WEIGHTED TRIGONOMETRICAL APPROXIMATION ON R^1 WITH APPLICATION TO THE GERM FIELD OF A STATIONARY GAUSSIAN NOISE

BY

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Notation

$f^*(\gamma)$ $(\gamma = a+ib)$ denotes the regular extension of $f^*(a) = f(a)^*$ so that $f^*(\gamma) = (f(\gamma^*))^*$, $(\gamma^* = a - ib)$.

$\int$ stands for $\int_{-\infty}^{+\infty}$.

$\int_1$ and the like stand for $\int_1^{\infty}$, etc.

$e(\gamma)$ means e^{γ}.

1 *a*. Introduction (weighted trigonometric approximation)

Given a non-trivial, even, non-negative, Lebesgue-measurable weight function $\Delta = \Delta(a)$ with $\int \Delta < \infty$, let Z be the (real) Hilbert space $L^2(R^1, \Delta\, da)$ of Lebesgue-measurable functions f with

$$f^*(-a) = f(a), \qquad \|f\| = \|f\|_\Delta = \left(\int |f|^2 \Delta\right)^{\frac{1}{2}} < \infty$$

subject to the usual identifications, and putting Z^{cd} = *the span* (*in* Z) *of* $e(iat)$ $(c \leqslant t \leqslant d)$, introduce the following subspaces of Z:

(a) $Z^- = Z^{-\infty 0}$,

(b) $Z^+ = Z^{0\infty}$,

(c) $Z^{+/-}$ = the projection of Z^+ onto Z^-,

(d) $Z^{\cdot}$ = the class of entire functions $f = f(\gamma)$ $(\gamma = a + ib)$ with

$$\overline{\lim_{R\uparrow\infty}} R^{-1} \max_{0\leqslant\theta\leqslant 2\pi} \lg |f(Re^{i\theta})| \leqslant 0,$$

which, restricted to the line $b = 0$, belong to Z,

(e) $Z^{0+} = \bigcap_{\delta>0} Z^{0\delta}$,

(f) $Z_{\cdot}$ = the span of $(ia)^d$, $d = 0, 1, 2$, etc., $\int a^{2d}\Delta < \infty$,

(g) $Z^{-\infty} = \bigcap_{t<0} Z^{-\infty t}$.

$Z^{-\infty\infty} = Z$ since $f \in Z$ implies $f\Delta \in L^1(R^1)$, and in that case $f\Delta = 0$ if $\int f\Delta\, e(-iat) = 0$ $(t \in R^1)$; the functions $f \in Z^{\cdot}$ are of 0 (minimal) exponential type, so-called.

$Z^{\cdot}$ is either dense in Z or a *closed* subspace of Z; the second alternative holds in the case of a *Hardy weight*:

$$\int \frac{\lg \Delta}{1+a^2} > -\infty,$$

and under this condition

$$Z^- \supset Z^{+/-} \supset Z^- \cap Z^+ \supset Z^{0+} = Z^{\bullet} \supset Z_{\bullet}.$$

Given a Hardy weight Δ, the problem is to decide if some or all of the above subspaces coincide; for instance, as it turns out, $Z^{+/-} = Z^{\bullet}$ if and only if $\Delta^{-1} = |f|^2$ with f entire of minimal exponential type, while $Z^{\bullet} = Z^{0+}$ for the most general Hardy weight.

$Z \neq Z^-$ in the Hardy case, while in the non-Hardy case $Z = Z^- \cap Z^+ = Z^{-\infty}$, and, if $\Delta \in \downarrow$ also, then $Z = Z^{0+}$ too. ($\Delta \in \downarrow$ means that $\Delta(a) \geqslant \Delta(b)$ for $0 \leqslant a < b$.)

$Z^{+/-}$ and Z^{0+} receive special attention below for reasons explained in the next part of the introduction.

S. N. Bernstein's problem of finding conditions on a weight $\Delta \leqslant 1$ so that each continuous function f with $\lim_{|a| \uparrow \infty} |f| \Delta = 0$ should be close to a polynomial p in the sense that $|f-p|\Delta$ be small, is similar to the problem of deciding if $Z_{\bullet} = Z$ or not, and it turned out that S. N. Mergelyan's solution of Bernstein's problem [10] and I. O. Hačatrjan's amplification of it [5] could be adapted to the present case.

1 *b*. Introduction (probabilistic part)

$\Delta\, da$ can be regarded as the spectral weight of a centered Gaussian motion with *sample paths* $t \to x(t) \in R^1$, *universial field* B, *probabilities* $P(\mathrm{B})$, and *expectations* $E(f)$:

$$E[x(s)x(t)] = \int e^{ia(t-s)} \Delta.$$

Bring in the (real) Hilbert space Q which is the closed span of $x(t)$ $(t \in R^1)$ under the norm $\|f\| = [E(f^2)]^{\frac{1}{2}}$ and map $x(t) \to e(iat) \in Z$. Q is mapped 1:1 onto Z, inner products being preserved, and with the notations Q^{cd} = *the span of* $x(t)$ $(c \leqslant t \leqslant d)$ and B^{cd} = *the smallest Borel subfield of* B *measuring* $x(t)$ $(c \leqslant t \leqslant d)$, a perfect correspondence is obtained between

(a) Z^-, $Q^- = Q^{-\infty 0}$, and $\mathrm{B}^- = \mathrm{B}^{-\infty 0}$ = *the past,*
(b) Z^+, $Q^+ = Q^{0\infty}$, and $\mathrm{B}^+ = \mathrm{B}^{0\infty}$ = *the future,*
(c) $Z^{+/-}$, the projection $Q^{+/-}$ of Q^+ onto Q^-, and $\mathrm{B}^{+/-}$ = *the smallest splitting field of past and future,*

(d) Z^{0+}, $Q^{0+}=\bigcap_{\delta>0} Q^{0\delta}$, and $\mathsf{B}^{0+}=\bigcap_{\delta>0}\mathsf{B}^{0\delta}=$ *the germ*,

(e) $Z_{\bullet}$, $Q_{\bullet}=$ the span of $x^{(d)}(0)$, $d=0,1,2$, etc., $E[x^{(d)}(0)^2]<\infty$, and the associated field $\mathsf{B}_{\bullet}$,

(*f*) $Z^{-\infty}$, $Q^{-\infty}=\bigcap_{t<0} Q^{-\infty t}$, and $\mathsf{B}^{-\infty}=\bigcap_{t<0}\mathsf{B}^{-\infty t}=$ *the distant past.*

B^{-}, B^{+}, B^{0+}, etc. do not just include the fields of Q^{-}, Q^{+}, Q^{0+}, etc., but for instance, if $f\in Q$ is measurable over B^{0+}, then it belongs to Q^{0+}; the proof of this fact and its analogues is facilitated by use of the lemma of Tutubalin-Freĭdlin [11]: if the field A is part of the smallest Borel field containing the fields of B and C and if C is independent of A and B then $\mathsf{A}\subset\mathsf{B}$.

$\mathsf{B}^{+/-}$ ($=$the splitting field) needs some explanation. Given a pair of fields such as B^{-} ($=$the past) and B^{+} ($=$the future), a field $\mathsf{A}\subset\mathsf{B}^{-}$ is said to be a splitting field of B^{-} and B^{+}, if, conditional on A, B^{+} is independent of B^{-}. B^{-} is a splitting field, and as is not hard to prove, a smallest splitting field exists, coinciding in the present (Gaussian) case with the field of the projection $Q^{+/-}$ (see H. P. McKean, Jr. [9] for the proof). $\mathsf{B}^{+/-}$ and so also $Z^{+/-}$ is a measure of the dependence of the future on the past.

Because $Z^{\bullet}=Z^{0+}$ for a Hardy weight, the condition $\Delta^{-1}=|f|^2$ (f entire of minimal exponential type) for $Z^{+/-}=Z^{\bullet}$ is equivalent in the Hardy case to the condition that the motion split over its germ ($\mathsf{B}^{+/-}=\mathsf{B}^{0+}$); this is the principal result of this paper from a probabilistic standpoint. Tutubalin-Freĭdlin's result [11] that if $\Delta\geqslant|a|^{-d}$ as $|a|\uparrow\infty$ for some $d\geqslant 2$ then $\mathsf{B}^{0+}=\mathsf{B}_{\bullet}$, is the sole fact about B^{0+} that has been published to our knowledge.

2. Hardy functions

An even Hardy weight Δ can be expressed as $\Delta=|h|^2$, h belonging to the Hardy class H^{2+} of functions $h=h(\gamma)$ ($\gamma=a+ib$) regular in the half plane ($b>0$) with $h^*(-a)=h(a)$ and $\int|h(a+ib)|^2\,da$ bounded ($b>0$); such a Hardy function satisfies

$$\lim_{b\downarrow 0}\int|h(a+ib)-h(a)|^2\,da=0 \quad\text{and}\quad \int|h(a+ib)|^2\,da\leqslant\int|h(a)|^2\,da \quad (b>0).$$

Hardy functions can also be described as the (regular) extensions into $b>0$ of the Fourier transforms of functions belonging to $L^2(R^1,dt)$ vanishing on the left half line ($t\leqslant 0$). According to Beurling's nomenclature, each Hardy function comes in 2 pieces: an *outer* factor o with

$$\lg |o(\gamma)| = \frac{1}{\pi}\int \frac{b}{(c-a)^2+b^2} \lg |h(c)|\, dc \quad (\gamma = a+ib)$$

and an *inner* factor j with

$$|j(\gamma)| \leqslant 1 \ (b>0), \qquad |j(\gamma)| = 1 \ (b=0);$$

the complete formula for the outer factor of h is

$$o(\gamma) = e\left[\frac{1}{\pi i}\int \frac{\gamma c - 1}{\gamma + c} \lg |h(c)| \frac{dc}{1+c^2}\right].$$

$Z^+ h = H^{2+}$, i.e., $e(i\gamma t)h$ $(t \geqslant 0)$ spans out the whole of H^{2+}, if and only if h is outer. H^{2-} stands for the analogous Hardy class for $b<0$. $L^2(R^1, da)$ is the (perpendicular) direct sum of H^{2-} and H^{2+}. Hardy classes $H^{1\pm}$ are defined in the same manner except that now it is $\int |h(a+ib)|da$ that is to be bounded. H^{1+} can be described as those functions h belonging to $L^1(R^1, da)$ with $\int e(-iat)h da = 0$ $(t \leqslant 0)$; it is characteristic of the moduli of such functions that $\int \lg |h|/(1+a^2) > -\infty$ (see [7] for proofs and additional information).

3. Discussion of $Z^- \supset Z^{+/-} \supset Z^- \cap Z^+$

Given Δ as in 1 a, Hardy or not, the inclusions $Z \supset Z^- \supset Z^{+/-} \supset Z^- \cap Z^+$ are obvious, so the problem is to decide in what circumstances some or all of the above subspaces coincide. As it happens,

(a) *either* $\int \lg \Delta/(1+a^2) = -\infty$ *and* $Z = Z^- \cap Z^+ = Z^{-\infty}$
or $\int \lg \Delta/(1+a^2) > -\infty$ *and* $Z \neq Z^- \neq Z^- \cap Z^+$;

in the second (Hardy) case, $\int \lg \Delta/(1+a^2) > -\infty$, $\Delta = |h|^2$ with h outer belonging to H^{2+}, and the following statements hold:

(b) $Z^- \neq Z^{+/-}$ *if and only if* $\mathfrak{j} = h/h^*$, *restricted to the line, coincides with the ratio of* 2 *inner functions,*

(c) $Z^{+/-} = Z^- \cap Z^+$ *if and only if* $\mathfrak{j} = h/h^*$, *restricted to the line, coincides with an inner function.*

(a) goes back to Szegö; the rest is new.

Proof of (a) adapted from [7]. $Z \neq Z^-$ implies that for the coprojection f of $e(ias)$ upon Z^-, $f\Delta \neq 0$ for some $s > 0$. Because the projection belongs to Z^-, $e(-ias)fe(iat) \in Z^-$ $(t \leq 0)$ and so is perpendicular (in Z) to f; also, f is perpendicular to $e(iat)$ $(t \leq 0)$, so

$$\int e^{ias}|f|^2 \Delta e^{-iat}\, da = \int f \Delta e^{-iat}\, da = 0 \quad (t \leq 0).$$

But in view of $\int |f| \Delta \leq \|f\|_\Delta (\int \Delta)^{\frac{1}{2}} < \infty$, it follows that $f\Delta$ belongs to the Hardy class H^{1+}, whence $\int \lg (|f|\Delta)/(1+a^2) > -\infty$. But also $\int \lg (|f|^2 \Delta)/(1+a^2) < \infty$ since $f \in Z$, and so $\int \lg \Delta/(1+a^2) > -\infty$, as stated. On the other hand, $\int \lg \Delta/(1+a^2) > -\infty$ implies $\Delta = |h|^2$ with h outer belonging to H^{2+}, and $Z \neq Z^-$ follows: indeed, since Δ is even, $h^*(-a) = h(a)$, and since $h^2 \in H^{1+}$,

$$\int e^{-iat} h^2\, da = \int e^{-iat}\, \mathfrak{j} \Delta\, da = 0 \quad (t \leq 0) \quad (\mathfrak{j} = h/h^*),$$

stating that $\mathfrak{j} \in Z$ is perpendicular to Z^-. $Z^- \neq Z^- \cap Z^+$ follows, since, in the opposite case, $Z^- \subset Z^+$ so that $Z^+ = Z$ and hence also $Z^- = Z$, against the fact that Δ is a Hardy weight. $Z^{-\infty} = \bigcap_{t<0} Z^{-\infty t} = Z$ follows in the non-Hardy case.

Proof of (b). Given $\int \lg \Delta/(1+a^2) > -\infty$, let $\Delta = |h|^2$ with h outer as before and prepare 3 simple lemmas.

$Z^+ h = H^{2+}$ since h is outer as stated in 2.

$Z^- h = \mathfrak{j} H^{2-}$ because $Z^- h^* = (Z^+ h)^* = (H^{2+})^* = H^{2-}$.

$Z^{+/-} h = \mathfrak{j} p \mathfrak{j}^{-1} H^{2+}$, p being the projection in $L^2(R^1)$ upon H^{2-}; indeed, $\mathfrak{j} p \mathfrak{j}^{-1}$ is a projection and coincides with the identity just on $\mathfrak{j} H^{2-}$.

Coming to the actual proof of (b), if the inclusion $Z^- \supset Z^{+/-}$ is proper, then $Z^- h = \mathfrak{j} H^{2-}$ contains a function $f = \mathfrak{j}(j_2 o_2)^*$ perpendicular to $Z^{+/-} h = \mathfrak{j} p \mathfrak{j}^{-1} H^{2+}$, j_2 being an inner and $o_2 \in H^{2+}$ an outer function. Because $\mathfrak{j} p \mathfrak{j}^{-1} = 1$ on $\mathfrak{j} H^{2-}$, it follows that f is perpendicular in $L^2(R^1)$ to H^{2+}, so $f \in H^{2-}$, i.e., $f = (j_1 o_1)^*$, j_1 being an inner and $o_1 \in H^{2+}$ an outer function; in brief, $\mathfrak{j}(j_2 o_2)^* = (j_1 o_1)^*$. Because $|o_1| = |o_2|$ on the line $b = 0$, the outer factors can be cancelled, proving that $\mathfrak{j} = j_2/j_1$. On the other hand, if $\mathfrak{j} = j_2/j_1$, then $f = \mathfrak{j}(j_2 h)^* \neq 0$ belongs to $\mathfrak{j} H^{2-} = Z^- h$. Also $f = (j_1 h)^* \in H^{2-}$ so that f is perpendicular in $L^2(R^1)$ to H^{2+}, and since $f \in \mathfrak{j} H^{2-}$, it must be perpendicular to $\mathfrak{j} p \mathfrak{j}^- H^{2+} = Z^{+/-} h$ also. $Z^- \neq Z^{+/-}$ follows, completing the proof.

Proof of (c). $Z^- \neq Z^- \cap Z^+$ in the Hardy case, so if $Z^{+/-} = Z^- \cap Z^+$, then $Z^- \neq Z^{+/-}$, and according to (b), $\mathfrak{j} = h/h^*$ is a ratio j_2/j_1 of inner functions with no common factor. $f \in Z^- h = \mathfrak{j} H^{2-}$ is perpendicular in $L^2(R^1)$ to $Z^{+/-} h = \mathfrak{j} p \mathfrak{j}^{-1} H^{2+}$ if and only if $\mathfrak{j}^{-1} f \in H^{2-}$ is perpendicular to $p \mathfrak{j}^{-1} H^{2+}$, or, and this is the same, to $\mathfrak{j}^{-1} H^{2+}$, and so, computing annihilators in $\mathfrak{j} H^{2-}$, $(Z^{+/-} h)^0 = \mathfrak{j} H^{2-} \cap H^{2-}$. Now $f \in \mathfrak{j} H^{2-} \cap H^{2-}$ can be expressed as $(j_2/j_1) j_3^* o_3^* = j_4^* o_4^*$ and the outer factors have to match, so $j_2 j_4 = j_1 j_3$, and since j_1 and j_2 have no common factors, j_1 divides j_4 [1, p. 246] and $f \in \mathfrak{j} H^{2-} \cap (1/j_1) H^{2-}$. Because $j_1^* H^{2-} \subset H^{2-}$, $Z^{+/-} h$ can now be identified as $[\mathfrak{j} H^{2-} \cap (1/j_1) H^{2-}]^0 = \mathfrak{j} H^{2-} \cap (1/j_1) H^{2+}$, the annihilator being computed in $\mathfrak{j} H^{2-}$; this is because $(1/j_1) H^{2-} = \mathfrak{j} j_2^* H^{2-} \subset \mathfrak{j} H^{2-}$ and $(1/j_1) H^{2-} \oplus \mathfrak{j} H^{2-} \cap (1/j_1) H^{2+}$ is a perpendicular splitting of $\mathfrak{j} H^{2-}$. But according to this identification, if $Z^{+/-} = Z^- \cap Z^+$, then $\mathfrak{j}(j_1 h)^* = (1/j_1) h \in Z^{+/-} h \subset Z^+ h = H^{2+}$, and h being outer, it follows that j_1 has to be constant, completing half the proof; the opposite implication is obvious using the above identification of $Z^{+/-} h$ in conjunction with $(Z^- \cap Z^+) h = \mathfrak{j} H^{2-} \cap H^{2+}$.

