POINTWISE LIMITS FOR SEQUENCES OF CONVOLUTION OPERATORS

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§ 0. Introduction

(0.1) This paper had its origin in an effort to obtain pointwise inversion formulae for Fourier transforms on a locally compact Abelian group. Does there exist a process for recapturing almost everywhere a function from its Fourier transform? Mean convergence of summability processes for Fourier transforms is of course well known and almost obvious (see for example [12], (20.15)). The whole point of the present paper is to replace mean convergence by pointwise convergence almost everywhere.

In § 1 we present a general theorem on pointwise limits of sublinear operators. Section 2 is concerned with differentiation of indefinite integrals and measures on a class of locally compact groups. In § 3, we obtain single convergence theorems and inversion formulae on the same class of groups. In § 4 we give an analogue of the martingale convergence theorem for singular convolution operators. We combine the foregoing results in § 5 to give *iterated* limit processes for inverting Fourier and Fourier-Stieltjes transforms on an arbitrary locally compact Abelian group or compact group.

(0.2) We follow the notation and terminology of [12] with the following additions. The term "neighbourhood of a point" means "a set whose interior contains that point". Let X be a locally compact Hausdorff space. A *positive Radon measure on* X is a set function ι on all subsets of X as defined in [12], § 11. Measurability of a subset of X for ι is as defined in [12], (11.28). For a measure ϱ that is in $\mathbf{M}(X)$ or is a positive Radon measure on X, and a locally ϱ-integrable function f on X, the symbol $f\varrho$ denotes the measure $A \rightarrow \int_A f\,d\varrho$. For a positive real number p and X and ι as just described, $\mathfrak{L}_{p,\,\mathrm{loc}}(X,\iota)$ is the set of all functions f on X such that $f\xi_F \in \mathfrak{L}_p(X,\iota)$ for all compact sets $F \subset X$.

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All topological groups considered in this paper are assumed to satisfy Hausdorff's separation axiom. For a locally compact Abelian group G with character group $\mathbf{X}$, and $f \in \mathfrak{L}_1(G)$, the Fourier transform $\hat{f}$ on $\mathbf{X}$ is defined by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\,d\lambda(x) \quad \text{for} \quad \chi \in \mathbf{X}.$$

Haar measure on $\mathbf{X}$ will always be denoted by the symbol θ, and the factor of proportionality for θ will be adjusted to λ in such a way that the Fourier inversion formula

$$h(x) = \int_{\mathbf{X}} \hat{h}(\chi)\chi(x)\,d\theta(\chi)$$

holds for every $h \in \mathfrak{L}_1(G)$ whose Fourier transform $\hat{h}$ is in $\mathfrak{L}_1(\mathbf{X})$. In (5.5) we construct a particular pair of such measures λ and θ.

For a locally compact group G, the expressions a.e. and l.a.e. mean almost everywhere for a left Haar measure on G and locally almost everywhere for a left Haar measure on G, respectively. Where measures other than Haar measures are meant, they will be specified.

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§ 1. A theorem on pointwise limits of operators

The main result of this section is Theorem (1.6). It and its immediate consequence (1.7) are essential for the results of §§ 2, 3, and 5. We were led to Theorem (1.6) by examining (5.6.1) and (5.6.2) of the classical monograph [15]. These theorems are in turn based upon a theorem of S. Saks. (See [15], (1.5.8) and [19].)

The notation and terminology of (1.1)–(1.3) will be used throughout the present section.

(1.1) Let $(S, \mathfrak{M}, \mu)$ be a countably additive measure space, i.e., S is a set, $\mathfrak{M}$ is a σ-algebra of subsets of S, and μ is a function on $\mathfrak{M}$ into or onto $[0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n=1}^{\infty} M_n) = \sum_{n=1}^{\infty} \mu(M_n)$ whenever $(M_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of $\mathfrak{M}$.

Since $(S, \mathfrak{M}, \mu)$ need not be σ-finite, we agree that a subset N of S is *null* if $N \in \mathfrak{M}$ and $\mu(N) = 0$, and that N is *locally null* if $N \in \mathfrak{M}$ and $\mu(N \cap F) = 0$ for every $F \in \mathfrak{M}$ such that $\mu(F) < \infty$. Every null set is locally null, and the converse is true if $(S, \mathfrak{M}, \mu)$ is σ-finite.

As usual, a property of points of S is said to hold μ-a.e. (or μ-l.a.e.) if the set of points of S not possessing the given property is null (or locally null).

(1.2) Given $(S, \mathfrak{M}, \mu)$ as in (1.1), the symbol $\mathfrak{F}=\mathfrak{F}(S, \mathfrak{M}, \mu)$ will denote the set of all $\mathfrak{M}$-measurable functions on S into or onto $[0, \infty]$. With the usual conventions, the functions $f+g$ and αf are defined for $f, g \in \mathfrak{F}$ and α any nonnegative real number. Also, with the usual order on $[0, \infty]$, suprema of subsets of $\mathfrak{F}$ are definable as elements of $[0, \infty]^S$. In case the family in question is countable, the supremum belongs to $\mathfrak{F}$.

(1.3) The symbol E will denote a real, semimetrisable topological vector space, i.e., E has a countable neighbourhood base at 0. We will consider operators P from E into $\mathfrak{F}$ which are sublinear in the sense that

$$P(\alpha f) \leqslant |\alpha| \cdot Pf \quad \text{and} \quad P(f+g) \leqslant Pf+Pg$$

for $f, g \in E$ and α any real number.

The operator P is said to be *continuous in measure* if the relation $\lim_{n\to\infty} f_n = f$ in E implies that $\lim_{n\to\infty} Pf_n = Pf$ in measure, i.e., for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu[\{s \in S : |Pf_n(s) - Pf(s)| > \varepsilon\}] = 0.$$

Since

$$|Pf - Pf_n| \leqslant P(f - f_n),$$

it is evident that P is continuous in measure if and only if $\lim_{n\to\infty} Pf_n = 0$ in measure whenever $\lim_{n\to\infty} f_n = 0$ in E.

Recall the well-known fact ([10], p. 93) that if a sequence $(h_n)_{n=1}^{\infty}$ converges in measure to 0 on a σ-finite measure space, then some subsequence of $(h_n)_{n=1}^{\infty}$ converges to 0 μ-a.e.

(1.4) LEMMA. *Suppose that A is a countable set and that $(P_\alpha)_{\alpha\in A}$ is a family of operators from E into F. Suppose also that:*

(i) *for each $\alpha \in A$, the relation $\lim_{n\to\infty} f_n = f$ in E entails the existence of a subsequence (f_{n_k}) of (f_n) such that $\lim_{k\to\infty} P_\alpha f_{n_k} = P_\alpha f$ μ-a.e. Now define the operator P from E into $\mathfrak{F}$ by*

$$Pf(s) = \sup_{\alpha\in A} P_\alpha f(s) \quad (f \in E,\ s \in S).$$

For positive real numbers p and q, write

$$S_p(f) = \{s \in S : Pf(s) > p\} \text{ and } E_{p,q} = \{f \in E : \mu(S_p(f)) \leqslant q^{-1}\}.$$

Then $E_{p,q}$ is a closed subset of E.

Remarks. In (i), the subsequence may depend upon α. It is clear that condition (i) is satisfied if each P_α is continuous in measure and $(S, \mathfrak{M}, \mu)$ is σ-finite.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence of points in $E_{p,q}$ converging in the topology of E to $f \in E$. It is trivial that

$$\sup_{\alpha \in A} P_\alpha f_n(s) \leq p \quad \text{for} \quad s \in S_p(f_n)', \tag{1}$$

and

$$\mu[S_p(f_n)] \leq q^{-1}. \tag{2}$$

We may identify A with the set of positive integers. Then, using (i) and the diagonal process, we extract from (f_n) a subsequence (f_{n_k}), independent of α, such that for each $\alpha \in A$, we have

$$\lim_{k \to \infty} P_\alpha f_{n_k}(s) = P_\alpha f(s) \quad \mu\text{-a.e.}$$

Consequently there exists a null set N such that

$$\lim_{k \to \infty} P_\alpha f_{n_k}(s) = P_\alpha f(s \quad \text{for} \quad \alpha \in A \quad \text{and} \quad s \in N'. \tag{3}$$

Write $S_p = \overline{\lim_{k \to \infty}} S_p(f_{n_k})$; (2) shows that

$$\mu(S_p) \leq \overline{\lim_{k \to \infty}} \mu[S_p(f_{n_k})] \leq q^{-1}.$$

If $s \in S_p'$, then $s \in S_p(f_{n_k})'$ for infinitely many values of k, say for $k_1 < k_2 < k_3 < \ldots$; (1) yields

$$P_\alpha f_{n_{k_j}}(s) \leq p \quad \text{for} \quad \alpha \in A \quad \text{and} \quad j = 1, 2, 3, \ldots.$$

So for $s \in (S_p \cup N)'$, (3) shows that $P_\alpha f(s) \leq p$ for $\alpha \in A$, and therefore $Pf(s) \leq p$ for $s \in (S_p \cup N)'$. The relations

$$\mu(S_p \cup N) = \mu(S_p) \leq q^{-1}$$

show that $\mu[S_p(f)] \leq q^{-1}$, which shows that $f \in E_{p,q}$. □

(1.5) THEOREM. *Suppose that* $\mu(S) < \infty$, *that* $(P_\alpha)_{\alpha \in A}$ *is as in* (1.4), *that* (1.4.i) *holds, and that* E *is of the second category. Suppose also that*

(i) $Pf(s)$ *is finite* μ*-a.e. for every* $f \in E$.

Then for every positive integer q, *there is a neighbourhood* U_q *of* 0 *in* E *and a positive real number* C_q *such that*

(ii) $\mu[\{s \in S : Pf(s) > C_q\}] \leq q^{-1}$ *for* $f \in U_q$.

In particular, P *is continuous in measure.*

Proof. The sets $E_{p,q}$ of (1.4) are closed in E. We will first show that for every q,

$$E = \bigcup \{E_{p,q} : p = 1, 2, 3, \ldots\}. \tag{1}$$

For $f \in E$, (i) asserts that $Pf(s) < \infty$ for $s \in N'$, where N is null. Since N' is the union of the sets $(S_p(f))'$, for $p = 1, 2, 3, \ldots$, we see at once that $0 = \lim_{p \to \infty} \mu(S_p(f))$, and so p can be chosen (depending on q) such that $\mu(S_p(f)) \leq q^{-1}$. For any such p, f is in $E_{p,q}$, and (1) is established.

Since E is of the second category, it follows that to each q there corresponds a positive integer p_q, an element $f_q \in E$, and a symmetric neighbourhood U_q of 0 in E, such that

$$f_q + U_q \subset E_{p_q, q}. \tag{2}$$

Consider any element f of U_q. We can write

$$f = \tfrac{1}{2}(f_q + f) - \tfrac{1}{2}(f_q - f) = \tfrac{1}{2}f' - \tfrac{1}{2}f'',$$

where f' and f'' each belong to $E_{p_q, q}$ by virtue of (2). Each P_α being sublinear, the same is true of P, so that

$$Pf \leqslant \tfrac{1}{2}Pf' + \tfrac{1}{2}Pf''.$$

From this and from the definition of the sets $E_{p,q}$, it follows readily that $Pf(s) \leqslant p_q$ save on a set $T_q(f) \in \mathfrak{M}$ such that $\mu[T_q(f)] \leqslant 2q^{-1}$. This yields the relation (ii), if we replace q by $2q$ and take $C_q = p = p_{2q}$.

Finally, suppose that $\lim_{n\to\infty} f_n = 0$ in E. If δ is a positive real number and $f \in C_q^{-1}\delta U_q = V_{q,\delta}$, we can write $f = C_q^{-1}\delta g$ for some $g \in U_q$. Then (ii) shows that $Pf(s) \leqslant \delta$ save on a set of measure at most q^{-1}. Since $V_{q,\delta}$ is a neighbourhood of 0 in E, f_n belongs to $V_{q,\delta}$ for all $n \geqslant n(q, \delta)$. Thus it appears that $\lim_{n\to\infty} Pf_n = 0$ in measure, so that P is continuous in measure. □

(1.6) THEOREM. *Let E be of the second category. Let $(P_\alpha)_{\alpha\in A}$ be a countable net of sublinear operators from E into $\mathfrak{F}$ satisfying* (1.4.i) *and* (1.5.i). *Let E_0 be the set of f in E for which $\lim_{\alpha\in A} P_\alpha f(s) = 0$ μ-l.a.e. Then E_0 is a closed vector subspace of E.*

Proof. The set E_0 is a vector subspace of E because each P_α is sublinear. To prove that E_0 is closed in E, it is simple to see that we may suppose that $\mu(S) < \infty$. For, suppose that the result has been established in this case. Take any $F \in \mathfrak{M}$ such that $\mu(F) < \infty$. All of our hypotheses remain satisfied if $(S, \mathfrak{M}, \mu)$ is replaced by $(F, \mathfrak{M}^*, \mu^*)$, where $\mathfrak{M}^* \subset \mathfrak{M}$ consists of all sets of the form $M \cap F$ with $M \in \mathfrak{M}$, and μ^* is the restriction of μ to $\mathfrak{M}^*$. Hence if $f \in \bar{E}_0$ (the closure in E of E_0), it will follow that

$$\lim_{\alpha\in A} P_\alpha f(s) = 0$$

for each $s \in F$ except for the points of a null set $N_F \subset F$. So the set N of points $s \in S$ for which the relation $\lim_{\alpha\in A} P_\alpha f(s) = 0$ does not hold is locally null, and f is therefore in E_0. In view of this, we will suppose throughout the rest of the proof that $\mu(S) < \infty$.

Consider any $f \in \bar{E}_0$ and choose from E_0 a sequence (f_n) converging in E to f. By (1.5), the functions $h_n = P(f - f_n)$ converge to 0 in measure as $n \to \infty$. Hence there exists a subsequence (h_{n_k}) and a null set $N \in \mathfrak{M}$ such that $\lim_{k\to\infty} h_{n_k}(s)=0$ for all $s \in N'$. For all α and all s, we have

$$P_\alpha f(s) \leqslant P_\alpha (f - f_{n_k})(s) + P_\alpha f_{n_k}(s) \leqslant h_{n_k}(s) + P_\alpha f_{n_k}(s).$$

Since f_{n_k} is in E_0, there is a null set N_1 such that $\lim_{\alpha \in A} P_\alpha f_{n_k}(s)=0$ for all k and all $s \in N_1'$. If $s \in (N \cup N_1)'$ and $\varepsilon > 0$, first choose and fix $k = k(\varepsilon, s)$ such that $h_{n_k}(s) \leqslant \frac{1}{2}\varepsilon$. Having fixed this k, select an $\alpha(k, \varepsilon) = \alpha(s, \varepsilon)$ such that $P_\alpha f_{n_k}(s) \leqslant \frac{1}{2}\varepsilon$ for $\alpha \geqslant \alpha(s, \varepsilon)$. Then we have $P_\alpha f(s) \leqslant \varepsilon$ if $s \in (N \cup N_1)'$ and $\alpha \geqslant \alpha(s, \varepsilon)$. Since $N \cup N_1$ is null, it is clear that $f \in E_0$. □

We close this section with a special case of (1.6) needed in § 3.

