}^0$ generated by them are identical provided they be both augmented (by all $\mathbf{P}$-null sets).*

Proof. If $\Lambda \in \mathfrak{F}^0$, it is well-known that there is a countable set $\{t_n\}$, $n \in \mathbf{N}$, such that $\Lambda \in \mathfrak{F}\{x(t_n),\ n \in \mathbf{N}\}$. For each n, $\mathbf{P}\{x(t_n) = \tilde{x}(t_n)\} = 1$ and so $\mathbf{P}\{\forall n \in \mathbf{N}: x(t_n) = \tilde{x}(t_n)\} = 1$. It follows that Λ differs from a set in $\mathfrak{F}\{\tilde{x}(t_n),\ n \in \mathbf{N}\}$ by a null set, hence $\Lambda \in \tilde{\mathfrak{F}}^0$ and the proof is finished.

As a consequence of the lemma, quantities such as those mentioned above are determined by $\prod$ in the sense that for two processes having the same transition matrix $\prod$, these quantities have the same values. However, even in the simplest cases involving no boundary, an explicit expression may be hard to come by, for example the taboo probability ${}_0p_{0j}(t)$ used in the proof of Theorem 14.3. Here indeed lies the great advantage of the probabilistic method, by relying on the sample functions of the process rather than its transition matrix. However, we are now pushing matters in the opposite direction.

To recapitulate from the beginning of the story (see [I; §§ 2–4]), Q is the initial derivative matrix $(p'_{ij}(0))$, supposed to be conservative; given Q, $\Phi(\cdot)$ can be constructed by a purely analytic iteration procedure (given 25 years ago by Feller) as the minimal solution of the Kolmogorov differential equations, both backward and forward. Q also determines (it is trivially equivalent to) the jump matrix $\mathsf{P} = (r_{ij})$ where $r_{ij} = (1 - \delta_{ij}) q_{ij} q_i^{-1}$. Now it is necessary to confront the boundary. Under Assumption B, that the passable part of the boundary be atomic, each boundary atom a, $a \in \mathbf{A}$, corresponds to an essentially uniquely determined atomic almost closed set A^a of $\mathbf{I}$, and we have (cf. [1; § I.17]):

$$L_i^a(\infty) = \lim_{n \to \infty} \mathsf{P}_i\{\chi_n \to a\} = \lim_{n \to \infty} \sum_{j \in A^a} r_{ij}^{(n)},$$

where $\{\chi_n\}$ is the jump chain whose probability behavior is controlled by its transition matrix P. The above is substantially Feller's definition of a sojourn solution [6]. There is another way of identifying $\{L^a(\infty),\ a \in \mathbf{A}\}$: they are the extreme points of the cone of solutions of the equation:

$$Qe = 0$$

with $e \leqslant 1$; see the discussion around Theorem 4.2 of [I]. This is also due to Feller.

Next, ξ^a may be identified by Theorem 4.5 of [I]:

$$\xi_j^a(t) = \lim_{\chi_n \to a} p_{\chi_n j}(t) \tag{18.1}$$

with probability one, for each j and $t>0$. The preceding formula, like the one above for $L^a(\infty)$, uses the jump chain and its boundary behavior. This seems inevitable as somehow the fact of approaching the boundary at a specified atom must be expressed, nevertheless there is the question of the "neatest" way of doing so. One may remark in passing that the Martin boundary theory (see [8]) yields a similar but less precise expression for L^∞ as follows:

$$\frac{L_i^a(\infty)}{L_0^a(\infty)} = \lim_{\chi_n \to a} \frac{\sum_{n=0}^{\infty} r_{i\chi_n}^{(n)}}{\sum_{n=0}^{\infty} r_{0\chi_n}^{(n)}}$$

where 0 is some fixed state.

Given ξ^a and $\prod$ the probability measure $P^a\{...\}$ on the Borel field $\mathfrak{F}\{x_t, t>0\}$ is uniquely determined, since if $0<t_1<...<t_l$, and $j_n \in \mathbf{I}$ we have

$$\mathbf{P}^a\{x(t_n) = j_n, 1 \leqslant n \leqslant l\} = \xi_{j_1}^a(t_1) \prod_{\nu=1}^{l-1} p_{j_\nu j_{\nu+1}}(t_{\nu+1} - t_\nu).$$

This may be extended to include the sets $\{x(0) = j\}$, since

$$\mathbf{P}^a\{x(0) = j\} = \lim_{t \downarrow 0} \mathbf{P}^a\{x(t) = j\}.$$

Analytically, the identification given for $\{\xi^a, a \in \mathbf{A}\}$ is sufficient to yield the following useful uniqueness theorem (cf. Reuter [14]).

