ON DEFORMATIONS OF DISCONTINUOUS GROUPS

BY

A. ANDREOTTI and E. VESENTINI

Brandeis University *Harvard University* (1)

Let D be a product of irreducible bounded symmetric domains in the complex number space and let Γ be a properly discontinuous group on D with the property that $\operatorname{vol}(D/\Gamma)$ is finite.

If one excludes that D has any components of complex dimension 1, it is generally suspected (cf. [15]) that any such group must be commensurable to an arithmetic group.

In particular, if this is the case, there will be no other families of discontinuous groups containing Γ except those obtained by operating on Γ by a family of inner automorphisms of the Lie group $G = \operatorname{Aut}(D)$

Under the more stringent assumption that D/Γ be compact, this has proved to be the case in [15] and in [7] as a consequence of a more general rigidity theorem. That result has been extended by A. Weil [21] to the case of all "reasonable" semisimple Lie groups (i.e., a semisimple Lie group without compact components whose Lie algebra has no simple factor of dimension 3).

In the case where D/Γ is not compact it has been stated by A. Selberg (in a conversation with one of the authors at the international congress in Stockholm) that at least the following should be true:

Let us suppose that Γ_1 and Γ_2 are two properly discontinuous groups on D and suppose that (a) Γ_1 is an arithmetic group, (b) there exist fundamental domains F_1, F_2 for Γ_1, Γ_2 respectively, such that, outside of a compact set $K \subset D$, $F_1 - F_1 \cap K = F_2 - F_2 \cap K$. Then Γ_2 must be commensurable with Γ_1.

Again, if this is the case, there will be only trivial families of discontinuous

(1) Supported by the National Science Foundation through research contracts at Brandeis University and Harvard University.

groups Γ, containing the arithmetic group Γ_1 and keeping the part at infinity of D/Γ rigid.

This statement can be formulated in precise terms following a pattern similar to that used by A. Weil in [20]. We do this in § 2, where we introduce the notion of a family of uniformizable structures on D, rigid at infinity.

In this paper we show that any such family is a locally trivial family provided one of the fibers D/Γ behaves at infinity as if Γ was an arithmetic group. Precisely we will assume that (i) D has no component of complex dimension 1, (ii) Γ is finitely generated, (iii) D/Γ is a strongly pseudoconcave space. Both of the latter two conditions are satisfied by arithmetic groups. The concavity assumption has been verified in particular cases in [1], [18] and by K. G. Ramanathan (unpublished). It has been established in general by A. Borel (unpublished).

The main tools of the proof are the following:

(α) If D/Γ is strongly pseudoconcave, then for a sufficiently large *relatively compact* open set $B \subset D/\Gamma$ its counter image $\tilde{B}$ in D has D as envelope of holomorphy.

(β) On any quotient of D by a properly discontinuous group Γ, the tangent bundle, with respect to the Bergmann metric, is W-elliptic in degree (01).

From (β) one first deduces that, by a generalization of the differential geometric methods used in the compact case (and because the family is rigid at infinity), one can locally deform trivially inside the family any compact subset of any fiber. The assumption (α) is what makes it possible to "extend" this partial trivialization to the family itself.

We remark that in the framework of this theory complex analytic families and differentiable families behave quite differently (cf. Theorem 2).

We have assumed throughout this paper that all discontinuous groups considered act without fixed points on D. This is no restriction since assumptions (ii) and (iii) are stable by commensurability and by virtue of a theorem of A. Selberg ([15] p. 154).

§ 1. On pseudoconcave manifolds

1. *Preliminaries.* (a) Let X be a complex manifold of pure complex dimension n.

A real valued C^∞ function Φ on X is said to be *strongly q-pseudoconvex* at the point $x_0 \in X$ if the Levi form

$$\mathcal{L}(\Phi) = \sum \left(\frac{\partial^2 \Phi}{\partial z_\alpha \, \partial \bar{z}_\beta} \right)_{x_0} u^\alpha \bar{u}^\beta$$

(where z_α are local holomorphic coordinates at x_0) has $n-q$ positive eigenvalues at least.