(1.7) THEOREM. *Let G be a locally compact group with a left Haar measure λ and let p be a real number $\geqslant 1$. Suppose that $(K_n)_{n=1}^{\infty}$ is a sequence of functions such that $\Delta^{-1/p'}K_n$ is in $\mathfrak{L}_1(G)$ and having the following two properties:*

(i) $\lim_{n\to\infty} f * K_n(x) = f(x)$ *for each* $x \in G$ *and each* $f \in \mathfrak{C}_{00}(G)$;

(ii) $\sup_{n \geqslant 1} |f * K_n(x)| < \infty$ *l.a.e. for each* $f \in \mathfrak{L}_p(G)$.

Then the relation

(iii) $\lim_{n\to\infty} f * K_n(x) = f(x)$ *a.e. on* G

obtains for each $f \in \mathfrak{L}_p(G)$.

If G is σ-compact and each K_n has compact support, then one may replace $\mathfrak{L}_p(G)$ by $\mathfrak{L}_{p,\mathrm{loc}}(G)$ in (ii) *and* (iii).

Proof. For the first part of the theorem, we apply (1.6) as follows: $S = G$, $\mu = \lambda$, $A =$ positive integers, $E = \mathfrak{L}_p(G)$, and $P_n f(x) = |f * K_n(x) - f(x)|$. The space $\mathfrak{L}_p(G)$ is semimetrisable and complete, hence of the second category. Since

$$\|P_n f\|_p \leqslant (1 + \|\Delta^{-1/p'}K_n\|_1) \cdot \|f\|_p,$$

it is clear that (1.4.i) is fulfilled. Property (1.5.i) is immediate from (ii). On the other hand, (i) shows that the subspace E_0 defined in (1.6) contains $\mathfrak{C}_{00}(G)$. Since $\mathfrak{C}_{00}(G)$ is dense in $\mathfrak{L}_p(G)$ for $1 \leqslant p < \infty$, the space E_0 must exhaust $\mathfrak{L}_p(G)$, since, by (1.6), it is closed. (We can use a.e. rather than l.a.e. in (iii) because the $f * K_n$ collectively vanish outside of some σ-finite subset of G.)

The second part of the theorem follows in much the same fashion, except that now we take E to be $\mathfrak{L}_{p,\mathrm{loc}}(G)$, which is semimetrisable and complete for the topology of convergence in mean with index p on each compact subset of G. □

(1.8) *Note.* A number of classical theorems on pointwise convergence are immediate consequences of (1.7). For example, to show that the (C, α) means $(\alpha > 0)$ of a Fourier series converge almost everywhere to the original function (see for example [25], Vol. I, Ch. III, Th. (5.1)), it suffices to note that: the result is trivial for trigonometric polynomials; trigonometric polynomials are dense in $\mathfrak{L}_1(-\pi, \pi)$; and by a theorem of Hardy and Littlewood (see [25], Vol. I, Ch. IV, Th. (7.8)), the (C, α) means have a finite supremum almost everywhere. The same argument also proves convergence almost everywhere of Abel sums. The case of restricted $(C, 1)$ sums of Fourier series in several variables is dealt with as above by using [25], Vol. II, Ch. XVII, Lemma (3.11). For the Riesz means S_R^{δ} for Fourier series in several variables, the inequalities (D) and (D^*) in [21] show at once that pointwise convergence holds almost everywhere for the functions and δ's under consideration. N. J. Fine's theorem on pointwise (C, α) summability of Walsh-Rademacher series [8] is proved similarly from (1.7).

For $f \in \mathfrak{L}_p(R)$ $(1 \leqslant p < 2)$, Zygmund has proved that the integrals

$$(2\pi)^{-\frac{1}{2}} \int_{-a}^{a} f(x)\, e^{-ixy} dx = \hat{f}_a(y)$$

converge to the Fourier transform $\hat{f}(y)$ for almost all $y \in R$, as $a \to \infty$. This too follows at once from (1.6) and the fact that the integrals $\hat{f}_a(y)$ have finite supremum almost everywhere. For a discussion, see [25], Vol. II, Ch. XVI, Th. (3.14). A similar result holds also for Fourier integrals in several variables.

§ 2. Differentiation of indefinite integrals

Throughout this section, G will denote a locally compact group and λ a left Haar measure on G. We seek differentiation processes on G of the type

$$\lim_{k \to \infty} \frac{1}{\lambda(U_k)} \int_{xU_k} f d\lambda = f(x) \quad \text{a.e.},$$

$(U_k)_{k=1}^{\infty}$ being a fixed sequence of λ-measurable subsets of G, and f an absolutely integrable function on G. Sufficient but perhaps not necessary conditions on the sequence $(U_k)_{k=1}^{\infty}$ in order for such a formula to hold lead to the following definition.

(2.1) DEFINITION. By a *D-sequence in G* we mean a sequence $(U_n)_{n=1}^{\infty}$ of λ-measurable subsets of G of finite measure such that:

(i) $U_1 \supset U_2 \supset U_3 \supset \dots$;

(ii) there exists a positive real number C such that

$$0 < \lambda(U_n \cdot U_n^{-1}) < C\lambda(U_n) \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

A D-sequence $(U_n)_{n=1}^{\infty}$ is said to be *open, closed, compact, relatively compact*, or *Borel* if each U_n has the corresponding property. If a D-sequence $(U_k)_{k=1}^{\infty}$ is Borel and also has the property that every neighbourhood of e in G contains some U_k, then $(U_k)_{k=1}^{\infty}$ is called a *D'-sequence*. Let $(U_k)_{k=1}^{\infty}$ be a D'-sequence such that for each k, there is a λ-measurable set V_k such that

(iii) $$V_k \cup (V_k V_k^{-1}) \subset U_k \quad \text{and} \quad \lambda(U_k) < C'\lambda(V_k),$$

where C' is a fixed positive number. Then $(U_k)_{k=1}^{\infty}$ is called a *D''-sequence*.

(2.2) THEOREM.(1). *Let $(U_k)_{k=1}^{\infty}$ be any D-sequence in G. Denote by $\mathfrak{S}$ the system of all sets xU_k ($x \in G$, $k = 1, 2, 3, \ldots$), and let $\mathfrak{S}^{\dagger}$ be a subsystem of $\mathfrak{S}$. Let E be a subset*(2) *of G such that*

(i) $\lambda(EU_1) < \infty$;

(ii) *for each $x \in E$ there is at least one positive integer k (possibly depending on x) such that $xU_k \in \mathfrak{S}^{\dagger}$. Then there exists a finite or infinite sequence of pairwise disjoint sets $x_n U_{k_n} \in \mathfrak{S}^{\dagger}$ ($n = 1, 2, 3, \ldots$) such that $x_n \in E$ and*

(iii) $$\sum_{n=1}^{\infty} \lambda(U_{k_n}) \geq C^{-1}\lambda(E),$$

where C is as in (2.1.ii).

Proof. We define the points x_n and the positive integers k_n by induction. Let k_1 be the least positive integer k for which there is an $x \in E$ such that $xU_k \in \mathfrak{S}^{\dagger}$. Then choose any x_1 in E such that $x_1 U_{k_1} \in \mathfrak{S}^{\dagger}$. In general, suppose that $p \geq 1$, and that points $x_1, \ldots, x_p \in E$ and positive integers $k_1, \ldots, k_p$ have been chosen so that:

(a) $x_n U_{k_n} \in \mathfrak{S}^{\dagger}$ $(n = 1, \ldots, p)$;

(b) the sets $x_1 U_{k_1}, \ldots, x_p U_{k_p}$ are pairwise disjoint;

(c) if $p > 1$ and $1 < r \leq p$, then k_r is the smallest positive integer k such that there is an $x \in E$ for which $xU_k \in \mathfrak{S}^{\dagger}$ and xU_k is disjoint from $x_1 U_{k_1} \cup \ldots \cup x_{r-1} U_{k_{r-1}}$.

If $$E \subset \bigcup_{n=1}^{p} x_n U_{k_n} \cdot U_{k_n}^{-1},$$

then the process stops. Otherwise, we choose x_{p+1} and k_{p+1} as follows. Consider any point

(1) We were led to this theorem and its proof by Banach's proof of Vitali's covering theorem, as in [20], Ch. IV, § 3. See also [24]. These authors use metrics, which we do not, although in some of our applications we need the first countability axiom (equivalent to metrisability) for G.

(2) The set E need not be λ-measurable.

$$x \in E \cap (\bigcup_{n=1}^{p} x_n U_{k_n} \cdot U_{k_n}^{-1})'.$$

Then there is a positive integer $k \geq k_1$ such that $xU_k \in S^\dagger$. If xU_k intersects the set $\bigcup_{n=1}^{p} x_n U_{k_n}$, and if r is the least positive integer such that xU_k intersects $x_r U_{k_r}$, there are two possibilities.

(α) If $r=1$, then xU_k intersects $x_1 U_{k_1}$ and hence, because $k \geq k_1$, we have

$$x \in x_1 U_{k_1} U_k^{-1} \subset x_1 U_{k_1} U_{k_1}^{-1},$$

which is false.

(β) If $r>1$, then xU_k does not intersect $\bigcup_{n=1}^{r-1} x_n U_{k_n}$, and the inductive hypothesis (c) implies that $k \geq k_r$. Yet xU_k intersects $x_r U_{k_r}$, and so (since $k \geq k_r$) we have

$$x \in x_r U_{k_r} U_k^{-1} \subset x_r U_{k_r} U_{k_r}^{-1},$$

which is again false. We have thus proved that xU_k is disjoint from $\bigcup_{n=1}^{p} x_n U_{k_n}$. Now, amongst all of the sets $xU_k \in S^\dagger$ with $x \in E$ that are disjoint from $\bigcup_{n=1}^{p} x_n U_{k_n}$, there is a smallest corresponding value of k. We take k_{p+1} to be this smallest value of k, and we choose x_{p+1} as any element of E such that $x_{p+1} U_{k_{p+1}} \in S^\dagger$ and $x_{p+1} U_{k_{p+1}}$ is disjoint from $\bigcup_{n=1}^{p} x_n U_{k_n}$. If the process terminates at any stage, we get:

$x_1 U_{k_1}, \ldots, x_p U_{k_p} \in S^\dagger$ and $x_1, \ldots, x_p \in E$; the sets $x_n U_{k_n}$ are pairwise disjoint;

and

$$\mathrm{E} \subset \bigcup_{n=1}^{p} x_n U_{k_n} U_{k_n}^{-1}.$$

If the process is defined for all positive integers, we get:

$x_1 U_{k_1}, \ldots, x_n U_{k_n}, \ldots \in S^\dagger$ and $x_1, \ldots, x_n, \ldots \in E$; and the sets $x_n U_{k_n}$ are pairwise disjoint.

We will prove that in the second case, the inclusion

$$E \subset \bigcup_{n=1}^{\infty} x_n U_{k_n} U_{k_n}^{-1} = S \tag{1}$$

obtains. Let x be an arbitrary element of E. Then by (ii) there is a $k \geq k_1$ such that $xU_k \in S^\dagger$. We show first that xU_k intersects the set $\bigcup_{n=1}^{\infty} x_n U_{k_n}$. If xU_k does not intersect $\bigcup_{n=1}^{\infty} x_n U_{k_n}$, our construction shows that $k \geq k_n$ for all n. Since the sets $x_n U_{k_n}$ are λ-measurable and pairwise disjoint and are all contained in EU_1, hypothesis (i) implies that

$$\sum_{n=1}^{\infty} \lambda(U_{k_n}) = \sum_{n=1}^{\infty} \lambda(x_n U_{k_n}) \leq \lambda(EU_1) < \infty,$$

and so $\lambda(U_{k_n}) \to 0$ as $n \to \infty$. Now, if k_n did not go to infinity with n, then an infinite number of the k_n would be equal, and the equality $\lim_{n\to\infty}\lambda(U_{k_n})=0$ could not occur. Hence we have $\lim_{n\to\infty}k_n=\infty$, and the relation $k\geqslant k_n$ for all n is impossible. This proves that xU_k intersects $\bigcup_{n=1}^{\infty}x_nU_{k_n}$.

Let N be the smallest value of n such that xU_k intersects $x_nU_{k_n}$. If $N=1$, then we have $x\in x_1U_{k_1}U_k^{-1}\subset x_1U_{k_1}U_{k_1}^{-1}\subset S$, and (1) is established. If $N>1$, then xU_k is disjoint from the set $\bigcup_{n=1}^{N-1}x_nU_{k_n}$, so that we have $k\geqslant k_N$. Since xU_k intersects $x_NU_{k_N}$, it follows that

$$x\in x_NU_{k_n}U_k^{-1}\subset x_NU_{k_N}U_{k_N}^{-1}\subset S,$$

and so (1) is established in all cases.

The proof is now completed easily. From (1) and the left invariance of λ, we get

$$\lambda(E)\leqslant\bigcup_{n=1}^{\infty}\lambda(U_{k_n}\cdot U_{k_n}^{-1})\leqslant C\sum_{n=1}^{\infty}\lambda(U_{k_n}),$$

as asserted. □

It is widely known that covering theorems like (2.2) imply the existence of derivatives in one form and another. Our Theorem (1.6) is the abstract form of an argument used in many such existence proofs. In the two following theorems we present consequences of (2.2) that will enable us to apply (1.6).

(2.3) THEOREM. *Let $(U_k)_{k=1}^{\infty}$ be a Borel D-sequence in G, and let μ be a positive Radon measure on G. For $x\in G$, let*

$$\mu^*(x)=\sup\left\{\frac{\mu(xU_k)}{\lambda(U_k)}:\ k=1,2,3,\ldots\right\}.$$

Let E be a subset of G such that $\lambda(EU_1)<\infty$, and for $\alpha>0$, let $M_\alpha=\{x\in G:\mu^*(x)>\alpha\}$. Then

(i) $$\lambda(E\cap M_\alpha)\leqslant C\alpha^{-1}\mu((E\cap M_\alpha)\cdot U_1),$$

and equality holds if and only if $E\cap M_\alpha=\emptyset$. If $\bar U_1$ is compact, then μ^* is finite l.a.e., and a.e. if μ has σ-compact support.