THEOREM 18.1. *Suppose that there is a decomposition of the form*

$$\prod(t) = \Phi(t) + \sum_{a \in \mathbf{A}} \int_0^t y^a(t-s)\, dL^a(s), \tag{18.2}$$

where $\prod$, Φ, L *have previous meanings; and either for each* a, y^a *is measurable and* $\geqslant 0$, *or for each* a, $\int_0^\infty e^{-\lambda t} |y^a(t)|\, dt < \infty$ *for* $\lambda > 0$. *Then we have for each* a *and almost every* t:

$$y^a(t) = \xi^a(t).$$

If in addition $y^a(\cdot)$ *is right or left continuous, then there is equality above for every* t.

Proof. Taking Laplace transforms, we have

$$\hat{p}_{ij}(\lambda) = \hat{f}_{ij}(\lambda) + \sum_{b \in \mathbf{A}} \hat{L}_i^b(\lambda)\, \hat{y}_j^b(\lambda). \tag{18.3}$$

The alternative condition on the y^a's ensures that the Laplace transforms of the convolution in (18.2) is the product indicated under the sum in (18.3), if we observe that $0 \leq \Pi(t) - \Phi(t) \leq 1$. Now substitute χ_n for i in (18.3) and let $\chi_n \to a$ as in (18.1). As a consequence of Theorem 4.1 of [I], we have

$$\lim_{\chi_n \to a} \hat{L}^b_{\chi_n}(\lambda) = \delta^{ab}, \qquad \lim_{\chi_n \to a} \hat{f}_{\chi_n j}(\lambda) = 0;$$

this together with the Laplace transform of (18.1) yields at once

$$\hat{\xi}^a_j(\lambda) = 0 + \sum_b \delta^{ab} \hat{y}^b_j(\lambda) = \hat{y}^b_j(\lambda).$$

The assertions of the theorem follow from the uniqueness theorem for Laplace transforms and the continuity of $\xi^a_j(\cdot)$. Q.e.d.

Given ξ^a and $L^a(\infty)$, $a \in \mathbf{A}$, the other quantities can be expressed purely analytically without further intervention of the boundary. This will be exhibited, repeating previous formulae to avoid excessive cross references, in the following summing-up.

THEOREM 18.2. (1) *a is not a recurrent trap: $\delta^a = 1$.*

$$L^a(t) = [I - \Phi(t)]\, L^a(\infty);$$

$$1 + \sigma^{aa}(t) = \lim_{s \downarrow 0} \frac{1 - \langle \xi^a(s), L^a(t) \rangle}{1 - \langle \xi^a(s), L^a(\infty) \rangle};$$

$E^a(\cdot)$ is the unique solution of the integral equation

$$\forall t > 0: \int_0^t [\delta^a + \sigma^{aa}(t-s)]\, dE^a(s) = 1;$$

$$F^{ab}(\infty) = \lim_{s \downarrow 0} \frac{\langle \xi^a(s), L^b(\infty) \rangle}{1 - \langle \xi^a(s), L^a(\infty) \rangle}, \quad a \neq b;$$

$$\sigma^{ab}(t) = F^{ab}(\infty) - \lim_{s \downarrow 0} \frac{\langle \xi^a(s), L^b(t) \rangle}{1 - \langle \xi^a(s), L^a(\infty) \rangle}; \quad a \neq b;$$

$$\begin{aligned} \varrho^a_j(t) &= \xi^a_j(t) - \sum_{b \neq a} \int_0^\infty \xi^b_j(t-s)\, dF^{ab}(s) \quad && \text{if } j \in \mathbf{I}^a, \\ &= 0 && \text{if } j \notin \mathbf{I}^a; \end{aligned}$$

$$e^a = \int_0^\infty \varrho^a(t)\, dt;$$

$$\eta^a(t) = \frac{d}{dt} e^a [I - \Phi(t)];$$

(2) *a is a recurrent trap:* $\delta^a = 0$.

