NON-UNITARY DUAL SPACES OF GROUPS

BY

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1. Introduction

The infinite-dimensional unitary representations of an arbitrary locally compact group G have been extensively studied since 1947. For some purposes, however, the unitary restriction is very undesirable—for example, if we wish to carry out "analytic continuation" of representations of G. This paper investigates some general concepts concerning non-unitary representations. Extending the ideas of [3], we define a "non-unitary dual space" $\hat{G}$ of G. Roughly speaking, $\hat{G}$ is the space of all equivalence classes of irreducible (not necessarily either unitary or finite-dimensional) representations of G. It is not however a trivial matter to decide what we ought to mean by 'representation', 'irreducible', or 'equivalence class'. At first sight it might appear reasonable to restrict ourselves to representations living in a Banach space. We shall therefore begin with an example showing that Banach spaces form too narrow a framework if we have in mind analytic continuation of representations of general groups.

Let G be the Galilean group, that is, the three-dimensional nilpotent Lie group of all triples of real numbers, multiplication being given by $\langle a,b,c\rangle \langle a',b',c'\rangle = \langle a+a', b+b', c+c'-ab'\rangle$. The unitary representations of G are well known (see [15]). For each non-zero real number λ there is a unique (infinite-dimensional) irreducible unitary representation T^{λ} of G with the property that, for each real c, T^{λ} sends the central element $\langle 0,0,c\rangle$ of G into the scalar operator $e^{i\lambda c}\cdot 1$. One would hope by a process of "analytic continuation" to obtain non-unitary irreducible representations T^{λ} having the same property for *complex* λ. But we shall now show that such a T^{λ} could not live in a Banach space. Indeed: Let us write $\gamma_1(a)=\langle a,0,0\rangle$, $\gamma_2(b)=\langle 0,b,0\rangle$, $\gamma_3(c)=\langle 0,0,c\rangle$ (a,b,c real); and let us suppose that T is a homomorphism of G into the group of bounded invertible operators on some Banach space H such that $T_{\gamma_3(c)}=e^{i\lambda c}\cdot 1$ for all real c, where λ is a non-real complex number (and 1 is the identity operator on H). Since $\gamma_1(-1)\gamma_2(b)\gamma_1(1)=\gamma_3(b)\gamma_2(b)$, we have

$$T_{\gamma_1(-1)}\, T_{\gamma_2(b)}\, T_{\gamma_1(1)} = e^{i\lambda b}\, T_{\gamma_2(b)}$$

for all real b. From this follows $|e^{i\lambda b}|\ \|T_{\gamma_2(b)}\| \leqslant \|T_{\gamma_1(-1)}\|\ \|T_{\gamma_2(b)}\|\ \|T_{\gamma_1(1)}\|$, or $|e^{i\lambda b}| \leqslant \|T_{\gamma_1(-1)}\|\ \|T_{\gamma_1(1)}\|$ for all real b, which is impossible since λ is not real. Thus such a T cannot exist.

Instead of merely Banach representations, therefore, we shall follow [5] and consider the more general objects called linear system representations. These were introduced by Mackey in [16], and amount to representations in a locally convex linear topological space where the only interesting property of the topology is its continuous linear functionals. Among their advantages is the fact that the theory of non-unitary induced representations is most naturally formulated in terms of them (compare [16], § 8).

Using linear system representations it will be very easy to construct the T^{λ} required in the preceding example for non-real λ (see Appendix, Example 1).

Apart from a few superficial generalities, the theory of linear system representations of quite general locally compact groups G is as yet a closed book. Only for a certain special class of groups will we be able to obtain non-trivial results, namely, those having a "large" compact subgroup K (see below). Indeed, the "largeness" of K will reduce the study of $\hat{G}$ to the study of the finite-dimensional representations of certain subalgebras of the group algebra of G; and for these finite-dimensional representations we have available the results of [5].

Our paper is divided into thirteen sections and an appendix. In § 2 we recall the basic notions connected with linear system representations of an associative algebra A. The most useful concept of irreducibility for these seems to be that of topological complete irreducibility. Denoting by $\mathcal{J}(A)$ the family of all topologically completely irreducible linear system representations of A, we topologize $\mathcal{J}(A)$ with the so-called functional topology, and define two elements S and T of $\mathcal{J}(A)$ to be functionally equivalent if they are not distinguished by the functional topology. The space of equivalence classes in $\mathcal{J}(A)$ under functional equivalence is called the (functional) dual space $\hat{A}$ of A. (This dual space is a larger object than the $\hat{A}$ defined in [5], which consisted of the *algebraically* completely irreducible linear system representations.)

In § 3 we mention commutative algebras, and in § 4 we show how $\hat{A}$ is related to $\hat{I}$ or $(eAe)\hat{}$, I being a two-sided ideal of A and e an idempotent element of A. § 4 is a generalization of § 4 of [5].

Now let G be a locally compact group, and $M_0(G)$ the convolution algebra of measures on G with compact support. In § 5 we define linear system representations of G. These have "integrated forms" which are linear system representations of $M_0(G)$. Thus the definitions in § 2 applied to $A = M_0(G)$ can be pulled back to G; and we obtain the notion of the dual space $\hat{G}$ of G with its functional topology. This $\hat{G}$ is the main object of study in this paper. One asks such questions as the following: For what groups G is $\hat{G}$ locally compact? When does the functional topology of $\hat{G}$ coincide on unitary representations with the hull-kernel topology discussed (for example) in [3]?

In § 6 we digress somewhat to discuss a relation between representations which we call Naimark-relatedness. It was introduced by M. A. Naimark in his study of the Banach representations of the Lorentz group (see [17]), and is closely connected with functional equivalence. Unfortunately Naimark-relatedness is not in general an equivalence relation (see Appendix, Examples 3 and 4). It becomes one, however, if we restrict ourselves to what we call FDS representations, in which "enough" of the operators have finite-dimensional range. For FDS representations in $\hat{G}$, indeed, Naimark-relatedness turns out to be the same thing as functional equivalence.

To answer the questions about $\hat{G}$ raised above, it appears necessary to make some kind of finiteness assumption about the representations in $\hat{G}$; only then will the results in [5] on finite-dimensional representations of Banach algebras become available. Let us refer to an idempotent element μ of $M_0(G)$ as "small" if $T(\mu)$ is of bounded finite rank for all T in $\hat{G}$. The appropriate finiteness assumption seems to be roughly the existence of "enough" small idempotents in $M_0(G)$. If this holds, then of course all elements of $\hat{G}$ are FDS. In § 7 we show that if "enough" small idempotents exist and if one further condition holds ("local boundedness" of $\hat{G}$), then $\hat{G}$ is locally compact.

If K is a compact subgroup of G such that, for every irreducible representation D of K, the multiplicity of D in T is finite and bounded for all T in $\hat{G}$, we say that K is "large". (Our notion of "largeness" is a little stronger than that of Godement [8], who requires only that the multiplicity of D in T be finite for each T in $\hat{G}$.) If G has a large compact subgroup, $M_0(G)$ has enough small idempotents and hence $\hat{G}$ is locally compact. Further, in this situation we can relate the topology of $\hat{G}$ to the subalgebras $L^0(\delta)$ of Godement ([8], § 10); this is done in § 8. In § 9 we show that, if G has a large compact subgroup, the hull-kernel and the functional topologies coincide on irreducible unitary representations.

Among the groups which have a large compact subgroup we find the connected semisimple Lie groups with finite center [9] and the Euclidean groups [8]. We conjecture that the connected semisimple Lie groups G with infinite center, though they have no large compact subgroup, are nevertheless FDS (in the sense that all elements of $\hat{G}$ are FDS) and have locally compact duals. We do not know of any groups G without a large compact subgroup for which $M_0(G)$ has enough small idempotents.

§ 10 gives the condition for local compactness and local boundedness of the duals of Abelian groups.

Suppose now that G is a Lie group. In that case the measure algebra of G can be enlarged to the distribution algebra $D_0(G)$ (the algebra, under convolution, of all Schwartz distributions on G with compact support). It is shown in § 11 that each element T of $\hat{G}$ gives rise to a complex homomorphism γ_T of the center Z of $D_0(G)$, and that the map $T \to \gamma_T$ is continuous. A very important subalgebra of $D_0(G)$ is the enveloping algebra E of the Lie algebra of G. If G has a large compact subgroup, it is shown in § 12 that each element T of $\hat{G}$ gives rise in a natural way to an (algebraically) irreducible linear system representation $\tilde{T}$ of E, and that the map $T \to \tilde{T}$ is one-to-one and continuous (in the functional topologies of $\hat{G}$ and $\hat{E}$). It would be very interesting to know whether it is a homeomorphism.

In § 13 we relate the topology of $\hat{G}$ (or rather, of the subset of $\hat{G}$ corresponding to a fixed idempotent measure and a fixed "norm-function") to the topology of uniform convergence on compact sets of "generalized spherical functions" on G. In so doing we verify a conjecture of Godement.

The Appendix contains four examples and counter-examples.

The ideas of this paper suggest two plausible and interesting conjectures. First, could it be that for some classes of groups the generalization from Banach representations to linear system representations was unnecessary? To be more precise, let us say that the group G is Banach-representable if every class in $\hat{G}$ contains some Banach space representation of G. The Galilean group is certainly not Banach-representable (see Example 1 of the Appendix). However, we conjecture that every group with a large compact subgroup (perhaps even every FDS group) is Banach-representable.

As for the second conjecture, we notice from § 10 that an Abelian group G which is compactly generated has a locally compact dual $\hat{G}$. Could it be that *every* compactly generated group having a large compact subgroup has a locally compact dual?

Combining the results of [10] and [5] we can show that for connected semisimple matrix groups both the above conjectures are correct. These results will be published later, along with other facts about the topology of $\hat{G}$ when G is semisimple or Euclidean.

Here are a few words on notation. C denotes the complexes, and R the reals. $f|A$ means the restriction of the function f to A. If X is a complex linear space, $X^{\#}$ is the space of all complex linear functionals on X; if X is a Banach space, X^* is the space of all continuous elements of $X^{\#}$. $\operatorname{Dim}(X)$ is the dimension of X. Pairs are denoted by angular brackets $\langle , \rangle$.

By a *locally compact* space we mean a (not necessarily Hausdorff) topological space in which every point has a basis of compact neighborhoods.

2. The functional dual space of an algebra

A linear system H is a pair of complex linear spaces H_1, H_2 together with a duality between them, that is, a complex bilinear function $(\,|\,)$ on $H_1 \times H_2$ such that $(\xi | H_2)=0$ only if $\xi=0$ and $(H_1|\eta)=0$ only if $\eta=0$. An *isomorphism* F between two linear systems H and H' is a pair $\langle F_1, F_2\rangle$, where F_i is a linear isomorphism of H_i onto H_i' $(i=1,2)$, and $(\xi|\eta)=(F_1(\xi)\,|\,F_2(\eta))$ for all $\xi \in H_1$, $\eta \in H_2$. If $\langle H_1, H_2\rangle$ is a linear system, the locally convex topology of H_1 generated by the functionals $\xi \to (\xi|\eta)$ $(\xi \in H_1)$, where η runs over H_2, is called the *$\sigma(H)$-topology of H_1*; similarly we define the *$\sigma(H)$-topology of H_2*. If $K_i \subset H_i$ $(i=1,2)$, let

$$K_1^{\perp}=\{\eta \in H_2 \,|\, (K_1|\eta)=0\}, \quad K_2^{\perp}=\{\xi \in H_1 \,|\, (\xi|K_2)=0\}.$$

Then $K_1^{\perp\perp}=K_1$ if and only if K_1 is $\sigma(H)$-closed; similarly for K_2.

We write $\dim(H)$ for $\dim(H_1)$ $(=\dim(H_2))$ if the latter is finite.

If H and K are two linear systems, their *direct sum* $H\oplus K$ is defined as the pair $\langle H_1\oplus K_1,\ H_2\oplus K_2\rangle$, with the duality $(\xi\oplus u\,|\,\eta\oplus v)=(\xi|\eta)+(u|v)$.

Throughout this paper, A will be a fixed associative algebra over the complex field; $\tilde{A}$ will denote the "reverse algebra", having the same underlying linear space and with $(xy)_{\tilde{A}}=(yx)_A$.

A *representation of* A is a homomorphism T of A into the algebra of all linear endomorphisms of a complex linear space $H=H(T)$ (the *space of* T). By $\dim(T)$ we mean $\dim(H(T))$. *Equivalence* and *irreducibility* of T are to be understood in the purely algebraic sense (the latter being taken to exclude the case of a zero- or one-dimensional zero representation). Suppose that $H(T)\neq\{0\}$; and that, for any positive integer r and any $2r$ vectors $\xi_1,\dots,\xi_r$, $\eta_1,\dots,\eta_r$ such that the $\xi_1,\dots,\xi_r$ are linearly independent, there is an a in A such that $T(a)\xi_j=\eta_j$ $(j=1,\dots,r)$; then T is called *completely irreducible.*

A *linear system representation of A in* a linear system $H=\langle H_1,H_2\rangle$ is a pair $T=\langle T_1,T_2\rangle$, where T_1 is a representation of A in H_1, T_2 is a representation of $\tilde{A}$ in H_2, and $(T_1(a)\xi|\eta)=(\xi\,|\,T_2(a)\eta)$ for all $\xi\in H_1, \eta\in H_2$. We shall frequently write $H(T)=\langle H_1(T),H_2(T)\rangle$ for the linear system H. Two linear system representations T and T' of A, in H and H' respectively, are *equivalent* (in symbols $T\cong T'$) if H and H' are isomorphic under an isomorphism F satisfying $F_1\circ T_1(a)=T_1'(a)\circ F_1$ (and hence $F_2\circ T_2(a)=T_2'(a)\circ F_2$) for all a in A. Let T be a linear system representation of A in H such that $T_1(a)\neq 0$ for some a. T is *irreducible* [resp. *completely irreducible*] if T_i is irreducible [resp. completely irreducible] for each $i=1,2$. T is *topologically irreducible* if H_1 has no non-trivial $\sigma(H)$-closed T_1-stable subspaces (and hence H_2 has no non-trivial $\sigma(H)$-closed T_2-stable subspaces). T is *topologically completely irreducible* if, given any finite collection of linearly independent elements $\xi_1,\dots,\xi_n$ of H_1, any finite collection of linearly independent elements $\eta_1, \ldots, \eta_m$ of H_2, and any complex numbers r_{ij} $(i=1, \ldots, n; j=1, \ldots, m)$, there is an a in A such that $(T_1(a)\xi_i | \eta_j) = r_{ij}$ for all i and j. Equivalently, T is topologically completely irreducible if and only if, whenever $\xi_1, \ldots, \xi_n$, $\xi'_1, \ldots, \xi'_n$ are $2n$ vectors in H_1 such that the $\xi_1, \ldots, \xi_n$ are linearly independent, there is a net $\{a_\gamma\}$ of elements of A such that $T_1(a_\gamma)\xi_i \underset{\gamma}{\to} \xi'_i$ ($\sigma(H)$-wise) for each $i=1, \ldots, n$. There is of course a similar equivalent condition in terms of H_2 and T_2. Topological complete irreducibility will be for us the most important kind of irreducibility for linear system representations. It obviously implies topological irreducibility.

A finite-dimensional linear system representation $T = \langle T_1, T_2 \rangle$ is determined by its first term T_1; in this case we may fail to distinguish between T_1 and T.

If T is a linear system representation of A, the kernels of T_1 and T_2 are the same; we call either one $\mathrm{Ker}(T)$, the *kernel* of T.

If X is a Banach space, the pair $\langle X, X^* \rangle$ with the obvious duality is called the linear system *associated with* X. It follows easily from the uniform boundedness principle that the linear system $\langle X, X^* \rangle$ determines the norm of X to within equivalence. By a *Banach representation* of A *on* X, we mean a homomorphism S of A into the algebra of all bounded linear operators on X; X is called the *space* of S, and is denoted by $X(S)$. Each Banach representation S of A on X gives rise in an obvious manner to an *associated* linear system representation T of A on the linear system $H = \langle X, X^* \rangle$:

$$T_1(a) = S(a), \qquad T_2(a) = (S(a))^*.$$

We shall say that S is *topologically irreducible* (or *topologically completely irreducible*) if its associated linear system representation is so. Since norm-closure and weak closure of linear subspaces of $X(S)$ are the same, the conditions of weak denseness and weak closure in these definitions can be replaced by the corresponding norm-conditions, which we will not rewrite explicitly.

We shall frequently fail to distinguish between a Banach representation of A and its associated linear system representation.

Let $\mathcal{J}(A)$ denote the family of all equivalence classes of topologically completely irreducible linear system representations of A. We note that, if $T \in \mathcal{J}(A)$, $H_1(T)$ and $H_2(T)$ each contain a dense subset of cardinality no greater than that of A, and hence the $H_i(T)$ $(T \in \mathcal{J}(A); i=1, 2)$ are of bounded cardinality. It follows that $\mathcal{J}(A)$ is a set rather than merely a class, in the sense of von Neumann's set theory.

When no ambiguity can arise we shall frequently confuse classes in $\mathcal{J}(A)$ with representations in those classes.