Example. $h = (1 - i\gamma)^{-3/2}$ is outer belonging to H^{2+} and $Z^- = Z^{+/-}$; indeed, $[(1 + i\gamma)/(1 - i\gamma)]^{3/2} = j_2/j_1$ would mean that $j_1^2 [(1 + i\gamma)/(1 - i\gamma)]^3 = j_2^2$, and this would make j_2^2 have a root of *odd* degree at $\gamma = i$.

An outer function h belonging to H^{2+} is determined by its phase factor $\mathfrak{j} = h/h^*$ if and only if $\dim Z^- \cap Z^+ = 1$; indeed, if $\dim Z^- \cap Z^+ = 1$ and if o is an outer function belonging to H^{2+} with $o/o^* = \mathfrak{j}$, then $o \in \mathfrak{j} H^{2-} \cap H^{2+} = Z^- \cap Z^+ h$ and, as such, is a multiple of h. On the other hand, if $o/o^* = \mathfrak{j}$ implies $o = \text{constant} \times h$, then $\dim Z^- \cap Z^+ = 1$ because if o is the outer factor of $f \in Z^- \cap Z^+ h = \mathfrak{j} H^{2-} \cap H^{2+}$, then $o/o^* = \mathfrak{j}/j$ with j an inner multiple of the inner factor of f. $(j+1)o$ is outer [7, p. 76], and since $(j+1)o/(j+1)^* o^* = \mathfrak{j}$, it is a multiple of h. $i(j-1)o$ is likewise a multiple of h, and so o itself is a multiple of h, $j = 1$, and f too is a multiple of h.

4. Discussion of $Z^\bullet$

Before proving the rest of the inclusions $Z^- \cap Z^+ \supset Z^{0+} \supset Z^\bullet \supset Z_\bullet$, Mergelyan's solution of Bernstein's ploblem, and his proof also, is adapted to the present needs.

Given Δ, Hardy or not, let $Z^\bullet = Z^\bullet_\Delta$ be the class of entire functions f of minimal exponential type which, restricted to $b = 0$, belong to Z, let $\Delta^+ = \Delta(1 + a^2)^{-1}$, suppose $\int \Delta^+ = 1$, and putting

$$\sigma^\bullet(\gamma) = \textit{the least upper bound of } |f(\gamma)| : f \in Z^\bullet_{\Delta^+},\ \|f\|_{\Delta^+} \leq 1,$$

let us check that the following alternative holds:

either $\sigma^{\bullet} \equiv \infty (b \neq 0)$,

$$\sup \int \frac{\lg^{+} |f|}{1+a^2} = \int \frac{\lg \sigma^{\bullet}}{1+a^2} = \infty, \text{ for } f \in Z^{\bullet}_{\Delta^+} \text{ with } \|f\|_{\Delta^+} \leqslant 1,$$

and $Z^{\bullet}$ *is dense in* Z,

or $\lg \sigma^{\bullet}$ *is a continuous, non-negative, subharmonic function,*

$$\int \frac{\lg \sigma^{\bullet}}{1+a^2} < \infty,$$

$$\lg \sigma^{\bullet}(\gamma) \leqslant \frac{1}{\pi} \int \frac{b}{(c-a)^2+b^2} \lg \sigma^{\bullet}(c)\, dc \quad (\gamma = a+ib,\ b>0),$$

$$\overline{\lim_{R\uparrow\infty}}\, R^{-1} \max_{0\leqslant\theta<2\pi} \lg \sigma^{\bullet}(Re^{i\theta}) \leqslant 0,$$

and $Z^{\bullet}$ *is a closed subspace of* Z;

the second alternative must hold in the case of a Hardy weight as will be proved in 6*b*. Because $(f(\gamma^*))^* = f(-\gamma) \in Z^{\bullet}$ if $f \in Z^{\bullet}$,

$$\sigma^{\bullet}(\gamma) = \sigma^{\bullet}(\gamma^*) = \sigma^{\bullet}(-\gamma);$$

this fact is used without additional comment below.

Break up the proof into simple lemmas.

(a) $\sigma^{\bullet}(\gamma) \equiv \infty (b \neq 0)$ *if and only if* $Z^{\bullet}$ *is dense in* Z.

Proof of (a). $\sigma^{\bullet}(\beta) = \infty (\beta = a+ib,\ b \neq 0)$ implies that $f \in Z^{\bullet}_{\Delta^+}$ can be found with $\|f\|_{\Delta^+} \leqslant 1$, $|f(\beta)| > \delta^{-1}$, and hence

$$\left\| \frac{1}{c-\beta} + \frac{f-f(\beta)}{(c-\beta)f(\beta)} \right\|_{\Delta} = \left\| \frac{f}{(c-\beta)f(\beta)} \right\|_{\Delta} \leqslant |f(\beta)|^{-1} \left\| \frac{c-i}{c-\beta} \right\|_{\infty} \|f\|_{\Delta^+} < \text{constant} \times \delta.$$

Breaking up $[f-f(\beta)]/(\gamma-\beta)f(\beta)$ into the sum of its odd and even parts f_1 and f_2 and then into the sum (with coefficients of modulus 1) of 4 pieces:

$$f_{11} = \frac{1}{2}(f_1 + f_1^*), \quad f_{12} = \frac{i}{2}(f_1 - f_1^*), \quad f_{21} = \frac{i}{2}(f_2 + f_2^*), \quad f_{22} = \frac{1}{2}(f_2 - f_2^*),$$

each of which belongs to $Z^{\bullet}_{\Delta}$, it follows that if $g \in Z$ is perpendicular to $Z^{\bullet}_{\Delta}$, then $\int g\Delta/(c-\beta) = 0$ $(\beta = a+ib,\ b \neq 0)$, whence

$$\int \frac{b}{(c-a)^2+b^2}\, g\,\Delta\, dc = 0 \quad (b>0),$$

and $g\Delta = 0$ as desired. On the other hand, if $Z^{\bullet}_{\Delta}$ is dense in Z, then it is possible to find an entire function f of minimal exponential type with $\|1/(c-\beta)-f\|_{\Delta} < \delta$ ($\beta = a+ib$, $b \neq 0$). Bring in an entire function g with $[g-g(\beta)]/(\gamma-\beta)g(\beta) = -f$; then

$$\delta > \left\| \frac{g}{(c-\beta)\,g(\beta)} \right\|_{\Delta} \geqslant \textit{a positive constant depending upon } \beta \textit{ alone} \times \frac{\|g\|_{\Delta^+}}{|g(\beta)|},$$

and so
$$|g(\beta)| > \text{constant} \times \delta^{-1} \|g\|_{\Delta^+}.$$

g is now split into the sum (with coefficients of modulus 1) of 4 members g_{11}, g_{12}, g_{21}, g_{22} of $Z^{\bullet}_{\Delta^+}$, and it develops that

$$\begin{aligned}
\text{constant} \times \delta^{-1} \|g\|_{\Delta^+} < |g(\beta)| &\leqslant |g_{11}(\beta)| + |g_{12}(\beta)| + |g_{21}(\beta)| + |g_{22}(\beta)| \\
&\leqslant \sigma^{\bullet}(\beta)\,(\|g_{11}\|_{\Delta^+} + \|g_{12}\|_{\Delta^+} + \|g_{21}\|_{\Delta^+} + \|g_{22}\|_{\Delta^+}) \\
&\leqslant 2\sigma^{\bullet}\,(\|g_1\|_{\Delta^+} + \|g_2\|_{\Delta^+}) \leqslant 2\sqrt{2}\,\sigma^{\bullet}\,(\|g_1\|^2_{\Delta^+} + \|g_2\|^2_{\Delta^+})^{\frac{1}{2}} \\
&= 2\sqrt{2}\,\sigma^{\bullet}\,\|g\|_{\Delta^+},
\end{aligned}$$

making use of $\int g_1^* g_2 \Delta^+ = 0$. But since δ can be made small, $\sigma^{\bullet}(\beta)$ is in fact $=\infty$.

(b) *$Z^{\bullet}$ dense in Z implies*

$$\sup \int \frac{\lg^+ |f|}{1+a^2} = \int \frac{\lg \sigma^{\bullet}}{1+a^2} = \infty, \quad \textit{for } f \in Z^{\bullet}_{\Delta^+} \textit{ with } \|f\|_{\Delta^+} \leqslant 1.$$

Proof of (b). Given $f \in Z^{\bullet}_{\Delta^+}$, if $\beta = a+ib$ ($b>0$), then

$$\lg |f(\beta)| \leqslant \frac{1}{\pi} \int \frac{b}{(c-a)^2+b^2} \lg^+ |f(c)|\, dc$$

as follows from Nevanlinna's theorem [2:1.2.3] on letting $R \uparrow \infty$ and using

$$\overline{\lim_{R\uparrow\infty}}\, R^{-1} \max_{0\leqslant\theta<2\pi} \lg |f(Re^{i\theta})| \leqslant 0.$$

Now apply (a).

(c) $\lg \sigma^{\bullet}(\beta) \leqslant \dfrac{1}{\pi} \displaystyle\int \frac{b}{(c-a)^2+b^2} \lg \sigma^{\bullet}(c)\, dc \quad (\beta = a+ib,\ b>0).$

Proof of (c). Obvious from (b).

(d) *$Z^{\bullet}$ non-dense implies that $\sigma^{\bullet}$ is bounded in the neighborhood of each point $\beta = a + ib$ $(b > 0)$; in fact, if $Z^{\bullet}$ is non-dense* $\lg \sigma^{\bullet}$ *is a non-negative continuous subharmonic function* $(b \neq 0)$.

Proof of (d). Given $\beta = a + ib$ $(b > 0)$ and a point α near it, take $g \in Z^{\bullet}_{\Delta^+}$ with $\|g\|_{\Delta^+} \leqslant 1$ and $|g(\alpha)|$ close to $\sigma^{\bullet}(\alpha)$, and let $f = 1 + [(\gamma - \beta)/(\gamma - \alpha)][(g - g(\alpha))/g(\alpha)]$, observing that f need not belong to $Z^{\bullet}_{\Delta^+}$ since $f^*(-a) = f(a)$ can fail.

$$\|f\|_{\Delta^+} = \left\| \frac{\beta - \alpha}{c - \alpha} - \frac{\beta - \alpha}{c - \alpha} \frac{g}{g(\alpha)} + \frac{g}{g(\alpha)} \right\|_{\Delta^+} \leqslant \left\| \frac{\beta - \alpha}{c - \alpha} \right\|_{\infty} (1 + |g(\alpha)|^{-1}) + |g(\alpha)|^{-1},$$

and so, as in the second part of the proof of (a),

$$1 = |f(\beta)| \leqslant 2\sqrt{2}\, \sigma^{\bullet}(\beta)\, \|f\|_{\Delta^+} \leqslant 2\sqrt{2}\, \sigma^{\bullet}(\beta) \left[\left\| \frac{\beta - \alpha}{c - \alpha} \right\|_{\infty} (1 + |g(\alpha)|^{-1}) + |g(\alpha)|^{-1} \right],$$

proving that $\sigma^{\bullet}(\alpha)$ is bounded on a neighborhood of β if $\sigma^{\bullet}(\beta) < \infty$. Because $1 \in Z^{\bullet}_{\Delta^+}$, $\sigma^{\bullet} \geqslant 1$ ($\int \Delta^+ = 1$ is used at this place), so $\lg \sigma^{\bullet} \geqslant 0$, and since $\lg |f|$ is subharmonic for each $f \in Z^{\bullet}_{\Delta^+}$, $\lg \sigma^{\bullet}$ is also subharmonic. But now it follows that if $\sigma^{\bullet}(\beta) = \infty$ at one point $\beta = a + ib$ $(b > 0)$, then it is also ∞ at some point of each punctured neighborhood of β, and arguing as in the first part of the proof of (a) with f perpendicular to $Z^{\bullet}_{\Delta}$, $\int f\Delta/(c - \alpha)\, dc$ is found to vanish at some point of each punctured neighborhood of β and hence to be $\equiv 0$. $Z^{\bullet}$ dense in Z follows as before, so $Z^{\bullet}$ non-dense implies the (local) boundedness of $\sigma^{\bullet}$. It remains to prove that $\sigma^{\bullet}$ is continuous $(b \neq 0)$. On a small neighborhood of $\alpha = a + ib$, $|f|$ $(f \in Z^{\bullet}_{\Delta^+})$ lies under a universal bound, $\sigma^{\bullet}$. An application of Cauchy's formula implies that $|f'|$ lies under a universal bound on a smaller neighborhood of α, and so $|f(\beta_2) - f(\beta_1)|$ lies under a universal constant B times $|\beta_2 - \beta_1|$ as β_1 and β_2 range over this smaller neighborhood. But then

$$|f(\beta_2)| \leqslant |f(\beta_1)| + B|\beta_2 - \beta_1| < \sigma^{\bullet}(\beta_1) + B|\beta_2 - \beta_1|,$$

so that

$$\sigma^{\bullet}(\beta_2) \leqslant \sigma^{\bullet}(\beta_1) + B|\beta_2 - \beta_1|,$$

and interchanging the roles of β_1 and β_2 completes the proof of (d).

(e) *$Z^{\bullet}$ non-dense implies* $\int \lg^+ |f|/(1 + a^2) \leqslant \int \lg \sigma^{\bullet}/(1 + a^2) < \infty$.

Proof of (e). $Z^{\bullet}$ non-dense implies the existence of $g \in Z$ perpendicular to $Z^{\bullet}_{\Delta}$, and since, if $f \in Z^{\bullet}_{\Delta^+}$, $(f - f(\beta))/(\gamma - \beta)$ is the sum (with coefficients of modulus 1) of 4 members of $Z^{\bullet}_{\Delta}$,

$$\int \frac{g^* f}{c-\beta} \Delta = \int \frac{g^* \Delta}{c-\beta} f(\beta) \equiv \hat{g} f \quad (f \in Z^{\bullet}_{\Delta^+},\ b \neq 0).$$

Because $\hat{g}$ is regular and bounded $(b \geqslant 1)$, $\int \lg |\hat{g}(a+i)|/(1+a^2) > -\infty$; also

$$|\hat{g} f(a+i)| \leqslant \|g\|_{\Delta} \|f\|_{\Delta^+} \left\| \frac{c-i}{c-a-i} \right\|_{\infty},$$

so that $\sigma^{\bullet}(a+i) \leqslant$ constant$\times (1+a^2)^{\frac{1}{2}} |\hat{g}(a+i)|^{-1}$ and $\int \lg \sigma^{\bullet}(a+i)/(1+a^2) < \infty$. But as in the proof of (b),

$$\lg |f(a)| \leqslant \frac{1}{\pi} \int \frac{\lg \sigma^{\bullet}(c+i)}{(c-a)^2+1} dc \quad (f \in Z^{\bullet}_{\Delta^+}),$$

and so

$$\int \frac{\lg \sigma^{\bullet}(a)}{1+a^2} da \leqslant \int \lg \sigma^{\bullet}(c+i)\, dc \frac{1}{\pi} \int \frac{da}{1+a^2} \frac{1}{(c-a)^2+1} = 2 \int \frac{\lg \sigma^{\bullet}(c+i)}{c^2+4} dc < \infty,$$

as stated.