More generally a real valued continuous function Φ on X will be called *strongly q-pseudoconvex* at the point $x_0 \in X$ if

(i) we can find a neighborhood U of x_0 in X and finitely many real valued C^∞ functions in U, $\Phi_1, \dots, \Phi_k$, such that

$$\Phi(x) = \sup\,(\Phi_1(x), \dots, \Phi_k(x)) \quad \forall\, x \in U;$$

(ii) we can find a biholomorphic imbedding

$$\tau : D^{n-q} \to U$$

of the disk $\quad D^{n-q} = \{t = (t_1, \dots, t_{n-q}) \in \mathbf{C}^{n-q} \mid \sum t_i \bar{t}_i < 1\}$

such that $\quad \tau(0) = x_0,$

for each i, $1 \leqslant i \leqslant k$, the Levi form of $\Phi_i \circ \tau$ is positive non-degenerate at $t=0$.

This more general class of functions seems to appear naturally in the study of discontinuous groups which do not require, for the purpose we have in mind, any additional complication.

The following properties of strongly q-pseudoconvex functions will be of constant use in the sequel.

(α) *If Φ is a strongly q-pseudoconvex function at $x_0 \in X$, then there exists a neighborhood $U(x_0)$ of x_0 in X such that Φ is strongly q-pseudoconvex at each point $x \in U(x_0)$.*

(β) *Let Φ be strongly q-pseudoconvex at each point of the coordinate neighborhood U (where $z_1, \dots, z_n$ are the holomorphic coordinates). For any compact set $K \subset U$ we can find a constant $c(K) > 0$ with the following property:*

for any C^∞ real valued function α on U satisfying

$$\operatorname{supp}\,\alpha \subset K, \quad \sup_{x \in U} \sum_{\gamma, \beta} \left| \frac{\partial^2 \alpha}{\partial z_\gamma\, \partial \bar{z}^\beta}(x) \right| \leqslant c(K)$$

the function $\Phi + \alpha$ is again strongly q-pseudoconvex in U.

(γ) *If Φ is strongly q-pseudoconvex at x_0, $0 \leqslant q \leqslant n-1$, then for any neighborhood $U(x_0)$ of x_0 in X*

$$\sup_{U(x_0)} \Phi > \Phi(x_0),$$

(i.e., *Φ cannot have a relative maximum at x_0*).

The proof of these statements reduces to the case where Φ is a C^∞ function; in this case they are known and easy to prove.

(b) Let X be a complex manifold.

DEFINITION. *We say that X is strongly q-pseudoconcave if we can find a compact set $K \subset X$ and a continuous function $\Phi : X \to \mathbf{R}$ such that*

(i) *at each point of $X - K$, Φ is strongly q-pseudoconvex,*

(ii) *for any $c > \inf_X \Phi$ the sets*

$$B_c = \{x \in X \mid \Phi(x) > c\}$$

are relatively compact in X.

We will always assume that q ranges between 0 and $n-1$.

If X is strongly q-pseudoconcave, so is each one of its connected components.

The following lemma is a consequence of the maximum principle (γ):

LEMMA 1. *Let X be strongly q-pseudoconcave and connected $(0 \leqslant q \leqslant n-1)$; then*

(a) *for $\min_K \Phi > c > \inf_X \Phi$ the closure of B_c in X is*

$$\bar{B}_c = \{x \in X \mid \Phi(x) \geqslant c\},$$

(b) *there exists a c_1, $\min_K \Phi > c_1 > \inf_X \Phi$, such that for $c < c_1$ the sets B_c are connected.*

Proof of (a). One has in any case $\bar{B}_c \subset \{x \in X \mid \Phi(x) \geqslant c\}$. However, if $c < \min_K \Phi$, at the point x_0 of $\{\Phi(x) = c\}$ Φ is strongly q-pseudoconvex and thus, because of (γ), x_0 is an accumulation point of the set $\{\Phi(x) > c\}$.