Our next goal is to topologize $\mathcal{J}(A)$. Let T be any linear system representation of A. As in [5] we define $\Phi(T)$ to be the linear span (in $A^{\#}$) of the set of all functionals on A of the form $a \to (T_1(a)\xi \mid \eta)$, where $\xi \in H_1(T)$ and $\eta \in H_2(T)$. Since

$$\mathrm{Ker}(T) = \{a \in A \mid \phi(a) = 0 \text{ for all } \phi \text{ in } \Phi(T)\},$$

the pair $L = \langle A/\mathrm{Ker}(T), \Phi(T)\rangle$ is a linear system under the natural duality; and one may define a linear system representation $S = \langle S_1, S_2\rangle$ on L of the tensor product algebra $A \otimes \tilde{A}$ as follows:

$$S_1(a \otimes b)\,(c + \mathrm{Ker}(T)) = acb + \mathrm{Ker}(T), \tag{1}$$

$$(S_2(a \otimes b)\phi)\,(c) = \phi(acb) \tag{2}$$

$(a, b, c \in A;\ \phi \in \Phi(T))$.

LEMMA 1. *If $T \in \mathcal{J}(A)$, then S is topologically irreducible.*

Proof. Assume that $0 \neq \phi \in \Phi(T)$, $c \in A$, and that $\phi(acb) = 0$ for all a, b in A; it suffices to deduce from this that $c \in \mathrm{Ker}(T)$. Now ϕ can be written in the form

$$\phi(a) = \sum_{i=1}^{r} (T_1(a)\,\xi_i \mid \eta_i) \quad (a \in A),$$

where the ξ_i and the η_i are linearly independent in $H_1(T)$ and $H_2(T)$ respectively. By assumption

$$\sum_{i=1}^{r} (T_1(c)\,T_1(b)\,\xi_i \mid T_2(a)\,\eta_i) = 0 \tag{3}$$

for all a, b in A. Suppose $T_1(c) \neq 0$. Then by the topological complete irreducibility of T we can first choose b so that $\zeta = T_1(c)T_1(b)\xi_1 \neq 0$, and then choose a so that $(\zeta \mid T_2(a)\eta_1) \neq 0$ and $(T_1(c)T_1(b)\xi_i \mid T_2(a)\eta_i) = 0$ for each $i = 2, \dots, r$. But these relations contradict (3). So $T_1(c) = 0$ or $c \in \mathrm{Ker}(T)$. This proves the lemma.

We shall always give to $A^{\#}$ the topology of pointwise convergence on A. If $\mathcal{S}$ is a family of linear system representations of A we write $\Phi(\mathcal{S})$ for $\bigcup_{S \in \mathcal{S}} \Phi(S)$. By $(\Phi(\mathcal{S}))^-$ we mean the closure of $\Phi(\mathcal{S})$ in $A^{\#}$.

COROLLARY. *Let $\mathcal{S}$ be a family of linear system representations of A and T an element of $\mathcal{J}(A)$. If $\Phi(T) \cap (\Phi(\mathcal{S}))^- \neq \{0\}$, then $\Phi(T) \subset (\Phi(\mathcal{S}))^-$.*

Proof. Let S_2 be the representation of $\tilde{A} \otimes A$ on all of $A^{\#}$ defined by (2). For each α in $\tilde{A} \otimes A$, $S_2(\alpha)$ is continuous on $A^{\#}$ and leaves $\Phi(\mathcal{S})$ stable; hence $S_2(\alpha)$ leaves $(\Phi(\mathcal{S}))^-$ stable. Thus, if $0 \neq \phi \in \Phi(T) \cap (\Phi(\mathcal{S}))^-$, $S_2(\tilde{A} \otimes A)\phi \subset (\Phi(\mathcal{S}))^-$. But, by Lemma 1, $S_2(\tilde{A} \otimes A)\phi$ is dense in $\Phi(T)$. So $\Phi(T) \subset (\Phi(\mathcal{S}))^-$, and the corollary is proved.

If $S \subset \mathcal{J}(A)$, we define the *closure* of S to consist of all T in $\mathcal{J}(A)$ such that $\Phi(T) \subset (\Phi(S))^-$ (or equivalently, by the corollary, some non-zero ϕ in $\Phi(T)$ belongs to $(\Phi(S))^-$). Because of the above corollary, this closure satisfies the Kuratowski axioms, and so defines a topology for $\mathcal{J}(A)$.

Definition. The topology for $\mathcal{J}(A)$ defined by this closure operation is called the *functional topology.*

Definition. Two elements S and T of $\mathcal{J}(A)$ which are not separated by the functional topology will be called *functionally equivalent* (in symbols, $S \simeq T$).

LEMMA 2. *If S, T are in $\mathcal{J}(A)$, then $S \simeq T$ if and only if* $\mathrm{Ker}(S) = \mathrm{Ker}(T)$.

This is easily checked.

Definition. By the (*functional*) *dual space* $\hat{A}$ of A we shall mean the quotient space $\mathcal{J}(A)/\simeq$, that is, the space of all functional equivalence classes of elements of $\mathcal{J}(A)$. $\hat{A}$ will always be equipped with its *functional topology*, that is, the functional topology of $\mathcal{J}(A)$ "lifted" to $\hat{A}$.

If $\tau \in \hat{A}$, we shall write $\mathrm{Ker}(\tau)$ for the common kernel of all members of τ (see Lemma 2). One could of course identify τ with $\mathrm{Ker}(\tau)$, and regard $\hat{A}$ as the space of all kernels of elements of $\mathcal{J}(A)$. We shall not do this however.

By definition $\hat{A}$ is always a T_0-space.

Our present dual space $\hat{A}$ differs from that defined in [5]. The dual space of [5] consisted of the *algebraically* completely irreducible elements of $\mathcal{J}(A)$ (with the same functional topology).

3. Central characters and commutative algebras

Let us denote by Z the center of A. If $T \in \mathcal{J}(A)$, we have a complex homomorphism λ_T of Z such that $T_i(a) = \lambda_T(a) \cdot 1_i$ for each a in Z and $i = 1, 2$ (1_i being the identity operator on $H_i(T)$). This λ_T is the *central character* of T.

LEMMA 3. *The map $T \to \lambda_T$ ($T \in \mathcal{J}(A)$) is continuous in the functional topology of $\mathcal{J}(A)$ (the topology for the λ_T being that of pointwise convergence on Z).*

Proof. Let $T \in \mathcal{J}(A)$, and let $0 \neq \phi = \lim_\gamma \phi_\gamma \in \Phi(T)$, where $\phi_\gamma \in \Phi(T_\gamma)$, $T_\gamma \in \mathcal{J}(A)$. It will suffice to show that $\lambda_{T_\gamma} \to \lambda_T$ in Z^*. Choose a in A so that $\phi(a) \neq 0$. If $b \in Z$, we have $\phi(ab) = \lambda_T(b)\phi(a)$, $\phi_\gamma(ab) = \lambda_{T_\gamma}(b)\phi_\gamma(a)$. Thus, from $\phi_\gamma(a) \to \phi(a)$ and $\phi_\gamma(ab) \to \phi(ab)$ it follows that $\lambda_{T_\gamma}(b) \to \lambda_T(b)$. This proves the lemma.

In particular, if $S \simeq T$ $(S, T \in \mathfrak{J}(A))$, then $\lambda_S = \lambda_T$. So λ_τ can be considered as defined for classes τ in $\hat{A}$, and $\tau \to \lambda_\tau$ is continuous on $\hat{A}$ to $Z^{\#}$.

COROLLARY. *If A is commutative, $\mathfrak{J}(A)$ (or $\hat{A}$) can be identified with the space of all non-zero homomorphisms of A into C; and the functional topology of $\mathfrak{J}(A)$ coincides with the topology of pointwise convergence on A.*

This corollary is the special case $n=1$ of Theorem 1 of [6].

4. Restriction to ideals and subalgebras

If I is a subalgebra of A, and T is a linear system representation of A, we write $T|I$ for the linear system representation $\langle T_1|I, T_2|I\rangle$ of I. Clearly, if I is a two-sided ideal of A, $I \not\subset \mathrm{Ker}(T)$, and T is topologically irreducible, then $T|I$ is topologically irreducible. In fact we have:

LEMMA 4. *If $T \in \mathfrak{J}(A)$, I is a two-sided ideal of A, and $I \not\subset \mathrm{Ker}(T)$, then $T|I \in \mathfrak{J}(I)$.*

Proof. If T is finite-dimensional the result is obvious by Burnside's Theorem. Assume T is infinite-dimensional.

Suppose that $n = \max\{\mathrm{rank}(T_1(a)) \mid a \in I\} < \infty$. Clearly there are linearly independent vectors $\xi_0, \xi_1, \ldots, \xi_n$ in $H_1(T)$, linearly independent vectors $\xi_0', \xi_1', \ldots, \xi_n'$ in $H_1(T)$, and elements a, b of I, such that $T_1(a)\xi_0 = \xi_0'$ and $T_1(b)\xi_i = \xi_i'$ for each $i = 1, \ldots, n$. By the topological complete irreducibility of T there are nets $\{c_\gamma\}$, $\{d_\gamma\}$ of elements of A such that $T_1(c_\gamma)\xi_0 \to \xi_0$, $T_1(d_\gamma)\xi_0 \to 0$, and $T_1(c_\gamma)\xi_i \to 0$ and $T_1(d_\gamma)\xi_i \to \xi_i$ for each $i = 1, \ldots, n$. So $T_1(ac_\gamma + bd_\gamma)\xi_i \to \xi_i$ for all $i = 0, 1, \ldots, n$. Thus, for large γ, $T_1(ac_\gamma + bd_\gamma)$ has rank at least $n+1$, and $ac_\gamma + bd_\gamma \in I$. This contradicts our supposition about n. Thus we have shown that $\mathrm{rank}(T_1(a))$ will be arbitrarily large for suitable a in I.

Now let $\xi_1, \ldots, \xi_m$ be any finite number of linearly independent elements of $H_1(T)$, and $\xi_1', \ldots, \xi_m'$ any other elements of $H_1(T)$. By the preceding paragraph and the topological complete irreducibility of T, there are elements c of I and a of A such that the $T_1(ca)\xi_i$ $(i = 1, \ldots, m)$ are linearly independent. We may then choose a net $\{b_\gamma\}$ of elements of A so that $T_1(b_\gamma ca)\xi_i \to \xi_i'$ for each i. Since $b_\gamma ca \in I$, our lemma is proved.

Thus, if I is a two-sided ideal of A, the restriction map

$$T \to T|I \tag{4}$$

carries $(\mathfrak{J}(A))_I = \{T \in \mathfrak{J}(A) \mid I \not\subset \mathrm{Ker}(T)\}$ into $\mathfrak{J}(I)$.

LEMMA 5. *$(\mathfrak{J}(A))_I$ is open in $\mathfrak{J}(A)$; and the map (4) is a homeomorphism of $(\mathfrak{J}(A))_I$ into $\mathfrak{J}(I)$.*

Proof. It is clear that $(\mathcal{J}(A))_I$ is open, and that (4) is one-to-one. Since $\phi | I \in \Phi(T | I)$ whenever $\phi \in \Phi(T)$, (4) is evidently continuous.

Assume that $0 \neq \phi = \lim_\gamma \phi_\gamma \in \Phi(T | I)$, where $\phi_\gamma \in \Phi(T^\gamma | I)$; here T, T^γ are in $(\mathcal{J}(A))_I$. The continuity of the inverse of (4) will be proved if we exhibit functionals ψ and ψ_γ such that $0 \neq \psi = \lim_\gamma \psi_\gamma \in \Phi(T)$, $\psi_\gamma \in \Phi(T^\gamma)$. Choose b in I so that $\phi(b) \neq 0$. Then, by the topological complete irreducibility of T the functional ψ: $a \to \phi(ab)$ on A is non-zero. Evidently $\psi \in \Phi(T)$; similarly ψ_γ: $a \to \phi_\gamma(ab)$ is in $\Phi(T^\gamma)$. Since $\phi_\gamma \to \phi$ on I, we have $\psi_\gamma \to \psi$ on A, and our task is done.

In particular, if S, T are in $(\mathcal{J}(A))_I$, then $S \simeq T$ if and only if $S | I \simeq T | I$; and the map (4) lifts to a homeomorphism of $\hat{A}_I$ into $\hat{I}$, where we have put $\hat{A}_I = \{\tau \in \hat{A} \,|\, I \not\subset \operatorname{Ker}(\tau)\}$.

LEMMA 6. *The range of* (4), *considered as a homeomorphism of $\hat{A}_I$ into $\hat{I}$, is all of $\hat{I}$.*

Proof. Let S be an element of $\mathcal{J}(I)$, acting in a linear system H. Let K_i be the linear span of $\{S_i(a)\xi \,|\, a \in I, \xi \in H_i\}$ $(i = 1,2)$. Then K_i is S_i-stable and non-zero, hence $\sigma(H)$-dense in H_i. By the same argument as in the proof of Lemma 5 of [5], there is a linear system representation T of A acting in $K = \langle K_1, K_2 \rangle$ such that $T_i(a) = S_i(a) | K_i$ for $i = 1,2$ and all a in I. Since $S \in \mathcal{J}(I)$, $T \in \mathcal{J}(A)$. Further $T | I$ and S have the same kernel. Thus the class of T in $\hat{A}$ is carried by (4) into the class of S in $\hat{I}$.

Summarizing Lemmas 4, 5, and 6, we have:

THEOREM 1. *Let I be a two-sided ideal of A, and let $\hat{A}_I = \{\tau \in \hat{A} \,|\, I \not\subset \operatorname{Ker}(\tau)\}$. Then $\hat{A}_I$ is open in $\hat{A}$, and* (4) *lifts to a homeomorphism of $\hat{A}_I$ onto $\hat{I}$.*

Now suppose that e is a fixed idempotent element $(e^2 = e)$ of A. If T is a linear system representation of A satisfying $T_1(e) \neq 0$, let $H^e(T)$ be the linear system

$$\langle \operatorname{range}(T_1(e)), \operatorname{range}(T_2(e)) \rangle$$

(with the restricted duality of $H(T)$); and let T^e be the linear system representation of eAe on $H^e(T)$ defined by $T_i^e(b) = T_i(b) | \operatorname{range}(T_i(e))$ $(i = 1,2;\ b \in eAe)$. A standard argument ([8], Lemma 3) shows that if $T \in \mathcal{J}(A)$ then $T^e \in \mathcal{J}(eAe)$. Thus, putting

$$(\mathcal{J}(A))_e = \{T \in \mathcal{J}(A) \,|\, T_i(e) \neq 0\},$$

we obtain a map
$$T \to T^e \tag{5}$$
of $(\mathcal{J}(A))_e$ into $\mathcal{J}(eAe)$. If $S, T \in (\mathcal{J}(A))_e$, we have $\operatorname{Ker}(T^e) = \operatorname{Ker}(T) \cap eAe$ and $\operatorname{Ker}(T) = \{a \in A \,|\, ebace \in \operatorname{Ker}(T^e)$ for all b, c in $A\}$, and similarly for S; hence $S \simeq T$ if and only if $S^e \simeq T^e$. So (5) lifts to a one-to-one map of $\hat{A}_e$ into $(eAe)^\wedge$, where we have put $\hat{A}_e = \{\tau \in \hat{A} \,|\, e \notin \operatorname{Ker}(\tau)\}$.

LEMMA 7. *$\hat{A}_e$ is open in $\hat{A}$; and the map $\hat{A}_e \to (eAe)^\wedge$ lifted from* (5) *is a homeomorphism.*

Proof. Obviously $\hat{A}_e$ is open in $\hat{A}$. The continuity of (5) results from the fact that $\phi|(eAe) \in \Phi(T^e)$ whenever $\phi \in \Phi(T)$ $(T \in \hat{A}_e)$. To prove that the inverse of (5) is continuous, it suffices to show that if $\mathcal{S}$ is a subset of $(\mathcal{J}(A))_e$ closed in $(\mathcal{J}(A))_e$, then $\mathcal{S}^e = \{S^e | S \in \mathcal{S}\}$ is closed relative to $\{T^e | T \in (\mathcal{J}(A))_e\}$. Assume that $T \in (\mathcal{J}(A))_e$, $T^e \in (\mathcal{S}^e)^-$. Then there is a non-zero functional ϕ in $\Phi(T^e)$ such that $\phi = \lim_\gamma \phi_\gamma$, where $\phi_\gamma \in \Phi(\mathcal{S}^e)$. Setting $\psi(a) = \phi(eae)$, $\psi_\gamma(a) = \phi_\gamma(eae)$ $(a \in A)$, we have $\psi \in \Phi(T), \psi_\gamma \in \Phi(\mathcal{S})$, and $\psi_\gamma \to \psi$ on A. So $T \in \mathcal{S}^-$, whence $T \in \mathcal{S}$, or $T^e \in \mathcal{S}^e$. It follows that the inverse of (5) is continuous. The lemma is now proved.

We do not know whether the map $\hat{A}_e \to (eAe)^\wedge$ lifted from (5) is always *onto* $(eAe)^\wedge$. By Lemma 8 of [5], if the class τ in $(eAe)^\wedge$ contains an (algebraically) completely irreducible member, then τ belongs to the range of (5). In particular all finite-dimensional elements of $(eAe)^\wedge$ belong to the range of (5).

5. The functional dual space of a group

Throughout the rest of this paper G is a fixed locally compact group with unit e; $M_0(G)$ is the algebra (under convolution $*$) of all complex regular Borel measures on G with compact support; and λ is a left Haar measure on G. An element x of G will be identified with the unit mass at x; thus $G \subset M_0(G)$. The space $L(G)$ of all continuous complex functions on G with compact support becomes a subspace (in fact a two-sided ideal) of $M_0(G)$ when we identify f with $fd\lambda$.

We now define a linear system representation T of G. We wish every such T to possess an "integrated form" (a representation of $M_0(G)$). Rather than assume "completeness" of the underlying linear system, we shall put this requirement into the definition itself.