Proof. The function μ^* is Borel measurable, as is shown in [12], (20.9). Let $\mathcal{S}$ consist of all sets xU_k ($x\in G$, $k=1,2,\ldots$), and let $\mathcal{S}^\dagger$ consist of all sets xU_k with $x\in E$ and

$$\frac{\mu(xU_k)}{\lambda(U_k)}>\alpha. \tag{1}$$

If $x\in E\cap M_\alpha$, then (1) is true for some k, so that $xU_k\in\mathcal{S}^\dagger$. We also have $\lambda((E\cap M_\alpha)\cdot U_1)\leqslant\lambda(EU_1)<\infty$. Applying (2.2) to the set $E\cap M_\alpha$, we find pairwise disjoint sets $x_nU_{k_n}\in\mathcal{S}^\dagger$ ($n=1,2,3,\ldots$) such that $\sum_{n=1}^{\infty}\lambda(U_{k_n})\geqslant C^{-1}\lambda(E\cap M_\alpha)$. For each n, we have

$$\lambda(U_{k_n})<\alpha^{-1}\mu(x_nU_{k_n}).$$

Adding over n and noting the inclusions $x_n U_{k_n} \subset (E \cap M_\alpha) \cdot U_1$, we have

$$\lambda(E \cap M_\alpha) \leq C \sum_{n=1}^{\infty} \lambda(U_{k_n}) < C\alpha^{-1}\mu(\sum_{n=1}^{\infty} x_n U_{k_n}) \leq C\alpha^{-1}\mu((E \cap M_\alpha) \cdot U_1).$$

The possible equality in (i), and the last sentence of the theorem, are easily checked. □

(2.4) THEOREM. *Let $(U_k)_{k=1}^{\infty}$ be a D'-sequence in G, let μ be a positive Radon measure on G, and for $x \in G$, let*

$$\bar{\mu}(x) = \overline{\lim_{k\to\infty}} \frac{\mu(xU_k)}{\lambda(U_k)}.$$

For every compact subset F of G and every $\alpha > 0$, we have

(i) $$\lambda(\{x \in F : \bar{\mu}(x) > \alpha\}) \leq C\alpha^{-1}\mu(F).$$

Proof. Apply (2.3) to the D-sequence $(U_k)_{k=r}^{\infty}$, where r is an arbitrary positive integer, with $E = F$. This gives us

$$\lambda\left(\left\{x \in F : \sup_{k \geq r} \frac{\mu(xU_k)}{\lambda(U_k)} > \alpha\right\}\right) \leq C\alpha^{-1}\mu(F \cdot U_r). \tag{1}$$

Since
$$\bar{\mu}(x) \leq \sup_{k \geq r} \frac{\mu(xU_k)}{\lambda(U_k)},$$
(1) implies that
$$\lambda(\{x \in F : \bar{\mu}(x) > \alpha\}) \leq C\alpha^{-1}\mu(F \cdot U_r) \quad (r = 1, 2, 3, \ldots). \tag{2}$$

Since the U_r are ultimately very small sets, we have $\bigcap_{r=1}^{\infty} F \cdot U_r = F$, and so $\lim_{r\to\infty} \mu(F \cdot U_r) = \mu(F)$. Hence (2) implies (i). □

We now apply (1.6) and (2.3) to prove our differentiation formula.

(2.5) THEOREM. *Let $(U_k)_{k=1}^{\infty}$ be a D'-sequence. Then the equality*

(i) $$\lim_{k\to\infty} \frac{1}{\lambda(U_k)} \int_{xU_k} f d\lambda = f(x)$$

holds l.a.e. for each $f \in \mathfrak{L}_{1,\mathrm{loc}}(G)$, and a.e. for each $f \in \mathfrak{L}_1(G)$.

Proof. We may clearly suppose that each U_k is relatively compact. Hence the values of all of the functions

$$\frac{1}{\lambda(U_k)} \int_{xU_k} f d\lambda \quad (k = 1, 2, 3, \ldots)$$

at the points of any preassigned compact subset of G depend only on the values of f on some compact subset of G. Let $(A_\nu)_{\nu=1}^{\infty}$ be a sequence of nonvoid open subsets of G each having finite λ measure. Denote by $\mathfrak{E}$ the subspace of $\mathfrak{L}_{1,\mathrm{loc}}(G)$ consisting of those functions $f \in \mathfrak{L}_{1,\mathrm{loc}}(G)$ that vanish outside of $\bigcup_{\nu=1}^{\infty} A_\nu$. It obviously suffices to prove the theorem for functions in $\mathfrak{E}$. With the topology defined by the seminorms $f \to \int_{A_\nu} |f| d\lambda$ ($\nu = 1, 2, 3, \ldots$), $\mathfrak{E}$ is a complete, semimetrisable, topological vector space. Plainly $\mathfrak{E}_{00}(G)$ is dense in $\mathfrak{E}$. Moreover, (i) holds for all $x \in G$ if $f \in \mathfrak{E}_{00}(G)$, since U_k is ultimately very small.

We now appeal to Theorem (1.6), taking

$$P_k f(x) = \left| \frac{1}{\lambda(U_k)} \int_{xU_k} f d\lambda - f(x) \right| \leqslant \frac{1}{\lambda(U_k)} \int_{xU_k} |f| d\lambda + |f(x)|.$$

By (2.3), we see that $Pf(x) = \sup_{k \geqslant 1} P_k f(x)$ is finite a.e. for each $f \in \mathfrak{E}$. Since the equality $\lim_{k\to\infty} P_k f(x) = 0$ holds for all $x \in G$ if $f \in \mathfrak{E}_{00}(G)$, Theorem (1.6) implies that $\lim_{k\to\infty} P_k f(x) = 0$ a.e. for each $f \in \mathfrak{E}$, which is equivalent to (i). □

For the inversion theorems of §§ 3 and 5, we need a fact about singular measures and D'-sequences, which is proved from (2.4).

(2.6) THEOREM. *Let σ be a measure in $\mathbf{M}(G)$ that is singular with respect to λ, i.e., there is a λ-null set N such that $|\sigma|(N') = 0$. Let $(U_k)_{k=1}^{\infty}$ be a D'-sequence in G. Then we have*

(i) $$\lim_{k\to\infty} \frac{\sigma(xU_k)}{\lambda(U_k)} = 0 \quad \textit{a.e. on } G.$$

Proof. We may suppose that $\sigma \geqslant 0$, so that $|\sigma| = \sigma$. Let F be a compact subset of G, and let α be a positive number. Theorem (2.4) implies that

$$\lambda(\{x \in F : \bar{\sigma}(x) > \alpha\}) \leqslant C\alpha^{-1}\sigma(F). \tag{1}$$

Since σ is singular with respect to λ, there is a σ-compact λ-null set N such that N' is σ-null. Applying (1) to a compact subset F of N', we infer that $\lambda(\{x \in F : \bar{\sigma}(x) > \alpha\}) = 0$. This equality being true for every $\alpha > 0$, we have $\bar{\sigma}(x) = 0$ a.e. on N'. Since N is λ-null, (i) follows. □

Theorems (2.5) and (2.6) are generalisations to locally compact groups of the celebrated theorems of Lebesgue concerning differentiation of functions of finite variation on R: an absolutely continuous function is the integral of its derivative, and a singular function of finite variation has derivative 0 almost everywhere. Similar facts about measures on R^a and T^a are also well known. For treatments of these cases and for various applications, see [20], Ch. IV, [25] *passim*, [24], and [5].

For some of the convergence theorems of § 3, we need a generalisation of the Hardy-Littlewood maximal theorem, which holds for D-sequences.

(2.7) THEOREM. *Let $(U_k)_{k=1}^{\infty}$ be a relatively compact Borel D-sequence in G, let f be a function in $\mathfrak{L}_{1,\,\mathrm{loc}}^{+}(G)$, and define*

$$f^*(x)=\sup\left\{\frac{1}{\lambda(U_k)}\int_{xU_k} f\,d\lambda : k=1,2,3,\ldots\right\}.$$

For $\alpha>0$, let $E_\alpha=\{x\in G: f(x)>\alpha\}$ and $E_\alpha^*=\{x\in G: f^*(x)>\alpha\}$. Let E be a λ-measurable subset of G such that EU_1 is λ-measurable and $\lambda(EU_1)<\infty$. The following inequalities hold:

(i) $$\lambda(E\cap E_\alpha^*)\leqslant 2C\alpha^{-1}\int_{(EU_1)\cap E_{\frac{1}{2}\alpha}} f\,d\lambda;$$

(ii) $$\int_E f^{*p}\,d\lambda\leqslant\frac{2^pCp}{p-1}\int_{EU_1} f^p\,d\lambda\quad(1<p<\infty);$$

(iii) $$\int_E f^*\,d\lambda\leqslant 2\lambda(E)+2C\int_{EU_1} f\log^+ f\,d\lambda;$$

(iv) $$\int_E f^{*p}\,d\lambda\leqslant 2^p\left(1+\frac{Cp}{1-p}\right)\lambda(E)^{1-p}\left(\int_{EU_1} f\,d\lambda\right)^p\quad(0<p<1).$$

In particular, the function $f^*(x)$ is finite l.a.e. and is finite a.e. if f vanishes outside of a set that is σ-finite for λ.

Proof. Let $g(x)=f(x)$ if $f(x)>\frac{1}{2}\alpha$ and $g(x)=0$ if $f(x)\leqslant\frac{1}{2}\alpha$. Clearly $g\in\mathfrak{L}_{1,\,\mathrm{loc}}^{+}(G)$ and $f^*(x)\leqslant g^*(x)+\frac{1}{2}\alpha$. Thus $E_\alpha^*\subset\{x\in G: g^*(x)>\frac{1}{2}\alpha\}$. Applying (2.3) to the measure $\mu=g\lambda$, we find

$$\begin{aligned}\lambda(E\cap E_\alpha^*)&\leqslant\lambda(E\cap\{x\in G: g^*(x)>\tfrac{1}{2}\alpha\})\\&\leqslant C(2\alpha)^{-1}\int_{EU_1} g\,d\lambda=C(2\alpha)^{-1}\int_{(EU_1)\cap E_{\frac{1}{2}\alpha}} f\,d\lambda.\end{aligned}$$

This is (i).

Since $\lambda(E)\leqslant\lambda(EU_1)<\infty$, we can apply Fubini's theorem to write

$$\begin{aligned}\int_E f^*(x)^p\,d\lambda(x)&=\int_E\left\{\int_0^{f^*(x)} pt^{p-1}\,dt\right\}d\lambda(x)=\int_E\left\{\int_0^\infty pt^{p-1}\xi_{[0,f^*(x)]}(t)\,dt\right\}d\lambda(x)\\&=\int_E\left\{\int_0^\infty pt^{p-1}\xi_{E_t^*}(x)\,dt\right\}d\lambda(x)=\int_0^\infty pt^{p-1}\left\{\int_E\xi_{E_t^*}(x)\,d\lambda(x)\right\}dt\\&=\int_0^\infty pt^{p-1}\lambda(E\cap E_t^*)\,dt,\end{aligned}$$

that is,
$$\int_E f^{*p} d\lambda = \int_0^\infty p t^{p-1} \lambda(E \cap E_t^*)\, dt. \tag{1}$$

Here p is any positive number. The inequalities (ii)–(iv) are obtained from (i) and (1) by using Fubini's theorem and making reasonably obvious estimates. The details are similar to those in the classical case, and we omit them. (See for example [25], Vol. I, Ch. I, Theorem (13.13).) □

(2.8) COROLLARY. *Let $(U_k)_{k=1}^\infty$ be as in (2.7). For $1<p<\infty$, we have:*

(i) *if $f \in \mathfrak{L}_{p,\text{loc}}(G)$, then $f^* \in \mathfrak{L}_{p,\text{loc}}(G)$;*

(ii) *if $f \in \mathfrak{L}_p(G)$, then $f^* \in \mathfrak{L}_p(G)$;*

(iii) *if f is l.a.e. equal to a function in $\mathfrak{L}_p(G)$, then the same is true of f^*;*

(iv) *if $f \log^+ f \in \mathfrak{L}_{1,\text{loc}}(G)$, then $f^* \in \mathfrak{L}_{1,\text{loc}}(G)$.*

For $0<p<1$, we have:

(v) *if $f \in \mathfrak{L}_{1,\text{loc}}(G)$, then $f^* \in \mathfrak{L}_{p,\text{loc}}(G)$; for compact G, if $f \in \mathfrak{L}_1(G)$, then $f^* \in \mathfrak{L}_p(G)$.*

Proof. All of these assertions save (ii) follow at once from (2.7). To prove (ii), observe that if f vanishes outside of a set that is σ-finite for λ, then the same is true of f^*. In this case, if $\int_F f^{*p} d\lambda$ is bounded for all compact sets F, then f^* is in $\mathfrak{L}_p(G)$. □

The class of locally compact groups admitting D'-sequences has not been adequately identified. The referee has kindly pointed out to us that an infinite-dimensional torus $T^{\mathfrak{m}}$ admits no such sequence. This follows easily from the Brunn-Minkowski theorem (see e.g. [9], p. 187). The possibility of differentiation theorems like (2.5) and (2.6) on $T^{\mathfrak{m}}$ remains open, however, so far as we know. For some groups, D'- and even D''-sequences (which are useful for the constructions of § 3) are easily found, as follows.

(2.9) THEOREM. *Let G be a locally compact, 0-dimensional group with the first countability axiom. Then G admits a D''-sequence consisting of compact open subgroups, which is also a complete family of neighbourhoods of e.*

Proof. The group G has an open basis $(U_k)_{k=1}^\infty$ at e consisting of compact open subgroups; this is proved, for example, in [12] (7.7). We may plainly suppose that $U_1 \supset U_2 \supset \ldots$. It is trivial that $U_k = U_k U_k^{-1}$, so that $(U_k)_{k=1}^\infty$ is a D'-sequence. In the definition of D''-sequences (2.1) we can take $V_k = U_k$. □

(2.10) THEOREM. *Every Lie group G admits a D''-sequence.*

Proof. It is sufficient to find a descending sequence $(W_k)_{k=1}^\infty$ of compact neighbourhoods of e such that $\bigcap_{k=1}^\infty W_k = \{e\}$, and $\lambda(W_k W_k^{-1} W_k W_k^{-1}) \leqslant C\lambda(W_k)$, where $C>0$. Then we can take $U_k = W_k W_k^{-1}$ and $V_k = W_k$, making $(U_k)_{k=1}^\infty$ a D''-sequence with V_k as in (2.1.iii).

Let m be the dimension of G. Take a local coordinate system $(t_1, \ldots, t_m)$ with domain a relatively compact open neighbourhood N of e in G such that $t_j(e)=0$. The coordinate map

$$T : x \to (t_1(x), \ldots, t_m(x))$$

may be taken to be a homeomorphism of N onto all of R^m. For $f \in \mathfrak{C}_{00}(G)$ such that $f(N') \subset \{0\}$, the Haar integral of f is equal to

$$\int_G f d\lambda = \int_N f d\lambda = \int_{R^m} f \circ T^{-1}(\mathbf{x}) F(\mathbf{x}) d\mathbf{x},$$

where $d\mathbf{x}$ refers to m-dimensional Lebesgue measure and F is a strictly positive continuous function on R^m.

Let $Q_\varepsilon = \{(x_1, \ldots, x_m) \in R^m,\ |x_1| \leqslant \varepsilon, \ldots, |x_m| \leqslant \varepsilon\}$. A routine argument using the differentiability of the coordinate functions shows that we may take $W_k = T^{-1}(Q_{\varepsilon_k})$ for a certain sequence $\varepsilon_1 > \varepsilon_2 > \ldots$ having limit 0. □

(2.11) THEOREM. *Every finite-dimensional compact group G admits a D''-sequence.*

Proof. It is known that G is locally the product of a local Lie group and a 0-dimensional closed normal subgroup of G ([17], Th. 69). This allows us to combine (2.9) and (2.10) to produce a D''-sequence in G. □

§ 3. Single limits for operators $f \star K_n$

Let G be a metrisable group that admits an open D'-sequence and is either compact or locally compact Abelian. Then there is a pointwise summability method for Fourier transforms on G involving only a single limiting operation. The existence of such a method is equivalent to the convergence almost everywhere to $f(x)$ of $f \star K_n(x)$, where $(K_n)_{n=1}^{\infty}$ is a certain sequence of kernels (i.e. functions) on G. Similar results apply to Fourier-Stieltjes transforms. The kernels K_n can be constructed on a larger class of groups, as we now show.