Proof of (b). The sets $\bar{B}_c$ being compact, they have a finite number, $s(c)$ of connected components $K_i(c)$:

$$\bar{B}_c = K_1(c) \cup \dots \cup K_{s(c)}(c).$$

For $\min_K \Phi > \alpha > \beta > \inf_X \Phi$ we have $\bar{B}_\alpha \subset \bar{B}_\beta$ so that for any component $K_i(\alpha)$ there is a uniquely defined component $K_{\tau(i)}(\beta)$ containing it.

We first show that

$$\tau : \{1, \dots, s(\alpha)\} \to \{1, \dots, s(\beta)\}$$

is surjective so that $$s(\beta) \leqslant s(\alpha).$$

In fact if $j \notin \operatorname{Im} \tau$, then $K_j(\beta) \cap \bar{B}_\alpha = \emptyset$ so that

$$K_j(\beta) \subset \{x \in X \mid \alpha > \Phi(x) \geqslant \beta\}.$$

Let $m = \max_{K_j(\beta)} \Phi$ and let $x_0 \in K_j(\beta)$ be such that $\Phi(x_0) = m$. In any neighborhood $U(x_0)$ of x_0 there is at least one point p where $\Phi(p) > m$. By the very definition of m, $p \notin K_j(\beta)$, but since $m \geqslant \beta$, $p \in \bar{B}_\beta$ i.e., $p \in \bigcup_{i \neq j} K_i(\beta)$. This shows that x_0 must be in (the closure of) $\bigcup_{i \neq j} K_i(\beta)$, and this is not possible.

We can thus find c_1, $\min_K \Phi > c_1 > \inf_X \Phi$, so that for $\alpha \leqslant c_1$ the number of connected components of $\bar{B}_\alpha$ is a constant s_0 independent of α.

In order to prove that $s_0 = 1$ we make use of the assumption that X is connected. Let

$$\bar{B}_{c_1} = K_1(c_1) \cup \ldots \cup K_{s_0}(c_1).$$

Let $a \in K_1(c_1)$, $b \in K_{s_0}(c_1)$ and let $\gamma: I \to X$ where $I = \{t \in \mathbf{R} \mid 0 \leqslant t \leqslant 1\}$ be a path in X with end points $\gamma(0) = a$, $\gamma(1) = b$. Since γ is continuous, $\gamma(I)$ is compact and for $\alpha < \min_{\gamma(I)} \Phi$ we have $\gamma(I) \subset \bar{B}_\alpha$. This shows that a, b are in the same connected component $K_i(\alpha)$ of $\bar{B}_\alpha$. Since $\tau: \{1, \ldots, s(c_1)\} \to \{1, \ldots, s(\alpha)\}$ is bijective, we must have $K_1(c_1) = K_{s_0}(c_1)$, i.e., $s_0 = 1$.

We now remark that for $c < c_1$ the sets

$$B_c = \bigcup_{c_1 \geqslant c' > c} \bar{B}_{c'}$$

as an increasing union of connected sets, are connected.

We will also need the following

Lemma 2. *Let X be strongly q-pseudoconcave and connected $(0 \leqslant q \leqslant n-1)$; let $\pi: \tilde{X} \to X$ be the universal covering space of X.*

We assume that the fundamental group $\pi_1(X)$ of X is finitely generated.

Then there exists a c_2, $\min_K \Phi > c_2 > \inf_X \Phi$ such that for $c < c_2$ the sets

$$\pi^{-1}(B_c) \subset \tilde{X}$$

are connected.

Proof. Let

$$\gamma_i: I \to X \quad 1 \leqslant i \leqslant r,$$

where $I = \{t \in \mathbf{R} \mid 0 \leqslant t \leqslant 1\}$, be a set of generators for $\pi_1(X)$.

We select c_2 so that $c_2 < c_1$ (as defined in Lemma 1) and so that $\gamma_i(I) \subset B_{c_2}$ for $1 \leqslant i \leqslant r$.