(f) *If $Z^{\bullet}$ is non-dense in Z then it is a closed subspace of Z and*

$$\overline{\lim_{R \uparrow \infty}} R^{-1} \max_{0 \leqslant \theta < 2\pi} \lg \sigma^{\bullet}(Re^{i\theta}) \leqslant 0.$$

Proof of (f).

$$R^{-1} \lg \sigma^{\bullet}(Re^{i\theta}) \leqslant \frac{1}{\pi} \int \frac{\sin\theta\,(1+c^2)}{(c-R\cos\theta)^2 + R^2 \sin^2\theta} \frac{\lg \sigma^{\bullet}}{1+c^2} dc \quad (0 < \theta < \pi)$$

according to (d). A simple estimate, combined with $\sigma^{\bullet}(\gamma) = \sigma^{\bullet}(\gamma^*)$ verifies

$$\overline{\lim_{R \uparrow \infty}} R^{-1} \lg \sigma^{\bullet}(Re^{i\theta}) \leqslant 0 \quad (\theta = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4).$$

Phragmén-Lindelöf is now applied to each of the sectors between $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$; for instance, in the sector $[\pi/4, 3\pi/4]$, each $f \in Z^{\bullet}_{\Delta^+}$ with $\|f\|_{\Delta^+} \leqslant 1$ satisfies

$$|f(\gamma)\, e^{i\gamma 2^{\frac{1}{2}} \delta}| \leqslant |f(Re^{i\theta})|\, e^{-R\delta} \leqslant A \quad (\pi/4 \leqslant \theta \leqslant 3\pi/4)$$

$$|f(\gamma)\, e^{i\gamma 2^{\frac{1}{2}} \delta}| \leqslant \sigma^{\bullet}(Re^{i\theta})\, e^{-R\delta} \leqslant B \quad (\theta = \pi/4,\ 3\pi/4)$$

with a constant B not depending upon f, and so

$$|f(\gamma)\, e^{i\gamma(2\delta)^{\frac{1}{2}}}| \leqslant B \quad (\pi/4 \leqslant \theta \leqslant 3\pi/4),$$

or

$$\sigma^{\bullet}(Re^{i\theta}) \leqslant B e^{R(2\delta)^{\frac{1}{2}}} \quad (\pi/4 \leqslant \theta \leqslant 3\pi/4).$$

$Z^{\bullet}$ *closed* follows since $|f|$ $(f \in Z^{\bullet}_{\Delta})$ lies under a universal bound $(\sigma^{\bullet})$ on any bounded region of the plane.

Mergelyan's alternative is now proved; several additional comments follow.

Given $f \in Z^{\bullet}_{\Delta+}$, $(\gamma+i)^{-1} fh \in H^{2+}$ while $(\gamma+i)^{-1} \in H^{2+}$ is an outer function, so that

$$\lg |fh(i)|^2 \leqslant \frac{1}{\pi} \int \frac{\lg |(c+i)^{-1} fh|^2}{1+c^2} + \frac{1}{\pi} \int \frac{\lg |c+i|^2}{1+c^2}$$

$$= \frac{1}{\pi} \int \frac{\lg |fh|^2}{1+c^2} \leqslant \lg \left(\frac{1}{\pi} \int \frac{|f|^2 \Delta}{1+c^2} \right) = \lg \left(\frac{1}{\pi} \, \|f\|^2_{\Delta+} \right),$$

and so $\pi^{\frac{1}{2}} \sigma^{\bullet}(i) \leqslant |h(i)|^{-1}$. Now it is proved that this upper bound is attained if and only if h^{-1} *is entire of minimal exponential type*. Using the compactness that

$$\lim_{R \uparrow \infty} R^{-1} \max_{0 \leqslant \theta < 2\pi} \lg \sigma^{\bullet}(Re^{i\theta}) \leqslant 0$$

ensures, it is possible to choose $f \in Z^{\bullet}_{\Delta+}$ with $f(i) = \sigma^{\bullet}(i)$ and $\|f\|_{\Delta+} = 1$. As before,

$$|fh(i)|^2 \leqslant e^{\left[\frac{1}{\pi} \int \frac{\lg |f|^2 \Delta}{1+a^2}\right]} \leqslant \frac{1}{\pi} \int \frac{|f|^2 \Delta}{1+a^2} = \frac{1}{\pi},$$

so if $\pi^{\frac{1}{2}} \sigma^{\bullet}(i) = |h(i)|^{-1}$, then the converse of Jensen's inequality implies that fh is constant; the other implication is trivial.

$\sigma^{\bullet}(i)$ can also be computed from a Szegö minimum problem:

$$\frac{1}{\sigma^{\bullet}(i)^2} = \inf \int \frac{|1-f|^2 \Delta}{1+a^2}, \quad \text{for } f \in Z^{\bullet}_{\Delta+} \text{ with } f(i) = 0,$$

as the reader can easily check.

Because of the compactness of $Z^{\bullet}$ used above, it is possible in the non-dense case to find $f = f_{\gamma} \in Z^{\bullet}_{\Delta+}$ with $f(\gamma) = \sigma^{\bullet}(\gamma)$ and $\|f\|_{\Delta+} = 1$. f_{γ} is unique and is perpendicular (in $Z^{\bullet}_{\Delta+}$) to each $f \in Z^{\bullet}_{\Delta+}$ vanishing at γ. $f_{\alpha}(\beta)\,\sigma^{\bullet}(\alpha)$ acts as a Bergman reproducing kernel for $Z^{\bullet}_{\Delta+}$ since

$$\int f^*_{\alpha} [f - f(\alpha)] \Delta^+ = 0 \quad (f \in Z^{\bullet}_{\Delta+})$$

implies

$$\int f^*_{\alpha} f_{\Delta+} = f(\alpha) \int f^*_{\alpha} \Delta^+ = \frac{f(\alpha)}{\sigma^{\bullet}(\alpha)} \int |f_{\alpha}|^2 \Delta^+ = \frac{f(\alpha)}{\sigma^{\bullet}(\alpha)}.$$

5. Proof of $Z^{\cdot} \subset Z^{0+}$ (Δ Hardy or not)

To begin with, each $f \in Z^{\cdot}$ can be split into an even part $f_1 = \frac{1}{2}[f(\gamma)+f(-\gamma)] \in Z^{\cdot}$ and an odd part $f_2 \in Z^{\cdot}$; the proof is carried out for an even function $f \in Z^{\cdot}$ with Hadamard factorization

$$f(\gamma) = \gamma^{2m} \prod_{n=1}^{\infty}\left(1 - \frac{\gamma^2}{\gamma_n^2}\right),$$

the odd case being left to the reader. A simple estimate justifies us in ignoring the root of f at $\gamma = 0$; indeed $f_\delta = \delta^{2m}(1-\gamma^2/\delta^2)^m f/\gamma^{2m}$ is an even entire function of minimal exponential type, $|f_\delta/f|$ tends to 1 as $|\gamma| \uparrow \infty$ so that $f_\delta \in Z^{\cdot}$, and $\|f_\delta - f\|_\Delta$ tends to 0 as $\delta \downarrow 0$ so that if $f_\delta \in Z^{0+}$ then so does f.

Bring in the function

$$g(\gamma) = \prod_{|\gamma_n|<d}\left(1 - \frac{\gamma^2}{\gamma_n^2}\right) \prod_{n>d\delta}\left(1 - \frac{\gamma^2\delta^2}{n^2}\right),$$

depending upon a small positive number δ and a large integral number d. Given $\delta > 0$, $\varepsilon > 0$, and $A < \infty$, it is possible to find $d_1 = d_1(\delta, \varepsilon, A)$ and a universal constant B so that for each $d \geqslant d_1$,

(a) $|f-g| < \varepsilon \quad (|a| < A)$

(b) $|g| < B|f| \quad (A \leqslant |a| < d/2)$

(c) $|g| < B \quad (|a| \geqslant d/2)$

(d) $g \in L^2(R^1)$.

It is best to postpone the proof of (a), (b), (c), (d) and to proceed at once to the

Proof that $f \in Z^{0+}$. Using (a), (b), (c) above,

$$\|f-g\|_\Delta^2 < \varepsilon^2 \int \Delta + 2(B+1)^2 \int_A^{d/2} |f|^2 \Delta + 2\int_{d/2} (B+|f|)^2 \Delta$$

tends to 0 as $d \uparrow \infty$, $A \uparrow \infty$, and $\varepsilon \downarrow 0$ in that order. Because the entire function g differs from $\sin \pi\delta\gamma$ by a rational factor and, as such, is of exponential type $\pi\delta$, it follows from (d) in conjuction with the Paley–Wiener theorem that

$$g(a) = \int_{|t|<\pi\delta} e^{iat}\,\hat{g}(t)\,dt \ \text{ with } \int_{|t|<\pi\delta} |\hat{g}|^2\,dt < \infty.$$

But $\int_{|t|<\pi\delta} e(iat)\,\hat{g}\,dt \in Z^{|t|\leqslant\pi\delta}$, as is obvious upon noting the bound

$$\left\|\int_{|t|<\pi\delta} e^{iat}\,\hat{g}\,dt\right\|_{\Delta}^{2} \leqslant 2\pi\delta\int_{|t|<\pi\delta}|\hat{g}|^2\int\Delta$$

and so $f \in \bigcap_{\delta>0} Z^{|t|<\pi\delta} = Z^{0+}$ (see $6a$).

Coming to the proof of (a), (b), (c), (d) above, it is convenient to introduce

$$p(\gamma) = p_m(\gamma) = \pi\gamma\prod_{n=1}^{m}\left(1-\frac{\gamma^2}{n^2}\right)$$

and to check the existence of a universal constant B such that $Q \equiv |\sin(\pi a)/p(a)|$ is bounded as in

$$\text{(e)} \qquad Q/B < \begin{cases} e^{-a^2/m} & |a| < m \\ e^{-a/2} & m \leqslant |a| < 2m \\ e^{-m-2m\lg(a/m)} & |a| \geqslant 2m. \end{cases}$$

Proof of (e).

$Q = \prod_{n>m}(1-a^2/n^2)$ for $|a| < m$, and since $1-c \leqslant e\,(-c)$, $Q < e\,(-a^2/(m+1))$. Stirling's approximation is now used to estimate p below for $|a| \geqslant m$, removing first a factor $a-m$ in case $m \leqslant |a| < m+\frac{1}{2}$, and then $|\sin \pi a|$ is estimated above by 1 or on this range. On the other hand, if m is the biggest integer $<d\delta$ and if $|a|<d/2$, then $\delta|a|<m$ so that the first appraisal listed under (e) supplies us with the bound

$$Q(a\delta)=\prod_{n>d\delta}(1-a^2\delta^2/n^2)<Be^{-a^2\delta^2/(m+1)}<Be^{-a^2\delta/2d},$$

and it follows that

$$B|f(a)|>\prod_{|\gamma_n|<d}\left|1-\frac{a^2}{\gamma_n^2}\right|\prod_{n>d\delta}\left(1-\frac{a^2\delta^2}{n^2}\right)=|g|,$$

as desired.

Proof of (c) *and* (d). On the range $|a|\geqslant d/2$,

$$\begin{aligned}\lg\prod_{|\gamma_n|<d}\left|1-\frac{a^2}{\gamma_n^2}\right| &\leqslant \int_0^d \lg\left(1+\frac{a^2}{R^2}\right)\#(dR)\\ &=\#(d)\lg\left(1+\frac{a^2}{d^2}\right)+\int_0^d\frac{2a^2}{a^2+R^2}\frac{\#}{R}\,dR\\ &\leqslant 2\#(d)\lg(3|a|/d)+2\int_0^d\frac{\#}{R}\,dR\\ &=o[d+d\lg(|a|/d)]\end{aligned}$$

for large d, while according to (e), if $|a|\geqslant d/2$ and if m is the biggest integer $<d\delta$, then

$$Q(a\delta)<Be[-\tfrac{1}{2}d\delta(1+\lg(a/d))].$$

But then $|g|<B$ for large d as stated in (c), while for $d>8/\delta$

$$|g|<Be[-\tfrac{1}{4}d\delta(1+\lg(a/d))]\quad(|a|>d/2).$$

But for still larger d, $d\delta(1+\lg(a/d))-8\lg a>0$ for $a>d/2$, since the left side is positive at $a=d/2$ and increasing for $a>d/2$. Thus

$$|g|<B/a^2\quad(|a|>d/2)$$

so that $g\in L^2(R^1)$ as stated in (d).

6 a. Proof of $Z^-\cap Z^+\supset Z^{0+}$ (Δ Hardy or not)

Given $f\in Z^{0+}\subset Z^+$, then $e(-ia\delta)f\in Z^{-\delta 0}\subset Z^-$, and

$$\begin{aligned}\|(e^{-ia\delta}-1)f\|_\Delta &\leqslant \max_{|a|\leqslant n}|e^{-ia\delta}-1|\;\|f\|_\Delta+2\left(\int_{|a|>n}|f|^2\Delta\right)^{\frac{1}{2}}\\ &\leqslant n\delta\,\|f\|_\Delta+2\left(\int_{|a|>n}|f|^2\Delta\right)^{\frac{1}{2}}\end{aligned}$$

is small for $\delta=n^{-2}$ and $n\uparrow\infty$, so that $f\in Z^-$ also. Our proof justifies

$$Z^{0+}=\bigcap_{\delta<0} Z^{\delta 0}=\bigcap_{\delta>0} Z^{|t|<\delta};$$

this fact will be used without additional comment below.

6 *b*. Proof of $Z^{0+}=Z^{\cdot}$ (Δ Hardy)

$Z^{0+}\subset Z^{\cdot}$ is proved next for a Hardy weight Δ. Combined with the previous result $Z^{0+}\supset Z^{\cdot}$, this gives $Z^{0+}=Z^{\cdot}$.

Given $f\in Z^{0+}$, it is possible to find a finite sum

$$f_n=\sum c_k^n e(i\gamma t_k^n) \text{ with } 0\leqslant t_k^n<1/n,\ \|f-f_n\|_\Delta<1/n,$$

and hence

$$\|f_n\|_\Delta<1/n+\|f\|_\Delta\leqslant 1+\|f\|_\Delta .$$

Phragmén–Lindelöf is now applied to obtain bounds on $|f_n|$. Because $|f_n|$ is bounded ($b\geqslant 0$) and f_n is entire, $f_n h\in H^{2+}$, so

$$\int |f_n h(a+ib)|^2\, da \leqslant \int |f_n|^2 \Delta$$

is bounded ($b>0$, $n\geqslant 1$), and an application of Cauchy's formula to a ring supplies us with the bound

$$|f_n h|\leqslant B_1 \quad (b\geqslant 1,\ n\geqslant 1).$$

Also, $|e(-i\gamma/n) f_n|$ is bounded ($b<0$), so

$$|e^{-i\gamma/n} f_n h^*|\leqslant B_2 \quad (b\leqslant -1,\ n\geqslant 1)$$

with a similar proof. Next, the underestimate

$$\pi \lg |h(a+ib)|=\pi \lg |h^*(a-ib)|\geqslant \int \frac{b \lg^-|h|}{(c-a)^2+b^2}dc\geqslant B_3(1+a^2)\int \frac{\lg^-|h|}{1+c^2}dc$$

$$\geqslant B_4 |e^{-B_5\gamma^2}| \quad (1\leqslant b\leqslant 2,\ B_4>0)$$

justifies the bound

$$|g_n|\leqslant B_6 \text{ for } 1\leqslant b\leqslant 2,\ n\geqslant 1 \text{ with } g_n\equiv e(-B_5\gamma^2) f_n.$$

Because $|g_n|$ tends to 0 at the ends of the strip $|b|\leqslant 2$, it is bounded ($\leqslant B_6$) in the whole strip according to the maximum modulus principle. In particular, $|f_n|\leqslant B_7$ on the disc $|\gamma|\leqslant 2$. A second underestimate of $|h|$ is obtained from the Poisson integral for $\lg |h|$: $\lim_{R\uparrow\infty} R^{-1} \lg |h(Re^{i\theta})| = 0$ $(\theta = \pi/4, 3\pi/4)$, and it follows from the resulting bound

$$|f_n| \leqslant B_8\, e^{\delta R} \quad (R \geqslant 1,\ \theta = \pi/4,\ 3\pi/4)$$

and its companion

$$|e^{-i\gamma/n} f_n| \leqslant B_9\, e^{\delta R} \quad (R \geqslant 1,\ \theta = 5\pi/4,\ 7\pi/4)$$

combined with an application of Phragmén–Lindelöf to each of the 4 sectors between $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$, that

$$|f_n| \leqslant B_{10}\, e^{(\delta+1/n)R}.$$

But now it is legitimate to suppose that as $n \uparrow \infty$, f_n tends on the whole plane to an entire function f_∞; moreover, this function is specified on the line $b=0$ since $\|f_n - f\|_\Delta$ tends to 0 as $n \uparrow \infty$. Accordingly, the entire function f_∞ is an extension of f, and since $|f_\infty| \leqslant B_{10}\, e(\delta R)$, it is clear that $f \in Z^{\cdot}_\Delta$ as desired.

If Δ is non-Hardy then it is possible for Z^{0+} to contain $Z^{\cdot}$ properly. Indeed let $\Delta(a)$ be even, non-increasing for $a > 0$, and non-Hardy. Then, as will be proved in 8, $Z^{0+} = Z \dotplus Z^{\cdot}$.

Δ non-Hardy does not ensure that $Z^{\cdot}$ is dense in Z; in fact if $\int_{-1}^{1} \lg \Delta/1 + a^2) = -\infty$ while $\Delta \geqslant 1/a^2$ $(|a| \geqslant 1)$, then $f \in Z^{\cdot}$ satisfies $\int |f|^2/(1+a^2) < \infty$, and a simple application of Phragmén–Lindelöf implies that f is constant; in short, $\dim Z^{\cdot} = 1$.

6 *c*. A condition that $Z^- \cap Z^+ = Z^{\cdot}$ (Δ Hardy)

$Z^- \cap Z^+ = Z^{\cdot}$ if Δ is a Hardy weight and if $\int_{-d}^{+d} \Delta^{-1} < \infty$ $(d < \infty)$.