Definition. By a *linear system representation T of G on a linear system* $H(T) = \langle H_1, H_2 \rangle$, we mean a pair $\langle T_1, T_2 \rangle$, where

(i) T_1 [resp. T_2] is a homomorphism [resp. anti-homomorphism] of G into the group of invertible linear endomorphisms of H_1 [resp. H_2];

(ii) $(T_1(x)\xi | \eta) = (\xi | T_2(x)\eta)$ $\quad (x \in G, \xi \in H_1, \eta \in H_2)$;

(iii) $x \to (T_1(x)\xi | \eta)$ is continuous on G for each ξ in H_1 and η in H_2;

(iv) for each μ in $M_0(G)$, there exist (unique) linear endomorphisms $T_1(\mu)$ and $T_2(\mu)$ of H_1 and H_2 respectively such that, if $\xi \in H_1$ and $\eta \in H_2$,

$$(T_1(\mu)\xi | \eta) = \int_G (T_1(x)\xi | \eta)\, d\mu x = (\xi | T_2(\mu)\eta).$$

One should remark that, if $\langle H_1, H_2\rangle$ is the linear system $\langle X, X^*\rangle$ associated with a Banach space X, the above definition tallies with the usual definition of a Banach representation of G. Indeed, we define a *Banach representation of G on X* as a homomorphism S of G into the group of invertible bounded linear operators on X such that $x \to S_x\xi$ is norm-continuous on G for each ξ in X. Now it is a known but anonymous theorem (see [2], Theorem 2.8) that the definition is unchanged if we replace norm-continuity by weak continuity of $x \to S_x\xi$, that is, if we assume merely that $x \to \alpha(S_x\xi)$ is continuous on G for each ξ in X and α in X^*. Thus S is a Banach representation of G on X if and only if $x \to \langle S(x), (S(x))^*\rangle$ $(x \in G)$ is a linear system representation of G on the associated linear system $\langle X, X^*\rangle$. We shall refer to $x \to \langle S(x), (S(x))^*\rangle$ as the linear system representation of *G associated with S.*

We shall frequently fail to distinguish between a Banach representation of G and the associated linear system representation of G.

If T is a linear system representation of G, one easily checks that $\mu \to \langle T_1(\mu), T_2(\mu)\rangle$ (see (iv) of the definition) is a linear system representation of $M_0(G)$ on $H(T)$; call it the *integrated form* of T. We say that T is *topologically irreducible* [resp. *topologically completely irreducible*] if its integrated form is so. Two linear system representations S and T of G are *equivalent* [resp. *functionally equivalent*] if their integrated forms are so. We define $\mathcal{J}(G)$ [resp. $\hat{G}$] as the family of all equivalence classes [resp. functional equivalence classes] of topologically completely irreducible linear system representations of G. Thus we can regard $\mathcal{J}(G)$ and $\hat{G}$ as subspaces of $\mathcal{J}(M_0(G))$ and $(M_0(G))^\wedge$ respectively, and equip them with the relativized functional topology of $\mathcal{J}(M_0(G))$ and $(M_0(G))^\wedge$. So equipped, $\hat{G}$ will be called the (*functional*) *dual space* of G.

Recall that $L(G)$ is a two-sided ideal of $M_0(G)$; and note that the integrated form T of a non-zero linear system representation of G never vanishes on $L(G)$. In fact T is determined by $T|L(G)$. By Lemma 4 $T|L(G)$ is topologically completely irreducible whenever T is; and, by Lemma 5, $T \to T|L(G)$ is a homeomorphism of $\mathcal{J}(G)$ into $\mathcal{J}(L(G))$. Thus $\mathcal{J}(G)$ and $\hat{G}$ (with their functional topologies) can also be regarded as topological subspaces of $\mathcal{J}(L(G))$ and $(L(G))^\wedge$ respectively.

The functional topology of $\mathcal{J}(G)$, as defined above, has certain drawbacks. For one thing, on the subfamily of *unitary* representations it does not always coincide with the "hull-kernel" topology (studied in [3] and elsewhere). Consider for example the "$ax+b$" group G. This has precisely two distinct infinite-dimensional topologically completely irreducible unitary representations; call them S and T. Further S and T are distinguished by the hull-kernel topology (see [4], p. 263). However, it is easy to check that the integrated forms of S and T are both faithful on $M_0(G)$, and hence are not distinguished by the functional topology of $\mathcal{J}(G)$. Fortunately, for groups having large compact subgroups, it will turn out that the hull-kernel and functional topologies coincide on the unitary elements of $\mathcal{J}(G)$ (see § 9).

Naturally, the elements of $\mathcal{J}(G)$ which are Banach representations will be of particular interest to us. For handling these we shall need the idea of a norm-function. A *norm-function* on G is a positive-real-valued lower semi-continuous function α on G which is bounded on compact sets and satisfies $\alpha(xy) \leqslant \alpha(x)\alpha(y)$ for all x,y in G. If α and β are norm-functions so is $\max(\alpha,\beta)$; thus the norm-functions form an upward directed set. If α is a norm-function, $M_0(G)$ is a normed algebra under the norm $\| \;\|_\alpha$ defined by

$$\|\mu\|_\alpha = \int_G \alpha(x)\, d\,|\mu|\, x$$

($|\mu|$ being the total variation of μ).

If T is a Banach representation of G, the function α: $x \to \|T_x\|$ is a norm-function on G, and the integrated form of T is continuous on $M_0(G)$ with respect to $\| \;\|_\alpha$.

Definition. Suppose that α is a norm-function on G. Then $(\mathcal{J}(G))_\alpha$ will be the set consisting of those Banach representations T in $\mathcal{J}(G)$ such that, for some constant $k > 0$, we have $\|T_x\| \leqslant k\alpha(x)$ for all x in G. By $\hat{G}_\alpha$ we shall mean the family of classes τ in $\hat{G}$ which contain some element of $(\mathcal{J}(G))_\alpha$.

6. Naimark-equivalence and FDS representations

Closely connected with functional equivalence is the idea of Naimark-relatedness.

Let T be a linear system representation of A in $H = \langle H_1, H_2\rangle$; suppose K_i is a $\sigma(H)$-dense T_i-stable subspace of H_i $(i=1,2)$; then $K = \langle K_1, K_2\rangle$ is a linear system (with the duality restricted from H). Putting $S_i(a) = T_i(a)\,|\,K_i$ $(i=1,2,\, a \in A)$, we shall refer to $S = \langle S_1, S_2\rangle$, acting in K, as a *dense contraction* of T.

Note that a dense contraction of a dense contraction of T need not be a dense contraction of T (see Appendix, Example 2).

The following lemma is stated without proof in [16], § 8. We include the proof for completeness' sake.

LEMMA 8. *Let T and T' be two linear system representations of A, acting in linear systems H and H' respectively. Then the following two conditions are equivalent:*

(i) *T and T' have equivalent dense contractions;*

(ii) *There exists a $\sigma(H \oplus H')$-closed one-to-one linear map F of a $\sigma(H)$-dense T_1-stable subspace D of H_1 onto a $\sigma(H')$-dense T_1'-stable subspace D' of H_1', such that $FT_1(a)x = T_1'(a)Fx$ for all a in A and x in D.*

Proof. (I) Let T and T' have dense contractions S and S' acting in K and K' respectively; and suppose $S \cong S'$ under an equivalence $G = \langle G_1, G_2 \rangle$. Let F be the $\sigma(H \oplus H')$-closure of G_1 (considered as a subspace of $H_1 \oplus H_1'$). Now the equation $(\xi | \eta) = (\xi' | G_2(\eta))$ holds for all $\xi \oplus \xi'$ in G_1; hence it holds for all $\xi \oplus \xi'$ in F. Thus, if $(0 \oplus \xi') \in F$, we have $(\xi' | \eta') = 0$ for all η' in range$(G_2) = K_2'$, and hence $\xi' = 0$. Similarly $(\xi \oplus 0) \in F$ only if $\xi = 0$. So F is a $\sigma(H \oplus H')$-closed one-to-one linear map. Since G_1 is stable under the $\sigma(H \oplus H')$-continuous endomorphism $T_1(a) \oplus T_1'(a)$ of $H_1 \oplus H_1'$ (for each a), so is its closure F. This implies the remaining conditions on F required in (ii).

(II) Assume that (ii) holds. Let $F^0 = \{\xi' \oplus (-\xi) | \xi \oplus \xi' \in F\}$, and put

$$F^* = F^{0\perp} = \{(\eta' \oplus \eta) \in H_2' \oplus H_2 | (\xi | \eta) = (F(\xi) | \eta') \text{ for all } \xi \text{ in } D\}.$$

If $(0 \oplus \eta) \in F^*$, then $(D | \eta) = 0$, so that $\eta = 0$. Similarly $(\eta' \oplus 0) \in F^*$ only if $\eta' = 0$. So F^* is a one-to-one function. If $\xi' \in H_1'$ and $\xi' \in (\text{domain}(F^*))^\perp$, then $(\xi' \oplus 0) \in F^{*\perp} = F^0$ (the latter equality holding because F^0 is $\sigma(H' \oplus H)$-closed), whence $\xi' = 0$. So domain(F^*) is $\sigma(H')$-dense in H_2'. Similarly range(F^*) is $\sigma(H)$-dense in H_2. Since F^0 is stable under $T_1'(a) \oplus T_1(a)$, $F^* = F^{0\perp}$ is stable under $T_2'(a) \oplus T_2(a)$ for each a in A; consequently domain(F^*) and range(F^*) are stable under T_2' and T_2 respectively, and F^* is an equivalence between the restrictions of T_2' and T_2 to these subspaces. Thus, if we put $K_1 = D$, $K_2 = \text{range}(F^*)$, $K_1' = D'$, $K_2' = \text{domain}(F^*)$, the dense contractions of T and T' on $\langle K_1, K_2 \rangle$ and $\langle K_1', K_2' \rangle$ respectively are equivalent under $\langle F, F^{*-1} \rangle$.

Definition. Two linear system representations T and T' of A will be said to be *Naimark-related* if conditions (i) and (ii) of Lemma 8 hold.

Two linear system representations S and T of the locally compact group G are *Naimark-related* if their integrated forms (on $M_0(G)$) are Naimark-related. Since $L(G)$ is a two-sided ideal of $M_0(G)$ and contains an "approximate identity", one verifies easily that S and T are Naimark-related if and only if their integrated forms restricted to $L(G)$ are Naimark-related.

If S and T are Banach representations of A (or G), the conditions of weak denseness and weak closure in the definitions of Naimark-relatedness can be replaced by the corresponding norm-conditions.

In general Naimark-relatedness is not transitive, hence not an equivalence relation (see Appendix, Examples 3 and 4). Two Naimark-related linear system representations S and T of A clearly have the same kernel and hence, if they are in $\mathcal{J}(A)$, are functionally equivalent. It is conceivable (though quite unlikely) that, on $\mathcal{J}(A)$, functional equivalence coincides with the smallest equivalence relation containing Naimark-relatedness. (In

Example 4 of the Appendix we show that the two functionally though not unitarily equivalent infinite-dimensional irreducible unitary representations of the "$ax+b$" group can indeed be joined by a chain of Naimark-related pairs.)

The non-transitivity of Naimark-relatedness is due to the fact that different dense contractions of the same linear system representation may have small intersection. However, if there exists a "smallest" dense contraction, this cannot happen.

Definition. Let T be a linear system representation of A. A dense contraction of T acting in $\langle K_1, K_2\rangle$ will be called the *strictly smallest dense contraction* of T if, given any dense contraction T' of a dense contraction of T, with T' acting in $\langle L_1, L_2\rangle$, we have $K_1 \subset L_1$ and $K_2 \subset L_2$.

If a strictly smallest dense contraction exists, it is unique.

Let T and T' be two linear system representations of A, having strictly smallest dense contractions S and S' respectively. Then clearly T and T' are Naimark-related if and only if $S \cong S'$. Thus, restricted to the family of linear system representations which possess strictly smallest dense contractions, Naimark-relatedness becomes an equivalence relation, which we shall call *Naimark-equivalence.*

Our next job is to single out a useful class of linear system representations which possess strictly smallest dense contractions.

Let T be a linear system representation of A in H, and put

$$I(T) = \{a \in A \mid T_1(a) \text{ is of finite rank}\}.$$

(Note that $T_1(a)$ is of finite rank if and only if $T_2(a)$ is.) Let $H_i^0(T)$ ($i=1,2$) be the linear span of the ranges of the $T_i(a)$ with $a \in I(T)$. Evidently $H_i^0(T)$ is T_i-stable.

Definition. T is *finite-dimensionally spanned* (FDS for short) if $H_i^0(T)$ is $\sigma(H)$-dense in H_i for $i=1, 2$.

If T is FDS, then T restricted to $H^0(T) = \langle H_1^0(T), H_2^0(T)\rangle$ is a dense contraction of T, which we shall in future always denote by T^0.

LEMMA 9. *If T is* FDS, *T^0 is the strictly smallest dense contraction of T.*

Proof. Let T' be a dense contraction of T acting in $\langle K_1, K_2\rangle$. Since K_1 is $\sigma(H)$-dense in H_1 and $T_1(a)$ is $\sigma(H)$-continuous, $T_1(a)$ (K_1) is dense in range$(T_1(a))$, hence equal to range$(T_1(a))$, for each a in $I(T)$. It follows that $H_1^0(T) \subset K_1$; and similarly $H_2^0(T) \subset K_2$; in particular T' is FDS. Applying the same argument to a dense contraction of T' which acts in $\langle L_1, L_2\rangle$, we find that $H_i^0(T) \subset H_i^0(T') \subset L_i$. This proves the Lemma.

Thus, as we have observed, Naimark-relatedness is an equivalence relation when restricted to FDS linear system representations of A.

Note that, if T is topologically irreducible, it will be FDS provided that $T_1(a)$ is of non-zero finite rank for at least one a in A.

If T is FDS, any dense contraction of T is obviously FDS. But the converse is false; T may have an FDS dense contraction without itself being FDS (see Appendix, Example 2).

LEMMA 10. *If T is* FDS, *the following five conditions are equivalent:*

(i) *T is topologically irreducible,*
(ii) *T is topologically completely irreducible,*
(iii) *T_1^0 is irreducible,*
(iv) *T_2^0 is irreducible,*
(v) *T^0 is completely irreducible.*

Proof. We shall first show that (i) $\Rightarrow$ (iii). Suppose T is topologically irreducible. Let K_1 be a T_1-stable non-zero subspace of $H_1^0(T)$. Then K_1 is $\sigma(H)$-dense in H_1; so the restriction of T to $\langle K_1, H_2\rangle$ is a dense contraction of T. Consequently $H_1^0(T) \subset K_1$, or $H_1^0(T) = K_1$. Thus T_1^0 is irreducible.

Next we show that (iii) $\Rightarrow$ (i). Let K_1 be a non-zero T_1-stable subspace of H_1. For a in $I(T)$ we have $T_1(a)K_1 \subset K_1 \cap H_1^0(T)$, and $T_1(a)K_1 \neq \{0\}$ for some a in $I(T)$. So $K_1 \cap H_1^0(T)$ is a non-zero T_1^0-stable subspace of $H_1^0(T)$, and therefore, if T_1^0 is irreducible, $K_1 \supset H_1^0(T)$. So K_1 is $\sigma(H)$-dense in H_1. Therefore (iii) $\Rightarrow$ (i).

Similarly (i) $\Rightarrow$ (iv) $\Rightarrow$ (i). Thus (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Next we shall show that (i) $\Rightarrow$ (v) $\Rightarrow$ (ii). Since (ii) $\Rightarrow$ (i) trivially, the proof will then be complete.

Assume (i). Since (iii) and (iv) then hold, (v) will hold by Jacobson's Theorem ([13]), p. 28) if we show that the division algebra E_i of endomorphisms of $H_i^0(T)$ commuting with all $T_i^0(a)$ is finite-dimensional over the complexes, and hence coincides with the complexes. Assume then that E_1 is infinite-dimensional; and pick an infinite sequence $\{Q_n\}$ of elements of E_1 which are linearly independent over the complexes. Choose ξ in $H_1^0(T)$ and a in $I(T)$ so that $\eta = T_1^0(a)\xi \neq 0$. Then, for each n, $Q_n\eta = T_1^0(a)Q_n\xi \in \text{range}(T_1^0(a))$. But the latter is finite-dimensional, so the $Q_n\eta$ are not linearly independent; that is, there exist complex numbers $\{\lambda_n\}$ (some, but only finitely many, of which are non-zero) such that $(\sum_n \lambda_n Q_n)\eta = 0$. Since E_1 is a division algebra, the last equation implies $\sum_n \lambda_n Q_n = 0$, contradicting the independence of the Q_n. So E_1 is finite-dimensional, and T_1^0 is completely irreducible. Similarly T_2^0 is completely irreducible. Thus (i) $\Rightarrow$ (v).