For $c < c_2$ we set $\tilde{B}_c = \pi^{-1}(B_c)$. Since $\pi_1(\tilde{X}) = 0$, we have the exact sequence for the pair $(\tilde{X}, \tilde{B}_c)$:

$$0 \to \pi_1(\tilde{X}, \tilde{B}_c) \to \pi_0(\tilde{B}_c) \to 0.$$

Now we have $\pi_1(\tilde{X}, \tilde{B}_c) \simeq \pi_1(X, B_c)$ ([16], p. 266). Moreover, for the pair (X, B_c) we have the exact sequence

$$\pi_1(B_c) \to \pi_1(X) \to \pi_1(X, B_c) \to \pi_0(B_c).$$

By construction $\pi_1(B_c) \to \pi_1(X)$ is surjective; by Lemma 1 $\pi_0(B_c) = 0$. Hence

$$\pi_1(X, B_c) = 0 = \pi_1(\tilde{X}, \tilde{B}_c).$$

From this we obtain $\pi_0(\tilde{B}_c) = 0$ as we wanted.

2. *Analytic completions.* a) Let X be a complex manifold and let O denote the sheaf of germs of holomorphic functions on X.

Let A be an open subset of X. We say that X is an *analgtic completion* of A if the restriction map

$$r: H^0(X, O) \to H^0(\mathrm{A}, O)$$

is an isomorphism.

LEMMA 3. *Let X be an analytic completion of A and let Y be a holomorphically complete manifold. Then any holomorphic map*

$$f: A \to Y$$

extends, in a unique way, to a holomorphic map

$$\tilde{f}: X \to Y.$$

Proof. If $Y = \mathbf{C}$, this is the definition of an analytic completion. It follows that the lemma is true when $Y = \mathbf{C}^N$.

In general we may assume Y imbedded as a closed subset of $\mathbf{C}^N$. If $\mathfrak{J}(Y)$ is the sheaf of germs of holomorphic functions in $\mathbf{C}^N$, vanishing on Y, we have

$$Y = \{z \in \mathbf{C}^N \mid g(z) = 0 \quad \forall\, g \in H^0(\mathbf{C}^N, \mathfrak{J}(Y))\}.$$

Now f extends to a mapping $\tilde{f}$ of X into $\mathbf{C}^N$. Since for every $g \in H^0(\mathbf{C}^N, \mathfrak{J}(Y))$, $g \circ f = 0$ on A, then also $g \circ \tilde{f} = 0$ on X, and therefore the image of $\tilde{f}$ is in Y.

If the complex manifold A has an analytic completion X which is holomorphically complete, this, by the previous lemma, is unique (up to isomorphisms which is the identity on A). We say then that X is the *envelope of holomorphy* of A.

b) We want to prove the following

THEOREM 1. *Let X be a connected complex n-dimensional manifold. We assume that*

(i) *X is strongly q-pseudoconcave for some value of q with $0 \leqslant q \leqslant n-2$.*

(ii) *$\pi_1(X)$, the fundamental group of X, is finitely generated.*

Let $\pi: \tilde{X} \to X$ be the universal convering of X. Then (with the usual notations) we can find a constant $c_3 > \inf_X \Phi$ such that, for any $c < c_3$,

$$\pi^{-1}(B_c) \text{ has } \tilde{X} \text{ as an analytic completion.}$$

In particular:

If $\tilde{X}$ is holomorphically complete, then, for any $c < c_3$, $\tilde{X}$ is the envelope of holomorphy of $\pi^{-1}(B_c)$.

3. *Proof of Theorem 1.* The proof of this theorem is based on the following:

PROPOSITION 1. *Let $0 \leqslant q \leqslant n-2$ and let Φ be a strongly q-pseudoconvex function defined in a neighborhood U of the origin in $\mathbf{C}^n$.*

Let

$$Y = \{z \in U \mid \Phi(z) > \Phi(0)\}.$$

Then there exists a fundamental system of (closed) neighborhoods $\{Q_\nu\}_{\nu \in \mathbf{N}}$ of $0 \in \mathbf{C}^n$ such that

(i) *each Q_ν is connected, $\mathring{Q}_\nu \supset Q_{\nu+1}$, $\forall \nu \in \mathbf{N}$,*

(ii) *the natural restriction maps*

$$H^0(Q_\nu, \mathcal{O}) \to H^0(Q_\nu \cap Y, \mathcal{O})$$

are isomorphisms, for $\forall \nu \in \mathbf{N}$.