Proof. The idea is that $f \in Z^- \cap Z^+$ is regular for $b \neq 0$ and can be continued across $b = 0$ if Δ is not too small (see T. Carleman [3] for a similar argument).

Given $f \in Z^- \cap Z^+$, then $fh \in H^{+2}$, $\lim_{b\downarrow 0} f(a+ib) = f(a)$ except at a set of points of Lebesgue measure 0 [7, p. 123], and so the Lebesgue measure of

$$A \equiv (a: \sup_{0 \leqslant b < \delta} |f(a+ib)| > \varepsilon^{-1}, |a| < d)$$

tends to 0 as δ and $\varepsilon \downarrow 0$; it is to be proved that

$$\sup_{0 \leqslant b < \delta} \int_A |f(a+ib)|\, da$$

is small for small δ and ε for each $d < \infty$. Bring in the summable weight

$$B = \Delta^{-1} \qquad (|c| \leqslant 2d)$$
$$= (1+c^2)^{-1} \quad (|c| > 2d);$$

then for larged,

$$\left(\int_A |f(a+ib)|\,da\right)^2$$
$$\leqslant \int |fh(a+ib)|^2 da \int_A [\Delta(a+ib)]^{-1} da$$
$$\leqslant \|f\|_\Delta^2 \int_A da \; e\left[\frac{1}{\pi}\int_{|c|\leqslant 2d} \frac{b}{(c-a)^2+b^2} \lg \Delta^{-1} dc\right] e\left[\frac{1}{\pi}\int_{|c|>2d} \frac{b}{(c-a)^2+b^2} \lg \Delta^{-1} dc\right]$$
$$\leqslant 2\|f\|_\Delta^2 \int_A da\, e\left[\frac{1}{\pi}\int \frac{b}{(c-a)^2+b^2} \lg B\, dc\right]$$

and an application of Jensen's inequality implies

$$\sup_{0 \leqslant b < \delta} \left[\int_A |f(a+ib)|\,da\right]^2 \leqslant 2\,\|f\|_\Delta^2 \int B dc \sup_{0 \leqslant b < \delta} \int_A \frac{b}{(c-a)^2+b^2} \frac{da}{\pi}$$
$$\downarrow 2\,\|f\|_\Delta^2 \int_{\bigcap_{\delta,\varepsilon>0} A} B dc = 0 \quad (\delta, \varepsilon \downarrow 0).$$

Using this appraisal, it follows that

$$\lim_{b \downarrow 0} \int_{-d}^{+d} |f(a+ib) - f(a)|\,da = 0;$$

the analogous result for $b<0$ follows from a similar appraisal. Choose c so that $f(c+ib)$ tends boundedly to $f(c)$ as $b \downarrow 0$ and define

$$g(\gamma) = \int_c^a f(\xi+ib)\,d\xi + i\int_0^b f(c+i\eta)\,d\eta \quad (\gamma = a+ib).$$

g is regular $(b \neq 0)$ since $f \in Z^- \cap Z^+$ is such, it is continuous across $b=0$ and hence entire, so $f=g'$ is likewise entire, and all that remains to be proved is that f is of minimal exponential type.

Because $fh \in H^{2+}$, $\int |\lg|fh||/(1+a^2) < \infty$, and since $\lg^+|f| \leqslant \lg^+|fh| - \lg^-|h|$, the integral $\int |\lg|f||/(1+a^2)$ is also convergent; also, $\lg|fh|$ is smaller than its Poisson integral, so

$$\lg^+|f(Re^{i\theta})| \leqslant \frac{1}{\pi}\int \frac{R\sin\theta \lg^+|f(c)|\,dc}{R^2 - 2Rc\cos\theta + c^2} \quad (0<\theta<\pi),$$

$\lg|h|$ being expressible by its Poisson integral since h is an outer function. According to this bound,

$$\int_0^\pi \lg^+ |f(Re^{i\theta})|\, d\theta \leqslant \frac{2}{\pi}\int_0 \lg^+ |f(c)| \lg \left|\frac{R+c}{R-c}\right| \frac{dc}{c}$$

and

$$\int_R^{2R} dR \int_0^\pi d\theta \lg^+ |f(Re^{i\theta})| \leqslant \frac{2}{\pi}\int_0 \lg^+ |f(c)|\, dc \int_{R/c}^{2R/c} \lg\left|\frac{t+1}{t-1}\right| dt$$

$$< B_1(1+R^2)\int_0 \frac{\lg^+|f(c)|}{1+c^2},$$

as a simple appraisal justifies. A similar bound holds for $\lg^+|f|$ in the lower half plane $b<0$, so that

$$\int_R^{2R} dR \int_0^{2\pi} d\theta \lg^+ |f(Re^{i\theta})| < B_2(1+R^2),$$

and it follows that between each large R and its double $2R$ can be found an R_1 with

$$\int_0^{2\pi} \lg^+ |f(R_1 e^{i\theta})|\, d\theta < 2B_2 R_1.$$

An application of the Poisson–Jensen formula now supplies us with the bound

$$\lg^+|f| < B_3 R \quad (R \uparrow \infty),$$

and a second application of the fact that $\lg^+|f|$ is smaller than its Poisson integral supplies the additional information that

$$\overline{\lim_{R\uparrow\infty}}\, R^{-1} \lg^+ |f(Re^{i\theta})| \leqslant 0 \quad (\theta = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4).$$

Phragmén–Lindelöf is now applied to each of the 4 sectors between, with the result that

$$\overline{\lim_{R\uparrow\infty}}\, R^{-1} \max_{0\leqslant\theta<2\pi} \lg^+ |f(Re^{i\theta})| \leqslant 0,$$

and the proof is complete.

A second proof of $Z^{0+}\subset Z^{\cdot}$ can be based on the above; indeed, if $f\in Z^{0+}$ and if f_n is chosen as in $6b$, then

$$\int |(f-f_n)h(a+i)|^2 da \leqslant \|f-f_n\|_\Delta^2 < 1/n^2,$$

and so $f(a+i)\in Z^{0+}_{\Delta(a+i)}$ with $\Delta(a+i)=|h(a+i)|^2$. But $\Delta(a+i)$ is positive and continuous, so

$$Z^{0+}_{\Delta(a+i)} \subset Z^{-}_{\Delta(a+i)} \cap Z^{+}_{\Delta(a+i)} = Z^{\cdot}_{\Delta(a+i)},$$

proving that $f(a+i)$ is entire of minimal exponential type.

$Z^{\cdot} \neq Z^{-} \cap Z^{+}$ if, for instance, $\int_{-1}^{+1} \Delta/a^2 < \infty$; indeed in this case,

$$\frac{1}{\pm ia+\delta} = \int_0^\infty e^{-\delta t} e^{\pm iat}\, dt \in Z^{\pm} \quad (\delta > 0),$$

while
$$\left\| \frac{1}{ia \pm \delta} - \frac{1}{ia} \right\|_\Delta^2 \leqslant \delta^2 \int_{|a|>1} \Delta + \int_{|a|\leqslant 1} \frac{\Delta}{a^2} \frac{\delta^2}{a^2+\delta^2}$$

tends to 0 as $\delta \downarrow 0$, so that $1/ia \in Z^{-} \cap Z^{+}$.

The Hardy weight $\Delta = a^2 e(-2|a|^{-\frac{1}{2}})/(1+a^4)$ illustrates the point that $f \in Z^{-} \cap Z^{+}$ can be regular in the punctured plane but have an essential singular point at $\gamma = 0$. Define $f = \gamma^{-1} \cos(1/\gamma^{\frac{1}{2}})$; then $f_\delta = f(\gamma + i\delta)$ $(\delta > 0)$ is of modulus $\leqslant |a|^{-1} e(1/|a|^{\frac{1}{2}})$ on the line so that $\|f - f_\delta\|_\Delta$ tends to 0 as $\delta \downarrow 0$, while, as an application of the Paley–Wiener theorem justifies, $f_\delta = \int_0^\infty e(iat) \hat{f}_\delta(t)\, dt$ with $\hat{f}_\delta$ and $t\hat{f}_\delta \in L^2[0, \infty)$. $f_{0+} = f \in Z^{+}$ follows and a similar argument with $\delta < 0$ proves that $f \in Z^{-}$ also.

6 *d*. A condition that genus $Z^{\cdot}=0$ (Δ Hardy)

Each $f \in Z^{\cdot}$ *is of genus* 0 *and* $\int_1 \lg \max_{0 \leqslant \theta < 2\pi} |f(Re^{i\theta})|/R^2 < \infty$ *if* $\int_1 \lg^{-} \Delta(ib)/b^2 > -\infty$ *or, and this is the same, if* $\int_1 \lg^{-} \Delta \lg a/a^2 > -\infty$.

Proof. To begin with, $\int_1 \lg^{-} \Delta(ib)/b^2$ and $\int_1 \lg^{-} \Delta(a) \lg a/a^2$ converge and diverge together; indeed, since $\int_1 \lg^{+} \Delta(a) \lg a/a^2 \leqslant \int_1 \Delta < \infty$, the convergence of $\int_1 \lg^{-} \Delta(a) \lg a/a^2$ combined with the Poisson formula

$$\lg \Delta(ib) = \frac{1}{\pi} \int \frac{b}{a^2+b^2} \lg \Delta(a)\, da,$$

leads at once to the bound

$$\int_1 \frac{|\lg \Delta(ib)|}{b^2} \leqslant \frac{1}{\pi} \int |\lg \Delta(a)|\, da \int_1 \frac{db}{b(b^2+a^2)},$$

the second integral converging, since

$$\int_1 \frac{db}{b(b^2+a^2)} \sim \frac{\lg a}{a^2} \quad (a \uparrow \infty).$$

On the other hand, if $\int_1 \lg^{-} \Delta(ib)/b^2 > -\infty$, then $\int_1 \lg^{-} \Delta(a) \lg a/a^2$ is not smaller than a positive multiple of

$$\int_1 \lg^- \Delta(a)\,da \frac{1}{\pi}\int_1 \frac{db}{b(b^2+a^2)} \geqslant \int_1 \frac{db}{b^2}\frac{1}{\pi}\int \frac{b}{a^2+b^2}\lg^-\Delta(a)\,da$$

$$= \int_1 \frac{db}{b^2}\left(\lg\Delta(ib) - \frac{1}{\pi}\int\frac{b}{a^2+b^2}\lg^+\Delta(a)\,da\right)$$

$$\geqslant \int_1 \lg^-\Delta(ib)/b^2 - \text{ constant } \times \int \lg^+\Delta(a) > -\infty.$$

Given $\int_1 \lg^-\Delta(ib)/b^2 > -\infty$, if $f \in Z^{\cdot}_{\Delta}$, then f is of genus 0 and

$$\int_1 \lg \max_{0\leqslant\theta<2\pi} |f(Re^{i\theta})|/R^2 < \infty;$$

indeed, since $\Delta(ib)$ is bounded $(b \geqslant 1)$,

$$\begin{aligned}\Delta^o(b) = \Delta^o(-b) &= \Delta(ib)/b^2 \quad (b>1)\\ &= 1 \qquad\qquad\quad (0\leqslant b\leqslant 1)\end{aligned}$$

is a Hardy weight, and if $f \in Z^{\cdot}_{\Delta}$, then $|fh|$ is bounded $(b \geqslant 1)$, $|fh^*|$ is bounded $(b \leqslant -1)$, and $\int |f(ib)|^2\Delta^o\,db < \infty$, i.e., $f(i\gamma) \in Z^{\cdot}_{\Delta^o}$. But then $\int_1 |\lg|f(ib)||/b^2 < \infty$, and combining this with $\int_1 |\lg|f(a)||/a^2 < \infty$ and an application of Carleman's theorem, one finds that the sum of the reciprocals of the moduli of the roots of f has to converge [2; 2.3.14], i.e., that the genus of f is 0. Because $f^+ = f + f^* \in Z^{\cdot}_{\Delta}$ satisfies

$$\int_1 \lg^+|f^+(ib)|b^2 < \infty \quad\text{and}\quad \int_1 \lg^+|f^+(a)|/a^2 < \infty,$$

it is of genus 0. It is also even, so $\int_1 \lg \max_{0\leqslant\theta<2\pi} |f_+(Re^{i\theta})|/R^2 < \infty$ [2; 2.12.5]; the same holds for $f_- = f - f^* \in Z^{\cdot}_{\Delta}$ since γf_- is entire, even, and of genus 0, so

$$\int_1 \lg \max_{0\leqslant\theta<2\pi} |f(Re^{i\theta})|/R^2 < \infty,$$

as stated.

$\int \lg^-\Delta(ib)/b^2$ can diverge even though each $f \in Z^{\cdot}_{\Delta}$ is of genus 0, as can be seen from the Hardy weight Δ:

$$\begin{aligned}e^{a^{\frac{1}{2}}}\Delta &= 1 \text{ on } [0,1)+[2,3)+\text{etc.}\\ &= e[-a/\lg^2(a+1)] \text{ on } [1,2)+[3,4)+\text{etc.}\end{aligned}$$

Δ is Hardy since $(a\lg^2(a+1))^{-1}$ is summable, while

$$\int_1 \lg^-\Delta \lg a/a^2 \leqslant \sum_{\substack{d \text{ odd}\\ d\geqslant 1}} \int_d^{d+1} (a \lg(a+1))^{-1} = -\infty,$$

so that $\int_1 \lg\Delta(ib)/b^2 = -\infty$. Given $f \in Z^{\cdot}_{\Delta}$,

$$B_1 = \|f\|^2_\Delta > \int_{2d}^{2d+1} |f|^2 e^{-2a^{\frac{1}{2}}} > |f|^2 e^{-2a^{\frac{1}{2}}}$$

at some point $2d \leqslant a < 2d+1$ $(d \geqslant 0)$, so an application of the Duffin–Schaeffer theorem [2; 10.5.1] applied to $fe(-\gamma^{\frac{1}{2}})$ on the half plane $a \geqslant 0$ supplies us with the bound $|f|\, e(-a^{\frac{1}{2}}) < B_2$ on the half line $a \geqslant 0$. $|f|\, e(|a|^{\frac{1}{2}}) < B_3$ on the left half line for similar reasons. Phragmén–Lindelöf applied to $fe(-(2\gamma)^{\frac{1}{2}} e^{-i\pi/4})$ on the half plane $b \geqslant 0$ together with an analogous argument on $b > 0$ supplies the bound $|f| < B_4\, e[(2R)^{\frac{1}{2}}]$ on the whole plane, and it follows that f is of genus 0.

6 *e*. Rational weights

$\dim Z^{+/-} = d < \infty$ if and only if Δ is a rational function of degree $2d$.

See, for example, Hida [6] from whom the following proof is adapted.

Proof. $\dim Z^{+/-} = d < \infty$ implies $Z^{+/-} \neq Z$, so Δ is a Hardy weight and can be expressed as $|h|^2$ with h outer. Define the Fourier transform $\hat{f}(t) = (1/2\pi) \int e(-iat) f(a)\, da$ and note that if $\mathfrak{j} = h/h^*$ and if p is the projection upon H^{2-}, then $Z^{+/-} h = \mathfrak{j} p \mathfrak{j}^{-1} H^{2+}$ is of the same dimension d as

$$\begin{aligned}[p\mathfrak{j}^{-1} H^{2+}]^\wedge &= \text{span}\,[p\mathfrak{j}^{-1} e^{iat} h : t > 0]^\wedge = \text{span}\,[p e^{iat} h^* : t > 0]^\wedge \\ &= \text{span}\,[(e^{iat} h^*)^\wedge\, i(s) : t > 0] \\ &= \text{span}\,[(\hat{h}(t-s)\, i(s) : t > 0],\end{aligned}$$

where $i(s)$ is the indicator of $s \leqslant 0$. $[p\mathfrak{j}^{-1} H^{2+}]^\wedge$ has a unit perpendicular basis $f_1, \ldots, f_d$, and $\hat{h}(t-s) = c_1(t) f_1(s) + \ldots + c_d(t) f_d(s)$ $(s \leqslant 0)$ with (real) coefficients $c_1, \ldots, c_d$. Choose $g_1, \ldots, g_d \in C^\infty(-\infty, 0]$ vanishing near $-\infty$ and 0 with $\det[\int_{-\infty}^0 f_i g_j] \neq 0$; then

$$\sum_{i \leqslant d} c_i \int_{-\infty}^0 f_i g_j\, ds = \int_{-\infty}^0 \hat{h}(t-s)\, g_j\, ds \quad (j \leqslant d,\ t > 0),$$

so that $c_1, \ldots, c_d \in C^\infty(0, \infty)$, and it follows that $\hat{h} \in C^\infty(0, \infty)$ also. Given $0 < t_0 < \ldots < t_d$, a dependence with non-trivial (real) coefficients must prevail between $\hat{h}(t_0 - s), \ldots, \hat{h}(t_d - s)$ $(s \leqslant 0)$, and since $\hat{h} \in C^\infty(0, \infty)$, it is possible to find a differential operator D with constant (real) coefficients and degree $\leqslant d$ annihilating $\hat{h}$ on the half line $t > 0$. But this means that $\hat{h}$ is a sum of $\leqslant d$ terms $t^a e^{bt} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ct$, the permissible a filling out a series 0, 1, 2, etc., $b < 0$, and the trigonometrical factors either absent or both permissible.