$L^a(t)$ *as above;*

$$e_j^a = \int_0^\infty {}_0p_{0j}(t)\,dt, \quad 0 \textit{ fixed in } \mathbf{I}^a, \; j \in \mathbf{I}^a;$$

$\eta^a(t)$ *as above with the e just defined;*

$\sigma^{aa}(t) = \langle \eta^a(t), L^a(\infty) \rangle;$

$E^a(\cdot)$ *as above;*

$\varrho^a \equiv \xi^a; \; F^{ab}(\cdot) \equiv \sigma^{ab}(\cdot) \equiv 0; \quad a \neq b.$

As illustrations, the preceding theorem enables us to solve the following problems. The point is we can go from one set of analytical data to another without introducing new quantities.

Problem 1. Given a process constructed by Theorem 16.1, with $z^a = L^a(\infty)$, $a \in \mathbf{A}$; to find its canonical decomposition as given in Theorem 15.2.

The solution is immediate since the construction produces ξ^a, $a \in \mathbf{A}$, as well as $\prod$, from which we determine δ^a by putting $\delta^a = 0$ if and only if

$$\langle \xi^a(s), 1 - L^a(\infty) \rangle \equiv 0,$$

otherwise $\delta^a = 1$. Now in either case of Theorem 18.2, we obtain ϱ^a, E^a, η^a, F^{ab}, thus retrieving the canonical decomposition (15.6).

Problem 2. Given $L^a(\infty)$ and $\xi^a(\cdot)$, $a \in \mathbf{A}$, constructed or otherwise; to find the transition matrix of the process stopped at $\mathbf{A}_1$, a subset of $\mathbf{A}$.

If $\{x(t)\}$ is the original full process, the stopped process $\{\tilde{x}(t)\}$ is defined as follows. Recalling the definition of α^a in § 13, we put $\alpha^{\mathbf{A}_1} = \inf_{a \in \mathbf{A}_1} \alpha^a$, and

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if} \quad 0 \leqslant t \leqslant \alpha^{\mathbf{A}_1}, \\ \theta & \text{if} \quad t > \alpha^{\mathbf{A}_1}; \end{cases}$$

where θ is the adjoined absorbing state. For $\mathbf{A}_1 = \mathbf{A} - \{a\}$, the stopped process is just the a-process defined near the beginning of § 14, with the transition matrix $\prod^a$ given there. The general solution to Problem 2 should now be obvious. First we find ϱ^a and F^{ab} by Theorem 18.2; then we set

$$\tilde{\xi}^a = \varrho^a + \sum_{b \in \mathbf{A}_1} \int_0^t \tilde{\xi}^b(t-s)\, dF^{ab}(s), \quad a \in \mathbf{A}_1,$$

or its Laplace transform

$$\check{\xi}^a(\lambda) = \varrho^a(\lambda) + \sum_{b \in \mathbf{A}_1} \hat{F}^{ab}(\lambda)\check{\xi}^b(\lambda), \quad a \in \mathbf{A}_1.$$

Let the restriction of $\hat{F}(\lambda)$ to $\mathbf{A}_1 \times \mathbf{A}_1$ be $\hat{F}_1(\lambda)$, then $I - \hat{F}_1(\lambda)$ is invertible as in the proof of Theorem 15.2. Hence we may solve for $\check{\xi}$:

$$\check{\xi}(\lambda) = [I - \hat{F}_1(\lambda)]^{-1}\hat{\varrho}(\lambda);$$

and

$$\check{\prod}(\lambda) = \Phi(\lambda) + \sum_{a \in \mathbf{A}_1} \hat{L}^a(\lambda)\,\check{\xi}^a(\lambda)$$

is the Laplace transform of the required stopped transition matrix (without completion).

The procedure given in Theorem 18.2 is somewhat tedious to follow in practice (see the Example at the end of this section). Quicker results can be obtained by uncovering a certain linear transformation which reduces a given decomposition to the canonical form. In what follows we shall again omit the "$\hat{}$" on Laplace transforms.

THEOREM 18.3. *Let* $0 < \lambda < \infty$; *and*

$$\xi(\lambda) = M(\lambda)\eta(\lambda),$$

where

$$M(\lambda) = [I - \Omega + U(\lambda)]^{-1}$$

is any construction given by the first part of Theorem 16.1; and let $\tilde{M}$, $\tilde{\eta}$, $\tilde{\Omega}$, $\tilde{U}$ *be any other one. Then there exists a constant invertible matrix* C *such that*

$$\tilde{\eta}(\cdot) = C\eta(\cdot). \tag{18.4}$$

Proof. We have

$$M(\lambda)\eta(\lambda) = \xi(\lambda) = \tilde{M}(\lambda)\tilde{\eta}(\lambda). \tag{18.5}$$