(3.1) THEOREM. *Let G be a locally compact group admitting an open D'-sequence $(U_n)_{n=1}^{\infty}$. Then there is a sequence $(K_n)_{n=1}^{\infty}$ of functions on G with the following properties:*

(i) *K_n is continuous, nonnegative, and zero outside of U_n^{-1};*

(ii) *K_n is a finite linear combination of continuous positive-definite functions each of which vanishes outside of $(U_n U_n^{-1}) \cup U_n \cup U_n^{-1}$;*

(iii) $\int_G K_n d\lambda = 1$;

(iv) $\lim_{n\to\infty} f \star K_n(x) = f(x)$ *a.e. on G for each $f \in \mathfrak{L}_p(G)$* $(1 \leqslant p < \infty)$.

Define $f^{\#}(x)=\sup\{|f*K_n(x)|:n=1,2,3,\ldots\}$ *for* $f\in\mathfrak{L}_{1,\,\mathrm{loc}}(G)$. *Then:*

(v) $\int_G f^{\#p}d\lambda\leqslant\text{const. }\int_G|f|^p d\lambda$ *if* $1<p<\infty$;

(vi) $\int_E f^{\#}d\lambda\leqslant\text{const. }(\lambda(E)+\int_{EU_1}|f|\log^+|f|\,d\lambda)$ *if* E *is compact;*

(vii) $\int_E f^{\#p}d\lambda\leqslant\text{const. }\lambda(E)^{1-p}(\int_{EU_1}|f|\,d\lambda)^p$ *if* E *is compact and* $0<p<1$.

Proof. For each positive integer n, choose a compact set $H_n\subset U_n$ such that $\lambda(H_n)\geqslant\frac{1}{2}\lambda(U_n)$ and then choose a compact symmetric neighbourhood W_n of e such that $H_nW_n\subset U_n$ and $W_n^2\subset U_nU_n^{-1}$. Consider the function

$$K_n=\lambda(H_n)^{-1}\lambda(W_n)^{-1}\xi_{W_n}*\tilde{\xi}_{W_n}\,(^1) \tag{1}$$

Properties (i) and (iii) are obvious, and (ii) follows from the polar identity

$$4u*v^{\sim}=(u+v)*(u+v)^{\sim}-(u-v)*(u-v)^{\sim}+i(u+iv)*(u+iv)^{\sim}-i(u-iv)*(u-iv)^{\sim},$$

after a short computation. (Note that K_n has the form $u*v^{\sim}$ where u and v are bounded and vanish outside of compact sets.)

Properties (i) and (iii) show that $\lim_{n\to\infty}f*K_n(x)=f(x)$ for all $x\in G$ and all continuous functions f on G. Since $\mathfrak{C}_{00}(G)$ is dense in $\mathfrak{L}_p(G)$ for $1\leqslant p<\infty$, (iv) will follow from (1.7) as soon as we show that $f^{\#}(x)$ is finite a.e. on G for each $f\in\mathfrak{L}_p(G)$ $(1\leqslant p<\infty)$. This is an immediate corollary of (v) and (vii), which we now prove.

Let $\alpha_n=\sup\{\Delta(y):y\in U_n\}$. It is clear that $\alpha_n\to1$ as $n\to\infty$, and so $\alpha=\sup\{\alpha_n:n\geqslant1\}$ is finite. The definitions of $*$ and $\sim$ and a routine calculation show that

$$K_n\leqslant\frac{2\alpha}{\lambda(U_n)}\xi^{*}_{U_n}.$$

So for $f\in\mathfrak{L}^+_{1,\,\mathrm{loc}}(G)$, we obtain

$$f*K_n(x)\leqslant\frac{2\alpha}{\lambda(U_n)}f*\xi^{*}_{U_n}(x)=2\alpha\,\frac{1}{\lambda(U_n)}\int_{xU_n}f\,d\lambda. \tag{2}$$

Thus $f^{\#}(x)\leqslant2\alpha f^{*}(x)$, and (2.7) implies (v), (vi), and (vii). □

(3.2) COROLLARY. *Let G be a locally compact group admitting a D''-sequence as in* (2.1). *Then the sequence $(K_n)_{n=1}^{\infty}$ of functions of* (3.1) *can be constructed with all of the properties listed in* (3.1), *and with* (3.1.ii) *replaced by the stronger condition*

(i) *each K_n is positive-definite and vanishes outside of U_n^{-1}.*

Proof. Let $(U_n)_{n=1}^{\infty}$ be a D''-sequence in G and let $(V_n)_{n=1}^{\infty}$ be as in (2.1.iii). Let $K_n=(\lambda(V_n))^{-2}\tilde{\xi}_{V_n}*\xi_{V_n}$. Then argue as in (3.1). □

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(1) For a complex function φ on G, $\tilde{\varphi}$ is the function $x\to\Delta(x^{-1})\overline{\varphi(x^{-1})}$. See [12], p. 300 *et seq.* The function φ^{*} is defined by $\varphi^{*}(x)=\varphi(x^{-1})$.

(3.3) THEOREM. *Let G, $(U_n)_{n=1}^{\infty}$, and $(K_n)_{n=1}^{\infty}$ be as in* (3.1). *Let ϱ be any measure in* $\mathbf{M}(G)$, *with Lebesgue decomposition $f\lambda+\sigma$, where $f\in\mathfrak{L}_1(G)$ and σ is singular with respect to λ. Then*

(i) $$\lim_{n\to\infty} \varrho * K_n(x) = f(x) \quad a.e. \text{ on } G.$$

Proof. Apply (3.1) to $(f\lambda)*K_n$ together with (2.6) to $\sigma * K_n$. □

The kernels K_n of Theorems (3.1)–(3.3) can be chosen to be trigonometric polynomials if G is compact and to have Fourier transforms with compact supports if G is locally compact Abelian. This fact makes our final inversion theorems of § 5 more elegant than they would perhaps otherwise be and completes the analogy of our theory with the classical theory of pointwise summability for Fourier series. It seems therefore worthwhile to carry out the construction. A preliminary fact is needed.

(3.4) THEOREM. *Let G be a metrisable group that is either compact or locally compact Abelian. There exists a sequence $(u_n)_{n=1}^{\infty}$ of functions on G with the following properties:*

(i) *u_n is continuous, integrable, nonnegative, positive-definite, and central;*

(ii) *$\int_G u_n d\lambda = 1$ for all n;*

(iii) *each u_n has compact spectrum;* (1)

(iv) *if U is any neighbourhood of e in G, then $\lim_{n\to\infty}\int_{U'} u_n d\lambda = 0$;*

(v) *if $\mathfrak{S}$ denotes any one of the spaces $\mathfrak{L}_p(G)$ $(1\leqslant p<\infty)$ or $\mathfrak{C}^u(G)$ (the space of bounded uniformly continuous functions on G with the supremum norm), then $\lim_{n\to\infty} f*u_n = f$ in $\mathfrak{S}$ for each $f\in\mathfrak{S}$.*

Proof. Assertion (v) follows readily from (i), (ii), and (iv). We treat separately the cases (I) G is compact, and (II) G is locally compact Abelian.

(I) Suppose that G is compact. There exists a base $(U_n)_{n=1}^{\infty}$ at e formed of sets that are symmetric and invariant under all inner automorphisms of G. (This is immediate from [12], (4.9).) Take any $w_n\in\mathfrak{C}^+(G)$, vanishing on U_n', such that $\int_G w_n d\lambda = 1$. Put $w_n' = w_n * \tilde{w}_n$. Then w_n' is continuous, nonnegative, positive-definite, vanishes on $(U_n^2)'$, and has the property that $\int_G w_n' d\lambda = 1$. For each $a\in G$, the function $x\to w_n'(axa^{-1})$ is continuous, nonnegative, positive-definite, and vanishes on $(U_n^2)'$. The function $v_n(x)=\int_G w_n'(axa^{-1})da$ is therefore continuous, nonnegative, positive-definite, vanishes outside of U_n^2, has integral 1, and is in addition central. Since the sets U_n^2 form a neighbourhood base at e, (iv) is evident for $(v_n)_{n=1}^{\infty}$.

(1) By this we mean that u_n is a trigonometric polynomial if G is compact and that $\hat{u}_n$ has compact support if G is locally compact Abelian.

We will modify the functions v_n to obtain the functions u_n. Consider a set $\mathfrak{D}=\{D\}$ of continuous irreducible representations of G by unitary operators on (finite-dimensional) Hilbert spaces that are pairwise inequivalent and also complete. Let χ_D be the character of D. It is well known that $v_n(x)=\sum_{D\in\mathfrak{D}} c_n(D)\chi_D(x)$, where $c_n(D)\geq 0$ and $\sum_{D\in\mathfrak{D}} c_n(D)\chi_D(e)<\infty$. For each n, we can thus choose a finite partial sum, say v_n', of the series for v_n such that $\|v_n-v_n'\|_u\leq(2n)^{-1}$. If we set $v_n''=\frac{1}{2}(\overline{v_n'}+v_n')+(2n)^{-1}$, then v_n'' is clearly a continuous, nonnegative, positive-definite, central trigonometric polynomial, and $\|v_n-v_n''\|_u\leq n^{-1}$. This implies that $\lim_{n\to\infty}\int_G v_n''\,d\lambda=\lim_{n\to\infty}\int_G v_n\,d\lambda=1$. It therefore suffices to take

$$u_n(x)=\left[\int_G v_n''\,d\lambda\right]^{-1} v_n''(x)$$

in order to satisfy conditions (i)–(iv).

(II) Suppose now that G is locally compact Abelian. The character group $\mathbf{X}$ of G is σ-compact (see [12], (24.48)), and so there is an increasing sequence $(\mathbf{H}_n)_{n=1}^{\infty}$ of relatively compact open subsets of $\mathbf{X}$ such that

$$\lim_{n\to\infty}\frac{\theta(\mathbf{H}_n\cap(\chi\mathbf{H}_n))}{\theta(\mathbf{H}_n)}=1 \quad \text{for all} \quad \chi\in\mathbf{X}$$

(see [12], (18.13)). Define the function φ_n on $\mathbf{X}$ as

$$\varphi_n=(\theta(\mathbf{H}_n))^{-1}\xi_{\mathbf{H}_n}*\tilde{\xi}_{\mathbf{H}_n} \quad (n=1,2,3,\ldots).$$

It is clear that φ_n is continuous, nonnegative, and positive-definite, and that φ_n vanishes outside of the relatively compact open set $\mathbf{H}_n\mathbf{H}_n^{-1}$. Cauchy's inequality shows that $\|\varphi_n\|_u\leq 1$, and it is obvious that $\varphi_n(1)=1$. (We write the identity character in $\mathbf{X}$ as 1.) Furthermore we have

$$\varphi_n(\chi)=\frac{\theta((\chi^{-1}\mathbf{H}_n)\cap\mathbf{H}_n)}{\theta(\mathbf{H}_n)} \quad \text{for all} \quad \chi\in\mathbf{X},$$

so that $\lim_{n\to\infty}\varphi_n(\chi)=1$ for all $\chi\in\mathbf{X}$.

Finally, define u_n on G as the inverse Fourier transform

$$\check{\varphi}_n(x)=\int_{\mathbf{X}}\chi(x)\varphi_n(\chi)\,d\theta(\chi) \quad (n=1,2,3,\ldots).$$

It is then clear that u_n is in $\mathfrak{C}_0^+(G)\cap\mathfrak{L}_1^+(G)$ and that u_n is positive-definite. Since Fourier inversion holds everywhere for φ_n and u_n, we have $\hat{u}_n=\varphi_n$ everywhere on $\mathbf{X}$ and in particular $\int_G u_n\,d\lambda=\varphi_n(1)=1$. Thus (i), (ii), and (iii) hold for u_n. To prove (iv), we need only show that $\lim_{n\to\infty}\int_U u_n\,d\lambda=1$ for every neighbourhood U of e in G. Choose a positive-definite function $h\in\mathfrak{C}_{00}^+(G)$ vanishing on U' and such that $h(e)=1$. Parseval's identity implies that

$$\int_G hu_n d\lambda = \int_{\mathbf{X}} \hat{h}\hat{u}_n d\theta = \int_{\mathbf{X}} \hat{h}\varphi_n d\theta. \tag{1}$$

Since $h \in \mathfrak{L}_1(\mathbf{X})$ and φ_n converges boundedly to 1 everywhere on $\mathbf{X}$, we can take limits in (1) to write $\lim_{n\to\infty} \int_G hu_n d\lambda = \int_{\mathbf{X}} \hat{h} d\theta = h(e) = 1$. The desired relation (iv) now follows easily. □

We now modify the kernels K_n.

(3.5) THEOREM. *Let G be metrisable and either compact or locally compact Abelian. Suppose that G admits an open D''-sequence. Then there is a sequence $(K_n)_{n=1}^{\infty}$ of functions on G having all of the properties set down in* (3.1) *and* (3.3) *except for* (3.1.i) *and* (3.1.ii). *These are replaced by:*

(i) *each K_n is continuous, nonnegative, positive-definite, central, and has a compact spectrum;*

(ii) *for every neighbourhood U of e, $\lim_{n\to\infty} \int_{U'} K_n d\lambda = 0$.*

Proof. First construct a sequence $(K_n^0)_{n=1}^{\infty}$ according to Theorem (3.2), so that $(K_n^0)_{n=1}^{\infty}$ satisfies (3.2.i), (3.1.iii)–(3.1.vii), and (3.3.i). Suppose that $(u_k)_{k=1}^{\infty}$ is as in (3.4). By (3.4.v) we can for each n choose $k_n \geq n$ and so large that

$$K_n = K_n^0 * u_{k_n}$$

satisfies

$$\|K_n - K_n^0\|_1 \leq n^{-2}. \tag{1}$$

The properties of the functions K_n^0 and u_k show that (3.5.i) and (3.1.iii) hold for these kernels K_n.

To prove (3.1.iii) for our present K_n, take any $f \in \mathfrak{L}_p(G)$. The inequality (1) implies that

$$\|f * K_n - f * K_n^0\|_p \leq \|f\|_p \cdot \|K_n - K_n^0\|_1 \leq n^{-2}\|f\|_p,$$

so that

$$\sum_{n=1}^{\infty} \|f * K_n - f * K_n^0\|_p < \infty,$$

and hence

$$\lim_{n\to\infty} |f * K_n(x) - f * K_n^0(x)| = 0 \text{ a.e. on } G,$$

and now (3.1.iv) follows from (3.1.iv) for K_n^0.