We shall next prove that (v) $\Rightarrow$ (ii). Assume (v); and let $\xi_1, \ldots, \xi_n$ be linearly independent elements of H_1. Since $H_2^0(T)$ is $\sigma(H)$-dense in H_2, there are elements $\eta_1, \ldots, \eta_n$ in $H_2^0(T)$ such that the matrix $\{(\xi_i|\eta_j)\}_{i,j=1,\ldots,n}$ has non-zero determinant. Now it follows from (v) that the restriction T' of T^0 to the ideal $I(T)$ is completely irreducible. Indeed, since (v) $\Rightarrow$ (iii) trivially and since $I(T) \not\subset \operatorname{Ker}(T_1)$, T' must be irreducible; hence, by the implication (iii) $\Rightarrow$ (v) applied to T', T' is completely irreducible. Thus there exists an element a of $I(T)$ such that $T_2(a)\eta_j = \eta_j$ for all j; and the matrix of numbers $(T_1(a)\xi_i|\eta_j) = (\xi_i|T_2(a)\eta_j) = (\xi_i|\eta_j)$ has non-zero determinant. Therefore the $T_1(a)\xi_i$ $(i = 1, 2, \ldots, n)$ are linearly independent elements of $H_1^0(T)$. Now suppose $\xi_1', \ldots, \xi_n'$ are any elements of H_1. By (v) applied to the $T_1(a)\xi_i$, we can find a net $\{b_\gamma\}$ of elements of A such that

$$T_1(b_\gamma a)\,\xi_i = T_1(b_\gamma)\;T_1(a)\,\xi_i \underset{\gamma}{\rightarrow} \xi_i'$$

for each i. By the arbitrariness of the ξ_i and the ξ_i', this proves that T is topologically completely irreducible. So (v) $\Rightarrow$ (ii).

Note that Theorem 6 of [8] and its consequences are a special case of the construction of T^0 from T.

LEMMA 11. *Two topologically irreducible* FDS *linear system representations S and T of A are Naimark-equivalent if and only if* $\operatorname{Ker}(S) = \operatorname{Ker}(\mathrm{T})$.

Proof. Clearly $\operatorname{Ker}(S) = \operatorname{Ker}(S^0)$, and likewise for T. Hence by Lemma 10 it is sufficient to assume that S and T are completely irreducible, and to show that $\operatorname{Ker}(S) = \operatorname{Ker}(T)$ if and only if $S \cong T$.

Let $a \in A$. I claim that $S_1(a)$ is of rank 1 if and only if $\{b \in A \,|\, S_1(aba) = 0\}$ is of co-dimension 1 in A.

Indeed, assume that $S_1(a)$ has range $C\xi$ $(0 \neq \xi \in H_1(S))$. Then it is easy to see that there is a non-zero linear functional λ on A such that $S_1(aba) = \lambda(b)S_1(a)$ $(b \in A)$. In particular it follows that $\{b \,|\, S_1(aba) = 0\}$ has co-dimension 1. Now assume that the rank of $S_1(a)$ is greater than 1; and let u and u' be elements of $H_1(S)$ such that $\xi = S_1(a)u$ and $\xi' = S_1(a)u'$ are linearly independent. By the complete irreducibility of S_1, there are elements b, c of A such that $S_1(b)\xi = u$, $S_1(b)\xi' = 0$, $S_1(c)\xi = 0$, $S_1(c)\xi' = u'$. It follows that $S_1(aba)u = \xi$, $S_1(aba)u' = 0$, $S_1(aca)u = 0$, $S_1(aca)u' = \xi'$. Thus the operators $S_1(aba)$ and $S_1(aca)$ are linearly independent; and therefore the kernel of the map $b \rightarrow S_1(aba)$ $(b \in A)$ must have co-dimension greater than 1. This proves the above claim.

To prove the lemma we must show that $\operatorname{Ker}(S) = \operatorname{Ker}(T)$ only if $S \cong T$. Assume then that $\operatorname{Ker}(S) = \operatorname{Ker}(T)$. Since S is FDS, there exists an a in A such that $S_1(a)$ (and hence also $S_2(a)$) has rank 1. By the preceding claim and the fact that $\operatorname{Ker}(S) = \operatorname{Ker}(T)$, this implies that $T_1(a)$ and $T_2(a)$ likewise have rank 1. Let ξ, η, ξ', η' be non-zero vectors in the ranges of $S_1(a)$, $S_2(a)$, $T_1(a)$, and $T_2(a)$ respectively; and put $p(b) = (S_1(b)\xi \mid \eta)$, $p'(b) = (T_1(b)\xi' \mid \eta')$ $(b \in A)$. If $b \in A$, we have $p_1(b) = 0 \Leftrightarrow 0 = (S_1(ba)u \mid S_2(a)v) = (S_1(aba)u \mid v)$ for all u in $H_1(S)$ and v in $H_2(S) \Leftrightarrow S_1(aba) = 0$; that is, $p(b) = 0 \Leftrightarrow aba \in \mathrm{Ker}(S)$. The same holds for p'. Thus p and p' are non-zero and have the same kernel, whence $p' = k \cdot p$ for some non-zero complex constant k. By Proposition 2 of [5], this implies that $S \cong T$. The proof is now complete.

COROLLARY 1. *For* FDS *elements of* $\mathcal{J}(A)$, *functional equivalence and Naimark-equivalence are the same.*

COROLLARY 2. *If* S, T *are functionally equivalent elements of* $\mathcal{J}(A)$ *and* T *is* FDS, *then* S *is also* FDS.

Proof. By Lemma 10, S is FDS if and only if $S_1(a)$ is of rank 1 for some a in A. Now it is easy to see that the claim asserted in the second paragraph of the proof of Lemma 11 is true whenever $S \in \mathcal{J}(A)$, regardless of whether S is initially assumed to be FDS or not. Thus, choosing a so that $T_1(a)$ is of rank 1, we see that $S_1(a)$ is also of rank 1, and hence that S is FDS.

In view of Corollary 2, we can refer to a class τ in $\hat{A}$ as FDS if some (hence all) of its members are FDS.

A linear system representation T of G is FDS if its integrated form is FDS, or equivalently, if its integrated form restricted to $L(G)$ is FDS.

Definition. The locally compact group G will be said to be FDS if every element of $\mathcal{J}(G)$ is FDS.

By the above Corollary 1, for FDS groups functional equivalence in $\mathcal{J}(G)$ can always be replaced by Naimark-equivalence.

7. Groups with enough small idempotents

Let μ be an idempotent element of $M_0(G)$; we shall write $L^\mu(G)$ for the subalgebra $\mu * L(G) * \mu$ of $L(G)$. If T is a linear system representation of G, form the linear system representation T^μ of $\mu * M_0(G) * \mu$ on $H^\mu(T)$ as in § 4 (identifying T with its integrated form). We shall denote the restriction of T^μ to $L^\mu(G)$ by $T^{(\mu)}$. By § 4, if $T \in \mathcal{J}(G)$ and $T_1(\mu) \neq 0$, then $T^{(\mu)} \in \mathcal{J}(L^\mu(G))$.

LEMMA 12. *Let* μ *be an idempotent element of* $M_0(G)$, *and* m *a positive integer. The following conditions* (i)–(iv) *are equivalent:*

(i) *Every topologically completely irreducible linear system representation of* $L^\mu(G)$ *is of finite dimension* $\leqslant m$;

(ii) *There are enough representations of* $L^{\mu}(G)$ *of finite dimension* $\leqslant m$ *to distinguish points of* $L^{\mu}(G)$;

(iii) *if* T *is any topologically completely irreducible linear system representation of* G, *then* $\dim(H^{\mu}(T)) \leqslant m$;

(iv) *there are enough linear system representations* T *of* G *satisfying* $\dim(H^{\mu}(T)) \leqslant m$ *to separate points of* $L(G)$.

Proof. That (ii) implies (i) follows from Kaplansky's theory of polynomial identities (see [8], Lemma 1). The remark preceding this lemma shows that (i) implies (iii). That (iii) implies (iv) follows from the fact that points of $L(G)$ are separated by the topologically completely irreducible Banach representations of G (in fact by the unitary ones; see [7]). The construction of $T^{(\mu)}$ from T shows that (iv) implies (ii).

Definition. An idempotent element μ of $M_0(G)$ which for some m satisfies conditions (i)–(iv) of Lemma 12 will be said to be *small.*

Notice that if μ is small, and γ is an idempotent in $M_0(G)$ satisfying $\gamma \leqslant \mu$ (that is, $\gamma * \mu = \mu * \gamma = \gamma$), then by (iii) of Lemma 12 γ is also small. This observation makes the term 'small' appropriate. Likewise, if γ and μ are small idempotents with $\gamma * \mu = \mu * \gamma = 0$, then $\gamma + \mu$ is a small idempotent.

Suppose now that μ is a fixed small idempotent in $M_0(G)$, and m is the positive integer of Lemma 12. Denoting by $\hat{G}^{(\mu)}$ the set of all classes τ in $\hat{G}$ such that $\mu \notin \mathrm{Ker}(\tau)$, and noting that $L^{\mu}(G)$ is an ideal of $\mu * M_0(G) * \mu$, we have by Lemmas 5 and 7:

LEMMA 13. *$\hat{G}^{(\mu)}$ is an open subset of $\hat{G}$; and the map*

$$T \to T^{(\mu)} \tag{6}$$

is a homeomorphism of $\hat{G}^{(\mu)}$ into $(L^{\mu}(G))^{\wedge}$.

We recall from Lemma 12 (i) that all elements of $(L^{\mu}(G))^{\wedge}$ are of dimension no greater than m.

What can we say about the range of the mapping (6)?

LEMMA 14. *If S is a finite-dimensional irreducible representation of $L^{\mu}(G)$ which is continuous with respect to $\| \ \|_{\alpha}$ for some norm-function α, there is a topologically completely irreducible Banach representation T of G such that*

(i) *the integrated form of T on $M_0(G)$ is continuous with respect to $\| \ \|_{\alpha}$;*

(ii) $T^{(\mu)} \cong S$.

(*Note that, in view of* (ii), *T is* FDS.)

This is proved in the course of the proof of sufficiency for Theorem 8 of [8].

Lemma 14 can be restated in the following form:

LEMMA 15. *For each norm-function α, the image of $\hat{G}^{(\mu)} \cap \hat{G}_\alpha$ under the mapping* (6) *consists precisely of those elements of $(L^\mu(G))^\frown$ which are continuous with respect to $\| \ \|_\alpha$.*

Definition. G is said to have *enough small idempotents* if, for each topologically completely irreducible linear system representation T of G, there is a small idempotent element μ of $M_0(G)$ such that $T_1(\mu) \neq 0$.

If this condition holds, G is FDS.

LEMMA 16. *If G has enough small idempotents, then $\hat{G}_\alpha$ is a closed locally compact subspace of $\hat{G}$ for each norm-function α.*

Proof. Let α be a norm-function. Let μ be any small idempotent. Let $L^\mu_\alpha(G)$ be the completion of $L^\mu(G)$ with respect to $\| \ \|_\alpha$. Since every finite-dimensional irreducible representation of $L^\mu_\alpha(G)$ is continuous ([5], Proposition 10), and hence determined by its (irreducible) restriction to $L^\mu(G)$, we may regard $(L^\mu_\alpha(G))^\frown$ as a subset of $(L^\mu(G))^\frown$. In fact, by Remark 2 of § 8 of [5], the embedding is topological.

By Lemma 15, the mapping (6) carries $\hat{G}_\alpha \cap \hat{G}^{(\mu)}$ onto $(L^\mu_\alpha(G))^\frown$. Since (6) is a homeomorphism (Lemma 13), and since $(L^\mu_\alpha(G))^\frown$ is a closed and locally compact subset of $(L^\mu(G))^\frown$ ([5], Proposition 14 and Theorem 6), it follows that $\hat{G}_\alpha \cap \hat{G}^{(\mu)}$ is a closed and locally compact subset of $\hat{G}^{(\mu)}$. Now the $\hat{G}^{(\mu)}$ (μ running over all small idempotents) form an open covering of $\hat{G}$ (open by Lemma 13, a covering since G has enough small idempotents). Hence since $\hat{G}_\alpha \cap \hat{G}^{(\mu)}$ is closed in $\hat{G}^{(\mu)}$ for each such μ, $\hat{G}_\alpha$ is closed in $\hat{G}$; and since $\hat{G}_\alpha \cap \hat{G}^{(\mu)}$ is locally compact for each such μ, $\hat{G}_\alpha$ is locally compact.

LEMMA 17. *If G satisfies the second axiom of countability and has enough small idempotents, then $\hat{G}_\alpha$ satisfies the second axiom of countability for all norm-functions α.*

Proof. The hypothesis implies that $L^\mu_\alpha(G)$ (defined as in the preceding proof) is separable for each small idempotent μ and each norm-function α, and hence (by [5], Theorem 5) that $L^\mu_\alpha(G)^\frown$ satisfies the second axiom of countability. Thus, by the argument of the preceding proof, $\hat{G}_\alpha \cap \hat{G}^{(\mu)}$ satisfies the second axiom of countability for each such μ and α. The proof will therefore be complete if we show that $\hat{G}$ can be covered by countably many $\hat{G}^{(\mu)}$ (the μ being small idempotents).

Let $\{D_n\}$ be an increasing sequence of compact subsets of G such that every compact set is contained in some D_n. Let $p_n(f) = \sup_{x \in D_n} |f(x)|$ for each f in the space $C(G)$ of all continuous complex functions on G; and let

$$S_{n,q}=\{\mu\in M_0(G)\,|\,|\mu(f)|\leqslant qp_n(f)\text{ for all }f\text{ in }C(G)\}.$$

On each $S_{n,q}$ the topology of pointwise convergence on $C(G)$ is metrizable and separable. Since the $S_{n,q}$ $(n, q=1, 2, \dots)$ form a countable covering of $M_0(G)$, it follows that, to every subset S of $M_0(G)$ there is a countable subset S' of S which is pointwise dense in S (that is, in the topology of pointwise convergence on $C(G)$).

In particular there is a countable family W' of small idempotents which is pointwise dense in the set W of all small idempotents. If T is an element of $\hat{G}$, then, since G has enough small idempotents, $(T_1(\mu)\xi\,|\,\eta)\neq 0$ for some μ in W, some ξ in $H_1(T)$, and some η in $H_2(T)$. But then the denseness of W' in W implies that μ may be chosen to lie in W'. Thus $T_1(\mu)\neq 0$ for some μ in W', and we have $\bigcup_{\mu\in W'}\hat{G}^{(\mu)}=\hat{G}$. The proof of Lemma 17 is now complete.

Now there is no a priori reason to suppose that $\bigcup_\alpha \hat{G}_\alpha=\hat{G}$. Even if $\bigcup_\alpha \hat{G}_\alpha=\hat{G}$, the fact that each $\hat{G}_\alpha$ is locally compact (Lemma 16) does not imply that $\hat{G}$ is locally compact, as we shall see for Abelian groups in § 10. Let us make the following definitions.

Definition. G will be called *Banach-representable* if every class in $\hat{G}$ contains some Banach representation.

Definition. Assume that G has enough small idempotents. We shall say that G has a *locally bounded dual* (or, less logically, that $\hat{G}$ is *locally bounded*) if, for each T in $\hat{G}$, there exists a norm-function α such that $\hat{G}_\alpha$ is a neighborhood of T in $\hat{G}$.

If G has a locally bounded dual it is obviously Banach-representable. The converse is false, since an Abelian group is Banach-representable but need not have a locally bounded dual (see § 10). In Example 1 of the Appendix we shall show that the Galilean group (see the Introduction) is not Banach-representable.

We conjecture that all FDS groups are Banach-representable, but we cannot prove it.

THEOREM 2. *Assume that G has enough small idempotents and a locally bounded dual. Then $\hat{G}$ is locally compact. If in addition G is compactly generated and satisfies the second axiom of countability, then $\hat{G}$ satisfies the second axiom of countability.*

Proof. The first statement is evident from Lemma 16. The second statement will follow from Lemma 17 if we can show that there exist countably many norm-functions $\{\alpha_n\}$ such that

$$\bigcup_n \hat{G}_{\alpha_n}=\hat{G}.$$

To establish this, choose a compact neighborhood U of e such that $\bigcup_{p=1}^{\infty} U^p=G$. For each positive integer n, define α_n to be the supremum of all the norm-functions β on G satisfying $\beta(x)\leqslant n$ for all x in U. Since $\bigcup_{p=1}^{\infty}U^p=G$, $\alpha_n(x)<\infty$ for each x; and it is easy to see that α_n is itself a norm-function. Clearly, to every norm-function α, there is an n such that $\alpha \leqslant \alpha_n$ (take $n \geqslant \sup_{x \in U} \alpha(x)$), and hence $\hat{G}_\alpha \subset \hat{G}_{\alpha_n}$. It follows that

$$\bigcup_n \hat{G}_{\alpha_n} = \bigcup_\alpha \hat{G}_\alpha = \hat{G}.$$

The proof is complete.

Conjecture. Every compactly generated locally compact group which has enough small idempotents has a locally bounded dual.

8. Large compact subgroups

Let us fix a compact subgroup K of G. Identifying each f in $L(K)$ with the measure $f(u)du$ on K (du being normalized Haar measure on K) and hence with a measure in $M_0(G)$ (the "injection" of $f(u)du$ into G), we shall regard $L(K)$ as a subalgebra of $M_0(G)$ under convolution.

Definition. K is a *large* compact subgroup of G if every idempotent element of $L(K)$ is small (with respect to G).

Since every idempotent in $L(K)$ is contained in a central idempotent, and every central idempotent is a finite sum of minimal central idempotents, K will be large if its minimal central idempotents are small in G (see the paragraphs preceding Lemma 13).

By $\hat{K}$ we shall mean as usual the family of all equivalence classes of irreducible (finite-dimensional) Banach representations of K. If $D \in \hat{K}$, let

$$\psi_D(u) = (\dim D)\ (\mathrm{Trace}(D_u))^-$$

($u \in K$; $^-$ means complex conjugate). The ψ_D are precisely the minimal central idempotents of $L(K)$. Further, if T is a linear system representation of G in H, then the range of $T_1(\psi_D)$ is the *D-subspace of* T_1, that is, the sum of all those T_1-stable subspaces of H_1 on which $T_1 | K$ acts equivalently to D. It follows that K is a large compact subgroup of G if and only if, for each D in $\hat{K}$, there is a positive integer m_D such that, for any topologically completely irreducible linear system representation T of G, the multiplicity of D in $T_1 | K$ is no greater than m_D.