Δ rational of degree $\leqslant 2d$ follows at once upon taking the inverse Fourier transform. On the other hand, if Δ is rational of degree $2d$, then it is a Hardy weight $|h|^2$ with h outer, h is also rational (of degree d), $\hat{h}$ is a sum of terms $t^a e^{bt} \begin{smallmatrix}\cos\\ \cdot\\ \sin\end{smallmatrix} ct$ as above, the number of them coinciding with $\deg h$ and the trigonometrical factors either absent or present in pairs, and $\dim Z^{+/-} = d$ follows from dim span $[\hat{h}(t-s)\, i(s) : t > 0] = d$.

Δ rational of degree $2d$ implies that

(a) $h = p_0 p_1 / p_2$, p_0, p_1, p_2 *being polynomials in* $i\gamma$ *with roots on the line in the case of* p_0 *and in the open half plane* $b < 0$ *in the case of* p_1 *and* p_2, *and of degrees* d_0, d_1, d_2 $(=d)$ *with* $d_0 + d_1 < d_2$,

(b) $Z^{\bullet} = Z^{0+} = Z. =$ *polynomials in* $i\gamma$ *of degree* $< d_2 - d_1 - d_0$,

(c) $Z^- \cap Z^+ = 1/p_0 \times$ *polynomials in* $i\gamma$ *of degree* $< d_2 - d_1$,

(d) $Z^{+/-} = 1/p_0 p_1^* \times$ *polynomials in* $i\gamma$ *of degree* $< d_2 (=d)$,

esp.,

(e) $Z^{\bullet} = Z^- \cap Z^+$ *if and only if* h *has no roots on* $b = 0$,

(f) $Z^- \cap Z^+ = Z^{+/-}$ *if and only if* h *has no roots in* $b < 0$,

(g) $Z^{+/-} = Z^- \cap Z^+ = Z^{\bullet} - Z^{0+} = Z.$ *if and only if* h *has no roots at all.*

Proof of (a). Obvious.

Proof of (b). $f \in Z^{\bullet}_{\Delta}$ implies $\int |f|^2/(1+a^2)^d < \infty$, and a simple application of Phragmén–Lindelöf implies that f is a polynomial; the bound on its degree is obvious.

Proof of (c). $f \in Z^- \cap Z^+$ implies $p_0 f \in Z^-_{\Delta^o} \cap Z^+_{\Delta^o}$ $(\Delta^o = |p_1/p_2|^2)$, and since Δ^o is bounded from 0 on bounded intervals, $p_0 f \in Z^{\bullet}_{\Delta^o}$ (Section 6 c). But then $p_0 f$ has to be a polynomial as in the proof of (b) above, the bound on the degree of this polynomial is obvious, and the rest of the proof is a routine application of $Z^- \cap Z^+ h = \mathfrak{j} H^{2-} \cap H^{2+}$ $(\mathfrak{j} = h/h^*)$.

Proof of (d). Use the formula $Z^{+/-} h = \mathfrak{j} H^{2-} \cap (1/j_1) H^{2+}$ $(\mathfrak{j} = j_2/j_1)$ of Section 3 and match dimensions.

Proof of (e), (f), (g). Obvious.

7. A condition that $Z^{+/-} = Z^{\bullet}$ (Δ Hardy)

Given a Hardy weight $\Delta = |h|^2$ *(h outer),* $Z^{+/-} = Z^{\bullet}$ *if and only if* h *is the reciprocal of an entire function of minimal exponential type.*

Proof. Suppose h is the reciprocal of an entire function f of minimal exponential type; then $h=1/f$ implies $\int_{|a|<d}\Delta^{-1}<\infty$ $(d<\infty)$, so $Z^{\bullet}=Z^{-}\cap Z^{+}$ (6c), and to complete the proof of $Z^{+/-}=Z^{\bullet}$, it is enough to check that $\mathfrak{j}=h/h^{*}=f^{*}/f$ is an inner function (Section 3(c)). But $1/f=h$ being outer, it is root-free $(b\geqslant 0)$, and

$$\lg|f|=\frac{1}{\pi}\int\frac{b}{(c-a)^2+b^2}\lg|f|\,dc\quad(b>0),$$

while f^{*}, as an entire function of minimal exponential type with $\int\lg|f^{*}|/(1+a^2)<\infty$, satisfies

$$\lg|f^{*}|\leqslant\frac{1}{\pi}\int\frac{b}{(c-a)^2+b^2}\lg|f^{*}|\,dc\quad(b>0),$$

so f^{*}/f is regular $(b>0)$ with

$$|f^{*}/f|=1\quad(b=0)$$

$$|f^{*}/f|\leqslant e\left[\frac{1}{\pi}\int\frac{b}{(c-a)^2+b^2}\lg|f^{*}/f|\,dc\right]=1\quad(b>0),$$

i.e., f^{*}/f is inner.

On the other hand, if $Z^{+/-}=Z^{\bullet}$ and if p is the projection upon H^{2-}, then the projection of $e(iat)$ $(t>0)$ upon Z^{-}:

$$h^{-1}\mathfrak{j}p\mathfrak{j}^{-1}e^{iat}h\quad(\mathfrak{j}=h/h^{*})$$

$$=h^{-1}\mathfrak{j}p\,e^{iat}h^{*}$$

$$=h^{-1}\mathfrak{j}\frac{1}{2\pi}\int_{-\infty}^{0}e^{ias}ds\int e^{-ics}e^{ict}h^{*}\,dc$$

$$=h^{-1}\mathfrak{j}\frac{1}{2\pi}\int_{-\infty}^{0}e^{ias}\,ds\left(\int e^{-ic(t-s)}h\,dc\right)^{*}$$

$$=h^{-1}\mathfrak{j}\frac{1}{2\pi}\int_{-\infty}^{0}e^{ias}\,ds\,\hat{h}(t-s)\quad\left(\hat{h}=\frac{1}{2\pi}\int e^{-iat}h\,dt=\hat{h}^{*}\right),$$

belongs to $Z^{\bullet}$, and since its conjugate also belongs to $Z^{\bullet}$,

$$\frac{e^{-iat}}{2\pi h}\int_{t}^{\infty}e^{ias}\hat{h}\,ds\equiv f_t(a)\in Z^{\bullet}_{\Delta}\quad(t>0).$$

Choose $t>0$ belonging to the Lebesgue set of $\hat{h}$ so that $\lim_{\delta\downarrow 0}\delta^{-1}\int_t^{t+\delta}\hat{h}ds=\hat{h}(t)\neq 0$ and $\delta^{-1}\int_t^{t+\delta}|\hat{h}|\,ds$ is bounded as $\delta\downarrow 0$.

$$2\pi \|f_{t+\delta}-f_t\|_{\Delta^+} \leqslant \left\| (e^{-ia(t+\delta)}-e^{-iat}) \int_{t+\delta}^{\infty} e^{ias}\hat{h}ds \right\|_{1/(1+a^2)} + \left\| e^{-iat}\int_t^{t+\delta} e^{ias}\hat{h}ds \right\|_{1/(1+a^2)}$$

$$= \left\| (e^{ia\delta}-1) \int_{t+\delta}^{\infty} e^{ias}\hat{h}ds \right\|_{1/(1+a^2)} + \left\| \int_t^{t+\delta} e^{ias}\hat{h}ds \right\|_{1/(1+a^2)}$$

$$\leqslant \text{constant} \times \delta \left\| \int_{t+\delta}^{\infty} e^{ias}\hat{h}ds \right\|_1 + \int_t^{t+\delta} |\hat{h}|\, ds \left(\int \frac{da}{1+a^2} \right)^{1/2}$$

$$< \text{constant} \times \delta,$$

and it follows, thanks to the bound $\overline{\lim}_{R\uparrow\infty} R^{-1} \max_{0\leqslant\theta<2\pi} \lg \sigma^{\bullet}(Re^{i\theta}) \leqslant 0$, that $\delta^{-1}(f_{t+\delta}-f_t)$ can be made to tend on the whole plane to some $f^{\bullet} \in Z^{\bullet}_{\Delta^+}$ as $\delta \downarrow 0$ via some series $\delta_1 > \delta_2 >$ etc. Going back to the definition of $f_t \equiv f$, it develops that

$$-\hat{h}(t)/2\pi h(a) = [iaf + f^{\bullet}] \in Z^{\bullet}_{\Delta^+},$$

and the proof is complete.

8. A condition that $Z^{0+} = Z$

$Z^{0+} = Z$ if $\int_1 da/a^2 \lg \int_a \Delta e^{-2B} = -\infty$ with $0 \leqslant B \in \uparrow$, $\int_1 e^{-2B} < \infty$, and $\int_1 B/a^2 < \infty$.

Δ has to be non-Hardy for this integral to diverge since

$$\int_1 \frac{da}{a^2} \lg \int_a \Delta e^{-2B} \geqslant \int_1 \frac{da}{a^2} \lg \left[a^3 \int_a \frac{\Delta e^{-2B}}{c^3} \right]$$

$$= \int_1 \frac{\lg(2a)}{a^2} da + \int_1 \frac{da}{a^2} \lg \left[\frac{a^2}{2} \int_a \frac{\Delta e^{-2B}}{c^3} \right]$$

$$\geqslant \int_1 \frac{\lg(2a)}{a^2} da + \int_1 \frac{da}{a^2} \left[\frac{a^2}{2} \int_a \frac{\lg \Delta e^{-2B}}{c^3} \right]$$

$$\geqslant \int_1 \frac{\lg(2a)}{a^2} da + \frac{1}{2} \int_1 da \int_a \frac{\lg^- \Delta}{c^3} - \int_1 da \int_a \frac{B}{c^3}$$

$$> \text{constant} + \frac{1}{2} \int_1 \lg^- \Delta/a^2;$$

also, if $\Delta \in \downarrow$, then $\int_1 da\, a^{-2} \lg \int_a \Delta e^{-2B}$ and $\int_1 \lg^- \Delta/a^2$ converge or diverge together, since under this condition,

$$\int_1 \frac{da}{a^2} \lg \int_a \Delta e^{-2B} \leqslant \int_1 \frac{da}{a^2} \left(\lg \Delta + \lg \int_a e^{-2B} \right)$$
$$\leqslant \int_1 \lg \Delta/a^2 + \lg \int_1 \frac{da}{a^2} \int_a e^{-2B}$$
$$< \int_1 \lg \Delta/a^2 + \text{constant};$$

esp., if $\Delta \in \downarrow$, then $Z^{0+} = Z$ if and only if $\int_1 \lg \Delta/a^2 = -\infty$.

As to the proof of the original statement, if $\int_1 da\, a^{-2} \lg \int_a \Delta e^{-2B} = -\infty$ with B as above and if $Z^{0+} \neq Z$, then $Z^{|t|<\delta} \neq Z$ for small δ, and it is possible to find $f \in Z$ with $\int f e(iat) \Delta\, da = 0$ $(|t| < \delta)$. But

$$\int_a |f| \Delta e^{-B} \leqslant \|f\|_\Delta \left(\int_a \Delta e^{-2B} \right)^{\frac{1}{2}} \quad (a \geqslant 1),$$

so that
$$\int_1 \frac{da}{a^2} \lg \int_a |f| \Delta e^{-B} = -\infty,$$

and according to Levinson [8, p. 81], this cannot happen unless $f = 0$.

9. Discussion of Z.

I. O. Hačatrjan's contribution to the Bernstein problem [5] is adapted as follows.

Consider the span $Z. = Z._\Delta$ of (real) polynomials p of $i\gamma$ belonging to Z, let $\int a^{2d} \Delta < \infty$ $(d \geqslant 1)$, let $\sigma.(\gamma)$ be the least upper bound of $|p(\gamma)|$ for $p \in Z._{\Delta^+}$ with $\|p\|_{\Delta^+} \leqslant 1$, and let us prove that the following alternative holds:

either $\sigma. \equiv \infty$ $(b \neq 0)$,

$$\sup \int \frac{\lg^+ |p|}{1+a^2} = \int \frac{\lg \sigma.}{1+a^2} = \infty, \text{ for } p \in Z._{\Delta^+} \text{ with } \|p\|_{\Delta^+} \leqslant 1,$$

and $Z. = Z$,

or $\lg \sigma.$ *is a continuous, non-negative, subharmonic function,*

$$\int \frac{\lg \sigma.}{1+a^2} < \infty,$$

$$\lg \sigma.(\gamma) \leqslant \frac{1}{\pi} \int \frac{b}{(c-a)^2 + b^2} \lg \sigma.(c)\, dc \quad (\gamma = a + ib,\ b > 0),$$

$$\varlimsup_{R \uparrow \infty} R^{-1} \max_{0 \leqslant \theta < 2\pi} \lg \sigma.(Re^{i\theta}) \leqslant 0,$$

and $Z. \neq Z$;

in the second case, $Z. \subset Z^{\cdot}$, *the two coinciding if and only if* $\sigma. \equiv \sigma^{\cdot}$ $(b \neq 0)$.

Proof. The proof is identical to the discussion of $Z^{\bullet}$ (Section 4), excepting the final statement to which attention is now directed.

Given $\sigma_{\bullet} = \sigma^{\bullet} < \infty$ while $Z_{\bullet} \neq Z^{\bullet}$, then it would be possible to find $f \in Z^{\bullet}_{\Delta}$, $f \not\equiv 0$, with $\int f^* a^d \Delta = 0$ $(d \geqslant 0)$; this implies

$$\int f^* \frac{p - p(\beta)}{c - \beta} \Delta = 0 \quad (\beta = a + ib,\ b \neq 0),$$

and it follows that

$$\left| \int \frac{f^* \Delta}{c - \beta} \right| = \left| \int \frac{f^* \Delta p}{(c - \beta)\, p(\beta)} \right| \leqslant \left\| \frac{c - i}{c - \beta} f \right\|_{\Delta} |p(\beta)|^{-1} \|p\|_{\Delta^+} \quad (\beta = a + ib,\ b \neq 0),$$

esp.,

$$\left| \int \frac{f^* \Delta}{c - ib} \right| = o(\sigma_{\bullet}(ib)^{-1}) \quad \text{as } |b| \uparrow \infty.$$

Chose $g \in Z^{\bullet}_{\Delta^+}$; then $\int f^* g \Delta (c - \beta)^{-1}$ tends to 0 at both ends of $a = 0$ so that

$$\hat{g} \equiv \int f^* \frac{g - g(\beta)}{c - \beta} \Delta$$

satisfies

$$\begin{aligned} |\hat{g}(ib)| &\leqslant o(1) + |g(ib)| \left| \int \frac{f^* \Delta}{c - ib} \right| \\ &= o(1) + |g(ib)|\, o(\sigma_{\bullet}(ib)^{-1}) \\ &= o(1) + |g(ib)|\, o(\sigma^{\bullet}(ib)^{-1}) \\ &= o(1) \quad (|b| \uparrow \infty), \end{aligned}$$

and since $\hat{g}$ is entire of minimal exponential type, Phragmén–Lindelöf implies $\hat{g} \equiv 0$. But then $\int f^* g (c - \beta)^{-1} \Delta = g(\beta) \int f^* \Delta (c - \beta)^{-1} = 0$ if β is a root of $g \in Z^{\bullet}_{\Delta^+}$ $(b \neq 0)$, so taking $g = (\gamma - i) f \in Z^{\bullet}_{\Delta^+}$ and $\beta = i$, $\|f\|^2 = \int f^* g (c - i)^{-1} \Delta = 0$, and the proof is complete.

10 *a*. Special case ($1/\Delta = 1 + c_1 a^2 +$ etc.)

Hačatrjan [5] states the analogue for the Bernstein problem of the following result:

If $1/\Delta = 1 + c_1 a^2 + c_2 a^4 +$ *etc.* $(c_1, c_2, \text{etc.} \geqslant 0)$ *and if* $\int a^{2d} \Delta < \infty$ $(d \geqslant 0)$, *then either* Δ *is non-Hardy and* $Z_{\bullet} = Z$ *or* Δ *is Hardy and* $Z_{\bullet} = Z^{\bullet}$.

Proof. $p_d = \sum_{n \leqslant d} c_n \gamma^{2n}$ can be expressed as $|q_d|^2$, q_d being a polynomial in $i\gamma$ of degree d with no roots in the closed half plane $b \geqslant 0$. As $d \uparrow \infty$,

$$\lg |q_d(i)|^2 = \frac{1}{\pi}\int \frac{\lg |q_d|^2}{1+c^2} \uparrow \frac{1}{\pi}\int \frac{\lg \Delta^{-1}}{1+c^2}$$

while
$$\|q_d\|^2_{\Delta^+} = \frac{1}{\pi}\int \frac{p_d \Delta}{1+c^2} \uparrow 1,$$

so *either* $\int \lg \Delta/(1+c^2) = -\infty$, $\sigma.(i)=\infty$, and $Z.=Z$ *or* Δ is Hardy ($\Delta = |h|^2$ with h outer). Because $|q_d|^2 = p_d \leqslant \Delta^{-1}$, an application of Lebesgue's dominated convergence test shows that $h^{-1} = \lim_{d\uparrow\infty} q_d$ ($b \geqslant 0$) in the second case.