To prove (3.3.i) for our present K_n, it suffices to show that $\lim_{n\to\infty} \sigma * K_n(x) = 0$ a.e. We know that $\lim_{n\to\infty} \sigma * K_n^0(x) = 0$ a.e. On the other hand, we also have

$$\|\sigma * K_n - \sigma * K_n^0\|_1 \leq \int_G d|\sigma| \cdot \|K_n - K_n^0\|_1.$$

Now repeat the argument of the preceding paragraph. The proofs of (3.1.v)–(3.1.vii) for our present K_n run along similar lines, and are omitted.

It remains only to prove (ii). We write

$$\int_{U'} K_n d\lambda = \int_{U'} \left\{ \int_G K_n^0(y) u_{k_n}(y^{-1}x)\, dy \right\} dx$$
$$= \int_{U_n^{-1}} K_n^0(y) \left\{ \int_{U'} u_{k_n}(y^{-1}x)\, dx \right\} dy = \int_{U_n^{-1}} K_n^0(y) \left\{ \int_{y^{-1}U'} u_{k_n}(z)\, dz \right\} dy.$$

Choose the neighbourhood V of e so small that $U_n V \subset U$ (which is possible since the U_n form a base at e). Then we have

$$\int_{U'} K_n d\lambda \leqslant \int_{U_n^{-1}} K_n^0(y) \left\{ \int_{V'} u_{k_n}(z)\, dz \right\} dy = \int_{V'} u_{k_n}(z)\, dz.$$

The last integral tends to zero as $n \to \infty$, since $k_n \geqslant n$ and (3.4.iv) holds. □

(3.6) Corollary. *Let G be metrisable and locally compact Abelian and admit a D''-sequence. Let $\varrho \in \mathbf{M}(G)$ have the decomposition $\varrho = f\lambda + \sigma$, where $f \in \mathfrak{L}_1(G)$ and σ is singular with respect to λ. Let $(K_n)_{n=1}^{\infty}$ be the sequence constructed in* (3.5) *or* (3.2). *Then pointwise inversion of the Fourier-Stieltjes transform $\hat{\varrho}$ obtains:*

(i) $$\lim_{n\to\infty} \int_{\mathbf{X}} \hat{K}_n(\chi)\, \hat{\varrho}(\chi)\, \chi(x)\, d\theta(\chi) = f(x) \quad \textit{a.e. on } G.$$

Proof. Since $\hat{K}_n$ is in $\mathfrak{L}_1(\mathbf{X})$, we have

$$\int_{\mathbf{X}} \hat{K}_n(\chi)\, \hat{\varrho}(\chi)\, \chi(x)\, d\theta(x) = \varrho * K_n(x).$$

Now apply (3.5) or (3.2). □

(3.7) Corollary. *Let G be metrisable and compact and admit a D''-sequence. Let ϱ be as in* (3.6) *and $(K_n)_{n=1}^{\infty}$ as in* (3.5). *Let $\mathfrak{D}$ be as in* (3.4.I), *and $\hat{\varrho}(D)$ the operator $\int_G \overline{\bar{D}(x)}\, d\varrho(x)$, where $\bar{D}$ is the representation conjugate to D; $\hat{K}_n(D)$ is defined similarly. Then $\hat{K}_n(D)$ is a nonnegative multiple $\alpha_n(D) I$ of the identity operator, different from* 0 *for only finitely many D; and*

(i) $$\lim_{n\to\infty} \sum_{D \in \mathfrak{D}} d(D)\, \alpha_n(D)\, \mathrm{Tr}\,(\hat{\varrho}(D)\, D(x)) = f(x) \quad \textit{a.e. on } G.$$

Proof. The sum in the left side of (i) is $K_n * \varrho(x)$. Now apply (3.5). □

(3.8) *Examples.* (a) Corollary (3.6) may have some interest even in the classical cases $G = T^a$ $(a = 1, 2, 3, \ldots)$. Identify T with $]-\pi, \pi]$, take $U_n = [-n^{-1}, n^{-1}]$, and $V_n = [-(2n)^{-1}, (2n)^{-1}]$. Computing K_n as in (3.2), we find $K_n(x) = \max\{0, 2\pi n^2(n^{-1} - |x|)\}$. The

Fourier coefficients $\hat{K}_n(\chi)$ are $(\sin(\frac{1}{2}n^{-1}\chi))^2(\frac{1}{2}n^{-1}\chi)^{-2}$ $(\chi=\pm 1, \pm 2, \ldots)$, $\hat{K}_n(0)=1$. Thus we have

$$\lim_{n\to\infty} \varrho * K_n(x) = \lim_{n\to\infty} \{ \sum_{\chi=-\infty}^{\infty} \hat{\varrho}(\chi)(\sin(\tfrac{1}{2}n^{-1}\chi))^2(\tfrac{1}{2}n^{-1}\chi)^{-2} e^{i\chi x} \}$$
$$= \text{the Lebesgue–Radon–Nikodým derivative of } \varrho \text{ a.e. on } T.$$

That is, Fourier-Stieltjes transforms can be inverted pointwise by Riemann's method. For $G=T^a$, we use analogous U_n and V_n (hypercubes), and obtain $K_n^{(a)}(x_1, \ldots, x_a) = \prod_{l=1}^{a} K_n(x_l)$. Thus restricted Riemann summability obtains.

(b) New results also appear for $G=R^a$. First, the functions of (3.4.II) can be taken of class $\mathfrak{C}^\infty$. It suffices to replace φ_n by $\varphi_n * \psi$ where $\psi \in \mathfrak{C}^\infty$, ψ is nonnegative, positive-definite, and of integral 1. Then each $\hat{u}_n$ is in $\mathfrak{C}^\infty$, and so

$$\hat{K}_n = (K_n^0 * u_{k_n})^\wedge = \hat{K}_n^0 \cdot \hat{u}_{k_n} \in \mathfrak{C}^\infty;$$

since K_n^0 has compact support, its transform $\hat{K}_n^0$ is actually entire-analytic.

Take now any $f \in \mathfrak{L}_p(R^a)$, where $1 \leq p < \infty$. It has a distribution-valued Fourier transform $\hat{f}$ (which belongs to $\mathfrak{L}_{p'}(R^a)$ if $1 \leq p \leq 2$, but which is otherwise not necessarily a function at all). Since $K_n \in \mathfrak{L}_1(R^a)$ and $\hat{K}_n \in \mathfrak{C}^\infty$, there is no difficulty in showing that $(f * K_n)^\wedge = \hat{f} \cdot \hat{K}_n$, which is a distribution with compact support. By the uniqueness theorem for Fourier transforms,

$$f * K_n(x) = \langle e^{2\pi i \chi x}, \hat{K}_n(\chi)\hat{f}(\chi) \rangle;$$

the right side here is the restriction to R^a of an entire-analytic function of a complex variable. Theorem (3.5) implies that

$$f(x) = \lim_{n\to\infty} \langle e^{2\pi i \chi x}, \hat{K}_n(\chi)\hat{f}(\chi) \rangle \quad \text{a.e. on } R^a.$$

Naturally, if $f \in \mathfrak{L}_p(R^a)$ with $1 \leq p \leq 2$, then $\hat{K}_n \hat{f}$ is a function in $\mathfrak{L}_p(R^a)$ with compact support, and

$$f(x) = \lim_{n\to\infty} \int_{R^a} \hat{K}_n(\chi)\hat{f}(\chi) e^{2\pi i \chi x} d\chi \quad \text{a.e. on } R^a.$$

§ 4. Some limit theorems for singular convolution operators

For the inversion theorems of § 5, we require a result on pointwise convergence of Lebesgue–Radon–Nikodým derivatives. Our theorems generalise the results of Jerison and Rabson [14], and are adapted from a convergence theorem of Andersen and Jessen [1] and [2]. As Jerison and Rabson point out, these results are also related to the martingale convergence theorem (see [6], Chapter VII). There are a number of important differences, however, since our process is not exactly a martingale, and since we also deal with singular measures. It is therefore necessary to give the proofs in full. The case of singular measures has been dealt with by Boclé [3], Chapter I, but only for mean convergence and convergence in measure, with which we are not concerned.

(4.1) Let S be a set, $\mathfrak{M}$ a σ-algebra of subsets of S, and μ and η countably additive, nonnegative, extended real-valued measures on $\mathfrak{M}$. We will suppose that μ is σ-finite and that η is actually finite (this last condition can be relaxed, but for our purposes finiteness of η is the weakest reasonable restriction). It is classical that η admits a unique decomposition

(i) $$\eta = h\mu + \sigma,$$

where σ is nonnegative and singular with respect to μ, h is an $\mathfrak{M}$-measurable, nonnegative function, and $\int_S h\,d\mu$ is finite. The measures σ and $h\mu$ are carried by complementary sets, say B and B', respectively. The set B has μ-measure 0, and we can define $h(x)$ as $+\infty$ on B without disturbing the validity of (i). A function h for which (i) holds and for which $h(B) \subset \{+\infty\}$ will be termed a *Lebesgue-Radon-Nikodým derivative of η with respect to μ* (more briefly, an LRN *derivative of η with respect to μ*).

(4.2) THEOREM. *Let S, $\mathfrak{M}$, μ, and η be as in* (4.1). *An $\mathfrak{M}$-measurable, nonnegative, extended real-valued function h on S is an* LRN *derivative of η with respect to μ if and only if the following conditions obtain. For every positive number α, let*

$$D = \{x \in S : h(x) \leqslant \alpha\} \quad \text{and} \quad E = \{x \in S : h(x) \geqslant \alpha\}.$$

Then for all $A \in \mathfrak{M}$, the inequalities

(i) $$\eta(D \cap A) \leqslant \alpha\mu(D \cap A)$$

and

(ii) $$\eta(E \cap A) \geqslant \alpha\mu(E \cap A)$$

hold.

Proof. Suppose that h is an LRN derivative of η with respect to μ. Then we have

$$\eta(D \cap A) = \int_{D \cap} h\,d\mu + \sigma(D \cap A) \leqslant \int_{D \cap A} \alpha\,d\mu + 0 = \alpha\mu(D \cap A).$$

This is just (i); (ii) is proved similarly.

To prove the converse,[1] suppose that the decomposition (4.1.i) of η is $\eta = h_0\mu + \sigma_0$, where $\sigma_0(B_0') = 0$ and $\mu(B_0) = 0$. If h and h_0 are not equal μ-almost everywhere, then there

(1) This proof of the converse was kindly suggested by the referee.

are a subset F of B_0' and real numbers α' and α, $0<\alpha'<\alpha$, such that $0<\mu(F)<\infty$ and $h(x)\geqslant\alpha>\alpha'\geqslant h_0(x)$ for all $x\in F$, or there is an F such that $h_0(x)\geqslant\alpha>\alpha'\geqslant h(x)$ for all $x\in F$. In the first case, condition (ii) implies that

$$\eta(F)\geqslant\alpha\mu(F)>\alpha'\mu(F)\geqslant\int_F h_0\,d\mu=\eta(F),$$

a contradiction. The second case is likewise impossible in view of (i), and so $h=h_0$ μ-a.e. on S. For $\alpha>0$ and $D=\{x\in S:h(x)\leqslant\alpha\}$, (i) shows that

$$\sigma_0(D\cap B_0)=\eta(D\cap B_0)\leqslant\alpha\mu(D\cap B_0)=0.$$

Hence $\sigma_0(\{x\in S:h(x)<\infty\})=0$, and the uniqueness of (4.1.i) shows that $h(B_0)\subset\{+\infty\}$. □

(4.3) THEOREM. *Let G be a locally compact group. Let $(H_n)_{n=1}^{\infty}$ be a descending sequence of compact subgroups of G, with intersection H_ω. Let μ_n be normalised Haar measure on H_n ($n=1,2,3,\ldots,\omega$). Let ϱ be any measure in $\mathbf{M}^+(G)$. For $n=1,2,3,\ldots,\omega$,* write

(i) $$\varrho*\mu_n=h_n\lambda+\sigma_n,$$

where h_n is measurable for the σ-algebra $\mathfrak{B}_n$ of all Borel sets of the form AH_n and σ_n is defined on $\mathfrak{B}_n$ and is singular with respect to λ. Let

(ii) $$\underline{h}=\varliminf_{n\to\infty}h_n,\quad \bar{h}=\varlimsup_{n\to\infty}h_n.$$

Then the equalities

(iii) $$\underline{h}(x)=\bar{h}(x)=h_\omega(x)$$

hold for almost all $x\in G$.

Proof. Suppose that $r,s\in\{1,2,3,\ldots,\omega\}$ and that $r\leqslant s$. Let A be a Borel set such that $A=AH_r$. For $x\in G$ and $y\in H_s$, it is clear that $xy\in AH_r$ if and only if $x\in AH_r$. Therefore we have

$$\varrho*\mu_s(AH_r)=\int_G\int_{H_s}\xi_{AH_r}(xy)\,d\mu_s(y)\,d\varrho(x)=\int_G\xi_{AH_r}(x)\,d\varrho(x)=\varrho(AH_r).$$

In particular, for $s\geqslant r$ and $s'\geqslant r$, we have

$$\varrho*\mu_s(AH_r)=\varrho*\mu_s'(AH_r)=\varrho(AH_r). \tag{1}$$

Now let α be a positive real number, and let

$$D=\{x\in G:\underline{h}(x)\leqslant\alpha\}.$$

Let $(\alpha_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of real numbers with limit α. For every positive integer n, let

$$D_n=\{x\in G:\inf\{h_{n+1}(x),\ h_{n+2}(x),\ldots\}<\alpha_n\}.$$

Let
$$D_{n,1}=\{x\in G: h_{n+1}(x)<\alpha_n\}$$
and let
$$D_{n,p}=\{x\in G:\min\{h_{n+1}(x),\ldots,h_{n+p-1}(x)\}\geqslant\alpha_n \text{ and } h_{n+p}(x)<\alpha_n\},$$

for $p=2, 3, 4, \ldots$. It is clear that $D=\bigcap_{n=1}^{\infty} D_n$, that $D_n\supset D_{n+1}$, that $\bigcup_{p=1}^{\infty} D_{n,p}=D_n$, and that the sets $D_{n,p}$ are pairwise disjoint.

Now consider any Borel set of the form AH_s, where s is a positive integer. Since the functions h_r are by their construction constant on each left coset of H_r, the set $D_{n,p}\cap A$ is the union of left cosets of H_{n+p} if $n+1\geqslant s$, which we now suppose. From (1) we infer that

$$\varrho * \mu_\omega(D_n\cap A)=\sum_{p=1}^{\infty}\varrho*\mu_\omega(D_{n,p}\cap A)=\sum_{p=1}^{\infty}\varrho*\mu_{n+p}(D_{n,p}\cap A). \tag{2}$$

Since $D_{n,p}\cap A\in\mathcal{B}_{n+p}$, and since h_{n+p} is an LRN derivative of $\varrho*\mu_{n+p}$ with respect to λ on $\mathcal{B}_{n+p}$, (2) and (4.2) imply that

$$\varrho*\mu_\omega(D_n\cap A)\leqslant\sum_{p=1}^{\infty}\alpha_n\lambda(D_{n,p}\cap A)=\alpha_n\lambda(D_n\cap A). \tag{3}$$

Taking the limit as $n\to\infty$ on both sides of (3), we obtain

$$\varrho*\mu_\omega(D\cap A)\leqslant\alpha\lambda(D\cap A). \tag{4}$$

Next let $E=\{x\in G:\bar{h}(x)\geqslant\alpha\}$. The argument of the two preceding paragraphs can be repeated with obvious changes to show that

$$\varrho*\mu_\omega(E\cap A)\geqslant\alpha\lambda(E\cap A) \tag{5}$$

for Borel sets $A=AH_s$ $(s=1, 2, 3, \ldots)$.