Recalling the resolvent equation, valid for $0 < \lambda < \infty$, $0 < \mu < \infty$:

$$(\lambda - \mu)\eta(\lambda)\Phi(\mu) = \eta(\mu) - \eta(\lambda)$$

for η, and a similar one for $\tilde{\eta}$, we obtain from (18.5)

$$M(\lambda)[\eta(\mu) - \eta(\lambda)] = \tilde{M}(\lambda)[\tilde{\eta}(\mu) - \tilde{\eta}(\lambda)].$$

Cancelling against (18.5), we have

$$M(\lambda)\eta(\mu) = \tilde{M}(\lambda)\tilde{\eta}(\mu). \tag{18.6}$$

This being true for every $\lambda > 0$, $\mu > 0$, we may fix any $\lambda = \lambda_0 > 0$ above and put

$$C = \tilde{M}(\lambda_0)^{-1} M(\lambda_0)$$

to conclude (18.4).

In Theorems 18.4 and 18.5 below we shall assume that the set of $|\mathbf{A}|$ vectors $\{\eta^a(\mu),\ a \in \mathbf{A}\}$, each regarded as an element of the vector space $\mathcal{M}(\mathbf{I})$ mentioned in Definition 13.1 above, is linearly independent for some value of μ: $0 < \mu < \infty$. Then it follows at once from the resolvent equation for each $\eta^a(\cdot)$ that it is linearly independent for every such value of μ. Under this assumption the equation (18.6) implies that we have for every λ, $0 < \lambda < \infty$:

$$\tilde{M}(\lambda)^{-1} M(\lambda) = C,$$

where C is as in (18.4).

THEOREM 18.4. *If $I - \Omega$ is invertible for some Ω in Theorem 18.3, then it is invertible for every $\tilde{\Omega}$ and we have*

$$C = (I - \tilde{\Omega})(I - \Omega)^{-1}. \tag{18.7}$$

This situation obtains if and only if all boundary atoms are nonrecurrent.

Proof. By (16.6)–(16.9) we have $\langle \eta^a(\mu), 1 \rangle < \infty$ for each $\mu > 0$. Using the resolvent equation for η^a we have for each $\lambda > 0$:

$$u^{ab}(\lambda) = \lambda \langle \eta^a(\mu), z^b \rangle + (\mu - \lambda) \langle \eta^a(\mu), \lambda \Phi(\lambda) z^b \rangle;$$

hence $\lim_{\lambda \downarrow 0} u^{ab}(\lambda) = 0$ for each a and b by (16.3), namely $\lim_{\lambda \downarrow 0} U(\lambda) = 0$. Since $M(\lambda)^{-1} = I - \Omega + U(\lambda)$ we obtain

$$\lim_{\lambda \downarrow 0} M(\lambda)^{-1} = I - \Omega. \tag{18.8}$$

It follows from a well-known proposition of finite matrix theory that (18.8) implies

$$\lim_{\lambda \downarrow 0} M(\lambda) = (I - \Omega)^{-1}$$

in the sense that the existence of one member of the equation implies that of the other and also the equality of both. Hence if $I - \Omega$ is invertible, then

$$\lim_{\lambda \downarrow 0} \tilde{M}(\lambda) = \lim_{\lambda \downarrow 0} M(\lambda)\, C^{-1} = (I - \Omega)^{-1} C^{-1}$$

which implies that $I - \tilde{\Omega}$ is invertible and

$$(I - \tilde{\Omega})^{-1} = (I - \Omega)^{-1} C^{-1}$$

proving (18.7). On the other hand, for the canonical $\Omega = (F^{ab}(\infty))$, $I - \Omega$ is invertible unless there exists $\mathbf{A}_0 \subset \mathbf{A}$ such that $\Omega|_{\mathbf{A}_0}$ is stochastic. This is the case if and only if the boundary atoms in $\mathbf{A}_0$ are recurrent. Thus $I - \Omega$ is invertible if all the boundary atoms are nonrecurrent.

THEOREM 18.5. *If all boundary atoms are nonsticky, we have*

$$C=(I-\tilde{\Omega}+\tilde{U}(\infty))(I-\Omega+U(\infty))^{-1}.$$

This is proved in a similar way as the preceding theorem. We now give an example to show the possibility of distinct decompositions mentioned in § 11.