This proposition can be considered as a particular case of Theorem 10 of [2] (cf. Proposition 12 of [2]), However, here the function Φ is not required to be differentiable. This restriction can be easily removed since only properties $(\alpha), (\beta), (\gamma)$ listed in Section 1 are requested in the proof.

We give here an outline of a direct proof of this proposition which, we believe, will be easier to follow than that of Theorem 10 of [2]. We divide the proof into several steps.

Step 1. Set $p = n - q$ and let $\mathbf{C}^n = \mathbf{C}^p \times \mathbf{C}^q$. We denote by

$$\xi_\alpha = x_\alpha + iy_\alpha, \quad 1 \leqslant \alpha \leqslant p, \text{ the coordinates in } \mathbf{C}^p,$$

$$\eta_\beta = v_\beta + iw_\beta, \quad 1 \leqslant \beta \leqslant q, \text{ the coordinates in } \mathbf{C}^q.$$

We set $$Q^{2p-1} = \{\xi \in \mathbf{C}^p \mid x_1 = 0,\ |x_j| \leqslant 1,\ 2 \leqslant j \leqslant p,\ |y_k| \leqslant 1,\ 1 \leqslant k \leqslant p\},$$
$$Q^1 = \{\xi \in \mathbf{C}^p \mid |x_1| \leqslant 1,\ x_j = 0,\ 2 \leqslant j \leqslant p,\ y_k = 0,\ 1 \leqslant k \leqslant p\},$$
$$Q^{2q} = \{\eta \in \mathbf{C}^q \mid |v_\beta| \leqslant 1,\ |w_\beta| \leqslant 1 \ \text{ for } \ 1 \leqslant \beta \leqslant q\}.$$

Thus $$Q = Q^1 \times Q^{2p-1} \times Q^{2q}$$

represents the unit cube in $\mathbf{C}^n$. We suppose that, as in our proposition, $0 \leqslant q \leqslant n-2$.

LEMMA 4. *Let Φ be a strongly q-pseudoconvex function defined in an open neighborhood U of Q. We suppose that*

(i) *$\Phi \mid (\mathbf{C}^p \times \{\eta\}) \cap U$ is strongly 0-pseudoconvex for all $\eta \in \mathbf{C}^q$;*

(ii) *the set*
$$\{(\bigcup_{-1 \leqslant t \leqslant 0} \{t\} \times Q^{2p-1}) \cup (Q^1 \times \partial Q^{2p-1})\} \times Q^{2q}$$

and the closure $\bar{V}$ of the set
$$V = \{z \in Q \mid \Phi(z) < \Phi(0)\}$$
are disjoint.

Then $$H^0(Q, \mathcal{O}) \to H^0(Q - V, \mathcal{O})$$

is an isomorphism and $H^r(Q - V, \mathcal{O}) = 0$ *for* $0 < r < n - q - 1$.

Proof. (α). From the exact sequence
$$0 \to H^0_k(V, \mathcal{O}) \to H^0(Q, \mathcal{O}) \to H^0(Q - V, \mathcal{O}) \to H^1_k(V, \mathcal{O}) \to \ldots$$

(where the suffix k denotes cohomology with compact supports) since $H^s(Q, \mathcal{O}) = 0$ for $s \geqslant 1$, we see that the lemma is equivalent to the statement
$$H^r_k(V, \mathcal{O}) = 0 \quad \text{for} \quad r < n - q = p.$$

(β). First one establishes the lemma when $q = 0$, i.e., $p = n$, for instance by the following method due essentially to B. Malgrange:

We remark that $\mathring{V}$ is a domain of holomorphy; thus
$$H^r_k(\mathring{V}, \mathcal{O}) = 0 \quad \text{for} \quad r \neq n.$$

Let $$Q^1_\varepsilon = \{-1 \leqslant t \leqslant 1 + \varepsilon\}, \quad Q_\varepsilon = Q^1_\varepsilon \times Q^{2n-1},$$
$$V_\varepsilon = \{z \in Q_\varepsilon \mid \Phi(z) < \Phi(0)\}.$$

Let ϱ^{or} be a $\bar{\partial}$-closed (or) form, compactly supported in V. For $\varepsilon > 0$ sufficiently small, we may assume that

(i) $(Q^1_\varepsilon \times \partial Q^{2n-1}) \cap \bar{V}_\varepsilon = \emptyset$

(ii) ϱ^{or} is defined, $\bar{\partial}$-closed, compactly supported in V_ε.