From the last remark we deduce that, if K is large and L is a compact subgroup of G containing K, then L is likewise large. This makes the term 'large' appropriate.

Since, for every linear system representation T, $T_1 | K$ contains some D in $\hat{K}$, we have

LEMMA 18. *If G has a large compact subgroup K, then G has enough small idempotents.*

LEMMA 19. *If K is a large compact subgroup of G, and K_0 is a closed subgroup of K such that K/K_0 is finite, then K_0 is a large compact subgroup of G.*

Proof. By Frobenius' Reciprocity Theorem, there are only finitely many irreducible representations of K whose restrictions to K_0 contain a given irreducible representation of K_0. From this the lemma follows immediately.

Suppose now that K is a large compact subgroup of G. For each f in $L(G)$ let $f^0(x)=\int_K f(uxu^{-1})\,du$ $(x\in G)$. Thus $f\to f^0$ is an idempotent linear map of $L(G)$ onto the subalgebra I of $L(G)$ consisting of all f for which $f(uxu^{-1})=f(x)$ $(u\in K, x\in G)$. If $D\in\hat{K}$, we write $L^D(G)$ instead of $L^{(\varphi_D)}(G)$, and put $I^D=I\cap L^D(G)$. One verifies easily that $\{f^0\,|\,f\in L^D(G)\}=I^D$. We shall write $\hat{G}^{(D)}$ instead of $\hat{G}^{(\varphi_D)}$ (the subset of $\hat{G}$ consisting of those T such that $T_1|K$ contains D); and, for each T in $\hat{G}^{(D)}$ we shall write $T^{(D)}$ for $T^{(\varphi_D)}$, the corresponding finite-dimensional irreducible representation of $L^D(G)$ on $H^D(T)=H^{\varphi_D}(T)$.

It has been observed by Godement ([8], Lemma 9) that, if $D\in\hat{K}$ and $T\in\hat{G}^{(D)}$, the restriction of $T^{(D)}$ to I^D is a multiple of a (unique) irreducible representation of I^D which we shall denote by $\tilde{T}^{(D)}$. Thus we have a mapping

$$T\to\tilde{T}^{(D)} \tag{7}$$

of $\hat{G}^{(D)}$ into $(I^D)\hat{}$.

THEOREM 3. *Suppose that K is a large compact subgroup of G. Then, for each D in $\hat{K}$, the map* (7) *is a homeomorphism of $\hat{G}^{(D)}$ into $(I^D)\hat{}$. In particular,* (7) *is one-to-one.*

Proof. Since the map $T\to T^{(D)}$ is already known (Lemma 13) to be a homeomorphism of $\hat{G}^{(D)}$ into $(L^D(G))\hat{}$, it suffices to show that $T^{(D)}\to\tilde{T}^{(D)}$ is a homeomorphism.

Let $T\in\hat{G}^{(D)}$, $\mathcal{S}\subset\hat{G}^{(D)}$; and assume first that $T^{(D)}$ belongs to the functional closure of $\mathcal{S}^{(D)}=\{S^{(D)}\,|\,S\in\mathcal{S}\}$. Choose a functional ϕ on $L^D(G)$ associated with $T^{(D)}$ (see [5], § 1) which does not vanish on I^D; then there exists a net $\{\phi_\gamma\}$ of functionals on $L^D(G)$, each associated with some $S_\gamma^{(D)}$ $(S_\gamma\in\mathcal{S})$, which converges pointwise to ϕ. But then the $\phi_\gamma|I^D$ and $\phi|I^D$ are associated with $\tilde{S}_\gamma^{(D)}$ and $\tilde{T}^{(D)}$ respectively, and $\phi_\gamma|I^D\xrightarrow[\gamma]{}\phi|I^D$ pointwise on I^D. Thus $\tilde{T}^{(D)}$ belongs to the functional closure of $\tilde{\mathcal{S}}^{(D)}=\{\tilde{S}^{(D)}\,|\,S\in\mathcal{S}\}$; and we have proved that $T^{(D)}\to\tilde{T}^{(D)}$ is continuous.

Conversely, assume that $\tilde{T}^{(D)}$ belongs to the functional closure of $\tilde{\mathcal{S}}^{(D)}$. Then there is a non-zero functional ψ on I^D associated with $\tilde{T}^{(D)}$, and a net $\{\psi_\gamma\}$ of functionals on I^D each associated with some $\tilde{S}_\gamma^{(D)}$ in $\tilde{\mathcal{S}}^{(D)}$, such that $\psi_\gamma\to\psi$ pointwise on I^D. Define ϕ, ϕ_γ on $L^D(G)$ as follows: $\phi(f)=\psi(f^0)$, $\phi_\gamma(f)=\psi_\gamma(f^0)$ $(f\in L^D(G))$. Clearly $\phi_\gamma\to\phi\neq 0$ pointwise on $L^D(G)$. If $f\in L^D(G)$ and $T_f^{(D)}=0$, then

$$T_{(f^0)}^{(D)}=\int_K T_u T_f^{(D)} T_u^{-1}\,du=0;$$

and, since $T_{(f^0)}^{(D)}$ is a direct sum of copies of $\tilde{T}_{(f^0)}^{(D)}$, this implies $\tilde{T}_{(f^0)}^{(D)}=0$, whence $\psi(f^0)=0$, or

$\phi(f)=0$. Thus ϕ is associated with $T^{(D)}$; and likewise ϕ_γ is associated with $S_\gamma^{(D)}$. It follows that $T^{(D)}$ belongs to the closure of $\mathcal{S}^{(D)}$.

Since ϕ was defined from $\tilde{T}^{(D)}$, and in its turn determines $T^{(D)}$ ([5], Proposition 2), the map $T^{(D)} \to \tilde{T}^{(D)}$ is one-to-one. It follows from the preceding paragraph that $\tilde{T}^{(D)} \to T^{(D)}$ is continuous. So the proof of Theorem 3 is complete.

It can be shown, though we shall not need it here, that *every* finite-dimensional irreducible representation of I^D which is continuous with respect to $\| \ \|_\alpha$ for some norm-function α, is of the form $\tilde{T}^{(D)}$ for some T in $\hat{G}^{(D)} \cap \hat{G}_\alpha$.

9. Unitary representations

A unitary representation of G is a Banach representation T such that $X(T)$ is a Hilbert space and T_x is unitary for each x in G. Let $\hat{G}_{un}$ be the family of all unitary equivalence classes of topologically irreducible unitary representations of G. By the von Neumann double commuter theorem the elements of $\hat{G}_{un}$ are actually topologically completely irreducible; and by Mackey's form of Schur's Lemma (see [12]) Naimark-relatedness is the same as unitary equivalence for elements of $\hat{G}_{un}$. It follows that $\hat{G}_{un}$ may be regarded as a subset of $\mathcal{J}(G)$. In fact, if G is an FDS group, $\hat{G}_{un}$ may be regarded as a subset of $\hat{G}$ (namely, the set of those classes in $\hat{G}$ which contain some unitary representation).

Now in [3] we imposed a topology on $\hat{G}_{un}$ called the *hull-kernel* topology; it could be defined by means of uniform convergence on compact sets of functions of positive type associated with the representations (see [3], Theorem 1.5). For FDS groups it is natural to ask (a) whether the hull-kernel topology of $\hat{G}_{un}$ coincides with the relativized functional topology of $\hat{G}_{un}$ considered as a subset of $\hat{G}$, and (b) whether the functional topology of $\hat{G}$ can also be defined in terms of uniform convergence on compact sets of functions on G associated with the representations. We do not know the answers to these questions in general. However, if G has a large compact subgroup, the answer to both questions is 'yes'. Question (a) in this case will be answered in the next lemma. Question (b) is treated in § 13.

LEMMA 20. *If G has a large compact subgroup K, the relativized functional topology of $\hat{G}_{un}$ coincides with the hull-kernel topology.*

Proof. Let $C^*(G)$ be the group C^*-algebra of G (see for example § 3 of [3]); $L(G)$ is of course a dense subalgebra of $C^*(G)$. The left and right action of $M_0(G)$ on $L(G)$ is continuous with respect to the norm of $C^*(G)$, and so can be extended to $C^*(G)$. Thus, fixing $D \in \hat{K}$, we may form the closed $*$-subalgebra $C_D^*(G) = \psi_D * C^*(G) * \psi_D$ of $C^*(G)$. If $T \in \hat{G}_{un} \cap \hat{G}^{(D)}$, $T^{(D)}$ extends to an irreducible finite-dimensional $*$-representation of $C_D^*(G)$, which we shall identify with $T^{(D)}$.

Now, by an argument exactly similar to that of [5], § 4, we can prove that (a) $\hat{G}_{un} \cap \hat{G}^{(D)}$ is open in $\hat{G}_{un}$ in the hull-kernel topology, and (b) that $T \to T^{(D)}$ is a homeomorphism of $\hat{G}_{un} \cap \hat{G}^{(D)}$ into the set S of all irreducible *-representations of $C^*_D(G)$ with the hull-kernel topology. Comparing these facts with Lemma 13, we see that Lemma 20 will be proved if we show that the hull-kernel topology of S coincides with the functional topology of $(L^D(G))\hat{}$ relativized to S. But this follows from [5], Proposition 15, and [5], Remark 2 of § 8, if we remember that the elements of S are of bounded finite dimension.

We remark that Lemma 20 holds if we merely assume that G has enough small *self-adjoint* idempotents. We do not know whether it holds if G merely has enough small idempotents.

10. Abelian groups

In this section the group G is assumed to be Abelian. $\hat{G}$ consists of all continuous homomorphisms of G into the multiplicative group of non-zero complex numbers. Setting $\chi(f) = \int_G \chi(x) f(x)\, d\lambda x$, we may regard each χ in $\hat{G}$ as a multiplicative linear functional on $L(G)$. By the corollary of Lemma 3 and the paragraph dealing with $L(G)$ in § 5 we have:

LEMMA 21. *The functional topology of $\hat{G}$ is the topology of pointwise convergence on $L(G)$ of the corresponding functionals on $L(G)$.*

THEOREM 4. *The functional topology of $\hat{G}$ coincides with the topology of uniform convergence on compact subsets of G.*

Proof. It is obvious from Lemma 21 that the uniform-on-compacta topology contains the functional topology.

To prove the converse we invoke a well-known structure theorem for locally compact Abelian groups ([12], p. 389), to write

$$G = R^n \times H,$$

where R^n is the additive group of real n-space and H is a locally compact Abelian group having a compact open subgroup K. Now suppose that $\phi_\gamma \to \phi$ pointwise on $L(G)$ $(\phi_\gamma, \phi \in \hat{G})$, and put $\alpha = \phi | R^n$, $\beta = \phi | H$, $\alpha_\gamma = \phi_\gamma | R^n$, $\beta_\gamma = \phi_\gamma | H$. We have $\alpha(x) = e^{(u,x)}$, $\alpha_\gamma(x) = e^{(u^\gamma, x)}$ $(x \in R^n)$, where $u^\gamma, u \in C^n$, and $(u,x) = \sum_i u_i x_i$. Choose g in $L(H)$ such that $\beta(g) \neq 0$, and a differentiable function f in $L(R^n)$ such that $\alpha(f) \neq 0$. Integrating by parts, we find that

$$\alpha\left(\frac{\partial f}{\partial x_j}\right) = -u_j \alpha(f), \quad \alpha_\gamma\left(\frac{\partial f}{\partial x_j}\right) = -u_j^\gamma \alpha_\gamma(f).$$

Hence $$u_j^\gamma \alpha_\gamma(f)\,\beta_\gamma(g) = -\phi_\gamma\left(\frac{\partial f}{\partial x_j}\times g\right) \underset{\gamma}{\rightarrow} -\phi\left(\frac{\partial f}{\partial x_j}\times g\right) = u_j\alpha(f)\,\beta(g)$$

for each j; also $$\alpha_\gamma(f)\,\beta_\gamma(g) = \phi_\gamma(f\times g) \underset{\gamma}{\rightarrow} \phi(f\times g) = \alpha(f)\,\beta(g) \neq 0.$$

(Here $(f\times g)\,(xy) = f(x)g(y)$, $x\in R^n$, $y\in H$, and similarly for $(\partial f/\partial x_j)\times g$.) This implies that $u_j^\gamma \rightarrow u_j$ for each j, whence $u^\gamma \rightarrow u$ in C^n. Thus $\alpha_\gamma \rightarrow \alpha$ uniformly on compacta.

If f is as before and $h\in L(H)$, we have

$$\alpha_\gamma(f)\,\beta_\gamma(h) = \phi_\gamma(f\times h) \underset{\gamma}{\rightarrow} \phi(f\times h) = \alpha(f)\,\beta(h).$$

But by the preceding paragraph $\alpha_\gamma(f)\rightarrow\alpha(f)\neq 0$. Hence $\beta_\gamma(h)\rightarrow\beta(h)$; so $\beta_\gamma\rightarrow\beta$ pointwise on $L(H)$. Since the $\beta_\gamma|K$ are characters of the compact group K, this implies that, for all large enough γ, $\beta_\gamma\equiv\beta$ on K. Hence to prove that $\beta_\gamma\rightarrow\beta$ uniformly on compacta in H, we need only show that $\beta_\gamma(y) \underset{\gamma}{\rightarrow} \beta(y)$ for each y in H. Fix y in H, and let

$$f(yu) = \begin{cases} \overline{\beta(u)} & \text{for } u\in K, \\ 0 & \text{for } u\in H-K. \end{cases}$$

Then $f\in L(H)$; and we check that $\beta(f)=\beta(y)$, and $\beta_\gamma(f)=\beta_\gamma(y)$ for all large γ. But $\beta_\gamma(f)\rightarrow\beta(f)$; so $\beta_\gamma(y)\rightarrow\beta(y)$. Consequently $\beta_\gamma\rightarrow\beta$ uniformly on compacta in H.

From the last two paragraphs it follows that $\phi_\gamma\rightarrow\phi$ uniformly on compacta in G. Thus, for elements of $\hat{G}$, pointwise convergence on $L(G)$ implies uniform convergence on compacta in G; and the theorem is proved.

THEOREM 5. *Let G be a locally compact Abelian group. Then the following three conditions are equivalent:*

(i) *$\hat{G}$ is locally compact;*

(ii) *G has a locally bounded dual;*

(iii) *there exists a compact subset U of G such that, for every x in G, there is a positive integer n for which x^n belongs to the subgroup of G generated by U.*

Proof. We have already seen that (ii) implies (i).

Let U be as in (iii); and let k be any positive number. To prove (ii), it is enough (by Theorem 4) to find a norm-function α on G such that, if $\beta\in\hat{G}$ and $|\beta(x)|\leqslant k$ for all x in U, then $|\beta(x)|\leqslant\alpha(x)$ for all x in G. We may assume without cost that U is a neighborhood of e and that $U=U^{-1}$. For each x in G put $\alpha(x)=\sup_\beta|\beta(x)|$, where β runs over those elements of $\hat{G}$ such that $|\beta(y)|\leqslant k$ for all y in U. From (iii) we deduce that $\alpha(x)<\infty$ for all x, and that in fact α is a norm-function with the required property. Thus (iii) implies (ii).

Assume that (iii) is false. Let R^n, H, and K be as in the proof of Theorem 4, and put $D=G/(R^n\times K)$. It is easy to see from the failure of (iii) that the discrete group D has no finite maximal independent ([12], p. 441) subset. From this we shall now prove that $\hat{D}$ is not locally compact. Indeed: Let Γ be any finite subset of D; since no subset of Γ is maximal independent, there exists a w in $G-\Gamma$ such that, for all positive integers n, w^n does not belong to the subgroup of D generated by Γ. Thus for each positive integer m there is a χ'_m in $\hat{L}$ (L being the subgroup of D generated by w and Γ) such that $\chi'_m\equiv 1$ on Γ and $|\chi'_m(w)|>m$. It is well known and easily proved that χ'_m can be extended to an element χ_m of $\hat{D}$; and $\{\chi_m\}$ has no convergent subnet. By the arbitrariness of Γ this shows that $\hat{D}$ is not locally compact. Now $\hat{D}$ is a closed topological subspace of $\hat{G}$. It follows that $\hat{G}$ is not locally compact; and we have shown that (i) implies (iii). This completes the proof.

11. Lie groups

In this section G is assumed to be a Lie group. Let $C_0^\infty(G)$ be the space of all infinitely differentiable complex functions on G with compact support; and let $D_0(G)$ be the *distribution algebra* of G, that is, the associative algebra consisting of all distributions (in the sense of L. Schwartz) on G with compact support, with the operation of convolution defined by

$$(\alpha*\beta)(\phi)=\int_G\int_G \phi(xy)\,d\alpha x\,d\beta y$$

($\alpha,\beta\in D_0(G)$; $\phi\in C_0^\infty(G)$; we use the integral notation for distributions acting on functions). Evidently $C_0^\infty(G)\subset L(G)\subset M_0(G)\subset D_0(G)$; $C_0^\infty(G)$ is a two-sided ideal of $D_0(G)$. (See [8].)