Now in the second case, if $f \in Z^{\cdot}_{\Delta}$ is perpendicular to $Z._{\Delta}$, if $g \in Z^{\cdot}_{\Delta^+}$, and if

$$\hat{g}(\beta) \equiv \int f^* \frac{g - g(\beta)}{c-\beta}\Delta$$

as before, then

$$\left| q_d(ib) \int \frac{f^* \Delta\, dc}{c-ib}\right| = \left|\int \frac{f^* q_d \Delta\, dc}{c-ib}\right| \leqslant \|f\|_\Delta \left(\int \frac{|q_d|^2 \Delta\, dc}{c^2+b^2}\right)^{1/2} \leqslant \|f\|_\Delta \left(\int \frac{dc}{c^2+b^2}\right)^{1/2} = \|f\|_\Delta (\pi/b)^{1/2},$$

and so
$$|\hat{g}(ib)| \leqslant \left|\int \frac{f^* g \Delta\, dc}{c-ib}\right| + |g(ib)| \left|\int \frac{f^* \Delta\, dc}{c-ib}\right|$$
$$\leqslant \|f\|_\Delta \left(\int \frac{c^2+1}{c^2+b^2}|g|^2 \Delta^+ dc\right)^{1/2} + \inf_{d>0}\left|\frac{g(ib)}{q_d(ib)}\right| \left|\int \frac{f^* q_d \Delta\, dc}{c-ib}\right|$$
$$= o(1) + |gh(ib)|\, \|f\|_\Delta (\pi/b)^{1/2}.$$

Since the Poisson integral applies as an inequality to $\lg|(\gamma+i)^{-1} gh|$ and as an equality to $\lg|\gamma+i|$,

$$|gh(ib)|^2 \leqslant e\left[\frac{1}{\pi}\int \frac{b}{b^2+c^2}\lg |gh|^2\right] \leqslant \frac{1}{\pi}\int \frac{b(c^2+1)}{b^2+c^2}|g|^2 \Delta^+ = o(b),$$

and so $\lim_{b\uparrow\infty} |\hat{g}(ib)| = 0$. Repeating the proof as $b \downarrow -\infty$ justifies $\lim_{b\downarrow -\infty} |\hat{g}(ib)| = 0$, and now $\hat{g} = f = 0$ follows as in Section 9.

A special case of the above is the fact that if h is the reciprocal of an entire function and if the roots of h^{-1} fall in the sector $-3\pi/4 \leqslant \theta \leqslant -\pi/4$, then $Z.=Z^{\cdot}$; obvious improvements can be made, but $Z.=Z^{\cdot}$ does not hold without some condition on the roots of h^{-1} as the example of Section 11 proves.

As a second application, it will be proved that

$Z.=Z^{\cdot}$ *in case* $\Delta(a) = e(-2|a|^p)$ $(0<p<1)$;

similar but more complicated cases can be treated in the same fashion (see below).

Proof. It suffices to construct a weight $\Delta^o = (1 + c_1 a^2 + \text{etc.})^{-1}$ with non-negative coefficients, positive multiples of which bound Δ above and below. Define $\#(R) = [\theta R^p + 1/2]$ with an adjustable $\theta > 0$, the bracket denoting the integral part, and let

$$-\lg \Delta^o(a) = \int_0 \lg\left(1 + \frac{a^2}{R^2}\right) \#(dR) = 2a^2 \int_0 \frac{\#(R)\, dR}{(a^2 + R^2)\, R}$$

$$= \frac{2a^2}{p} \int_0 \frac{[\theta c + 1/2]\, dc}{(a^2 + c^{2/p})\, c} \quad (c = R^p)$$

$$= J_1 + J_2$$

with

$$J_1 = \frac{2a^2}{p} \int_0 \frac{[\theta c + 1/2] + 1/2 - (\theta c + 1/2)}{(a^2 + c^{2/p})\, c}\, dc$$

and

$$J_2 = \frac{2a^2 \theta}{p} \int_0 (a^2 + c^{2/p})^{-1}\, dc.$$

In J_2, subtitute $c = |a|^p t$ and let $\theta^{-1} = (2/p) \int_0 (1 + t^{2/p})^{-1}$, obtaining $J_2 = 2|a|^p$. Coming to J_1, note that the numerator under the integral sign is periodic and that its average over a period is 0, so that J_1 tends to a constant as $|a| \uparrow \infty$. J_1 is then bounded, so Δ is bounded above and below by positive multiples of Δ^o, and the proof is complete.

$Z_\bullet = Z^\bullet$ also holds in the more general case of a Hardy weight.

$$\Delta = \Delta(0)\, e\left(-\int_0^{|a|} \frac{\omega(c)}{c}\, dc\right)$$

provided $\omega \in \uparrow$ and $\omega(c) \lg c$ tends to ∞ as $c \uparrow \infty$.

Proof. Under the above condition it is possible, according to Y. Domar [4], to find a reciprocal weight $1/\Delta^o = 1 + c_1 a^2 + \text{etc.}$ with non-negative coefficients such that Δ is bounded above by a positive multiple of Δ^o and below by a positive multiple of $\Delta^\theta = \Delta^o(\theta a)$ with a constant depending upon $\theta > 1$ alone. Because

$$Z_{\bullet\Delta^\theta} = Z^\bullet_{\Delta^\theta} \supset Z^\bullet_\Delta,$$

each $f \in Z^\bullet_\Delta$ can be approximated in Z_{Δ^θ} by a polynomial p so as to have

$$\int |f(a/\theta) - p(a/\theta)|^2\, \Delta \leqslant \text{constant} \times \theta\, \| f - p \|^2_{\Delta^\theta}$$

small, and to complete the proof it suffices to check that $f_\theta(a) = f(a/\theta)$ tends to f in Z_Δ as $\theta \downarrow 1$. But this is obvious from the fact that

$$\|f_\theta\|_\Delta^2 = \theta \int |f|^2 \Delta(\theta a) \sim \|f\|_\Delta^2 \quad (\theta \downarrow 1)$$

while f_θ tends to f pointwise under a local bound.

By the same method it is easy to prove that if Δ has the above form with $\omega \in \uparrow$ and $\int_1 \omega/c^2 = \infty$ (non-Hardy case), then $Z_\bullet = Z$.

Domar's paper was brought to our notice through the kindness of Professor L. Carleson.

10 *b*. A special case ($\Delta = e^{-2|a|^{\frac{1}{2}}}$)

$\Delta = \exp(-2|a|^{\frac{1}{2}})$ falls under the discussion of $10a$, but it is entertaining to check $Z^\bullet = Z_\bullet$ from scratch using the following special proof.

$\Delta = |h|^2$ with

$$h = e[-(2\gamma)^{\frac{1}{2}} e^{-i\pi/4}] = \int_0^\infty e^{i\gamma t} \frac{e^{-1/2t}}{(2\pi t^3)^{\frac{1}{2}}} dt,$$

and h is outer since

$$\lg|h(i)| = -2^{\frac{1}{2}} = \frac{1}{\pi} \int \frac{\lg|h|}{1+a^2}$$

(see [7, p. 62]).

Given $f \in Z_\Delta^\bullet$, a simple application of Phragmén–Lindelöf supplies us with the bound

$$f(\gamma) \leqslant Be[(\sqrt{2}+\delta)\sqrt{R}] \quad (\delta > 0);$$

hence, $|f(\gamma^2)| \leqslant Be[(\sqrt{2}+\delta)R]$, and according to Pólya's theorem [2; 5.3.5],

$$f(\gamma^2) = \int e^{\gamma w} g = \int e^{-\gamma w} g = \int \cosh(\gamma w)\, g\, dw,$$

i.e.,

$$f(\gamma) = \int \cosh(\sqrt{\gamma} w)\, g\, dw,$$

the integral being extended over $|w| = 2^{\frac{1}{2}} + \delta$ and g being regular outside $|w| = 2^{\frac{1}{2}}$ and at ∞. Accordingly, if $f \in Z^\bullet$ is perpendicular to $Z_\bullet$, then

$$
\begin{aligned}
0&=\int f a^d \Delta\, da=\int g\, dw \int \cosh(\sqrt{aw})\, a^d \Delta\, da\\
&=\int g\left[\int_0^\infty \cosh(\sqrt{aw})\, a^d e^{-2a^{\frac12}}+\int_0^\infty \cos(\sqrt{aw})\,(-a)^d e^{-2a^{\frac12}}\right]\\
&=\int g D^{2d}\left[\int_0^\infty \cosh(\sqrt{aw})\, e^{-2a^{\frac12}}+\int_0^\infty \cos(\sqrt{aw})\, e^{-2a^{\frac12}}\right]\\
&=2\int g D^{2d+1}\left[\int_0^\infty \sinh(aw)\, e^{-2a}+\int_0^\infty \sin(aw)\, e^{-2a}\right]\\
&=\int g D^{2d+1}\left[\frac{1}{2-w}-\frac{1}{2+w}+\frac{1}{2i+w}-\frac{1}{2i-w}\right]\\
&=\int g D^{2d+1}\frac{16w}{16-w^4}.
\end{aligned}
$$

Because $\int e^{\gamma w} g=f(\gamma^2)$ is an even function, $\int g w^d=0$ (d odd) and since $w/(16-w^4)$ is a sum of powers w^d ($d\equiv 1(4)$), it follows that

$$0=\int g D^d\left[\frac{1}{2-w}-\frac{1}{2+w}+\frac{1}{2i+w}-\frac{1}{2i-w}\right]\quad (d\geqslant 0),$$

and so

$$
\begin{aligned}
0&=\int g\left[\frac{1}{2-w+t}-\frac{1}{2+w-t}+\frac{1}{2i+w-t}-\frac{1}{2i-w+t}\right]dw\\
&=g(t+2)+g(t-2)-g(t-2i)-g(t+2i)
\end{aligned}
$$

for small $|t|$.

Draw four circles, each of radius $2^{\frac12}$, having centers at 2, $2i$, -2 and $-2i$ respectively. The circles with centers at 2 and $2i$ are tangent at A, which is $1+i$. The circles with centers at 2 and $-2i$ are tangent at B, which is $1-i$. The point C is $-3+i$ and lies on the circle with center at -2. Using this diagram depicting 4 discs on each of which just one of the summands can be singular, it follows that $g(t-2)=-g(t+2)+g(t-2i)+g(t+2i)$ can be singular only at A and B since the second member is non-singular on the rest of $|t-2|\leqslant 2^{\frac12}$. Now if $g(t-2)$ is singular at A, then $g(t+2)$ is singular at $C=A-4$ and that is impossible, so $g(t-2)$ cannot be singular at A, nor, for similar reasons, at B. But then g is entire, and by Cauchy's theorem, $f(\gamma^2)=\int \cosh(\gamma w)\, g=0$, completing the proof.

$Z^-=Z^{+/-}+Z^-\cap Z^+=Z^\bullet=Z^{0+}=Z.$ can be proved at little extra cost. $Z^-\cap Z^+=Z^\bullet$ is obvious from Section 6, and so it suffices to prove that $\mathfrak{j}=h/h^*=e[2i\,\mathrm{sgn}(a)\,|a|^{\frac12}]$ is not a ratio j_2/j_1 of inner functions (Section 3). But in the opposite case, $\mathfrak{j}f\in H^{2+}$ ($f=j_1 h$), so that

$$0=\frac{1}{2}\int e^{-iat}\,j f\,da \quad (t<0)$$

$$=\operatorname{Re}\left[\int_0^\infty e^{-iat}\,e^{2ia^{\frac{1}{2}}} f\,da\right]=\operatorname{Im}\left[\int_0^\infty e^{bt}\,e^{(2b)^{\frac{1}{2}}(i-1)} f(ib)\,db\right],$$

since

$$\left|\int_0^{\pi/2} e^{-iRe^{i\theta}t}\,e^{2iR^{\frac{1}{2}}e^{i\theta/2}} f(Re^{i\theta})\,Re^{i\theta}\,id\theta\right|\leqslant\int_0^{\pi/2} e^{R\sin\theta t}\,e^{-2R^{\frac{1}{2}}\sin\theta/2}\,e^{-(2R)^{\frac{1}{2}}\cos(\theta/2-\pi/4)}\,Rd\theta$$

tends to 0 as $R\uparrow\infty$. Because $f=f^*$ $(a=0)$,

$$0=\operatorname{Im}\left[e^{(2b)^{\frac{1}{2}}(i-1)} f(ib)\right]=\sin(2b)^{\frac{1}{2}}\,e^{-(2b)^{\frac{1}{2}}} f(ib) \quad (b\geqslant 0),$$

and that is absurd.

An entertaining illustration of the delicacy of the projection $Z^{+/-}$ is thus obtained. $Z^{+/-}\neq Z^{\cdot}$ as was just proved, so naturally the condition that $Z^{+/-}=Z^{\cdot}$, to wit, that $\Delta=|f|^{-2}$ with f entire of minimal exponential type, does not hold. But as proved in $10a$, $e(-2|a|^{\frac{1}{2}})$ is bounded above and below by positive multiples of such a weight.

11. An example (Δ Hardy, $\dim Z_{\cdot}=\infty$, $Z^{\cdot}=Z^{0+}\neq Z_{\cdot}$)

A weight Δ exists with the following properties:

(a) $\int \lg\Delta/(1+a^2)>-\infty$, *i.e., Δ is a Hardy weight,*

(b) $\int a^{2d}\Delta<\infty$ $(d\geqslant 0)$, *i.e.,* $\dim Z_{\cdot}=\infty$,

(c) $Z_{\cdot}\neq Z^{\cdot}=Z^{0+}$.

Consider for the proof

$$\delta_n=1/\sinh\pi n,\quad \gamma_{+n}=n^2-i\delta_n,\quad \gamma_{-n}=-n^2-i\delta_n,$$

$$1/h(\gamma)=\prod_{|n|>0}\left(1-\frac{\gamma}{\gamma_n}\right),\quad \Delta=|h|^2,$$

$$f=\frac{\sin\pi\sqrt{\gamma}\,\sinh\pi\sqrt{\gamma}}{\pi^2\gamma}=\prod_{n\geqslant 1}\left(1-\frac{\gamma^2}{n^4}\right),\quad\text{and}\quad g=f/(1-\gamma^2)=\prod_{n\geqslant 2}\left(1-\frac{\gamma^2}{n^4}\right),$$

and break up the proof into a series of simple lemmas.

(a) $0<B_1<|fh|<B_2$ *if* $|\gamma\pm n^2|\geqslant\frac{1}{2}$ $(n\geqslant 1)$, *while* $0<B_3<|fh|\,|(\gamma-\gamma_{\pm n})/(\gamma\mp n^2)|<B_4$ *if* $|\gamma\pm n^2|<\frac{1}{2}$; *a similar appraisal holds with h^* in place of h.*

(b) $g\in Z^{\cdot}_{\Delta}$.

(c) *Δ is a Hardy weight and* $\int a^{2d}\Delta<\infty$ $(d\geqslant 0)$.

(d) $g\notin Z_{\cdot\Delta}$.

Proof of (a). Obvious.

Proof of (b). g is entire of minimal exponential type with $g^*(-a)=g(a)$, so it is enough to check that $\|g\|_\Delta<\infty$. But (a) supplies us with the bound $|fh|<B_5$, so $|gh|<B_5/(1-a^2)$, and since $|gh|<B_6$ for small $|a|$, $\|g\|_\Delta<\infty$.

Proof of (c). h^{-1} is entire and free of roots in the closed half-plane $b\geqslant 0$, and $\Delta(a+ib)\in\downarrow$ as a function of $b>0$, so it suffices to check

$$\int_8 a^{2d}\Delta \leqslant \sum_{n=3}^{\infty}\int_{n^2-n+\frac14}^{n^2+n+\frac14} a^{2d}\Delta<\infty \quad (d\geqslant 0).$$

But on $|a-n^2|<\frac12$,

$$\Delta<B_4^2|f|^{-2}\frac{(a-n^2)^2}{(a-n^2)^2+\delta_n^2}, \qquad \frac{|a-n^2|}{|f|}=\frac{\pi^2 a}{\sinh \pi\sqrt{a}}\left|\frac{a-n^2}{\sin\pi\sqrt{a}}\right|<B_7 n^3 e^{-\pi n},$$

and hence
$$a^{2d}\Delta<B_8\frac{n^{2d+6}\,e^{-2\pi n}}{(a-n^2)^2+\delta_n^2}$$

on this range, while on the rest of $n^2-n+\frac14\leqslant a<n^2+n+\frac14$,

$$a^{2d}\Delta<(n+1)^{2d}B_2^2|f|^{-2}<B_9 n^{2d+6}\,e^{-2\pi n},$$

so that

$$\int_{n^2-n+\frac14}^{n^2+n+\frac14} a^{2d}\Delta<B_{10}\left[n^{2d+6}\,e^{-2\pi n}\int\frac{da}{a^2+\delta_n^2}+n^{2d+7}\,e^{-2\pi n}\right]<B_{11}n^{2d+7}\,e^{-\pi n},$$

which is the general term of a convergent sum.