Let α be a C^∞ function of t such that

$$\alpha = \begin{cases} 1 & \text{if } t < 1 + \varepsilon/3 \\ 0 & \text{if } t > 1 + 2\varepsilon/3. \end{cases}$$

Then setting

$$\mu^{or+1} = \bar{\partial}\alpha \wedge \varrho^{or},$$

we have

$$\operatorname{supp} \mu^{or+1} \subset \{1 + \varepsilon/3 \leqslant t \leqslant 1 + 2\varepsilon/3\} \times \mathring{Q}^{2n-1}.$$

Actually, if

$$P_\varepsilon = \{1 + \varepsilon/4 < t < 1 + \varepsilon\} \times \mathring{Q}^{2n-1},$$

then

$$\operatorname{supp} \mu^{or+1} \subset\subset P_\varepsilon \cap V_\varepsilon.$$

Let U be a neighborhood of Q_ε which is a domain of holomorphy.

In our situation one has ([5], Lemma 29) that

$$\tau: H^r_k(P_\varepsilon \cap V_\varepsilon, O) \to H^r_k(U, O)$$

is an injective map. Since $\tau\{\mu^{or+1}\} = 0$ by its very construction, it follows then that there exists a form σ^{or} of type (or), with compact support in $P_\varepsilon \cap V_\varepsilon$ such that

$$\mu^{or+1} = \bar{\partial}\sigma^{or}$$

i.e.

$$\bar{\partial}(\alpha\varrho^{or} - \sigma^{or}) = 0.$$

Now the form $\alpha\varrho^{or} - \sigma^{or}$ is compactly supported and $\bar{\partial}$-closed in $\mathring{V}_\varepsilon$. By the remark made at the beginning one sees that:

If $r \leqslant n-1$, then there is a form γ^{or-1} compactly supported in $\mathring{V}_\varepsilon$ such that

$$\alpha\varrho^{or} - \sigma^{or} = \bar{\partial}\gamma^{or-1}.$$

By restriction to V, we then obtain

$$\varrho^{or} = \bar{\partial}\gamma^{or-1}$$

and supp γ^{or-1} is compact in V.

(γ). One establishes then the same result for the case of a product of the situation considered in (β) by a cube (i.e,, one proves the lemma in the case in which Φ is independent of the variables η).

This can be done by means of the Künneth formula.

(δ). In the general case we consider the projection π of V onto the space Q^{2q} of the η parameters. Let ψ be the family of closed subsets F of V on which $\pi|F$ is a proper map. Let $\mathcal{H}^s_\psi(O)$ be the s-th direct image of O with supports ψ. Then $H^r_k(V,O)$ is the limit of a spectral sequence whose term $E_2^{r,s}$ is given by

$$E_2^{r,s}=H^r(Q^{2q-2},\mathcal{H}^s_\psi(O)).$$

If $r+s<p-1$, then $s<p-1$, but from (γ) one deduces that

$$\mathcal{H}^s_\psi(O)=0 \quad \text{for} \quad s<p-1.$$

It follows then that $\bigoplus_{r+s\leqslant p-1} E_2^{r,s}=0$ and thus that

$$H^r_k(V,O)=0 \quad \text{if} \quad r<p-1.$$

This completes the proof of the lemma.