If A is a Borel set, if $A=AH_s$ $(s=1, 2, 3, \ldots)$ and $\underline{h}(x)\geqslant\alpha$ for $x\in A$, then it is clear that $A=E\cap A$ and so (5) holds. Similarly, if $\bar{h}(x)\leqslant\alpha$ for $x\in A$, then $A=D\cap A$ and (4) holds. To apply (4.2), consider any Borel set $A=AH_\omega$. Then we have

$$\begin{aligned}\varrho*\mu_\omega(D\cap(AH_\omega))&=\sup\{\varrho*\mu_\omega(F): F \text{ is compact and } F\subset D\cap(AH_\omega)\}\\ &=\sup\{\varrho*\mu_\omega(FH_\omega): F \text{ is compact and } FH_\omega\subset D\cap(AH_\omega)\}.\end{aligned} \tag{6}$$

It is easy to see that
$$FH_\omega=\bigcap\{FH_n: n=1, 2, 3, \ldots\} \tag{7}$$

if F is compact. For an arbitrary $\varepsilon>0$, choose a compact F as in (6) such that

$$\varrho*\mu_\omega(D\cap(AH_\omega))-\varepsilon<\varrho*\mu_\omega(FH_\omega). \tag{8}$$

Then (4) implies that

$$\begin{aligned}\varrho * \mu_\omega(FH_\omega) &= \varrho * \mu_\omega(D \cap (FH_\omega)) = \lim_{n\to\infty} \varrho * \mu_\omega(D \cap (FH_n)) \\ &\leqslant \alpha \lim_{n\to\infty} \lambda(D \cap (FH_n)) = \alpha\lambda(D \cap (FH_\omega)) \leqslant \alpha\lambda(D \cap (AH_\omega)).\end{aligned} \tag{9}$$

Relations (8) and (9) imply that

$$\varrho * \mu_\omega(D \cap (AH_\omega)) \leqslant \alpha\lambda(D \cap (AH_\omega)). \tag{10}$$

In the same way we apply (7) and (5) to show that

$$\varrho * \mu_\omega(E \cap (AH_\omega)) \geqslant \alpha\lambda(E \cap (AH_\omega)). \tag{11}$$

From (10) and (11), the relations (4.2.i) and (4.2.ii) follow at once, for both of the functions $\underline{h}$ and $\bar{h}$. Theorem (4.2) shows that $\underline{h}$ and $\bar{h}$ are LRN derivatives of $\varrho * \mu_\omega$ with respect to λ, both of these measures being restricted to the σ-algebra $\mathcal{B}_\omega$. (Note that the finite measures $\varrho * \mu_n$ $(n = 1, 2, 3, \ldots)$ are carried by a single σ-compact open and closed subset S of G. Thus for the purpose of applying (4.2) we can restrict our attention to the set S, on which λ is σ-finite.)

Since the decomposition (4.1.i) is unique, we have therefore proved that

$$\int_A \underline{h}\,d\lambda = \int_A \bar{h}\,d\lambda = \int_A h_\omega\,d\lambda \tag{12}$$

for all sets $A \in \mathcal{B}_\omega$. (We define $\underline{h}(x)$, $\bar{h}(x)$, and $h_\omega(x)$ as 0 on S'.) If $\bar{h}(x) \neq \underline{h}(x)$ on a set A not of λ-measure 0, then (12) would fail for a set A in $\mathcal{B}_\omega$, since $\underline{h}$ and $\bar{h}$ are $\mathcal{B}_\omega$-measurable. Similarly we see that $\bar{h}(x) = h_\omega(x)$ for λ-almost all $x \in G$. □

Some consequences of (4.3) will be used in § 5.

(4.4) Theorem. *Let G, H_n, and μ_n be as in (4.3). Let f be a λ-integrable, Borel measurable function on G. Then*

(i) $$\lim_{n\to\infty} f * \mu_n(x) = f * \mu_\omega(x)$$

for λ-almost all $x \in G$.

Proof. Recall ([12], (20.9.ii)) that the function $f * \mu_n$ is defined by

$$f * \mu_n(x) = \int_{H_n} f(xy^{-1})\,\Delta(y^{-1})\,d\mu_n(y) = \int_{H_n} f(xy^{-1})\,d\mu_n(y),$$

and is an LRN derivative of the λ-absolutely continuous measure $(f\lambda) * \mu_n$. Then (4.3) shows that the functions

$$\underline{f}(x) = \varliminf_{n\to\infty} f * \mu_n(x) \quad \text{and} \quad \bar{f}(x) = \varlimsup_{n\to\infty} f * \mu_n(x)$$

are equal λ-a.e. to the function $f * \mu_\omega(x)$. □

(4.5) COROLLARY. *Let G, H_n, and μ_n be as in* (4.3). *Suppose that $H_\omega = \{e\}$. Then*

$$\lim_{n\to\infty} f * \mu_n(x) = f(x) \quad \lambda\text{-a.e. in } G.$$

Proof. This follows from (4.4) and the fact that μ_ω in the present case is the unit measure ε_e, for which $f * \varepsilon_e = f$. □

(4.6) THEOREM. *Let G, H_n, and μ_n be as in* (4.5). *Let ϱ be a measure in $\mathbf{M}^+(G)$ such that ϱ has a λ-absolutely continuous part equal to zero. Let h_n be an* LRN *derivative of $\varrho * \mu_n$ with respect to λ. Then*

$$\lim_{n\to\infty} h_n(x) = 0 \quad \lambda\text{-a.e. in } G.$$

Proof. As in (4.5), we have $\varrho * \mu_\omega = \varrho$, and the function 0 on G is an LRN derivative of ϱ with respect to λ. Now apply (4.3). □

(4.7) *Note.* All of (4.3)–(4.6) remain valid if the convolutions $\varrho * \mu_n$ are all replaced by $\mu_n * \varrho$ and $f * \mu_n$ by $\mu_n * f$. The Borel sets AH_n need only be replaced by $H_n A$ in the proof of (4.3).

(4.8) *Example.* Let G be a locally compact, 0-dimensional Abelian group. Let $\mathbf{X}$ denote as usual the character group of G. Let $(H_n)_{n=1}^\infty$ be any decreasing sequence of compact open subgroups of G, and as above let $H_\omega = \bigcap_{n=1}^\infty H_n$. Normalised Haar measure μ_n on H_n is $\lambda(H_n)^{-1}\xi_{H_n}\lambda$ for $n < \omega$, and $\hat{\mu}_n$ is the characteristic function of the annihilator $\mathbf{Y}_n$ of H_n in $\mathbf{X}$. Define Haar measure λ on G so that $\lambda(H_1) = 1$ and Haar measure θ on $\mathbf{X}$ so that $\theta(\mathbf{Y}_1) = 1$.

Now let ϱ be any measure in $\mathbf{M}(G)$. Then it is easy to see that

$$\text{(i)} \qquad \int_{\mathbf{Y}_n} \hat{\varrho}(\chi)\,\chi(x)\,d\theta(\chi) = \int_{\mathbf{X}} \hat{\varrho}(\chi)\,\hat{\mu}_n(\chi)\,\chi(x)\,d\theta(\chi) = \varrho * \left(\frac{1}{\lambda(H_n)}\xi_{H_n}\right)(x),$$

for all $x \in G$. The function $\varrho * \left(\frac{1}{\lambda(H_n)}\xi_{H_n}\right)(x)$

is an LRN derivative of the λ-absolutely continuous measure $\varrho * \mu_n$. Thus Theorem (4.3) and (i) show that

$$\text{(ii)} \qquad \lim_{n\to\infty} \int_{\mathbf{Y}_n} \hat{\varrho}(\chi)\,\chi(x)\,d\theta(\chi) = h_\omega(x)$$

for λ-almost all $x \in G$, where h_ω is an LRN derivative of $\varrho * \mu_\omega$ with respect to λ.

§ 5. Pointwise summability methods for arbitrary locally compact Abelian groups and compact groups

The main results of this section are (5.7) and (5.11), which give *iterated* limit processes for recapturing $f \in \mathfrak{L}_1(G)$ from $\hat{f}$. We do not know if a single limit process exists for every locally compact Abelian group or compact group.

We begin with some needed facts about measures on groups and quotient groups.

(5.1) THEOREM. *Let G be a locally compact group and H a compact normal subgroup of G. Let λ be a left Haar measure on G and ν a left Haar measure on the group G/H. Let τ be the natural mapping of G onto G/H: $\tau(x) = xH \in G/H$. If $\varphi \in \mathfrak{L}_1(G/H)$, the function $\varphi \circ \tau$ is necessarily λ-measurable. For given λ, the measure ν can be chosen so that*

$$\text{(i)} \qquad \int_{G/H} \varphi(xH)\, d\nu(xH) = \int_G \varphi \circ \tau(x)\, d\lambda(x) \quad \textit{for all} \quad \varphi \in \mathfrak{L}_1(G/H).$$

If G is compact and $\lambda(G) = 1$, then $\nu(G/H) = 1$.

Proof. Consider first a function $\varphi \in \mathfrak{C}_{00}(G/H)$. The function φ vanishes outside of a compact subset $\{xH : x \in F\}$ of G/H. By [12], (5.24.b), we may suppose that F is compact in G. Thus $\varphi \circ \tau$ vanishes outside of the compact subset FH of G and is in $\mathfrak{C}_{00}(G)$. Since $\varphi \circ \tau = 0$ only if $\varphi = 0$, the functional

$$\varphi \rightarrow \int_G \varphi \circ \tau(x)\, d\lambda(x) \tag{1}$$

is strictly positive on $\mathfrak{C}_{00}(G/H)$. For $a \in G$, we have $({}_{aH}\varphi) \circ \tau(x) = {}_a(\varphi \circ \tau)(x)$, and so the functional (1) is left invariant on $\mathfrak{C}_{00}(G/H)$. That is, (1) is a left Haar integral on $\mathfrak{C}_{00}(G/H)$, which is to say that we can choose ν so that (i) holds for $\varphi \in \mathfrak{C}_{00}(G/H)$.

Now consider a compact subset B of G/H that is the intersection of a countable number of open sets. It is easy to see that there is a decreasing sequence $(\varphi_n)_{n=1}^{\infty}$ of functions in $\mathfrak{C}_{00}(G/H)$ such that $\lim_{n\to\infty} \varphi_n = \xi_B$. We then have

$$\nu(B) = \lim_{n\to\infty} \int_{G/H} \varphi_n\, d\nu = \lim_{n\to\infty} \int_G \varphi_n \circ \tau\, d\lambda = \lambda(\tau^{-1}(B)). \tag{2}$$

Let $\mathcal{A}$ be the family of all Borel subsets A of G/H for which

$$\nu(A) = \lambda(\tau^{-1}(A)). \tag{3}$$

The family $\mathcal{A}$ is obviously closed under the formation of countable unions of increasing sequences and of countable pairwise disjoint unions. Also, if A_1, A_2 are in $\mathcal{A}$, if $A_1 \subset A_2$, and if $\nu(A_1)$ is finite, the set $A_2 \cap A_1'$ is in $\mathcal{A}$. Since G/H contains a σ-compact open subgroup, it follows readily from this and (2) that $\mathcal{A}$ contains all Baire sets in G/H. (Baire sets are defined as in [12], (11.1).)

Let P be a σ-bounded subset of G/H such that $\nu(P)=0$. The Kakutani-Kodaira theorem (see [12], (19.30)) implies that there is a Baire set Q such that $Q \supset P$ and $\nu(Q)=0$. For this Q, we have

$$0=\nu(Q)=\lambda(\tau^{-1}(Q)) \geqslant \lambda(\tau^{-1}(P)).$$

That is,

$$\lambda(\tau^{-1}(P))=\nu(P)=0. \tag{4}$$

The Kakutani-Kodaira theorem also shows that for every σ-bounded ν-measurable set A there is a Baire set $B \supset A$ such that $\nu(B \cap A')=0$. The relation (3) follows for the set A. In particular, (3) holds for all compact sets, therefore for all open sets, all sets of ν-measure 0, and for all ν-measurable sets of finite ν-measure. From this (i) follows readily. □

(5.2) The case of (5.1) in which $\lambda(H)$ is positive deserves special comment. In this case (5.1.4) implies that only the void set in G/H has ν-measure zero. That is, G/H is discrete, and so H is open. This fact also follows at once from the identity $\xi_H * \xi_H = \xi_H$ and the fact that $\xi_H * \xi_H$ is continuous. In fact, if a subgroup of G contains a λ-measurable subset of finite positive measure, then the subgroup is open. Let R_d be the group R with the discrete topology. The subgroup $\{0\} \times R_d$ in $R \times R_d$ is an example of a closed, nonopen, locally λ-null, non λ-null subgroup.

(5.3) THEOREM. *Let G, H, λ, ν, and τ be as in* (5.1). *Let f be a Borel measurable function in $\mathfrak{L}_1(G)$ that is constant on cosets of H: $f(ax)=f(ay)$ if $a \in G$ and $x, y \in H$. Let $f^\dagger$ be the function on G/H such that $f^\dagger(aH)=f(a)$ for all $a \in G$, so that $f^\dagger \circ \tau = f$. Then $f^\dagger$ is in $\mathfrak{L}_1(G/H)$ and*

(i) $$\int_{G/H} f^\dagger \, d\nu = \int_G f d\lambda.$$

This theorem is proved from (5.1) by routine arguments. We omit the proof. The following result is also easy to establish and is presented without proof.

(5.4) THEOREM. *Let G, H, λ, ν, and τ be as in* (5.1). *Let ϱ be a measure in $\mathbf{M}(G)$. Consider the linear functional*

(i) $$\varphi \to \int_G (\varphi \circ \tau)\, d\varrho,$$

defined for $\varphi \in \mathfrak{C}_{00}(G/H)$. This functional is a bounded linear functional on $\mathfrak{C}_{00}(G/H)$, and so there is a (unique) measure $\varrho^\dagger$ in $\mathbf{M}(G/H)$ such that

(ii) $$\int_G (\varphi \circ \tau)\, d\varrho = \int_{G/H} \varphi d\varrho^\dagger \quad \textit{for} \quad \varphi \in \mathfrak{C}_0(G/H).$$

For every Borel measurable function g on G/H that is in $\mathfrak{L}_1(G/H, \varrho^\dagger)$, *we have*

$$\text{(iii)} \qquad \int_{G/H} g\,d\varrho^\dagger = \int_G (g\circ\tau)\,d\varrho.$$

Theorems (5.1), (5.3), and (5.4) appear in a modified, and more general, form in [4], p. 75, Théorème 1 and pp. 81–82, Exercice 1. See also [22] and [23].

To prove our theorems on pointwise summability for Fourier and Fourier–Stieltjes transforms, we also need some group-theoretic facts.