Proof of (d). $g\in Z_{\cdot\Delta}$ implies the existence of polynomials $p_\delta\in Z_{\cdot\Delta}$ with $\|g-p_\delta\|_\Delta<\delta$. p_δ can be supposed even since g is such; also, as $\delta\downarrow 0$, p_δ tends to g on the whole plane under a local bound ($\sigma_\cdot<\infty$), so that $p_{0+}(0)=g(0)=1$, and according to Hurwitz's theorem, the roots of p_δ tend to the roots $\pm 2^2$, $\pm 3^2$, etc. of g. Rotate the roots of p_δ onto the line $b=0$ and put its bottom coefficient $=1$, defining a new polynomial q_δ with $|q_\delta|\leqslant|p_\delta/p_\delta(0)|$ $(b=0)$ and $\|q_\delta\|_\Delta\leqslant\|p_\delta\|_\Delta/|p_\delta(0)|$ bounded as $\delta\downarrow 0$; it is this boundedness of $\|q_\delta\|_\Delta$ that leads to a contradiction.

Evaluate $\int q_\delta^2 h^*$, integrating about the semicircle $Re^{i\theta}$ $(-\pi/2\leqslant\theta\leqslant\pi/2)$ and then down along the segment joining iR to $-iR$ with R half an odd integer. Bound the integral on the arc with the aid of $|fh^*|<B_2$ and let $R\uparrow\infty$, obtaining

$$\frac{1}{2\pi}\int q_\delta^2 h^*(ib)\,db=\sum_{n=1}^{\infty}\frac{q_\delta^2(\gamma_n^*)}{(1/h^*)'(\gamma_n^*)}\equiv Q_\delta.$$

Because $h^*(ib) > 0$ and $|q_\delta(ib)| \geqslant |p_\delta(ib)/p_\delta(0)|$, an application of Fatou's lemma combined with $|fh^*| > B_1 > 0$ justifies the under-estimate:

$$Q_{0+} \geqslant \frac{1}{2\pi} \int g^2 h^*(ib) > B_{13} \int_1 f(ib)/b^4 > B_{14} \int_1 e^{\pi(2b)^{\frac{1}{2}}}/b^5 = \infty.$$

Q_δ is now estimated again with the contradictory result that it is bounded as $\delta \downarrow 0$.

$\int (q_\delta h)^2 = 0$, the integral being taken around the arc $Re^{i\theta}(0 \leqslant \theta \leqslant \pi/2)$, down the segment joining iR to 0, and thence out along the segment joining 0 to R with R half an odd integer. Bound the integral along the arc as before and let $R \uparrow \infty$, obtaining

$$\int_0^\infty (q_\delta h)^2 (ib) = -i \int_0^\infty (q_\delta h)^2 (a) \leqslant \| q_\delta \|_\Delta^2 < B_{15},$$

the first integrand being positive.

$\int (q_\delta h)^2 (\gamma - \gamma_n)/(\gamma - \gamma_n^*)$ is now evaluated along the same curve, giving

$$-\int_0^\infty (q_\delta h)^2 (ib) \frac{ib - \gamma_n}{ib - \gamma_n^*} - i \int_0^\infty (q_\delta h)^2 (a) \frac{a - \gamma_n}{a - \gamma_n^*} = 4\pi i \delta_n (q_\delta h)^2 (\gamma_n^*);$$

this supplies the bound

$$4\pi\delta_n |q_\delta h(\gamma_n^*)|^2 \leqslant \int_0^\infty (q_\delta h)^2 (ib) + \int_0^\infty |q_\delta h|^2 (a) < 2B_{15} = B_{16},$$

and it follows that

$$Q_{0+} < B_{16} \sum_{n=1}^\infty e^{\pi n} \left| \frac{h^{-2}(\gamma_n^*)}{(1/h^*)'(\gamma_n^*)} \right|.$$

But, since

$$|(\gamma - \gamma_n^*) h^*| < B_4 \frac{|\gamma - n^2|}{|f|} \text{ near } \gamma = \gamma_n^*,$$

$$|(1/h^*)'(\gamma_n^*)|^{-1} \leqslant 2B_4 e^{-\pi n}/|f(\gamma_n^*)|,$$

while

$$|h(\gamma_n^*)|^{-2} < 4B_3^{-2} |f(\gamma_n^*)|^2,$$

and combining these bounds leads at once to the desired contradiction:

$$Q_{0+} < B_{17} \sum_{n=1}^\infty |f(\gamma_n^*)| < B_{18} \sum_{n=1}^\infty n^{-3} < \infty.$$

$Z^{\cdot}$ is sometimes closed under $f \to {}^{\backprime}f = if'$, but this can fail; indeed in the above case,

$$\Delta > \frac{B_3^2}{|f|^2} \frac{(a-n^2)^2}{(a-n^2)^2+\delta_n^2} > B_{19} \frac{n^6 \delta_n^2}{(a-n^2)^2+\delta_n^2} \quad (|a-n^2| < \sqrt{\delta_n}),$$

while on the same range, $\quad |{}^{\backprime}g| > B_{20}\, e^{\pi n}\, n^{-7}$

so that $\|{}^{\backprime}g\|_\Delta = \infty$ because

$$\int_{n^2-\delta_n^{\frac{1}{2}}}^{n^2+\delta_n^{\frac{1}{2}}} \frac{n^6 \delta_n^2\, e^{2\pi n - 14}}{(a-n^2)^2+\delta_n^2} > B_{21} n^{-8} \int_{-\delta_n^{\frac{1}{2}}}^{+\delta_n^{\frac{1}{2}}} \frac{1}{a^2+\delta_n^2} > B_{22} n^{-8}\, e^{\pi n} \quad (n \uparrow \infty)$$

is the general term of a divergent sum.

12. Hardy weights with arithmetical gaps

Consider a weight Δ that bounds above a decreasing Hardy weight $|h|^2$ (h outer) on an arithmetical series of intervals:

$$|a-(2n-1)c| < d \quad (0 < d < c,\ n = 0,\ \pm 1,\ etc.)$$

but is otherwise unspecified. Then

(a) *$Z^{\cdot}$ is a closed subspace of Z,*

(b) *$Z^{\cdot} \supset Z^{0+}$, and hence in accordance with Section 5, $Z^{\cdot} = Z^{0+}$.*

As an application, it is easy to derive the lemma of Tutubalin–Freĭdlin [11]: that if $\Delta \geqslant |a|^{-2m}$ $(m > 0)$ far out, then $Z^{0+} = Z_{\cdot}$; indeed, according to (b), $f \in Z^{0+}$ is an entire function of minimal exponential type, and since $\infty > \int |f|^2/(1+a^2)^m$, a simple application of Phragmén–Lindelöf implies that f is a polynomial (of degree $< m$). Actually, it is enough to have $\Delta \geqslant |a|^{-2m}$ on an arithmetical series of intervals, as the reader can easily check using (b) and the Duffin–Schaeffer theorem [2; 10.5.1].

Proof of (a). Similar to that of (b).

Proof of (b). $f \in Z^{0+}$ implies the existence of a sum f_δ of trigonometrical functions $e(iat)$ with $|t| < \delta$, real coefficients, and $\|f - f_\delta\|_\Delta < \delta$, and it follows that

$$B_1 > \|f_\delta\|_\Delta^2 \geqslant \int_{(2n-1)c-d}^{(2n-1)c+d} |f_\delta h|^2 \geqslant 2d\, |f_\delta h(a_n)|^2$$

for some $|a_n - (2n-1)c| < d$ with a constant B_1 not depending upon δ. Bring in an entire function g of exponential type $\leqslant \varepsilon$ with $|g|<|h|$ far out on $b=0$ and $|g|\geqslant \frac{1}{2}$ on the two $45°$ lines: to be explicit, let

$$g(\gamma)=e^{-\frac{2}{3}}\prod_{n=n_1}^{\infty}\cos(\gamma/\gamma_n)$$

with

$$1<\gamma_1<\gamma_2<\text{etc.}$$

and

$$\#(R)=\sum_{\gamma_n<R}1=0 \qquad (R<1)$$

$$=\left[3\int_1^R \frac{\lg|h|^{-1}}{A}\,dA\right] \qquad (R\geqslant 1),$$

the bracket denoting the integral part and $|h(1)|$ being supposed $\leqslant 1$, choose n_1 so that

$$|g(\gamma)|\leqslant \prod_{n=n_1}^{\infty} e^{R/\gamma_n}=e\left[R\int_C \frac{\#(dB)}{B}\right] \quad (C=\gamma_{n_1},\ |\gamma|=R)$$

$$\leqslant e\left[R\int_C \frac{\#(B)}{B^2}\right]<e\left[3R\int_C \frac{dB}{B^2}\int_1^B \frac{\lg|h|^{-1}}{A}\,dA\right]$$

$$=e\left[3R\frac{1}{C}\int_1^C \frac{\lg|h|^{-1}}{A}\,dA+3R\int_C^{\infty}\frac{\lg|h|^{-1}}{B^2}\,dB\right]$$

$$<e^{\varepsilon R},$$

and use the obvious $|\cos a|<e(-a^2/3)$ $(|a|\leqslant 1)$ to bound $|g(a)|$ for large $|a|$ as follows:

$$e^{\frac{2}{3}}|g(a)|\leqslant \prod_{\gamma_n\geqslant |a|} e^{-a^2/3\gamma_n^2}=e\left[-\frac{a^2}{3}\int_{|a|}\frac{\#(dR)}{R^2}\right]$$

$$=e\left[\frac{a^2}{3}\int_{|a|}\frac{\#(R)-\#(|a|)}{R^3}\right]$$

$$<e\left[\frac{a^2}{2}\int_{|a|}\int_{|a|}^R \frac{-\lg|h|^{-1}}{A}\,dA\,\frac{dR}{R^3}+\frac{a^2}{3}\int_{|a|}\frac{dR}{R^3}\right]$$

$$=e\left[-\frac{a^2}{2}\int_{|a|}\frac{\lg|h|^{-1}}{R^3}\,dR+\tfrac{2}{3}\right]$$

$$\leqslant \frac{a^2}{2}\int_{|a|}|h|\frac{dR}{R^3}e^{\frac{2}{3}}$$

$$\leqslant |h|\,e^{\frac{2}{3}}.$$

$f_\delta g$ is then entire of exponential type $\delta+\varepsilon$ and $|f_\delta g(a_n)|<B_2$ with a constant B_2 not depending upon δ. An application of the Duffin–Schaeffer theorem [2; 10.5.3] implies $|f_\delta g|<B_3$ on the whole line $b=0$ if $\delta+\varepsilon$ is small enough, B_3 being likewise independent of δ. Phragmén–Lindelöf now implies that $|f_\delta g|<B_3 e[(\delta+\varepsilon)R]$, and since $|g|\geqslant\frac{1}{2}$ on the two 45° lines, $|f_\delta|<2B_3 e[(\delta+\varepsilon)R]$ there. Phragmén–Lindelöf is now applied to each of the 4 sectors between the 45° lines; this supplies us with the bound $|f_\delta|<2B_3 e[2(\delta+\varepsilon)R]$, establishing the compactness of f_δ as $\delta\downarrow 0$, and it follows that each limit function f_{0+} is entire of exponential type $\leqslant 2\varepsilon$ with $\|f-f_{0+}\|_\Delta=0$. But this means that f is the restriction to $b=0$ of an entire function of exponential type $\leqslant 2\varepsilon$, and since ε can be made as small as desired, $f\in Z^{\boldsymbol{\cdot}}_\Delta$, and the proof is complete.

13. Entire functions of positive type

Given a Hardy weight $\Delta=|h|^2$ and a positive number ϱ, let $Z^{\boldsymbol{\cdot}\varrho}$ be the class of entire functions $f=f(\gamma)$ of exponential type $\leqslant\varrho$:

$$\overline{\lim_{R\uparrow\infty}}\, R^{-1}\max_{0\leqslant\theta<2\pi}\lg|f(Re^{i\theta})|\leqslant\varrho,$$

which, restricted to the line $b=0$, belong to Z. Then

$$Z^{\boldsymbol{\cdot}\varrho}=Z^{|t|\leqslant\varrho+}=\bigcap_{\varrho'>\varrho}Z^{|t|\leqslant\varrho'}.$$

Proof. We first prove the inclusion

$$Z^{\boldsymbol{\cdot}\varrho}\supset Z^{|t|\leqslant\varrho+}.$$

If $f\in Z^{|t|\leqslant\varrho+}$, then it is possible to find (real) sums of trigonometrical functions:

$$f_n(\gamma)=\sum_{k\leqslant n}c_k^n\, e(i\gamma t_k^n)$$

with $|t_k^n|<\varrho+1/n$ and $\|f-f_n\|_\Delta<1/n$. Given $\delta>1/n$, $f_n e[i\gamma(\varrho+\delta)]h$ belongs to H^{2+}, and much as in Section 6*b*,

$$|f_n h|<B_1 e^{(\varrho+\delta)R}\quad(b\geqslant 1),\qquad |f_n h^*|<B_2 e^{(\varrho+\delta)R}\quad(b\leqslant -1),$$

and

$$|f_n|<B_3\quad(|\gamma|\leqslant 2)$$

with constants B_1, B_2, B_3 not depending upon n. An appraisal of h on $\theta=\pi/4$, $3\pi/4$ and of h^* on $\theta=5\pi/4$, $7\pi/4$ leads to

$$|f_n|<B_4\, e^{(\varrho+2\delta)R}$$

much as in Section 6 *b*, B_4 being likewise independent of n, and since $\|f-f_n\|_\Delta < 1/n$, it follows that as $n \uparrow \infty$, f_n tends on the whole plane to an entire function f_∞ of exponential type $\leqslant \varrho$, coinciding with f on $b=0$. But then $f \in Z^{\cdot\varrho}$, and the inclusion is proved.

As in Section 5, it suffices for the proof of the opposite inclusion:

$$Z^{\cdot\varrho} \subset Z^{|t|\leqslant\varrho+}$$

to consider *even* functions $f \in Z^{\cdot\varrho}$ with Hadamard factorization

$$f(\gamma) = \prod_{n=1}^{\infty}\left(1-\frac{\gamma^2}{\gamma_n^2}\right).$$

Because

$$\lg^+|f(a)|^2 \leqslant \lg^+(|f(a)|^2\Delta) - \lg^-\Delta \leqslant |f(a)|^2\Delta - \lg^-\Delta,$$

f satisfies

$$\int \frac{\lg^+|f(a)|}{1+a^2} < \infty;$$

it follows that

$$\overline{\lim_{R\uparrow\infty}}\, R^{-1}\lg|f(Re^{i\theta})| \leqslant \varrho\,|\sin\theta|$$

[8, p. 27] and that the roots of f in the half-plane $a>0$ have a density $D \leqslant \varrho/\pi$:

$$\lim_{n\to\infty} n/|\gamma_n| = D$$

[8, Theorem VIII]. Also, it is permissible to assume that the roots of f are real. Consider for the proof

$$f_1(\gamma) = \prod_{n=1}^{d}\left(1-\frac{\gamma^2}{\gamma_n^2}\right) f_2(\gamma) \quad \text{with} \quad f_2(\gamma) = \prod_{n>d}\left(1-\frac{\gamma^2}{|\gamma_n^2|}\right).$$

Then $|f_1(a)| \leqslant |f(a)|$ and the roots of $f_2(\gamma)$ have the same density D; this implies [2; 8.2.1] that f_2 is of type πD. Hence f_1 is also of type πD and so $f_1 \in Z^{\cdot\varrho}$. But then $(\gamma^2-1)^d f_2 \in Z^{\cdot\varrho}$, so $(\gamma^2-1)^n f_2 \in Z^{\cdot\varrho}$ $(n \leqslant d)$. All these functions have real zeros and hence we may assume them in $Z^{|t|\leqslant\varrho+}$. f_1 is a sum of these, so $f_1 \in Z^{|t|\leqslant\varrho+}$, and since $\|f-f_1\|_\Delta$ is small for large d it follows that $f \in Z^{|t|\leqslant\varrho+}$ also. From here on the roots of f are real: $0<\gamma_1\leqslant\gamma_2\leqslant$ etc.

Given $\varrho' > \varrho$, let us grant the existence of an entire function g of exponential type $\leqslant \varrho'$ with $\|f-g\|_\Delta$ as small as desired and $g \in L^2(R^1)$. As in Section 5, an application of the Paley–Wiener theorem implies $f \in Z^{|t|\leqslant\varrho'}$, and $f \in Z^{|t|\leqslant\varrho+}$ follows. Accordingly, it suffices to produce such an entire function g.

Given a small positive number $\varepsilon < 1$, define

$$\delta = (\varepsilon/8)^2, \quad D_* = D - \delta/2, \quad D^* = D + \delta/2,$$

$$g_1(\gamma) = \prod_{\gamma_n \leqslant d} \left(1 - \frac{\gamma^2}{\gamma_n^2}\right), \quad g_2(\gamma) = \prod_{n > D^* d} \left(1 - \frac{D^{*2}\gamma^2}{n^2}\right), \quad g_3(\gamma) = \prod_{n > \varepsilon d} \left(1 - \frac{\varepsilon^2 \gamma^2}{n^2}\right),$$

and let us check the following lemmas leading to the properties of $g = g_1 g_2 g_3$ needed for the proof of $f \in Z^{|t| \leqslant \varrho +}$ indicated above; in the lemmas, c_1, c_2, etc. denote positive constants depending upon ε alone, and it is understood that if ε and/or d is unspecified, then ε has to be small enough and d large enough, the smallest admissible d depending in general upon ε. At a first reading, just note the statements of lemmas (a)–(g) and then turn to (h).