Step 2. Let Φ be a strongly q-pseudoconvex function defined in a neighborhood U of the origin in $\mathbf{C}^n$. Without loss of generality, taking U sufficiently small and by a convenient choice of coordinates we may assume that on U,

$$\Phi=\sup(\varphi_1,\dots,\varphi_k),$$

where the φ's are C^∞ functions in U satisfying the conditions:

(i) $$\varphi_1(0)=\dots=\varphi_k(0)=\Phi(0)=0;$$

(ii) for $1\leqslant s\leqslant k$ the Levi forms at the origin

$$\mathfrak{L}(\varphi_s)_0=\sum_1^{n-q}\left(\frac{\partial^2\varphi_s}{\partial z_\alpha\,\partial\bar z_\beta}\right)_0 z_\alpha\bar z_\beta$$

on the space $\mathbf{C}^{n-q}=\{z_{n-q+1}=0,\dots,z_n=0\}$ are positive non-degenerate.

We restrict φ_s to $\mathbf{C}^{n-q}$; we then can write

$$\varphi_s|\mathbf{C}^{n-q}=2\,\mathrm{Re}\,g_s+\mathfrak{L}(\varphi_s)_0+O(3),$$

where $O(3)$ is small of third order and where

$$g_s=\sum_1^{n-q}{}_\alpha\left(\frac{\partial\varphi}{\partial z_\alpha}\right)_0 z_a+\frac{1}{2}\sum_1^{n-q}{}_{\alpha\beta}\left(\frac{\partial^2\varphi}{\partial z_\alpha\,\partial z_\beta}\right)_0 z_\alpha z_\beta.$$

We distinguish the following possibilities:

(a) all g_s are identically zero or one g_s is not identically zero but has non-vanishing differential at 0, $(dg_s)_0 \neq 0$;

(b) not all g_s are identically zero but their differentials all vanish at 0, $(dg_s)_0 = 0 \ \forall s$.

Case (a). We can find a holomorphic function f on $\mathbf{C}^{n-q}$ such that

$$f(0)=0, \ (df)_0 \neq 0,$$

$$\{z \in \mathbf{C}^{n-q} \cap U \mid \operatorname{Re} f(z) \geq 0\} \subset \{z \in U \mid \Phi(z) \geq 0\}.$$

Without restriction we may assume $-f \equiv z_1$.

As in Step 1 we denote by $\xi_\alpha = x_\alpha + iy_\alpha$ the first $p = n-q$ coordinates in $\mathbf{C}^n$, and by $\eta_\beta = v_\beta + iw_\beta$ the last q coordinates.

We set

$$\begin{aligned} Q^{2p-1}(\varepsilon) &= \{\xi \in \mathbf{C}^p \mid x_1 = 0, \ |x_j| \leq \varepsilon, \ |y_k| \leq \varepsilon, \ j \geq 2, \ k \geq 1\}, \\ Q^1(\delta) &= \{\xi \in \mathbf{C}^p \mid |x_1| \leq \delta, \ x_j = 0, \ y_k = 0, \ j \geq 2, \ k \geq 1\}, \\ Q^{2q}(\varrho) &= \{\eta \in \mathbf{C}^q \mid |v_\beta| \leq \varrho, \ |w_\beta| \leq \varrho, \ 1 \leq \beta \leq q\}. \end{aligned}$$

We can first find $\varepsilon > 0$, $\sigma > 0$ such that

$$\{t\} \times Q^{2p-1}(\varepsilon) \subset U \quad \text{for} \quad |t| \leq \sigma$$

$$\{t\} \times Q^{2p-1}(\varepsilon) \subset Y = \{z \in U \mid \Phi(z) > 0\} \quad \text{for} \quad -\sigma < t < 0.$$

Then we can find δ, $0 < \delta < \sigma$ such that

$$\{t\} \times \partial Q^{2p-1}(\varepsilon) \subset Y \quad \text{for} \quad |t| \leq \delta.$$

Then we can find $\varrho > 0$ such that

$$Q^1(\delta) \times Q^{2p-1}(\varepsilon) \times Q^{2q}(\varrho) \subset U$$

and

$$\{(\bigcup_{-\delta \leq t \leq -\delta/2} \{t\} \times Q^{2p-1}(\varepsilon)) \cup (Q^1(\delta) \times \partial Q^{2p-1}(\varepsilon))\} \times Q^{2q}(\varrho) \subset Y. \tag{*}$$