Example. Let $\mathbf{A}=\{a,b\}$; given z^a and z^b with $z^a+z^b=z$, suppose that η^a,η^b are linearly independent and such that

$$\begin{array}{lll} c^a=0, & u^{aa}(\infty)=+\infty, & u^{ab}(\infty)=\tfrac{1}{2}, \\ c^b=1, & u^{ba}(\infty)=0, & u^{bb}(\infty)<\infty; \\ \Omega^{ab}=1, & \Omega^{ba}=0. & \end{array}$$

These choices are consistent with the conditions of construction; in particular the numerical values of $u^{ab}(\infty)$ and c^b may be fixed by a proportional constant factor.

We have

$$I-\Omega+U(\lambda)=\begin{pmatrix}1+u^{aa}(\lambda) & -1+u^{ab}(\lambda)\\ 0 & 1+u^{bb}(\lambda)\end{pmatrix}.$$

$$M(\lambda)=[I-\Omega+U(\lambda)]^{-1}=\begin{pmatrix}E^a(\lambda) & E^a(\lambda)\,E^b(\lambda)\,[1-u^{ab}(\lambda)]\\ 0 & E^b(\lambda)\end{pmatrix},$$

where $E^c(\lambda)=[1+u^{cc}(\lambda)]^{-1}$ for $c=a,b$. Thus

$$M(\lambda)\,U(\lambda)=\begin{pmatrix}E^a(\lambda)\,u^{aa}(\lambda) & E^a(\lambda)\,u^{ab}(\lambda)+E^a(\lambda)\,E^b(\lambda)\,[1-u^{ab}(\lambda)]\,u^{bb}(\lambda)\\ 0 & E^b(\lambda)\,u^{bb}(\lambda)\end{pmatrix}.$$

Noting that

$$\langle\xi^a(\lambda),z^b\rangle=\sum_c M^{ac}(\lambda)\,\langle\eta^c(\lambda),z^b\rangle=\sum_c M^{ac}(\lambda)\,\lambda^{-1}u^{cb}(\lambda)=\lambda^{-1}M(\lambda)\,u(\lambda)\,|^{ab},$$

we can compute from the preceding matrix to obtain

$$\lim_{\lambda\uparrow\infty}\frac{\langle\xi^a(\lambda),z^b\rangle}{1-\langle\xi^a(\lambda),z^a\rangle}=\frac{u^{ab}(\infty)+u^{bb}(\infty)}{1+u^{bb}(\infty)}=u<1,$$

$$\lim_{\lambda\uparrow\infty}=\frac{\langle\xi^b(\lambda),z^a\rangle}{1-\langle\xi^b(\lambda),z^b\rangle}=0.$$

Thus the canonical $\tilde{\Omega}=(F^{ab}(\infty))$ is $\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix}$ not the constructed $\Omega=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$.

Theorem 18.4 is applicable and we obtain

$$C = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1-u \\ 0 & 1 \end{pmatrix},$$

$$\tilde{\eta}^a = \eta^a + (1-u)\,\eta^b, \qquad \tilde{\eta}^b = \eta^b.$$

The canonical decomposition is therefore

$$[C(I - \Omega + U(\lambda))]^{-1}\, C\eta(\lambda) = [I - \tilde{\Omega} + \tilde{U}(\lambda)]^{-1}\tilde{\eta}(\lambda).$$

These computations can also be made directly by Theorem 18.2.

§ 19. Complements

The first complement concerns nonsticky atoms, for which many of the results and their derivations can be simplfied. While our general discussion includes both sticky and nonsticky atoms it is worthwhile to examine for a little the simpler case where the intuitive meaning is more easily recognized and the analysis follows a smoother pattern. Indeed the sticky case may be regarded as a suitable limiting phenomenon, closely related to the passage of compound Poisson laws to an infinitely divisible law, which deserves further investigation (cf. in this connection a conjecture of Reuter [14], since proved by Kingman [III]).

There are several ways of direct handling of a nonsticky atom a. One is to begin with the Lemma in § 14 and observe the special form of $E^a(\cdot)$ as indicated in Theorem 9.1 of [I]. A more instructive approach, however, is to begin with the basic interpretation of $E^a(\cdot)$ given in (17.1). Recall from Theorem 12.5 that if a is nonsticky and the process starts at a, then the successive times at which it is reached is a sequence of optional random variables

$$0 = \tau_0 < \tau_1 < \tau_2 < \ldots,$$

the number of which being finite or infinite with probability one according as a is nonrecurrent or recurrent, by Theorem 12.1. As usual we define all the non-existing τ's as $+\infty$.