(a) *g is an entire function of exponential type $\pi(D^* + \varepsilon) \leqslant \varrho + \pi(\delta/2 + \varepsilon)$.*

Proof of (a). Obvious.

(b) *$|f - g|$ tends to 0 as $d \uparrow \infty$ independently of $\varepsilon (< 1)$ and of $|a| \leqslant A$ for each $A > 0$.*

Proof of (b).

$$e(-2A^2\varepsilon^2/n^2) \leqslant 1 - a^2\varepsilon^2/n^2 \leqslant 1 \quad (|a| \leqslant A)$$

for $n > \varepsilon d$ and $d > 2A$, so that as $d \uparrow \infty$

$$e(-2A^2 \sum_{n > \varepsilon d} \varepsilon^2 n^{-2}) \leqslant g_3(a) \leqslant 1$$

is close to 1 independently of $\varepsilon (< 1)$ and of $|a| \leqslant A$.

(c) *$|g| \leqslant B|f|$ for $|a| \leqslant d/2$, B being the universal constant involved in the appraisal* (e) *of Section 5.*

Proof of (c). Because the roots of f have density D,

$$n/D^* < \gamma_n < n/D_* \quad (n \geqslant n_0)$$

with n_0 depending only upon D_* and D^* and so only upon ε. Given $d > n_0$ and $0 \leqslant a \leqslant d/2$, if δ is so small that $D^*/D_* < 2$, then

$$|f/g_1| = \prod_{\gamma_n > d} \left(1 - \frac{a^2}{\gamma_n^2}\right) > \prod_{n > D_* d} \left(1 - \frac{D^{*2}a^2}{n^2}\right)$$

so that

$$|f/g_1 g_2| > \prod_{D_* d < n \leqslant D^* d} \left(1 - \frac{D^{*2}a^2}{n^2}\right),$$

and since, in this product,

$$D^{*2}a^2/n^2 < \frac{(D+\delta/2)^2}{4(D-\delta/2)^2} < \tfrac{1}{2}$$

for small δ, the bound $1-c > e(-2c)$ $(0 < c \leqslant \frac{1}{2})$ implies

$$|f/g_1 g_2| > e[-2a^2 D^{*2} \sum_{D_* d < n \leqslant D^* d} n^{-2}] > e[-3a^2(D^* - D_*)/d] = e(-3a^2\delta/d).$$

On the other hand, the appraisal (e) of Section 5 implies

$$g_3 < Be(-a^2\varepsilon/d) \quad (0 \leqslant a \leqslant d/2),$$

and since $3\delta < \varepsilon$ for small ε, the desired bound follows.

(d) $|g| < c_1 \quad (d/2 < |a| \leqslant D_* d/D^*)$.

Proof of (d). Given $d > 2n_0$ with n_0 as in the proof of (c), it is possible to find c_2 and c_3 depending upon $n_0 = n_0(\varepsilon)$ (and so upon ε) such that

$$|g_1| < c_1 a^{c_2} \prod_{n < D_* a} \left(\frac{D^{*2}a^2}{n^2} - 1\right) \prod_{D^* a < n < D_* d} \left(1 - \frac{D_*^2 a^2}{n^2}\right)$$

for $d/2 < a \leqslant D_* d/D^*$. Define $c_3 = c_1/(\pi D^*)$; then

$$|g_1 g_2| < c_3 a^{c_2 - 1} |\sin \pi D^* a| \, J_1/J_2 J_3,$$

$$J_1 = \prod_{D^* a < n < D_* d} \frac{n^2 - D_*{}^2 a^2}{n^2 - D^{*2}a^2}, \quad J_2 = \prod_{D_* a \leqslant n \leqslant D^* a} \left(\frac{a^2 D^{*2}}{n^2} - 1\right), \quad J_3 = \prod_{D_* d \leqslant n \leqslant D^* d} \left(1 - \frac{D^{*2}a^2}{n^2}\right).$$

J_1 is supposed non-void since the proof simplifies in the opposite case; also, it is supposed below that the smallest integer $n_1 > D^* a$ does not exceed $D^* a + \frac{1}{2}$, the discussion of J_1 being simpler and that of J_2 just a little more complicated if $n_1 > D^* a + \frac{1}{2}$. Bring out the leading factor of J_1:

$$\frac{n_1^2 - D_*{}^2 a^2}{n_1^2 - D^{*2}a^2} < \frac{n_1 - D_* a}{n_1 - D^* a} \leqslant \frac{1 + a\delta}{n_1 - D^* a} < \frac{e^{a\delta}}{n_1 - D^* a};$$

the product of other factors of J_1 does not exceed

$$\prod_{D^*a+\frac{1}{2}<n<D_*d} \frac{n-D_*a}{n-D^*a} = e\left[\sum_{D^*a+\frac{1}{2}<n<D_*d} \lg\left(1+\frac{a\delta}{n-D^*a}\right)\right]$$

$$< e\left[2\int_0^{D_*d-D^*a} \lg(1+a\delta/c)\,dc\right]$$

$$< e\left[2a\delta\int_0^{D^*/\delta} \lg(1+1/c)\,dc\right]$$

since $D_*d<2D^*a$, and using the bound $\lg(1+1/c)<1/c$, it follows that

$$J_1 < e\left[2a\delta\left(\int_0^1 \lg(1+1/c)\,dc + \lg D^*/\delta\right)\right]\frac{e^{a\delta}}{n_1-D^*a} < \frac{e^{a\delta^{\frac{1}{2}}}}{n_1-D^*a}$$

for small δ. Stirling's approximation is now applied to obtain an underestimate of J_2 for small δ, using $D^*a-(n_1-1)>\frac{1}{2}$:

$$J_2 > \prod_{D_*a\leqslant n\leqslant D^*a} \frac{D^*a-n}{n} > \frac{\Gamma(a\delta)}{(D^*a)^{a\delta+1}}$$

$$> c_4(a\delta)^{a\delta-\frac{1}{2}}\,e^{-a\delta}(D^*a)^{-a\delta-1}$$

$$> c_4(D^*a)^{-\frac{3}{2}}(\delta/eD^*)^{a\delta}$$

$$= c_4(D^*a)^{-\frac{3}{2}}\,e\left[-a\delta\left(\lg\frac{D^*}{\delta}+1\right)\right]$$

$$> c_4(D^*a)^{-\frac{3}{2}}\,e^{-a\delta^{\frac{1}{2}}}$$

with a universal constant c_4. Similarly

$$J_3 \geqslant \prod_{D_*d\leqslant n\leqslant D^*d}\left(\frac{n-aD^*}{n}\right) \geqslant \frac{\Gamma(D^*(d-a))}{\Gamma(D_*d-aD^*+1)\,(D^*d)^{\delta d+1}}$$

$$\geqslant c_5\,\frac{[D^*(d-a)]^{D^*(d-a)-\frac{1}{2}}\,e^{-D^*(d-a)}}{(D_*d-aD^*)^{D_*d-aD^*+\frac{1}{2}}\,e^{-D_*d+aD^*}\,(D^*d)^{\delta d+1}},$$

$$\geqslant c_5\,\frac{e^{-\delta d}}{(D_*d-aD^*)\,D^*d}\left[\frac{D^*(d-a)}{D_*d-aD^*}\right]^{D_*d-aD^*-\frac{1}{2}}\left(\frac{d-a}{d}\right)^{\delta d}$$

$$\geqslant c_5\,\frac{e^{-\delta d}}{D^*D_*d^2}\left(1-\frac{a}{d}\right)^{\delta d} \geqslant c_5\,e^{-2\delta a}\,a^{-2}\left(1-\frac{D_*}{D^*}\right)^{\delta d}\Big/(4D^*D_*)$$

$$\geqslant c_5 a^{-2}\,e[-2\delta a-\delta d\lg(D^*/\delta)]/(4D^*D_*) \geqslant c_5 a^{-2}\,e(-\sqrt{\delta}a)/(4D^*D_*)$$

with a universal constant c_5. Combining the bounds for J_1, J_2, J_3 and using $0<n_1-D^*a\leqslant\frac{1}{2}$, it follows that

$$|g_1 g_2| < c_6 a^{c_1+3} \left|\frac{\sin \pi D^* a}{n_1 - D^* a}\right| e^{3a\delta^{\frac{1}{2}}} < c_7 a^{c_1+3} e^{3a\delta^{\frac{1}{2}}} < c_7 e^{4a\delta^{\frac{1}{2}}}$$

with c_7 depending upon ε alone, d being increased if need be so as to achieve $a^{c_1+3} < e(a\delta^{\frac{1}{2}})$. But now the familiar appraisal (e) of Section 5 implies

$$|g_3| < Be^{-4a\delta^{\frac{1}{2}}},$$

and so
$$|g| = |g_1 g_2 g_3| < Bc_7 \equiv c_1,$$

completing the proof of (d).

(e) $|g| < c_8 \quad (D_* d/D^* < |a| \leqslant d)$.

Proof of (e).
$$|g_1| < c_9 a^{c_{10}} \prod_{n < D_* a} \left(\frac{D^{*2} a^2}{n^2} - 1\right)$$

for $D_* d/D^* < a \leqslant d$ with constants c_9 and c_{10} depending upon $n_0 = n_0(\varepsilon)$ alone, so

$$|g_1 g_2| < c_{11} a^{c_{10}} |\sin \pi D^* a| / J_4$$

with
$$J_4 = \prod_{D_* a \leqslant n \leqslant D^* d} \left|1 - \frac{D^{*2} a^2}{n^2}\right| \geqslant \prod_{D_* a \leqslant n \leqslant D^* d} \left|1 - \frac{D^* a}{n}\right|$$

$$\geqslant \left|\frac{n_2 - D^* a}{n^2}\right| \frac{\Gamma(D^* d - D_* a)\,\Gamma(a\delta)}{(D^* d)^{D^* d - D_* a + 3}},$$

n_2 being determined from $-\frac{1}{2} < n_2 - D^* a \leqslant \frac{1}{2}$. Both gamma functions contribute to this underestimate if, as is supposed below, $D^* a$ is not too close to $D_* a$ or to $D^* d$; the appraisal of J_4 is similar in the opposite case. Stirling's approximation is now applied to obtain

$$J_4 > c_{12} |n_2 - D^* a| (D^* d)^{-5} J_5 J_6$$

with
$$J_5 = e\left[-D^* d \left(\frac{d-a}{d}\right) \lg\left(\frac{d}{d-a}\right)\right]$$

and
$$J_6 = e\left[-D^* d \left(\frac{a\delta}{D^* d}\right) \lg\left(\frac{D^* d}{a\delta}\right)\right].$$

Because $d - a \leqslant d(1 - D_*/D^*) = d\delta/D^*$ and $a\delta \leqslant d\delta$, both J_5 and J_6 are bigger than $e(-a\delta^{\frac{1}{2}})$ for small δ, so

$$J_4 > c_{13} |n_2 - D^* a| a^{-5} e(-3a\sqrt{\delta}),$$

and the proof is completed as in (d) above.

(f) $|g| < c_{14}$ $(d < |a| \leqslant 2d)$.

Proof of (f).
$$|g_1| < c_{15}\, a^{c_{16}} \prod_{n<D_* d} \left(\frac{D^{*2}a^2}{n^2} - 1\right) e^{2n\delta}$$

for $d < a \leqslant 2d$, the exponential accounting for the factors of

$$\prod_{D_* d \leqslant n \leqslant D^* d} \left(\frac{D^{*2}a^2}{n^2} - 1\right)$$

that exceed 1; the rest of the proof is similar to but simpler than that of (e).

(g) $|g| < c_{17}$ $(|a| > 2d)$, *and* $g \in L^2(R^1)$.

Proof of (g).
$$|g_1| < c_{18} a^{c_{19}} \prod_{n \leqslant D^* d} \left(\frac{D^{*2}a^2}{n^2} - 1\right)$$

for $a > 2d$, so
$$|g_1 g_2| < c_{20} a^{c_{19}} |\sin \pi D^* a| \leqslant c_{20} a^{c_{19}},$$

and using the familiar appraisal (e) of Section 5 to bound g_3, it develops that

$$|g| < B c_{21} a^{c_{22}} e^{-\varepsilon d(1+2(\lg a/d))}.$$

But
$$d \lg (a/d) > \frac{d \lg 2}{\lg (2d)} \lg a \quad (a > 2d),$$

and so
$$|g| < c_{23}\, a^{c_{22} - 2\varepsilon d \lg 2/\lg (2d)}$$

is bounded $(a > 2d)$ and belongs to $L^2(R^1)$ if d is large enough.

(h) $\|f-g\|_\Delta$ *can be made as small as desired by appropriate choice of* ε *and* d.

Proof of (h).

$$\tfrac{1}{2} \|f-g\|_\Delta^2 \leqslant \int_0^A |f-g|^2 \Delta + (2B+1)^2 \int_A^{d/2} |f|^2 \Delta + \int_{d/2}^{\infty} (c_{24} + |f|)^2 \Delta$$

with an adjustable number A, a universal constant B, and c_{24} (= the greatest of c_1, c_8, c_{14}, c_{17}) depending upon ε alone, *provided* ε is small enough and $d(>2A)$ is large enough, the smallest admissible d depending upon ε. A is now chosen so large that $(2B+1)^2 \int_A^\infty |f|^2 \Delta < 1/n$ and then ε is chosen so small that $c_{24} = c_{24}(\varepsilon) < \infty$ and d is made so big that neither $\int_0^A |f-g|^2 \Delta$ nor $\int_{d/2}^\infty (c_{24} + |f|)^2 \Delta$ exceeds $1/n$, with the result that $\|f-g\|_\Delta^2 < 6/n$.

14. Another condition for $Z^{+/-} = Z^{0+}$ (Δ Hardy)

Because $Z^{|t| \leqslant \varrho +}$ is closed so is $Z^{\cdot\varrho}$, but it is possible to go another step and prove that,

if $$\sigma^{\cdot\varrho}(\gamma) = \sup |f(\gamma)| \quad f \in Z^{\cdot\varrho}_{\Delta_+}, \ \|f\|_{\Delta^+} \leqslant 1,$$

then $\lg \sigma^{\cdot\varrho}$ *is a non-negative, continuous subharmonic function such that*

$$\overline{\lim_{R \uparrow \infty}} R^{-1} \max_{0 \leqslant \theta < 2\pi} \lg \sigma^{\cdot\varrho}(Re^{i\theta}) = \varrho.$$

Proof. Only the last statement needs a proof. Given $f \in Z^{\cdot\varrho}_{\Delta^+}$, $(\gamma + i)^{-1} e^{i\gamma\varrho} fh \in H^{2+}$, and so

$$\lg \left| \frac{e^{i\gamma\varrho} fh}{\gamma + i} \right| \leqslant \frac{1}{\pi} \int \frac{b\, dc}{(c-a)^2 + b^2} \lg \frac{|fh|}{|a+i|} \quad (\gamma = a + ib, \ b > 0);$$

this leads at once to

$$\lg [e^{-b\varrho} \sigma^{\cdot\varrho}(\gamma)] \leqslant \frac{1}{\pi} \int \frac{b\, dc}{(c-a)^2 + b^2} \lg \sigma^{\cdot\varrho}$$

since $h(\gamma)/(\gamma + i)$ is outer. $\int \lg \sigma^{\cdot\varrho}/(1 + a^2) < \infty$ is now proved as in Section 4 (e), and it follows that

$$\overline{\lim_{R \uparrow \infty}} R^{-1} \lg \sigma^{\cdot\varrho}(Re^{i\theta}) \leqslant \varrho |\sin \theta|$$

for $\theta = \pi/4$, $3\pi/4$; the same holds by a similar argument for $\theta = 5\pi/4$, $7\pi/4$. An application of Phragmén–Lindelöf as in Section 4 (f) completes the proof that $\sigma^{\cdot\varrho}$ is of type $\leqslant \varrho$, and that the equality must hold follows since $e(-i\gamma\varrho) \in Z^{\cdot\varrho}_{\Delta^+}$.

As an application of the bound for $\sigma^{\cdot\varrho}$, it will be proved that if $Z^{|t| \leqslant \varrho+} \supset Z^{+/-}$, and indeed if the projection of $e(ias)$ upon Z^- belongs to $Z^{|t| \leqslant \varrho+}$ for *a single* $s > 0$, then $Z^{+/-} = Z^{0+}$. Suppose that projection belongs to $Z^{|t| \leqslant \varrho+}$ for a single $s > 0$; then it does so far a whole (bounded) interval of s with a larger ϱ, and selecting such an s from the Lebesgue set of

$$h = \frac{1}{2\pi} \int_0 e^{-iat} \hat{h}(a)\, da$$

and arguing as in Section 7 with $\sigma^{\cdot\varrho}$ in place of $\sigma^{\cdot}$, it is found that h^{-1} is an entire function of exponential type $\leqslant \varrho$. But then $\mathfrak{j} = h/h^*$ is inner as in Section 7 so that $Z^{+/-} = Z^- \cap Z^+$; also $Z^- \cap Z^+ = Z^{\cdot}$ since $1/\Delta$ is locally summable (Section 6 *c*), and so $Z^{+/-} = Z^{\cdot} = Z^{0+}$ as stated.

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