(5.5) Consider an arbitrary locally compact Abelian group G. According to a well-known structure theorem (see, for example, [12], (24.30)), G is topologically isomorphic with $R^a\times G_0$, where a is a nonnegative integer and G_0 is a locally compact Abelian group containing a compact open subgroup J_0.

Let $\mathbf{X}$ denote the character group of G. Then $\mathbf{X}$ has the form $R^a\times\mathbf{X}_0$, where $\mathbf{X}_0$ is the character group of G_0. Let $\mathbf{\Lambda}_0$ be the annihilator in $\mathbf{X}_0$ of the subgroup J_0 of G_0. It is easy to see that $\mathbf{\Lambda}_0$ is a compact open subgroup of $\mathbf{X}_0$.

For inverting Fourier transforms, it is convenient to make specific choices of Haar measure λ on G and Haar measure θ on $\mathbf{X}$. There is one and only one Haar measure λ_0 on G_0 for which $\lambda_0(J_0)=1$, and we take this measure λ_0 on the factor G_0. Let λ_1 denote the measure on R^a that is $(2\pi)^{-\frac{1}{2}a}$ times ordinary a-dimensional Lebesgue measure. Haar measure λ on G is then defined as the product measure $\lambda_1\times\lambda_0$. On $\mathbf{X}$ we construct the measure θ as follows. Let θ_0 be the Haar measure on $\mathbf{X}_0$ for which $\mathbf{\Lambda}_0$ has measure 1. Then θ is defined as $\lambda_1\times\theta_0$. It is known [11], and is easy to verify, that this choice of λ and θ produces equality in Plancherel's theorem, and so is appropriate for pointwise summability processes on G and $\mathbf{X}$. In the sequel, we will always take the above λ and θ, the subgroup J_0 being chosen once and for all.

(5.6) THEOREM. *The notation is as in* (5.5). *Suppose that there exists a compact subgroup* $\{0\}\times H$ *of* $\{0\}\times G_0$ *in* $G=R^a\times G_0$ *such that* G_0/H *is first countable. Then there is a decreasing sequence* $(H_n)_{n=1}^\infty$ *of compact subgroups of* $\{0\}\times G_0$ *such that* $\bigcap_{n=1}^\infty H_n=\{0\}\times H$ *and such that the group* G/H_n *contains an open subgroup of the form* $R^a\times T^{b_n}\times F_n$. *Here* $(b_n)_{n=1}^\infty$ *is a nondecreasing sequence of nonnegative integers, and* F_n *is a finite Abelian group, for* $n=1, 2, 3, \ldots$.

Proof. Consider first the group G/H, which obviously is topologically isomorphic with $R^a\times(G_0/H)$. The subgroup $\{0\}\times(G_0/H)$ of G/H contains the compact open subgroup $\{0\}\times(J_0/H)$, which is first countable because G/H is first countable. Let $\mathbf{Y}$ be the character group of $\{0\}\times(J_0/H)$. Since $\{0\}\times(J_0/H)$ is first countable, $\mathbf{Y}$ is a countable discrete group.

Suppose first that $\mathbf{Y}$ is finitely generated. Then $\mathbf{Y}$ has the form $Z^b \times F$, where b is a nonnegative integer and F is a finite Abelian group. The group $\{0\} \times (J_0/H)$ thus has the form $T^b \times F$, and so G/H has the form $R^a \times T^b \times F$. In this case we take all of the groups H_n equal to $\{0\} \times H$.

Suppose next that $\mathbf{Y}$ is not finitely generated. In this case, it is simple to verify that $\mathbf{Y}$ is the union of an increasing sequence $(\mathbf{\Delta}_n)_{n=1}^\infty$ of finitely generated subgroups. Then $\mathbf{\Delta}_n$ has the form $Z^{b_n} \times F_n$ for $n = 1, 2, 3, \ldots$. It is clear that $(b_n)_{n=1}^\infty$ is a nondecreasing sequence of nonnegative integers. Let $\{0\} \times M_n$ be the annihilator of $\mathbf{\Delta}_n$ in $\{0\} \times (J_0/H)$. The quotient group $(\{0\} \times (J_0/H))/(\{0\} \times M_n)$ is the character group of $\mathbf{\Delta}_n$ and so has the form $T^{b_n} \times F_n$. We have thus produced a continuous open homomorphism of G onto $R^a \times T^{b_n} \times F_n$, which is indicated schematically as follows:

$$G = R^a \times G_0 \to (R^a \times G_0)/(\{0\} \times H) = R^a \times (G_0/H) \to R^a \times ((G_0/H)/M_n).$$

We denote this homomorphism by φ_n, and we define H_n as the kernel of the homomorphism φ_n. The group $R^a \times ((G_0/H)/M_n)$ contains $R^a \times ((J_0/H)/M_n)$ as an open subgroup, and this last group has the form $R^a \times T^{b_n} \times F_n$. Since M_n is a compact subgroup of J_0/H, it is easy to see from [12], (5.24.b) that H_n is a compact open subgroup of $\{0\} \times G_0$. Our construction also makes it clear that $(H_n)_{n=1}^\infty$ is a decreasing sequence of subgroups. It remains only to prove that $\bigcap_{n=1}^\infty H_n = \{0\} \times H$. This follows at once from the fact that $\bigcap_{n=1}^\infty \{0\} \times M_n$ is the group identity in $\{0\} \times (J_0/H)$, which in turn is a consequence of the equality $\bigcup_{n=1}^\infty \mathbf{\Delta}_n = \mathbf{Y}$. □

We can now state and prove our main theorems on pointwise summability.

(5.7) THEOREM. *Let G be a locally compact Abelian group, with character group $\mathbf{X}$. Let $\mathbf{Y}$ be any σ-compact open subgroup of $\mathbf{X}$. There is a double sequence $(K_{m,n})_{m=1,}^\infty{}_{n=1}^\infty$ of functions on G with the following properties.*

(i) *Each $K_{m,n}$ is nonnegative, uniformly continuous, positive-definite, and in $\mathfrak{L}_1(G)$.*

(ii) *Each Fourier transform $\hat{K}_{m,n}$ is nonnegative, vanishes outside of $\mathbf{Y}$, and has compact support.*

(iii) *For every $f \in \mathfrak{L}_1(G)$ such that $\hat{f}$ vanishes outside of $\mathbf{Y}$, we have*

$$\lim_{n\to\infty} \left\{ \lim_{m\to\infty} \int_{\mathbf{X}} \hat{f}(\chi)\, \hat{K}_{m,n}(\chi)\, \chi(x)\, d\theta(\chi) \right\} = f(x)$$

for almost all $x \in G$.

Proof. We use the notation of (5.5). It is a routine matter to verify that $\mathbf{Y}$ is contained in a subgroup of $\mathbf{X}$ of the form $R^a \times \mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is a subgroup of $\mathbf{X}_0$ that is the union of a countable number of cosets of $\mathbf{\Lambda}_0$. We may thus suppose that $\mathbf{Y} = R^a \times \mathbf{\Sigma}$. Now consider the annihilator in G of the subgroup $\mathbf{Y}$. This subgroup of G has the form $\{0\}\times H$, where H is a compact subgroup of J_0. The quotient group $(R^a\times G_0)/(\{0\}\times H)$ is the character group of $\mathbf{Y}$; since $\mathbf{Y}$ is σ-compact, $(R^a\times G_0)/(\{0\}\times H)$ is first countable (see [12], (24.48)).

Thus we can apply (5.6) to G and its subgroup $\{0\}\times H$. We now write H_ω for $\{0\}\times H$. Let $\mathbf{Y}_n$ be the annihilator in $\mathbf{X}$ of H_n $(n=1, 2, 3, ..., \omega)$. Then each $\mathbf{Y}_n$ has the form $R^a\times\mathbf{\Sigma}_n$ where $\mathbf{\Sigma}_n$ is a countable union of cosets of $\mathbf{\Lambda}_0$. Also we have

$$\mathbf{Y}_1\subset\mathbf{Y}_2\subset\ldots\subset\mathbf{Y}_n\subset\ldots,\quad\text{and}\quad\bigcup_{n=1}^{\infty}\mathbf{Y}_n=\mathbf{Y}=\mathbf{Y}_\omega.$$

Note also the important fact that G/H_n contains an open subgroup of the form $R^a\times T^{b_n}$.

Let μ_n be normalised Haar measure on H_n, and regard μ_n as a measure in $\mathbf{M}(G)$ $(n=1, 2, 3, ..., \omega)$. It is clear that $\hat{\mu}_n=\xi_{\mathbf{Y}_n}$. Thus if $f\in\mathfrak{L}_1(G)$ and $\hat{f}$ vanishes on $\mathbf{Y}'_\omega$, then

$$\hat{f}=\hat{f}\xi_{\mathbf{Y}_\omega}=\hat{f}\hat{\mu}_\omega=(f*\mu_\omega)^\wedge.$$

The uniqueness theorem for Fourier transforms implies that $f=f*\mu_\omega$ in $\mathfrak{L}_1(G)$, i.e., $f(x)=f*\mu_\omega(x)$ for almost all $x\in G$.

Consider next the group G/H_n, for $n=1, 2, 3, \ldots$. Let ν_n be the Haar measure on G/H_n defined in (5.1). Since G/H_n contains an open subgroup of the form $R^a\times T^{b_n}$, we can apply Theorem (3.5) to G/H_n and assert the existence of a sequence $(P_{m,n})_{m=1}^{\infty}$ of functions on G/H_n with the following properties.

(1) Each $P_{m,n}$ is nonnegative, uniformly continuous, positive-definite, and in $\mathfrak{L}_1(G/H_n)$.

(2) Each Fourier transform $\hat{P}_{m,n}$ (which is defined on the subgroup $\mathbf{Y}_n$ of $\mathbf{X}$) is nonnegative and has compact support in $\mathbf{Y}_n$.

(3) For every $g\in\mathfrak{L}_1(G/H_n)$, we have

$$\lim_{m\to\infty}\int_{\mathbf{Y}_n}\hat{P}_{m,n}(\chi)\,\hat{g}(\chi)\,\chi(xH_n)\,d\theta(\chi)=\lim_{m\to\infty}\int_{G/H_n}P_{m,n}(xy^{-1}H_n)\,g(yH_n)\,d\nu_n(yH_n)=g(xH_n)$$

for all $xH_n\in G/H_n$ except perhaps those in a set A of ν_n-measure 0.

With regard to (3), to the appeal to Theorem (3.5) should be added the remark that the equalities

$$\int_{\mathbf{Y}_n}\hat{P}_{m,n}(\chi)\,\hat{g}(\chi)\,\chi(xH_n)\,d\theta(\chi)=\int_{G/H_n}P_{m,n}(xy^{-1}H_n)\,g(yH_n)\,d\nu_n(yH_n)\tag{4}$$

for $m,n=1, 2, 3, \ldots$ depend upon our choice of λ and θ and upon the definition of ν_n in (5.1). Let τ_n be the natural mapping of G onto G/H_n. Then

$$\int_{\mathbf{Y}_n} \hat{P}_{m,n}(\chi)\,\hat{g}(\chi)\,\chi(xH_n)\,d\theta(\chi)$$

$$= \int_{\mathbf{Y}_n} \int_{G/H_n} P_{m,n} \ast g(uH_n)\,\overline{\chi(uH_n)}\,d\nu_n\,(uH_n)\,\chi(x)\,d\theta\,(\chi)$$

$$= \int_{\mathbf{Y}_n} \int_G (((P_{m,n} \ast g)\,\bar{\chi}) \circ \tau_n)\,(u)\,d\lambda(u)\,\chi(x)\,d\theta(\chi)$$

$$= \int_G \int_{\mathbf{Y}_n} ((P_{m,n} \ast g) \circ \tau_n)^\wedge\,(\chi)\,\chi(x)\,d\theta(\chi)\,d\lambda(x). \tag{5}$$

The inner integral in the last expression of (5) is equal to $(P_{m,n} \ast g) \circ \tau_n(x)$ (convolution in G/H_n) because $((P_{m,n} \ast g) \circ \tau_n)^\wedge = \hat{P}_{m,n}\hat{g}$ is absolutely integrable on $\mathbf{Y}_n$, and λ and θ have been chosen so that pointwise inversion is valid for functions in $\mathfrak{L}_1(G)$ whose Fourier transforms are absolutely integrable. Thus the left side of (4) is equal to

$$\int_G (P_{m,n} \ast g) \circ \tau_n(x)\,d\lambda(x),$$

and this integral is, in view of (5.1), equal to

$$\int_{G/H_n} (P_{m,n} \ast g)\,d\nu_n.$$

This establishes (4). We define $K_{m,n}$ as the function $P_{m,n} \circ \tau_n$ on G, and claim that the functions $K_{m,n}$ satisfy (i)–(iii). Assertion (i) follows at once from (1).

To prove (ii), consider first any character $\chi \in \mathbf{Y}_n$. Both $K_{m,n}$ and $\bar{\chi}$ are constant on the cosets of H_n, and we use (5.3) to write

$$\int_G K_{m,n}(x)\,\overline{\chi(x)}\,d\lambda(x) = \int_G P_{m,n} \circ \tau_n(x)\,\overline{\chi(x)}\,d\lambda(x)$$

$$= \int_{G/H_n} P_{m,n}(xH_n)\,\overline{\chi(xH_n)}\,d\nu_n(xH_n) = \hat{P}_{m,n}(\chi).$$

Suppose next that $\chi \in \mathbf{X} \cap \mathbf{Y}_n'$, i.e., that $\chi(a) \neq 1$ for some $a \in H_n$. Then we have

$$\int_G K_{m,n}(x)\overline{\chi(x)}\,d\lambda(x) = \int_G K_{m,n}(ax)\,\overline{\chi(ax)}\,d\lambda(x) = \overline{\chi(a)} \int_G K_{m,n}(x)\,\overline{\chi(x)}\,d\lambda(x),$$

and so $\hat{K}_{m,n}(\chi) = 0$. That is, $\hat{K}_{m,n}$ is equal to $\hat{P}_{m,n}$ on $\mathbf{Y}_n$ and vanishes elsewhere on $\mathbf{X}$. This proves (ii), in view of (2) and the fact that Haar measure on $\mathbf{Y}_n$ is the restriction to $\mathbf{Y}_n$ of Haar measure on $\mathbf{Y}$.

The last paragraph also shows that $\hat{K}_{m,n}\cdot\hat{\mu}_n=\hat{K}_{m,n}$. The uniqueness theorem for Fourier transforms shows that $K_{m,n}$ and $K_{m,n}*\mu_n$ are equal almost everywhere on G. Since $K_{m,n}$ is uniformly continuous, $K_{m,n}*\mu_n$ is continuous, and so we have $K_{m,n}=K_{m,n}*\mu_n$ everywhere on G.