Let T be a linear system representation of G on the linear system $H(T)=\langle H_1,H_2\rangle$. We shall denote by $H_i^\infty(T)$ ($i=1,2$) the linear span of the $T_i(f)\xi$ ($f\in C_0^\infty(G)$, $\xi\in H_i$). Since $C_0^\infty(G)$ contains an approximate identity for G, and $H_i^\infty(T)$ is stable under the integrated form of T_i, T restricted to $H^\infty(T)=\langle H_1^\infty(T),H_2^\infty(T)\rangle$ is a dense contraction of (the integrated form of) T; call this dense contraction T'. By the argument of Lemma 5 of [5], T' can be extended uniquely to a linear system representation T^∞ of $D_0(G)$ on $H^\infty(T)$. This T^∞ will be called the *distribution form* of T. The linear system $H^\infty(T)$ on which T^∞ acts is the *Gårding system* for T.

One verifies without difficulty that, if $\alpha\in D_0(G)$, $\xi\in H_1^\infty(T)$, and $\eta\in H_2(T)$, then $x\to(T_1(x)\xi\,|\,\eta)$ is infinitely differentiable on G and

$$(T_1^\infty(\alpha)\xi\,|\,\eta)=\alpha_x(T_1(x)\xi\,|\,\eta)$$

(where 'α_x' means that α acts on the expression which follows it considered as a function of x).

LEMMA 22. *If K_i is a $\sigma(H^\infty(T))$-dense subspace of $H_i^\infty(T)$ which is stable under the restriction to $C_0^\infty(G)$ of the integrated form of T_i $(i=1, 2)$, then K_i is $\sigma(H(T))$-dense in H_i.*

Proof. Without loss of generality take $i=1$. Assume that $\eta \in H_2$ and that $(\xi | \eta)=0$ for all ξ in K_1; we must show that $\eta=0$. Now for each f in $C_0^\infty(G)$, $T_2(f)\eta \in H_2^\infty(T)$, and, for all ξ in K_1, $(\xi | T_2(f)\eta)=(T_1(f)\xi | \eta)=0$ (since $T_1(f)\xi \in K_1$). Since K_1 is $\sigma(H^\infty(T))$-dense, this implies that $T_2(f)\eta=0$; and the latter holds for all f in $C_0^\infty(G)$. Since $C_0^\infty(G)$ contains an approximate identity, this gives $\eta=0$.

COROLLARY 1. *T is topologically irreducible [resp. topologically completely irreducible, resp.* FDS] *if and only if T^∞ is topologically irreducible [resp. topologically completely irreducible, resp.* FDS].

Proof. It is evident that if T has any one of these properties then so does T^∞. The converse follows easily from Lemma 22.

COROLLARY 2. *If S and T are linear system representations of G, then S and T are Naimark-related [resp. have the same kernel in $M_0(G)$] if and only if S^∞ and T^∞ are Naimark-related [resp. have the same kernel in $D_0(G)$].*

Proof. The statement about Naimark-relatedness is a routine consequence of Lemma 22. The statement about kernels is almost evident.

By Corollaries 1 and 2, the map $T \to T^\infty$ carries $\mathcal{J}(G)$ into $\mathcal{J}(D_0(G))$, and lifts to a one-to-one map of $\hat{G}$ into $(D_0(G))^\wedge$.

LEMMA 23. *The map lifted from $T \to T^\infty$ is a homeomorphism of $\hat{G}$ into $(D_0(G))^\wedge$.*

Proof. Since $C_0^\infty(G)$ is a two-sided ideal of both $M_0(G)$ and $D_0(G)$, the lemma results from a double application of Theorem 1.

Let Z be the center of $D_0(G)$. For each T in $\mathcal{J}(G)$, let γ_T be the central character (see § 3) of the element T^∞ of $\mathcal{J}(D_0(G))$. We shall refer to γ_T as the *central character* of T. By Lemmas 3 and 23 we have:

LEMMA 24. *The map $T \to \gamma_T$ $(T \in \mathcal{J}(G))$ is continuous with respect to the functional topology of $\mathcal{J}(G)$ and (for $Z^\#$) the topology of pointwise convergence on Z.*

Since the hull-kernel topology of $\hat{G}_{un}$ (see § 9) contains the relativized functional topology of $\hat{G}_{un}$, Lemma 24 strengthens and generalizes a result of Bernat and Dixmier [1].

In particular, it follows from Lemma 24 that two functionally equivalent elements of $\mathcal{J}(G)$ have the same central character. Thus we may speak of the central character γ_τ of a class τ in $\hat{G}$, and the map $\tau \to \gamma_\tau$ is continuous.

It is curious to observe that, if G is the "$ax+b$" group (see Example 4 of the Appendix), Z is trivial (i.e. $Z=C\cdot 1$). Indeed, let R be the regular representation of G. Then R is the direct sum of $\aleph_0$ copies of S and $\aleph_0$ copies of T, where S and T are the two infinite-dimensional irreducible unitary representations of G. Since S and T are functionally equivalent (see § 5), they have the same central character; and hence R^∞_z is a scalar operator for each z in Z. Thus $\{R^\infty_z \mid z\in Z\}$ is of dimension 1. Since R^∞ is faithful on $D_0(G)$ it follows that Z is of dimension 1. We leave it to the reader to fill in the details of this argument.

We conclude this section with a remark on Banach representations. Let S be a Banach representation of G on a Banach space X. It is well known that S gives rise in a canonical manner to a representation S^∞ of $D_0(G)$ not merely on the Gårding subspace (the linear span of the $S_f\xi$, $\xi\in X$, $f\in C^\infty_0(G)$) but on the larger space X_0 of all C^∞ vectors (that is, vectors ξ for which $x\rightarrow S_x\xi$ is infinitely differentiable on G).

LEMMA 25. *If S is a topologically completely irreducible Banach representation of G on a Banach space X, then for each α in Z, S^∞_α is the scalar operator $\gamma_S(\alpha)\cdot 1$ on the space X_0 of all C^∞ vectors for S.*

Note: γ_S means the same as γ_T, where T is the linear system representation associated with S.

Proof. By what we have already shown for T, S^∞_α is the scalar operator $\gamma_S(\alpha)\cdot 1$ on the Gårding subspace. Our result now follows by a trivial generalization of the proof of Lemma 32 of [9].

12. The enveloping algebra and infinitesimal equivalence

In this section G will be a fixed *connected* Lie group with unit e, and K a fixed connected compact subgroup of G. Let g be the (real) Lie algebra of G (consisting of the *left*-invariant vector fields) and E the enveloping algebra of the complexification g_c of g. It is well known that E may be identified with the subalgebra of $D_0(G)$ consisting of those distributions whose closed supports are contained in $\{e\}$ (see [11], Chapter II).

Let $\hat{K}$ and the ψ_D $(D\in\hat{K})$ be as in § 8.

Definition. A linear system representation T of G will be called *K-finite* if $T_1(\psi_D)$ (and hence also $T_2(\psi_D)$) is of finite rank for each D in $\hat{K}$.

That is, T is K-finite if and only if each D in $\hat{K}$ has finite multiplicity in T_1.

A K-finite linear system representation T is certainly FDS.

Throughout the rest of this section we shall denote range$(T_i(\psi_D))$ by $H^D_i(T)$ (for $D\in\hat{K}$), and shall put $H^f_i(T)=\sum_{D\in\hat{K}}H^D_i(T)$.

LEMMA 26. *If T is a K-finite linear system representation of G, then*

(i) $H_i^f(T) \subset H_i^\infty(T) \quad (i=1,2)$;

(ii) *$H_i^f(T)$ is stable under $T_i^\infty(a)$ for each $i=1,2$ and each a in E.*

Proof. Since $H_i^\infty(T)$ is $\sigma(H(T))$-dense in $H_i(T)$, and stable under $T_i(\psi_D)$,

$$H_i^\infty(T) \cap H_i^P(T)$$

is dense in $H_i^P(T)$. But the latter is finite-dimensional. Therefore $H_i^P(T) \subset H_i^\infty(T)$, and (i) is proved.

(ii) is proved by the same argument as Lemma 26 of [8].

The $H_i^f(T)$ of Lemma 26 is evidently $\sigma(H(T))$-dense in $H_i(T)$. Let us write $H^f(T)$ for the linear system $\langle H_1^f(T), H_2^f(T)\rangle$ with the restricted duality of $H(T)$, and $\tilde{T}$ for the linear system representation of E on $H^f(T)$ given by $\tilde{T}_i(a) = T_i^\infty(a) \mid H_i^f(T)$.

Definition. For each K-finite linear system representation T of G, $\tilde{T}$ will be called the *restricted infinitesimal form* of T.

Before embarking on the next result we need the notion of an analytic vector in the space of a linear system representation. Of course, we shall not be able to generalize Nelson's result ([18]) and state that analytic vectors are dense in the space of an arbitrary linear system representation. Nevertheless the weaker argument of Godement in [8] gives us the following definition and lemma which are sufficient for our purposes.

Definition. If T is a linear system representation of G, the vector ξ in $H_1(T)$ is *analytic* (*for* T) if the function $x \to (T_1(x)\xi \mid \eta)$ is analytic on G for all η in $H_2(T)$. Similarly a vector η in $H_2(T)$ is *analytic* (*for* T) if $x \to (T_1(x)\xi \mid \eta)$ is analytic on G for all ξ in $H_1(T)$.

Note: 'Analytic on G' means of course with respect to the *real* analytic structure of G, even if G has a complex analytic structure.

LEMMA 27. *If T is a K-finite linear system representation of G, then every vector in $H_i^f(T)$ $(i=1,2)$ is analytic for T.*

The proof is identical with that of Lemma 17 of [8].

Definition. Two K-finite linear system representations S and T of G are *infinitesimally equivalent* if their restricted infinitesimal forms $\tilde{S}$ and $\tilde{T}$ are (algebraically) equivalent.

The following lemma is proved for unitary representations in [8], § 23.

LEMMA 28. *Two K-finite linear system representations S and T of G are infinitesimally equivalent if and only if they are Naimark-equivalent.*

Proof. (I). Assume that $\tilde{S}$ and $\tilde{T}$ are equivalent under an isomorphism $\Phi = \langle \Phi_1, \Phi_2 \rangle$ of $H^f(S)$ onto $H^f(T)$.

First, I claim that $\Phi_i(H_i^D(S)) = H_i^D(T)$ for each D in $\hat{K}$. Indeed, let Z_K be the center of the enveloping algebra E_K of the complexification k_c of the Lie algebra k of K. Each D in $\hat{K}$ gives rise to the complex homomorphism λ_D of Z_K such that $D_a = \lambda_D(a) \cdot 1$ for all a in Z_K. Since K is a compact connected Lie group, we know ([19], Exposé 18) that $\lambda_D = \lambda_{D'} \Rightarrow D \cong D'$. Thus, if $D \in \hat{K}$, a vector ξ in $H_1^f(S)$ belongs to $H_1^D(S)$ if and only if

$$T_1^\infty(a)\,\xi = \lambda_D(a)\,\xi$$

for all a in Z_K; and similarly for $H_1^f(T)$. Since $Z_K \subset E$ and Φ_1 is an equivalence of $\tilde{S}_1$ and $\tilde{T}_1$, this implies that $\Phi_1(H_1^D(S)) = H_1^D(T)$. Similarly $\Phi_2(H_2^D(S)) = H_2^D(T)$. This proves the claim. (See also [8], just before the Appendix.)

Now let F be the $\sigma(H(S) \oplus H(T))$-closure of Φ_1 considered as a subspace of $H_1(S) \oplus H_1(T)$. I claim that F is a one-to-one function; that is, $(0 \oplus \xi') \in F \Rightarrow \xi' = 0$ and $(\xi \oplus 0) \in F \Rightarrow \xi = 0$. Assume that $(0 \oplus \xi') \in F$. That means that there is a net $\{\xi_j\}$ of elements of $H_1^f(S)$ such that $\xi_j \to 0$ (in $\sigma(H(S))$) and $\Phi_1 \xi_j \to \xi'$ (in $\sigma(H(T))$). Now let D be any element of $\hat{K}$. Since $S_1(\psi_D)$ is continuous, we have $S_1(\psi_D)\xi_j \to 0$. But since $H_1^D(S)$ is finite-dimensional, $\Phi_1 | H_1^D(S)$ is continuous; hence the preceding limit statement implies that $\Phi_1 S_1(\psi_D)\xi_j \to 0$. But by the last paragraph $\Phi_1 S_1(\psi_D)\,\xi_j = T_1(\psi_D)\,\Phi_1\,\xi_j$; so $T_1(\psi_D)\,\Phi_1\,\xi_j \to 0$. But $\Phi_1\,\xi_j \to \xi'$; therefore $T_1(\psi_D)\Phi_1\xi_j \to T_1(\psi_D)\xi'$, and we have $T_1(\psi_D)\xi' = 0$ for all D in $\hat{K}$. But this implies $\xi' = 0$; and we have shown that $(0 \oplus \xi') \in F \Rightarrow \xi' = 0$. Similarly $(\xi \oplus 0) \in F \Rightarrow \xi = 0$. This proves the claim.

Now let $\xi \in H_1^f(S)$, $\eta \in H_2^f(S)$, $\xi' = \Phi_1 \xi$, $\eta' = \Phi_2 \eta$. I claim that $(S_1(x)\xi | \eta) = (T_1(x)\xi' | \eta')$ for all x in G. Indeed, since G is connected and since by Lemma 27 these functions are analytic in x, it will be enough to show that all their derivatives at e coincide, in other words that $\alpha_x(S_1(x)\xi | \eta) = \alpha_x(T_1(x)\xi' | \eta')$ for all α in E. But by the remark preceding Lemma 22 and the definition of Φ, $\alpha_x(S_1(x)\xi | \eta) = (\tilde{S}_1(\alpha)\xi | \eta) = (\tilde{T}_1(\alpha)\xi' | \eta') = \alpha_x(T_1(x)\xi' | \eta')$; and the claim is proved.

Integrating the last claim we find that $(S_1(f)\xi | \eta) = (T_1(f)\Phi_1\xi | \Phi_2\eta)$ $(f \in L(G),\ \xi \in H_1^f(S),\ \eta \in H_2^f(S))$. Let us temporarily denote by M the linear span of the $\psi_D * f$ $(f \in L(G),\ D \in \hat{K})$. Then $S_1(f)$ leaves $H_1^f(S)$ stable for each f in M; and so the last equation shows that $\Phi_1 S_1(f)\xi = T_1(f)\Phi_1\xi$ $(f \in M,\ \xi \in H_1^f(S))$. Now, for an arbitrary f in $L(G)$, we can clearly choose a net $\{f_j\}$ of elements of M such that

$$S_1(f_j)\,\xi \xrightarrow[j]{} S_1(f)\,\xi \quad \text{and} \quad T_1(f_j)\,\Phi_1\xi \xrightarrow[j]{} T_1(f)\,\Phi_1\xi.$$

Since $\Phi_1 \subset F$ and F is closed we conclude that $S_1(f)\xi \in \text{domain}\,(F)$ whenever $f \in L(G)$ and $\xi \in H_1^f(S)$, and $FS_1(f)\xi = T_1(f)F\xi$. Again applying the continuity of $S_1(f)$ and $T_1(f)$ and the closure of F, we find that, for each f in $L(G)$, the domain and range of F are stable under $S_1(f)$ and $T_1(f)$ respectively, and

$$F \circ S_1(f) = T_1(f) \circ F \text{ on domain}(F).$$

This proves that S and T are Naimark-equivalent.

(II). Now assume that S and T are Naimark-equivalent. Then (see § 6) the strictly smallest dense contractions S^0 and T^0 of the integrated forms of S and T are equivalent under an isomorphism F of $H^0(S)$ onto $H^0(T)$. Evidently $F_i(H_i^D(S)) = H_i^D(T)$ for each D in $\hat{K}$, and so $F_i(H_i^f(S)) = H_i^f(T)$. Let $\Phi_i = F_i \mid H_i^f(S)$. I claim that Φ sets up an equivalence of $\tilde{S}$ and $\tilde{T}$. Indeed, if $\xi \in H_1^f(S)$ and $\eta \in H_2^f(S)$, we have $(S_1(x)\xi \mid \eta) = (T_1(x)F_1\xi \mid F_2\eta)$ for all x in G. Both sides are C^∞ in x; so, applying the element α of E to both sides, we find that $(\tilde{S}_1(\alpha)\xi \mid \eta) = (\tilde{T}_1(\alpha)F_1\xi \mid F_2\eta)$. But $\tilde{S}_1(\alpha)$ and $\tilde{T}_1(\alpha)$ leave $H_1^f(S)$ and $H_1^f(T)$ respectively stable. Thus it follows from the last equation that $\Phi_1 \circ \tilde{S}_1(\alpha) = \tilde{T}_1(\alpha) \circ \Phi_1$ for all α in E. Similarly $\Phi_2 \circ \tilde{S}_2(\alpha) = \tilde{T}_2(\alpha) \circ \Phi_2$. Thus $\tilde{S}$ and $\tilde{T}$ are equivalent.

This completes the proof of the Lemma.

The following lemma strengthens Theorem 16 of [8].