It is clear from the meaning of L^{aa} in Theorem 17.2 that $L^{aa}(0+) = 0$ for a nonsticky a, and that $L^{aa}(\cdot)$ is a probability distribution function if and only if a is a recurrent trap. If we write $L^{(n)aa}$ for the n-fold convolution of L^{aa} with itself, with $L^{(0)aa} = \varepsilon$ the unit mass at 0, we have by the Strong Markov property:

$$\mathbf{P}^a\{\tau_n < \beta^a < \tau_{n+1};\ \tau_n \leqslant t\} = L^{(n)aa}(t)\,[1 - L^{aa}(\infty)].$$

Now it follows from (17.1) that

$$E^a(t) = \sum_{n=0}^{\infty} \mathbf{P}^a\{\tau_n < \beta^a < \tau_{n+1}, \tau_n \leqslant t\} = \sum_{n=0}^{\infty} L^{(n)aa}(t)\,[1 - L^{aa}(\infty)]. \tag{19.1}$$

It is clear that $E^a(t)$ is just the expected number of times of reaching a in time t. [When a is sticky, $E^a(t)$ can be shown to be the expected "local time at a", but this interpretation seems less convenient than that given in (17.1).] Letting $t \downarrow 0$ in Corollary 1 to Theorem 14.7, we obtain

$$E^a(0)\,[\delta^a + \sigma^{aa}(0)] = 1$$

so that
$$E^a(0) = 1 - L^{aa}(\infty) = \frac{1}{\delta^a + \sigma^{aa}(0)}.$$

It is easy to express F^{ab} in terms of L^{ab}:

$$F^{ab}(t) = L^{ab}(t) + \int_0^t F^{ab}(t-s)\,dL^{aa}(s), \quad a \neq b,$$

or in terms of Laplace transforms:

$$\hat{F}^{ab}(\lambda) = \frac{\hat{L}^{ab}(\lambda)}{1 - \hat{L}^{aa}(\lambda)}. \tag{19.2}$$

In particular
$$F^{ab}(\infty) = \frac{L^{ab}(\infty)}{1 - L^{aa}(\infty)} = \frac{L^{ab}(\infty)}{E^a(0)}.$$

On the other hand, we have by (17.19), (14.35) and (17.9):

$$F^{ab}(0) = L^{ab}(0+) = E^a(0)\,[F^{ab}(\infty) - \sigma^{ab}(0)] = E^a(0)\,d^{ab}.$$

Next we have, by (17.18), (14.17), and (14.27),

$$\mathring{L}^{ab}(t) = \lim_{s \downarrow 0} \int_0^s \langle \eta^a(s-u), L^b(t) \rangle\, dE^a(u) = E^a(0) \int_0^t \theta^{ab}(s)\,ds.$$

Recalling (5.7) on p. 40 of [I], in our present notation:

$$\zeta_i^a(t) \overset{\text{def}}{=} \mathbf{P}^a\{\alpha > 0;\, x(t) = i\},$$

it is clear that
$$\varrho_i^a(t) = \zeta_i^a(t) + \int_0^t \varrho_i^a(t-s)\,dL^{aa}(s).$$

Namely, $\zeta_i^a(t)$ represents the probability, starting at a, of $x(t) = i$ without having reached any boundary atom before time t, while $\varrho_i^a(t)$ that of the same without reaching any boundary atom *except* a. In Laplace transforms, the equation above is

$$\hat{\varrho}^a(\lambda) = \frac{\hat{\zeta}^a(\lambda)}{1 - \hat{L}^{aa}(\lambda)}. \tag{19.3}$$

Substituting (19.2) and (19.3) into (15.2), we obtain

$$\hat{\xi}^a(\lambda) = \frac{1}{1-\hat{L}^{aa}(\lambda)}\{\hat{\zeta}^a(\lambda) + \sum_{b\in \mathbf{A}-\{a\}} \hat{L}^{ab}(\lambda)\,\hat{\xi}^b(\lambda)\}; \tag{19.4}$$

or clearing of fractions:

$$\hat{\xi}^a(\lambda) = \hat{\zeta}^a(\lambda) + \sum_{b\in \mathbf{A}} \hat{L}^{ab}(\lambda)\,\hat{\xi}^b(\lambda),$$

which is (5.20) of [I].