Next let f be any function in $\mathfrak{L}_1(G)$ ($\hat{f}$ need not vanish on $\mathbf{Y}'$). For an arbitrary $x\in G$, we compute as follows:

$$\int_{\mathbf{X}}\hat{f}(\chi)\hat{K}_{m,n}(\chi)\chi(x)\,d\theta(\chi)=\int_{\mathbf{Y}_n}\hat{f}(\chi)\hat{P}_{m,n}(\chi)\hat{\mu}_n(\chi)\chi(x)\,d\theta(\chi)$$

$$=\int_{G/H_n}(f*\mu_n)(xH_ny^{-1}H_n)P_{m,n}(yH_n)\,d\nu_n(yH_n). \tag{6}$$

(Since the function $f*\mu_n$ is constant on cosets of H_n, the expression $(f*\mu_n)(xH_ny^{-1}H_n)$ has an obvious meaning.) Theorem (5.3) shows that $f*\mu_n$, regarded as a function on G/H_n, is in $\mathfrak{L}_1(G/H_n)$. Accordingly we can combine (3) with (6) to write

$$\lim_{m\to\infty}\int_{\mathbf{X}}\hat{f}(\chi)\hat{K}_{m,n}(\chi)\chi(x)\,d\theta(\chi)=f*\mu_n(xH_n) \tag{7}$$

for all $xH_n\in G/H_n$ except for a set $\{xH_n:x\in A\}$ of ν_n-measure zero.

On the other hand, Theorem (4.4) shows that

$$\lim_{n\to\infty}f*\mu_n(x)=f*\mu_\omega(x) \tag{8}$$

for all $x\in G$ except for a set B such that $\lambda(B)=0$. Theorem (5.1) shows that $\lambda(AH_n)=0$. For all $x\in G\cap B'\cap\bigcap_{n=1}^{\infty}(AH_n)'$, (7) and (8) show that

$$\lim_{n\to\infty}\left\{\lim_{m\to\infty}\int_{\mathbf{X}}\hat{f}(\chi)\hat{K}_{m,n}(\chi)\chi(x)\,d\theta(\chi)\right\}=f*\mu_\omega(x). \tag{9}$$

As already noted, if $\hat{f}$ vanishes on $\mathbf{Y}'$, then $f*\mu_\omega(x)=f(x)$ almost everywhere on G, and so (9) proves (iii). □

Theorem (5.7) can in a certain sense be extended to Fourier-Stieltjes transforms.

(5.8) THEOREM. *All the notation is as in* (5.5)–(5.7). *Let ϱ be a measure in* $\mathbf{M}(G)$ *such that $\varrho*\mu_\omega$ is singular with respect to λ ($\varrho*\mu_\omega$ need not be continuous). Then we have*

(i) $$\lim_{n\to\infty}\lim_{m\to\infty}\int_{\mathbf{X}}\hat{\varrho}(\chi)\hat{K}_{m,n}(\chi)\chi(x)\,d\theta(\chi)=0$$

for almost all $x\in G$.

Proof. We may obviously suppose that ϱ is nonnegative. First write

$$\varrho * \mu_n = h_n \lambda + \sigma_n, \tag{1}$$

as in (4.3.i), where σ_n is defined on the σ-algebra $\mathcal{B}_n$ of (4.3) and h_n is a $\mathcal{B}_n$-measurable function. Thus h_n is constant on cosets of H_n, and the function $h_n^\dagger$ exists, as in (5.3). Let $(h_n\lambda)^\dagger$ be the measure on G/H defined as in (5.4). Then for $g \in \mathfrak{L}_1(G/H, (h_n\lambda))$, (5.4.iii) and (5.3) yield

$$\int_{G/H_n} g d(h_n\lambda)^\dagger = \int_G (g \circ \tau_n) h_n d\lambda = \int_G (g \circ \tau_n)(h_n^\dagger \circ \tau_n) d\lambda$$

$$= \int_G ((gh_n^\dagger) \circ \tau_n) d\lambda = \int_{G/H_n} gh_n^\dagger d\nu_n.$$

That is,

$$(h_n \lambda)^\dagger = h_n^\dagger \nu_n. \tag{2}$$

We now define the measure $\sigma_n^\dagger$ for Borel subsets A of G/H_n such that $\tau_n^{-1}(A)$ is a Borel set of the form BH_n, i.e., for all Borel subsets of G/H_n. For these sets, we write

$$\sigma_n^\dagger(A) = \sigma_n(\tau_n^{-1}(A)). \tag{3}$$

(The measure σ_n is not in general in $\mathbf{M}(G)$, since it is defined only on $\mathcal{B}_n$, a σ-algebra that may be a proper subfamily of the family of all Borel sets. However, (3) is well defined for all Borel sets in G/H, and the identity

$$\int_G (g \circ \tau_n) d\sigma_n = \int_{G/H_n} g d\sigma_n^\dagger \tag{4}$$

for all Borel measurable functions g on G/H_n is a trivial consequence of (3).)

Let B_n be a set in $\mathcal{B}_n$ of λ-measure 0 that carries the λ-singular measure σ_n. By (5.3), the set $\tau_n(B_n)$ has ν_n-measure 0. The measure $\sigma_n^\dagger$ being obviously carried by $\tau_n(B_n)$, we see that $\sigma_n^\dagger$ is ν_n-singular, and so we use (2) and (4) to decompose $(\varrho * \mu_n)^\dagger$ (which is defined exactly as in (5.4)) into

$$(\varrho * \mu_n)^\dagger = (h_n \lambda)^\dagger + \sigma_n^\dagger = h_n^\dagger \nu_n + \sigma_n^\dagger. \tag{5}$$

As in the proof of (5.7), we have:

$$\int_{\mathbf{X}} \hat{\varrho}(\chi) \hat{K}_{m,n}(\chi) \chi(x) d\theta(\chi)$$

$$= \int_{\mathbf{Y}_n} \hat{P}_{m,n}(\chi) \hat{\varrho}(\chi) \hat{\mu}_n(\chi) \chi(xH_n) d\theta(\chi) = (P_{m,n} * (\varrho * \mu_n)^\dagger)(xH_n)$$

$$= \int_{G/H_n} P_{m,n}(xy^{-1}H_n) h_n^\dagger(yH_n) d\nu_n(yH_n) + \int_{G/H_n} P_{m,n}(xy^{-1}H_n) d\sigma_n^\dagger(yH_n). \tag{6}$$

Since $\sigma_n^\dagger$ is ν_n-singular, the last integral in (6) has limit 0 as $m \to \infty$ (see Corollary (3.6)), except for xH_n in a set of ν_n-measure 0. By (3.6), the second to last integral in (6) has limit $h_n^\dagger(xH_n)$ for ν_n-almost all $xH_n \in G/H_n$. Thus we have

$$\lim_{m\to\infty} \int_{\mathbf{X}} \hat{\varrho}(\chi)\, \hat{K}_{m,n}(\chi)\, \chi(x)\, d\theta(\chi) = h_n^\dagger(xH_n) \tag{7}$$

except for a set $A_n \subset G/H_n$ such that $\nu_n(A_n) = 0$. Theorem (4.3) shows that

$$\lim_{n\to\infty} h_n(x) = 0 \tag{8}$$

except for a set $N \subset G$ of λ-measure 0. Since $h_n^\dagger \circ \tau_n = h_n$, we combine (7) and (8) to find that (i) holds for x not in $N \cup (\bigcup_{n=1}^{\infty} \tau_n^{-1}(A_n))$. Since this set has λ-measure 0 (5.3), the present theorem is proved. □

Theorems (5.7) and (5.8) can be combined as follows.

(5.9) THEOREM. *The notation is as in* (5.5)–(5.7). *Let* ϱ *be any measure in* $\mathbf{M}(G)$, *and let* h *be an* LRN *derivative of* $\varrho * \mu_\omega$ *with respect to* λ. *Then we have*

(i) $$\lim_{m\to\infty} \left\{ \lim_{n\to\infty} \int_{\mathbf{X}} \hat{\varrho}(\chi)\, \hat{K}_{m,n}(\chi)\, \chi(x)\, d\theta(\chi) \right\} = h(x)$$

for almost all $x \in G$. *If* $\hat{\varrho}$ *vanishes on* $\mathbf{Y}'$, *then* $\varrho * \mu_\omega = \varrho$ *and* h *is an* LRN *derivative of* ϱ *itself with respect to* λ.

Proof. All of this except for the last statement is immediate from (5.7) and (5.8). If $\hat{\varrho}$ vanishes on $\mathbf{Y}'$, then $\hat{\varrho} = \hat{\varrho}\hat{\mu}_\omega$, and so by the uniqueness theorem for Fourier-Stieltjes transforms, we have $\varrho = \varrho * \mu_\omega$. □

(5.10) *Examples*. (a) Let $\mathfrak{m}$ be an infinite cardinal number, and consider the group $T^{\mathfrak{m}}$, regarded as the group of all complex-valued functions of absolute value 1 defined on a set X of cardinal number $\mathfrak{m}$. The group operation is pointwise multiplication, and a generic neighbourhood of 1 is the set

$$\{x \in T^{\mathfrak{m}} : |x(t_j) - 1| < \varepsilon \text{ for } j = 1, 2, \ldots, m\};$$

here ε is an arbitrary positive real number and $\{t_1, t_2, \ldots, t_m\}$ is an arbitrary finite subset of X. The character group $Z^{\mathfrak{m}*}$ of $T^{\mathfrak{m}}$ is identified with the group of all integer-valued functions y on X such that $y(t) = 0$ except on a (y-dependent) finite subset of X. The value of y at $x \in T^{\mathfrak{m}}$ is $\prod_{t\in X} x(t)^{y(t)}$, the product actually being finite. The σ-compact subgroups $\mathbf{Y}$ and $\mathbf{Y}_n$ appearing in Theorem (5.6) are constructed as follows. Let $Q = \{t_1, t_2, \ldots, t_n, \ldots\}$ be a countably infinite subset of X (we will take $Q = X$ if $\mathfrak{m} = \aleph_0$). Let $Q_n = \{t_1, t_2, \ldots, t_n\}$ for

$n=1, 2, 3, \ldots$. Let $\mathbf{Y}$ be the set of all $y \in Z^{\mathfrak{m}*}$ such that $y(t)=0$ for $t \notin Q$, and let $\mathbf{Y}_n$ be the set of all $y \in Z^{\mathfrak{m}*}$ such that $y(t)=0$ for $t \notin Q_n$. The annihilator H_n of $\mathbf{Y}_n$ in $T^{\mathfrak{m}}$ is the set of all $x \in T^{\mathfrak{m}}$ such that $x(t)=1$ for all $t \in Q_n$, and the annihilator H of $\mathbf{Y}$ in $T^{\mathfrak{m}}$ is the set of all $x \in T^{\mathfrak{m}}$ such that $x(t)=1$ for all $t \in Q$. There are many choices open to us for the functions $K_{m,n}$ appearing in (5.7). For example, we can imitate the restricted $(C, 1)$ kernels on the n-dimensional torus T^n. In this case we define

$$\hat{K}_{m,n}(y) = \prod_{k=1}^{n} \max\left\{1 - \frac{|y(t_k)|}{m+1}, 0\right\}$$

for $y \in \mathbf{Y}_n$ and $\hat{K}_{m,n}(y)=0$ for $y \notin \mathbf{Y}_n$.

For this choice of $\hat{K}_{m,n}$ (and $K_{m,n}$), Theorems (5.7)–(5.9) hold. Other possible choices of $K_{m,n}$ will no doubt suggest themselves to the interested reader.

(b) M. Mahowald in [16] has described an analogue of Abel summability for $T^{\aleph_0}$, using a single limit instead of an iterated limit. His theorems are not stronger than ours, since they provide pointwise convergence only for functions in $\mathfrak{L}_\infty$ at points of continuity. Note that this can be obtained by using any approximate identity. Also Mahowald's computations (see for example [16], p. 355, lines 20–22) seem hard to follow, and his Theorem II conflicts with known properties of Sidon sets (see [18], Section 5.7, or [7], Theorem 1). An analogue of Abel summability for continuous functions on an arbitrary (finite dimensional) unitary group has been given by Hua [13]. Hua's treatment is not remarkable for obtaining pointwise convergence, as this is possible for all functions in $\mathfrak{L}_1(G)$ for any Lie group G ((3.7) and (2.10)), but for the explicit construction of summability kernels resembling the Abel factors r^n for the circle group.

(5.11) Theorems (5.7)–(5.9) have complete analogues for arbitrary compact infinite groups G. Suppose for simplicity that G is metrisable. Then it is known that the set $\mathfrak{D}$ of (3.4.I) is countably infinite: let us write $\mathfrak{D}=\{D_1, D_2, \ldots, D_n, \ldots\}$ and d_n for the degree of the representation D_n. Define subsets of $\mathfrak{D}$ by induction as follows. Let $D_{l_1}=D_1$. Suppose that $D_{l_1}, \ldots, D_{l_n}$ have been chosen. Let $\mathfrak{E}_n$ be the smallest subset of $\mathfrak{D}$ that contains $\{D_{l_1}, \ldots, D_{l_n}\}$ and is closed under the formation of conjugate representations and of irreducible components of tensor products. If $\mathfrak{E}_n=\mathfrak{D}$, the construction stops. Otherwise, let $D_{l_{n+1}}$ be the first element of $\mathfrak{D}$ that is not in $\mathfrak{E}_n$. Let

$$A_n=\{x \in G: D_{l_1}(x), \ldots, D_{l_n}(x) \text{ are all equal to the identity operator}\}.$$

Then A_n is a closed normal subgroup of G, and it is simple to verify that G/A_n is topologically isomorphic with a closed subgroup of the product $\mathbf{P}_{k=1}^{n} \mathfrak{U}(d_{l_k})$, where $\mathfrak{U}(d)$ is the group of $d \times d$ unitary matrices. If $\mathfrak{E}_n=\mathfrak{D}$, then $A_n=G$, and G is a Lie group. In this case, we can apply (3.7). Let $(K_m)_{m=1}^{\infty}$ be a sequence of functions on G as in (3.5). Then

(i) $$\varrho * K_m(x) = K_m * \varrho(x) = \sum_{j=1}^{r_m} \alpha_{m,j} d_j \operatorname{Tr}[\hat{\varrho}(D_j) D_j(x)]$$

and so

(ii) $$\lim_{m\to\infty} \sum_{j=1}^{r_m} \alpha_{m,j} d_j \operatorname{Tr}[\hat{\varrho}(D_j) D_j(x)] = h(x)$$

exists for almost all $x \in G$ and is an LRN derivative of ϱ with respect to λ.

If no $\mathfrak{E}_n$ is equal to $\mathfrak{D}$, then G is not a Lie group, and so far as we know an iterated limit is needed. It is essential to note that $\hat{\mu}_n(D)$ is the identity operator for all $D \in \mathfrak{E}_n$ and is 0 for all other D, μ_n being normalised Haar measure on A_n. We find in this case a double sequence $(K_{m,n})_{m=1}^{\infty}{}_{n=1}^{\infty}$ of summability kernels on G. Let h_n be an LRN derivative of $\varrho * \mu_n$ with respect to λ. Our final result is:

(iii) $$h(x) = \lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \{\lim_{m\to\infty} \varrho * \mu_n * K_{m,n}(x)\}$$

$$= \lim_{n\to\infty} \{\lim_{m\to\infty} \sum_{j=1}^{r_{m,n}} \alpha_{m,n,j} d_j \operatorname{Tr}[\hat{\varrho}(D_j) D_j(x)]\}.$$

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