LEMMA 29. *Let S be a K-finite linear system representation of G. Then the following conditions are equivalent:*

(i) *S is topologically irreducible;*

(ii) *$\tilde{S}_1$ is (algebraically) irreducible;*

(iii) *$\tilde{S}_2$ is irreducible;*

(iv) *$\tilde{S}$ is (algebraically) completely irreducible.*

Proof. We shall first show that (ii) $\Rightarrow$ (i). Assume that $\tilde{S}_1$ is irreducible, and let L be a closed non-zero S_1-stable subspace of $H_1(S)$. Let

$$L^f = \sum_{D \in \hat{K}} S_1(\psi_D)(L).$$

By Lemma 26 L^f is an $\tilde{S}_1$-stable subspace of $H_1^f(S)$. Since $\tilde{S}_1$ is irreducible, L^f is either $\{0\}$ or $H_1^f(S)$. The former is impossible, since $L \neq \{0\}$ and therefore $S_1(\psi_D)L \neq \{0\}$ for some D. Hence $L^f = H_1^f(S)$, which is dense in $H_1(S)$. So L is dense in $H_1(S)$, and therefore $L = H_1(S)$. It follows that (ii) $\Rightarrow$ (i). Similarly (iii) $\Rightarrow$ (i).

Since obviously (iv) $\Rightarrow$ (ii) and (iii), the proof will be complete if we show that (i) $\Rightarrow$ (iv). Let us therefore assume that S is topologically irreducible, and show that $\tilde{S}$ is completely irreducible.

We shall first prove that $\tilde{S}$ is irreducible. Let L be a non-zero $\tilde{S}_1$-stable subspace of $H_1^f(S)$. I claim that

$$L = \sum_{D \in \hat{K}} (H_1^D(S) \cap L).$$

Indeed: It will be enough to show that, if $D_1, \ldots, D_r$ are distinct elements of $\hat{K}$, $\xi_j \in H_1^{D_j}(S)$ for each j, and $\xi = \sum_{j=1}^r \xi_j \in L$, then each $\xi_j \in L$. Let Z_K and λ_D be as in (I) of the proof of Lemma 28. Since $\lambda_{D_1}, \ldots, \lambda_{D_r}$ are all distinct, we may choose an element a of Z_K such that $\lambda_{D_1}(a) = 1$ and $\lambda_{D_j}(a) = 0$ for all $j > 1$. Then $\xi_1 = \sum_j \lambda_{D_j}(a)\xi_j = \tilde{S}_1(a)\xi \in L$. Similarly $\xi_j \in L$ for all j, and the claim is proved.

We shall next show that the $\sigma(H(S))$-closure $\bar{L}$ of L is S_1-stable, and hence equal to $H_1(S)$. For this it is enough to take an element ξ of L and an element η of $H_2(S)$ such that $(L \mid \eta) = 0$, and to show that $(S_1(x)\xi \mid \eta) = 0$ for all x in G. For each α in the enveloping algebra E we have $\alpha_x(S_1(x)\xi \mid \eta) = (\tilde{S}_1(\alpha)\xi \mid \eta) = 0$ (since $\tilde{S}_1(\alpha)\xi \in L$). Therefore all the derivatives of $x \to (S_1(x)\xi \mid \eta)$ at e are 0. Since G is connected and $x \to (S_1(x)\xi \mid \eta)$ is analytic (Lemma 27), it follows that $(S_1(x)\xi \mid \eta) = 0$ for all x.

Assume that $L \neq H_1^f(S)$. Since $L = \sum_{D \in \hat{K}} (H_1^D(S) \cap L)$, there must exist a D in $\hat{K}$ and a non-zero η in $H_2^D(S)$ such that $(L \mid \eta) = \{0\}$. But this contradicts the fact that $\bar{L} = H_1(S)$. Therefore $L = H_1^f(S)$. Thus $\tilde{S}_1$ is irreducible. Similarly $\tilde{S}_2$ is irreducible. Hence $\tilde{S}$ is irreducible.

To show now that $\tilde{S}_1$ is completely irreducible, it will be sufficient to show that the division algebra $\mathcal{A}$ of all endomorphisms A of $H_1^f(S)$ commuting with each $\tilde{S}_1(a)$ $(a \in E)$ must be one-dimensional. Let A be such an endomorphism. I claim that each $H_1^D(S)$ is stable under A. Indeed, let $\xi \in H_1^{D_1}(S)$, and suppose $A\xi = \xi_1 + \xi_2 + \ldots + \xi_r$, where $\xi_j \in H_1^{D_j}(S)$, the $D_1, \ldots, D_r$ being distinct. Let Z_K and the λ_D be as in (I) of the proof of Lemma 28, and choose a in Z_K so that $\lambda_{D_1}(a) = 1$ and $\lambda_{D_j}(a) = 0$ for $j > 1$. Then

$$\xi_1 + \ldots + \xi_r = A\xi = A\tilde{S}_1(a)\,\xi = \tilde{S}_1(a)\,A\xi = \tilde{S}_1(a)\,(\xi_1 + \ldots + \xi_r) = \sum_{j=1}^r \lambda_{D_j}(a)\,\xi_j = \xi_1.$$

Thus $\xi_2 = \ldots = \xi_r = 0$, whence $A\xi \in H_1^{D_1}(S)$. It follows that $H_1^{D_1}(S)$ is stable under A, and the claim is proved. Let $0 \neq \xi \in H_1^D(S)$. The map $A \to A\xi$ $(A \in \mathcal{A})$ is linear and one-to-one (since $\mathcal{A}$ is a division algebra), and has finite-dimensional range by the above claim (since $H_1^D(S)$ is finite-dimensional). Therefore $\mathcal{A}$ is a finite-dimensional division algebra over the complexes, and so is one-dimensional. Thus $\tilde{S}_1$ is completely irreducible; and similarly for $\tilde{S}_2$. So $\tilde{S}$ is completely irreducible and the lemma is proved.

Let us now suppose that K is a fixed connected *large* compact subgroup of G. (We notice from Lemma 19 that if G, being a Lie group, has a large compact subgroup, then it has a large connected compact subgroup.) Then every topologically completely irreducible linear system representation T of G is K-finite; and by Lemmas 28 and 29 the equivalence class of $\tilde{T}$ belongs to $\hat{E}$ and depends only on the class in $\hat{G}$ to which T belongs. Thus, if T is a *class* in $\hat{G}$ we can unambiguously define $\tilde{T}$ as the class in $\hat{E}$ to which the restricted infinitesimal form of any representative of T belongs.

THEOREM 6. *The map $T \to \tilde{T}$ is a one-to-one continuous map of $\hat{G}$ into $\hat{E}$.*

Proof. We have seen that it is one-to-one. To prove its continuity, let $W \subset \hat{G}$ and let T belong to the functional closure of W. We must show that $\tilde{T}$ belongs to the functional closure of $\tilde{W} = \{\tilde{S} \mid S \in W\}$.

Choose a D in $\hat{K}$ which occurs in T; and let $\xi \in H_1^D(T)$, $\eta \in H_2^D(T)$, and $f \in C_0^\infty(G)$ be so chosen that (i) $f = \psi_D * f$, and (ii) $(T_1(f)\xi \mid \eta) \neq 0$; such a choice is clearly possible. We now set $\phi(\mu) = (T_1(\mu)\xi \mid \eta)$ $(\mu \in M_0(G))$. Then ϕ is a functional on $M_0(G)$ associated with the integrated form of T. Since $T \in \overline{W}$, there is a net $\{S^{(j)}\}$ of elements of W, and for each j a functional ϕ_j associated with $S^{(j)}$ such that $\phi_j \to \phi$ pointwise on $M_0(G)$ (see § 2).

Now $C_0^\infty(G)$ is an ideal of $D_0(G)$; hence for each a in E, $\psi_D * a * f$ belongs to $C_0^\infty(G)$, and it makes sense to define the functionals Φ, Φ_j on E as follows:

$$\Phi(a) = \phi(\psi_D * a * f),\ \Phi_j(a) = \phi_j(\psi_D * a * f) \quad (a \in E).$$

Note that $\Phi \neq 0$ (since $\Phi(1) = \phi(\psi_D * f) = (T_1(f)\xi \mid \eta) \neq 0$); and that $\Phi_j \to \Phi$ pointwise on E. Thus it remains only to show that Φ and Φ_j are associated with $\tilde{T}$ and $(S^{(j)})^\sim$ respectively. Consider the functional ϕ_j. It must be of the form

$$\phi_j(\mu) = \sum_{k=1}^{r} (S_1^{(j)}(\mu)\,\xi_k \mid \eta_k) \quad (\mu \in M_0(G)),$$

where the ξ_k and η_k belong to $H_1(S^{(j)})$ and $H_2(S^{(j)})$ respectively. Therefore, if $a \in E$,

$$\Phi_j(a) = \sum_{k=1}^{r} (S_1^{(j)}(\psi_D * a * f)\,\xi_k \mid \eta_k) = \sum_{k=1}^{r} ((S^{(j)})_1^\infty(a)\, S_1^{(j)}(f)\,\xi_k \mid S_2^{(j)}(\psi_D)\,\eta_k).$$

Since $f = \psi_D * f$, $S_1^{(j)}(f)\,\xi_k$ and $S_2^{(j)}(\psi_D)\,\eta_k$ are in $H_1^f(S^{(j)})$ and $H_2^f(S^{(j)})$ respectively, so that in the above formula $(S^{(j)})_1^\infty(a)$ can be replaced by $(S^{(j)})_1^\sim(a)$; and we see that Φ_j is associated with $(S^{(j)})^\sim$. Similarly Φ is associated with $\tilde{T}$. This completes the proof.

Remark. It would be very interesting to know whether the map $T \to \tilde{T}$ of Theorem 6 is a homeomorphism.

13. Generalized spherical functions

Once again let G be an arbitrary locally compact group. Let μ be a fixed small idempotent element of $M_0(G)$, and α a fixed norm-function on G. As usual $L_\alpha(G)$ is the completion of $L(G)$ with respect to $\| \ \|_\alpha$, $L^\mu(G) = \mu * L(G) * \mu$, and $L^\mu_\alpha(G)$ is the closure of $L^\mu(G)$ in $L_\alpha(G)$. For brevity we shall write A for $L^\mu_\alpha(G)$. Since μ is small, $\hat{A}^{(f)} = \hat{A}^{(n)}$ for some integer n (see Lemma 12; also [5], Proposition 10; recall that $\hat{A}^{(f)} = \{T \in \hat{A} \mid \dim(T) < \infty\}$).

LEMMA 30. *If $S \in \hat{A}^{(f)}$ and $\phi \in \Phi(S)$ (see § 2), there exists a unique continuous complex function γ on G such that $\phi(\mu * f * \mu) = \int_G \gamma(x) f(x)\, d\lambda x$ for all f in $L(G)$. For some constant $k > 0$ we have $|\gamma(x)| \leqslant k\alpha(x)$ for all x in G.*

Proof. The uniqueness of γ is evident. To prove its existence, we invoke Lemma 15 to obtain a Banach representation T in $\hat{G}^{(\mu)} \cap \hat{G}_\alpha$ such that $T^{(\mu)} \cong S$ and $\|T_x\| \leqslant k\alpha(x)$ $(x \in G)$. If $\xi_1, \ldots, \xi_r$ is a basis of range(T_μ), with dual basis $\xi'_1, \ldots, \xi'_r$, there will exist complex numbers c_{ij} $(i, j = 1, \ldots, r)$ such that

$$\phi(f) = \sum_{i,j=1}^{r} c_{ij}\, \xi'_i\, (T_f\, \xi_j)$$

for all f in $L^\mu(G)$. Now the function

$$\gamma(x) = \sum_{i,j=1}^{r} c_{ij}\, \xi'_i\, (T_\mu\, T_x\, \xi_j)$$

clearly has the required properties.

The function γ of Lemma 30 will be called a *generalized spherical function of type μ associated with S* (or *with T* if $T^{(\mu)} \cong S$). If $\gamma \neq 0$, γ uniquely determines S and hence the Naimark-equivalence class of T.

Our goal in this section is to express the functional topology of $\hat{G}^{(\mu)}$, so far as possible, in terms of the uniform-on-compacta convergence of generalized spherical functions associated with the elements of $\hat{G}^{(\mu)}$. Unfortunately we shall have to restrict our attention to $\hat{G}^{(\mu)} \cap \hat{G}_\alpha$, where α is a norm-function fixed as before.

If in Lemma 30 $\phi = \chi_S$ (the character of S), the resulting γ will be called ζ_S, the *spherical function of type μ associated with S* (or *with T* if $T^{(\mu)} \cong S$).

LEMMA 31. *Let $\{S_\nu\}$ be a net of elements of $\hat{A}^{(f)}$ such that $\chi_{S_\nu} \to \phi$ pointwise on A, where $\phi = \sum_{k=1}^{r} m_k \chi_{R_k}$, the $R_1, \ldots, R_r$ being pairwise inequivalent elements of $\hat{A}^{(f)}$ and the m_k being positive integers. Then there is a subnet $\{S'_\varrho\}$ of $\{S_\nu\}$, and for each ϱ there are $r+1$ generalized spherical functions $\beta^\varrho_1, \ldots, \beta^\varrho_r, \gamma^\varrho$ associated with S'_ϱ such that:*

(i) $$\sum_{k=1}^{r} m_k \beta_k^\varrho + \gamma^\varrho = \zeta_{S_\varrho'};$$

(ii) $$\beta_k^\varrho \underset{\varrho}{\to} \zeta_{R_k} \; (k=1, \ldots, r) \textit{ uniformly on compact subsets of } G;$$

(iii) $$\int_G \gamma^\varrho(x) f(x)\, d\lambda x \underset{\varrho}{\to} 0 \textit{ for all } f \textit{ in } L(G).$$

Proof. Passing to a subnet, we may as well assume (by [5], Proposition 10) that the S_ν are matrix representations of A all of the same dimension n, with

$$|(S_\nu(f))_{ij}| \leq k\, \|f\|_\alpha \tag{7}$$

for all i, j, and ν, and f in A; and that, for each f in A,

$$S_\nu(f) \underset{\nu}{\to} Q(f),$$

where Q is a matrix representation of A (of dimension n) in triangular block form. Along the diagonal Q contains just the R_k (R_k occurring m_k times), together with the zero representation occurring $n - \sum_{k=1}^{r} m_k \dim(R_k)$ times.

By the Extended Burnside Theorem there are elements b_k of $L^\mu(G)$ $(k=1,\ldots,r)$ such that, if $j, k = 1,\ldots,r$, $R_j(b_k)$ is the zero matrix if $j \neq k$ and the unit matrix if $j = k$.

For each ν and each $k = 1,\ldots,r$ we define the linear functional ξ_k^ν on A:

$$\xi_k^\nu(f) = \text{Trace } (S_\nu(b_k * f)) \quad (f \in A).$$

Then ξ_k^ν is associated with S_ν ([5], Lemma 1), and is bounded in norm uniformly in ν; for $f \in A$

$$\xi_k^\nu(f) \to \text{Trace } Q(b_k * f) = m_k \chi_{R_k}(f).$$

Thus, by Gelfand's Lemma,

$$\xi_k^\nu \underset{\nu}{\to} m_k \chi_{R_k} \text{ uniformly on norm-compact subsets of } A. \tag{8}$$

Now, if D is a compact subset of G and $f \in L(\mathrm{G})$, then $\{\mu * x * f * \mu \mid x \in D\}$ is a norm-compact subset of A; so by (8)

$$\xi_k^\nu(\mu * x * b_k) \to m_k \chi_{R_k}(\mu * x * b_k)$$

uniformly in x on compact subsets of G. Let T_k be a Banach representation of G in $\hat{G}^{(\mu)} \cap \hat{G}_\alpha$ with $T_k^{(\mu)} \cong R_k$. Since $R_k(b_k) = 1$, $T_k(b_k * \mu) = T_k(\mu)$ so that

$$\chi_{R_k}(\mu * x * b_k) = \text{Trace } (T_k(\mu * x * b_k * \mu)) = \text{Trace } (T_k(\mu * x * \mu)) = \zeta_{R_k}(x).$$

Thus
$$\xi_k^\nu(\mu * x * b_k) \to m_k \zeta_{R_k}(x) \tag{9}$$
uniformly in x on compact sets. Now it is easy to see that the left side of (9), as a function of x, is a generalized spherical function associated with S_ν.

In view of (9), defining
$$\gamma^\nu(x) = \zeta_{S_\nu}(x) - \sum_{k=1}^{r} \xi_k^\nu(\mu * x * b_k),$$
we shall have completed the proof of the lemma if we prove that (iii) holds. By the definition of a generalized spherical function,
$$\int \gamma^\nu(x) f(x)\, d\lambda x = \int \gamma^\nu(x) (\mu * f * \mu)(x)\, d\lambda x.$$
Hence it is enough to prove (iii) for all f in $L^\mu(G)$. For such f we have
$$\int \gamma^\nu(x) f(x)\, d\lambda x = \chi_{S_\nu}(f) - \sum_{k=1}^{r} \xi_k^\nu(f * b_k) = \chi_{S_\nu}(f) - \sum_{k=1}^{r} \chi_{S_\nu}(b_k * f * b_k)$$
$$\underset{\nu}{\to} \chi_Q(f) - \sum_{k=1}^{r} \chi_Q(b_k * f * b_k) = 0.$$
This completes the proof.

Theorem 7. *Let $\mathcal{S} \subset \hat{A}^{(f)}$, $T \in \hat{A}^{(f)}$. Then the following three conditions are equivalent:*

(i) *T belongs to the functional closure of $\mathcal{S}$;*

(ii) *every generalized spherical function on G of type μ associated with T can be approximated uniformly on compact sets by generalized spherical functions of type μ associated with $\mathcal{S}$* (i.e., *with some S in $\mathcal{S}$*);

(iii) *some non-zero generalized spherical function on G of type μ associated with T can be approximated uniformly on compact sets by generalized spherical functions of type μ associated with $\mathcal{S}$.*

Proof. One sees immediately that (ii) ⇒ (iii) ⇒ (i). We shall prove that (i) ⇒ (ii). Assume (i). Then it follows from Lemma 31 and [5], Theorem 3, that ζ_T is a uniform-on-compacta limit of generalized spherical functions associated with $\mathcal{S}$. Now, if $g \in L^\mu(G)$, the generalized spherical function γ_g of type μ corresponding to the functional $f \to \chi_T(f * g)$ $(f \in A)$ is given by $\gamma_g(x) = \int_G \zeta_T(xy) g(y)\, d\lambda y$ $(x \in G)$. Since ζ_T is a uniform-on-compacta limit of generalized spherical functions associated with $\mathcal{S}$, so is γ_g. But every generalized spherical function associated with T is a linear combination of such γ_g. Therefore (ii) holds. This completes the proof.