It now follows from

$$\frac{\hat{\zeta}^a(\lambda)}{1-\hat{L}^{aa}(\lambda)} = \hat{\varrho}^a(\lambda) = \hat{E}^a(\lambda)\,\hat{\eta}^a(\lambda) = \frac{1-L^{aa}(\infty)}{1-\hat{L}^{aa}(\lambda)}\hat{\eta}^a(\lambda) = \frac{E^a(0)}{1-\hat{L}^{aa}(\lambda)}\hat{\eta}^a(\lambda)$$

that

$$\zeta^a(\cdot) = E^a(0)\,\eta^a(\cdot).$$

Thus ζ^a is the part of $\varrho^a = E^a * \eta^a$ which arises from the mass of E^a at 0. The possibility of using ζ, which has an easy meaning, rather than η, accounts largely for the simplicity of the nonsticky case.

The general reduction sketched above shows the sense in which the development in § 14 is an essential extension of the "first approach" in [I, § 5] to the case where some atoms may be sticky.

The second complement concerns the "last exit time from a before time t," as distinguished from the "last exit time from a before switch" introduced in Definition 17.1.

Definition 19.1. For each a and $t \geq 0$:

$$\gamma_t^a(\omega) = \sup\{S^a(\omega) \cap [0,t]\} = \sup\{s: 0 \leq s \leq t;\, x(s,\omega) = a\}.$$

This is the obvious extension of the last exit time from an ordinary state i before time t ([1; p. 261] and [2]); see also Corollary to Theorem 12.4.

We have then

$$\mathbf{P}^a\{\gamma_t^a \leq s \leq t < \beta^a;\, x(t) = j\} = \sum_i \varrho_i^a(s)\, f_{ij}(t-s)$$
$$= \int_0^s \sum_i \eta_i^a(s-u)\, f_{ij}(t-s)\, dE^a(u) = \int_0^s \eta_j^a(t-u)\, dE^a(u). \tag{19.5}$$

Summing over j: $\qquad \mathbf{P}^a\{\gamma_t^a \leq s \leq t < \beta^a\} = \int_0^s \eta_*^a(t-u)\, dE^a(u).$

Thus we have in density form, for $0 \leq u \leq t$:

$$\mathbf{P}^a\{\gamma_t^a \in du;\, t < \beta^a\} = \eta_*^a(t-u)\, dE^a(u),$$

and $$\mathbf{P}^a\{\gamma_t^a \in du; t < \beta^a; x(t) = j\} = \frac{\eta_j^a(t-u)}{\eta_*^a(t-u)} \mathbf{P}^a\{\gamma_t^a \in du; t < \beta^a\}.$$

Putting $s = t$ in (19.5), we obtain

$$\varrho_j^a(t) = \int_0^t \eta_j^a(t-u)\, dE^a(u);$$

this then is the meaning of the fundamental formula (14.17).

It is interesting to compare this with Theorem 17.1 by calculating the following more specific probabilities:

$$\mathbf{P}^a\{\gamma_t^a \leqslant s \leqslant t < \beta^a; x(t) = j; \alpha_t \leqslant u; x(\alpha_t) = b\}$$
$$= \sum_i \varrho_i^a(s) f_{ij}(t-s) L_i^b(u-t) = \int_0^s \eta_j^a(t-r) L_j^b(u-t)\, dE^a(r);$$

$$\mathbf{P}^a\{\gamma_t^a \leqslant s \leqslant t < \beta^a; x(t) = j; \alpha_t = +\infty\}$$
$$= \sum_i \varrho_i^a(s) f_{ij}(t-s)[1 - L_j(\infty)] = \int_0^s \eta_j^a(t-r)[1 - L_j(u-t)]\, dE^a(r).$$

Summing over j, we have if $t \leqslant u$:

$$\mathbf{P}^a\{\gamma_t^a \leqslant s \leqslant t < \beta^a; \alpha_t \leqslant u; x(\alpha_t) = b\}$$
$$= \int_0^s [\sigma^{ab}(t-r) - \sigma^{ab}(u-r)]\, dE^a(r) = \int_0^s \int_{t-r}^{u-r} \theta^{ab}(v)\, dv\, dE^a(r),$$

or in density form, valid for each a and b (not necessarily distinct):

$$\mathbf{P}^a\{t < \beta^a; \gamma_t^a \in ds; \alpha_t \in du; x(\alpha_t) = b\} = E^a(ds)\, \theta^{ab}(u)\, du.$$

We have also

$$\mathbf{P}^a\{\gamma_t^a \leqslant s \leqslant t < \beta^a; \alpha_t = +\infty\} = \int_0^s \eta_*^a(\infty)\, dE^a(r) = c^a E^a(s).$$

These results explain quantitatively the second sentence of [I].