If $S_\nu \to S$ in $\hat{A}^{(f)}$, when can the *spherical* functoin of type μ associated with S be approximated uniformly on compacta by the *spherical* functions associated with the S_ν? The next theorem tells us that this is the case when S and S_ν all have the same dimension. Thus it verifies a conjecture of Godement ([8], § 15).

THEOREM 8. *Let m be a fixed positive integer. Then the map $S \to \zeta_S$ restricted to $\hat{A}_m$ (the subset of $\hat{A}^{(f)}$ consisting of representations of dimension exactly m) is a homeomorphism with respect to the functional topology of $\hat{A}_m$ and the topology of uniform convergence on compact sets for the ζ_S.*

Proof. The map $\zeta_S \to S$ is obviously continuous. To prove that $S \to \zeta_S$ is continuous, we choose an element T of $\hat{A}_m$ and a subset $\mathcal{S}$ of $\hat{A}_m$ containing T in its functional closure. We must show that ζ_T can be approximated uniformly on compacta by the ζ_S ($S \in \mathcal{S}$).

By [5], Proposition 10, we may choose a net $\{S_\nu\}$ of matrix representations belonging to $\mathcal{S}$, and a constant $k > 0$, such that

$$|(S_\nu(f))_{ij}| \leqslant k \|f\|_\alpha \qquad \text{(for all } i, j, \text{ and all } f \text{ in } A), \tag{10}$$

and

$$S_\nu(f) \underset{\nu}{\to} T(f) \tag{11}$$

(where T is taken to be a matrix representation). By Burnside's Theorem there is an element b of $L^\mu(G)$ such that

$$T(b) = 1. \tag{12}$$

Let $\xi_\nu(f) = \text{Trace}\,(S_\nu(b * f))$ $(f \in A)$. By (11) and (12), if $f \in L_\alpha(G)$,

$$\xi_\nu(\mu * f * \mu) \underset{\nu}{\to} \chi_T(b * f * \mu) = \chi_T(\mu * f * \mu). \tag{13}$$

Since the ξ_ν, as functionals on A, are norm-bounded uniformly in ν, Gelfand's Lemma applied to (13) shows that

$$\text{Trace}\,(S_\nu(b * x * b)) = \xi_\nu(\mu * x * b) \to \chi_T(\mu * x * b) = \zeta_T(x) \tag{14}$$

uniformly in x on compact subsets of G.

Let T_ν be a Banach representation in $\hat{G}^{(\mu)} \cap \hat{G}_\alpha$ such that $(T_\nu)^{(\mu)} \cong S_\nu$ (see Lemma 15). We shall now prove that

$$\text{Trace}\,(T_\nu(b * x * b) - T_\nu(\mu * x * \mu)) \underset{\nu}{\to} 0 \tag{15}$$

uniformly in x on compact subsets of G. Since Trace $(T_\nu(\mu * x * \mu)) = \zeta_{S_\nu}(x)$ and Trace $(T_\nu(b * x * b)) = $ Trace $(S_\nu(b * x * b))$, it will follow from (14) and (15) that $\zeta_{S_\nu} \to \zeta_T$ uniformly on compact sets, and the theorem will be proved.

To prove (15) we observe that

$$S_\nu(b*x*b)-S_\nu(\mu*x*\mu)=(S_\nu(b)-1)\,S_\nu(\mu*x*\mu)+S_\nu(\mu*x*\mu)\,(S_\nu(b)-1)$$
$$+(S_\nu(b)-1)\,S_\nu(\mu*x*\mu)\,(S_\nu(b)-1) \tag{16}$$

($S_\nu(\mu*x*\mu)$ means that we extend S_ν to a matrix representation of $\mu*M_0(G)*\mu$, and evaluate at $\mu*x*\mu$). Since $S_\nu(b)\to T(b)=1$, (15) will follow from (16) if we show that, for each compact subset D of G,

$$\{S_\nu(\mu*x*\mu)\,|\,\text{all } \nu, \text{ all } x \text{ in } D\}$$

is a bounded set of matrices. Let D′ be a compact neighborhood of D, and F the set of all non-negative elements f of $L(G)$ with supports contained in D' and with $\int f d\lambda=1$. Since for x in D $S_\nu(\mu*x*\mu)$ can be approximated by matrices $S_\nu(\mu*f*\mu)$ with $f\in F$, it is sufficient to show that

$$\{S_\nu(\mu*f*\mu)\,|\,\text{all } \nu, \text{ all } f \text{ in } F\} \tag{17}$$

is a bounded set of matrices. Let $l=\sup_{x\in D'}\alpha(x)$. Then $\|f\|_\alpha\leqslant l$ for all f in F, so that

$$\|\mu*f*\mu\|_\alpha\leqslant l\,\|\mu\|_\alpha^2 \quad (f\in F),$$

whence by (10) $|(S_\nu(\mu*f*\mu))_{ij}|\leqslant kl\|\mu_\alpha\|^2$ for all ν, all i, j, and all f in F. Thus (17) is a bounded set, and the proof of the theorem is complete.

Remark. In the context of Lemma 31, consider the statement that

$$\zeta_{S_\nu}\to\sum_{k=1}^{r} m_k\,\zeta_{R_k} \tag{18}$$

uniformly on compact sets. The only impediment to the validity of (18) is the presence of the γ^ϱ in Lemma 31. It might be conjectured that if all S_ν were of dimension n, and if $\sum_{k=1}^{r} m_k \dim(R_k)=n$, then the γ^ϱ could be taken to be 0, so that (18) would hold. We have seen of course in Theorem 8 that this is true if $r=1$, $m_1=1$. We do not know, however, if it is true in general. The only bar to carrying through the proof of Theorem 8 in this more general situation is this: Unless the T of the proof of Theorem 8 is irreducible, we may not be able to choose b in $L^\mu(G)$ to satisfy (12), but only to be triangular with ones on the diagonal. But there is one other case in which (12) can still be satisfied—namely, when S_ν and T are matrix *-representations of the *-algebra $L^\mu(G)$, i.e., they come from *unitary* representations of G. In that case T is a diagonal block matrix representation, and so b can be chosen to satisfy (12). Then the proof of Theorem 8 goes through as before.

Appendix

In this Appendix we give four counter-examples. The first shows that the Galilean group (defined in the Introduction) is not Banach-representable. In the second we exhibit a linear system representation T of a group such that T, although not FDS itself, has an FDS dense contraction. The third (essentially due to M. Tomita) consists of three (not topologically irreducible) Banach representations of a group for which Naimark-relatedness fails to be transitive. In the fourth we show that Naimark-relatedness fails to be transitive on arbitrary topologically completely irreducible linear system representations of groups.

We do not know of an example of the non-transitivity of Naimark-relatedness involving only topologically completely irreducible *Banach* representations of a group.

Example 1. Let G be the Galilean group defined in the Introduction. Let $L(R)$ be the space of all continuous complex functions with compact support on the real line R, and H the linear system $\langle L(R), L(R)\rangle$ with the duality $(f|g)=\int_{-\infty}^{\infty} f(t)g(t)\,dt$. Take a "non-unitary character" χ of R^2; that is, for some complex λ, μ we have $\chi(b,c)=\exp(\lambda b+\mu c)$ for all real b,c. We shall define a linear system representation U^χ of G on H as follows:

$$(U_1^\chi(a,b,c)f)(t)=\chi(b,c+bt)f(t-a),$$
$$(U_2^\chi(a,b,c)g)(t)=\chi(b,c+(t+a)b)f(t+a)$$

($\langle a,b,c\rangle\in G$; $f,g\in L(R)$; $t\in R$). One verifies that U^χ is indeed a linear system representation of G.

I claim that, if $\mu\neq 0$, U^χ is topologically completely irreducible. Indeed: Let $f_1,\dots,f_n$ be linearly independent elements of $L(R)$, and let $g_1,\dots,g_n$ be elements of $L(R)$ such that

$$\sum_{j=1}^{n}(U_1^\chi(a,b,c)f_j\,|\,g_j)=0 \tag{1}$$

for all $\langle a,b,c\rangle$ in G. The claim will be proved if we show that the g_j must all be 0. Now let a be a fixed real number, and define

$$F(z)=\int_{-\infty}^{\infty}e^{zt}\Big(\sum_{j=1}^{n}g_j(t)f_j(t-a)\Big)\,dt. \tag{2}$$

F is complex analytic, and by (1), $F(\mu b)=0$ for all real b. Since $\mu\neq 0$, this implies that $F\equiv 0$, whence $\sum_{j=1}^{n}g_j(t)f_j(t-a)=0$ for all real t and a. But this amounts to saying that

$$\sum_{j=1}^{n}g_j(t)f_j(s)=0$$

for all real t, s. From this and the linear independence of the f_j it follows that $g_j\equiv 0$ for all j. So the claim is proved.

On central elements $\langle 0,0,c\rangle$ of G, $U_1^\chi(0,0,c)$ and $U_2^\chi(0,0,c)$ are the scalar operators $e^{\mu c}\cdot 1$. Now we showed in the Introduction that a Banach representation cannot have this property unless μ is pure imaginary. Hence:

If μ is not pure imaginary, U^χ is a topologically completely irreducible linear system representation which is not functionally equivalent to any Banach representation of G.

Thus G is not Banach-representable.

Example 2. It is very simple to exhibit a linear system representation of an *algebra* which, though not FDS, has an FDS dense contraction. But it is slightly less simple to exhibit such a representation of a *group*. Here is an example for groups.

Let K be a two-element group $\{e, u\}$, I an infinite cyclic group with generator g, and G the *free* product of K and I (considered as discrete). Let X be the Banach space of all continuous complex functions on $[0, \pi]$, with the supremum norm. Let σ be the function in X given by $\sigma(x) = \sin x$, λ the element of X^* given by

$$\lambda(f) = \frac{2}{\pi}\int_0^\pi f(x) \sin x \, dx,$$

and P the idempotent operator on X given by $Pf = \lambda(f)\cdot\sigma$ $(f\in X)$. Let T be the Banach representation of G on X such that $T_u = 1 - 2P$ and $(T_g f)(x) = (\exp(ix))f(x)$ $(f \in X, 0 \leqslant x \leqslant \pi)$. If Y is the weakly dense subset of X^* consisting of all ϕ_g $(g \in X)$, where $\phi_g(f) = \int_0^\pi f(x)g(x)dx$, we note that Y is stable under T^*, and so the restriction T'' of $\langle T, T^*\rangle$ to $\langle X, Y\rangle$ is a dense contraction of the linear system representation $T' = \langle T, T^*\rangle$ associated with T.

Now $T_e - T_u$ has one-dimensional range $C\sigma$. Further it is easily seen that the linear span of the $(T_g)^n\sigma$ (n running over all integers) is weakly dense in X with respect to Y. Therefore T'' is FDS.

But I claim that T' is not FDS. Indeed, let $X_0 = \{f\in X \mid f(0) = 0\}$. Since X_0 is not weakly dense in X (with respect to X^*), it will be sufficient to show that if $\phi\in L(G)$ and T_ϕ has finite-dimensional range, then range$(T_\phi) \subset X_0$. Now clearly $T_\phi = T_\psi + A$, where $\psi\in L(I)$ and A is a linear combination of terms of the form B_1PB_2, where B_1 and B_2 leave X_0 stable. Obviously A has finite-dimensional range contained in X_0. Thus if T_ϕ has finite-dimensional range, the same is true of T_ψ; but this clearly implies $\psi = 0$. So, if T_ϕ has finite-dimensional range, we have range $(T_\phi) = $ range $(A) \subset X_0$.

Hence T' is not FDS, but has an FDS dense contraction T''.

Example 3. Let G be the additive group of integers, and S the unit circle $\{u\in C \mid |u| = 1\}$. Take two Borel measures μ_1 and μ_2 on S whose closed support is all of S but which are not both absolutely continuous with respect to each other. Let X be the Banach space of all continuous complex functions on S, with the supremum norm. Consider the Banach representations S, T^1, and T^2 of G on the Banach spaces X, $L_2(\mu_1)$, and $L_2(\mu_2)$ respectively, all given by the formula

$$(V_n f)(u) = u^n f(u) \quad (u \in S)$$

(where V is S, T^1, or T^2, and f runs over X, $L_2(\mu_1)$, or $L_2(\mu_2)$). Now the identity map of X into $L_2(\mu_1)$ is continuous and one-to-one, and intertwines S and T^1. Thus S and T^1 are Naimark-related. Similary S and T^2 are Naimark-related. But I claim that T^1 and T^2 are not Naimark-related. Indeed T^1 and T^2 are unitary, and are not unitarily equivalent (since μ_1 and μ_2 have different null sets). Now it is known that, for unitary representations of a group, Naimark-relatedness and unitary equivalence are the same ([14], Theorem 1.2; the latter is valid for closed unbounded intertwining operators). Therefore T^1 and T^2 are not Naimark-related. Thus S, T^1, and T^2 demonstrate the non-transitivity of Naimark-relatedness for Banach representations.

Example 4. Here is an example of the non-transitivity of Naimark-relatedness involving only topologically completely irreducible linear system representations.

Let G be the "$ax+b$" group, that is, the group of all pairs $\langle a,b \rangle$ where a,b are real and $a>0$, with $\langle a,b\rangle\langle a',b'\rangle = \langle aa', b+ab'\rangle$. If $0<\varrho<\pi/4$, $0<\sigma<\pi/4$, and λ is any non-zero complex number, we shall define a linear system representation $T = T^{\lambda,\varrho,\sigma}$ of G as follows: Let $K = K^{\varrho,\sigma}$ be the family of all entire functions f on C with the following two properties: (i) $f(0)=0$, (ii) there exist positive constants k, l (depending on f) such that $|f(re^{i\theta})| \leq le^{-kr^2}$ whenever $r \geq 0$ and $-\varrho \leq \theta \leq \sigma$. Clearly $\langle K,K\rangle$ is a linear system under the duality

$$(f|g) = \int_0^\infty x^{-1} f(x)\, g(x)\, dx.$$

Now for each $\langle a, b\rangle$ in G let

$$(T_1(a,b)f)(z) = e^{i\lambda zb} f(az), \tag{3}$$

$$(T_2(a,b)g)(z) = e^{i\lambda za^{-1}b} f(a^{-1}z) \tag{4}$$

($f \in K$, $z \in C$). Evidently $T = T^{\lambda,\varrho,\sigma} = \langle T_1, T_2\rangle$ is a linear system representation of G on $H(T) = \langle K,K\rangle$. By an argument exactly analogous to that of Example 1 of this Appendix we prove that $T^{\lambda,\varrho,\sigma}$ is topologically completely irreducible.

Now we verify that, if $-\varrho<\theta<\sigma$, $\varrho' = \varrho+\theta$, $\sigma' = \sigma-\theta$, and $\lambda' = \lambda e^{i\theta}$, then

$$T^{\lambda,\varrho,\sigma} \text{ and } T^{\lambda',\varrho',\sigma'} \text{ are equivalent} \tag{5}$$

under the isomorphism $F: K^{\varrho,\sigma} \to K^{\varrho',\sigma'}$ given by $(Ff)(z) = f(e^{i\theta}z)$. Furthermore, if $0<\varrho_0 \leq \varrho$,

$0<\sigma_0\leqslant\sigma$, then $T^{\lambda,\varrho,\sigma}$ is a dense contraction of $T^{\lambda,\varrho_0,\sigma_0}$. In fact, let L be the Hilbert space $L_2([0,\infty);x^{-1}dx)$ and, for non-zero real λ, let U^λ be the unitary representation of G on L defined by the formula (3) (with U^λ in place of T_1). Then it is easy to see that, for real λ, $T^{\lambda,\varrho,\sigma}$ is a dense contraction of U^λ. (Here we identify an entire function with its restriction to the positive real axis.) It is known that U^λ is topologically completely irreducible (this can be proved directly by the argument of Example 1), and that U^1 and U^{-1} are unitarily inequivalent.

Thus, if $0<\varepsilon<\pi/8$, $T^{1,\varepsilon,\varepsilon+(\pi/8)}$ and $T^{\exp(i\pi/8),\varepsilon+(\pi/8),\varepsilon}$ are dense contractions of U^1 and $T^{\exp(i\pi/8),\varepsilon,\varepsilon}$ respectively, and are equivalent by (5). Thus U^1 and $T^{\exp(i\pi/8),\varepsilon,\varepsilon}$ are Naimark-related. Similarly, each consecutive pair in the chain

$$U^1,\ T^{\exp(i\pi/8),\varepsilon,\varepsilon},\ T^{\exp(i\pi/4),\varepsilon,\varepsilon},\ldots,T^{\exp(7i\pi/8),\varepsilon,\varepsilon},\ U^{-1}$$

is Naimark-related. But we know that the end terms U^1 and U^{-1} are not unitarily equivalent, hence not Naimark-related (see Example 3). Hence Naimark-relatedness is not transitive for topologically completely irreducible linear system representations of G.

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