Finally, we shall give a description of the boundary behavior of sample functions in the general case discussed in this paper, namely under A, B′, C_1 and D. This will be seen to be the completed version of the description given on pp. 45–46 of [I] for the case where all boundary atoms are nonsticky. A comparison of the two will show again how the present approach does complete the previous one.

Beginning at time $\beta_0 = 0$ with the banner (boundary atom) z_0, let the successive times for changing banners be $\beta_0 \leqslant \beta_1 \leqslant \ldots \leqslant \beta_n \leqslant \ldots$ where the sequence is terminated at the first β_n which is $+\infty$, or continued indefinitely if they are all finite. Note that each β_n may equal

β_{n+1} with positive probability. Let the successive banners be $z_0, z_1, \ldots, z_n, \ldots$ so that any two consecutive ones are distinct and that between time β_n and β_{n+1} it is the banner z_n which is flying. The time-atom process

$$\begin{pmatrix} \beta_0, & \beta_1, & \ldots, & \beta_n, & \ldots \\ z_0, & z_1, & \ldots, & z_n, & \ldots \end{pmatrix}$$

with state space $\mathbf{A}\times(0,\infty)$ is a Markov process characterized as follows:

$$\mathbf{P}\left\{\begin{matrix} z_{n+1}=b \\ \beta_{n+1}\leqslant t \end{matrix}\,\middle|\, \begin{matrix} z_0, \ldots, z_{n-1}, z_n = a \\ \beta_0, \ldots, \beta_{n-1}, \beta_n \end{matrix}\right\} = F^{ab}(t);$$

$$\mathbf{P}\left\{\beta_{n+1} = +\infty \,\middle|\, \begin{matrix} z_1, \ldots, z_{n-1}, z_n = a \\ \beta_1, \ldots, \beta_{n-1}, \beta_n \end{matrix}\right\} = \varrho_*^a(\infty).$$

The process $\{z_n, n\geqslant 0\}$ is a discrete parameter Markov chain with $(F^{ab}(\infty))$, $(a,b)\in\mathbf{A}\times\mathbf{A}$, where $F^{aa}(\infty)\equiv 0$ for each a, as transition matrix. The banner process, defined on $[0,\infty)$ to be at z_n in the time interval $[\beta^n, \beta^{n+1})$, is a semi-Markovian process (see Pyke [V]). Between each change of banners, namely in each time interval (β_n, β_{n+1}), the ordinary states i line up under the banner z_n with the following probabilities:

$$\mathbf{P}\left\{x(t)=i, t<\beta_{n+1} \,\middle|\, \begin{matrix} z_0, \ldots, z_{n-1}, z_n = a \\ \beta_0, \ldots, \beta_{n-1}, \beta_n = s \end{matrix}\right\} = \varrho_i^a(t-s), \quad 0\leqslant s\leqslant t.$$

Summing over i, we have

$$\mathbf{P}\left\{t<\beta_{n+1} \,\middle|\, \begin{matrix} z_0, \ldots, z_{n-1}, z_n = a \\ \beta_0, \ldots, \beta_{n-1}, \beta_n = s \end{matrix}\right\} = \varrho_*^a(t-s), \quad 0\leqslant s\leqslant t.$$

We have therefore

$$\begin{aligned} \mathbf{P}\{\beta_n \leqslant t < \beta_{n+1}; z_k = a_k, 1\leqslant k\leqslant n \mid z_0 = a_0\} \\ &= (F^{a_0a_1} * F^{a_1a_2} * \ldots * F^{a_{n-1}a_n})(t) - \sum_{a_{n+1}\neq a_n} (F^{a_0a_1} * \ldots * F^{a_{n-1}a_n} * F^{a_na_{n+1}})(t) \\ &= \int_0^t \varrho_*^{a_n}(t-s)\, d(F^{a_0a_1} * \ldots * F^{a_{n-1}a_n})(s); \end{aligned}$$

and consequently

$$\mathbf{P}^{a_0}\{x(t)=i\} = \sum_{n=0}^{\infty} \sum_{a_1,\ldots,a_n} \int_0^t \varrho_i^{a_n}(t-s)\, d(F^{a_0a_1} * \ldots * F^{a_{n-1}a_n})(s),$$

where for $n=0$, the last written convolution is $\varepsilon(\cdot)$ as usual. This is the precise meaning of the canonical decomposition in the first equation of (15.6). The meaning of the second equation has just been given above.

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