Thus for $Q = Q^1(\delta) \times Q^{2p-1}(\varepsilon) \times Q^{2q}(\varrho)$ we can apply Lemma 1 and obtain the isomorphism

$$H^0(Q, O) \xrightarrow{\sim} H^0(Q \cap \overline{Y}, O).$$

Let $\alpha > 0$ be a C^∞ function on Q; for $\lambda > 0$ sufficiently small we set $\Phi' = \Phi \pm \lambda\alpha$. Then the corresponding set $Y' = \{z \in Q \mid \Phi'(z) > \Phi'(0)\}$ will satisfy again the condition (*). We then deduce that

$$H^0(Q, O) \simeq H^0(Q \cap Y, O) \simeq H^0(Q \cap \overline{Y}, O).$$

Since U is arbitrary, Proposition 1 is established in this case.

Case (b). We can find among the functions g_s a holomorphic function f on $\mathbf{C}^{n-q}$, *not identically zero,* such that

$$f(0)=0, \quad (df)_0=0$$

$$\{z \in \mathbf{C}^{n-q} \cap U \mid \operatorname{Re} f(z) \geqslant 0\} \subset \{z \in U \mid \Phi(z) \geqslant 0\}.$$

We may extend f to $\mathbf{C}^n$ by lifting f to $\mathbf{C}^n$ by the coordinate projection $\mathbf{C}^n \to \mathbf{C}^{n-q}$.

Let us assume that the closed polycylinder

$$P = \{z \in \mathbf{C}^n \mid \sup_{1 \leqslant \alpha \leqslant n} |z_\alpha| \leqslant 1\}$$

is contained in U and that $\max_P |f| = 1$. Let

$$Q = \{\xi \in \mathbf{C}^{n+1} \mid \sup_{0 \leqslant \alpha \leqslant n} |\xi_\alpha| \leqslant 1\}$$

and let

$$\tau : P \to Q$$

be the holomorphic mapping defined by the equations

$$\xi_0 = f(z), \ \ \xi_1 = z_1, \dots, \xi_n = z_n.$$

The image of τ is thus the submanifold of Q with the equation

$$g(\xi) \equiv \xi_0 - f(\xi_1, \dots, \xi_n) = 0.$$

Let

$$\tilde{\varphi}_s(\xi) = 2 \operatorname{Re} g(\xi) + \varphi_s(\xi_1, \dots, \xi_n) + |g(\xi)|^2, \quad 1 \leqslant s \leqslant k$$

and set

$$\tilde{\Phi} = \sup (\tilde{\varphi}_1, \dots, \tilde{\varphi}_k).$$

Then

$$\tilde{\Phi}(\xi_0, \dots, \xi_n) = \Phi(\xi_1, \dots, \xi_n) + 2 \operatorname{Re} g(\xi) + |g(\xi)|^2,$$

so that

$$\Phi = \tilde{\Phi} \circ \tau.$$

Consider the linear space $\mathbf{C}^{n-q+1} = \{\xi \in \mathbf{C}^{n+1} \mid \xi_{n-q} = \dots = \xi_n = 0\}$ and on it the functions

$$\xi_1, \dots, \xi_{n-q}, g.$$

Since $(df)_0 = 0$, these functions can be assumed as local coordinates near the origin.

Then each $\tilde{\varphi}_s$, restricted to $\mathbf{C}^{n-q+1}$, is strongly 0-pseudoconvex at the origin. Therefore $\tilde{\Phi}$ is a strongly q-pseudoconvex function in a neighborhood W of the origin in $\mathbf{C}^{n+1}$.

Let us suppose that $f = g_1$ (as we can by renumbering the functions φ_s). Then we have on $\mathbf{C}^{n-q+1}$

$$\begin{aligned}\tilde{\varphi}_1 &= 2\,\mathrm{Re}\,(\xi_0 - g_1) + 2\,\mathrm{Re}\,g_1 + \mathcal{L}(\varphi_s)_0 + |g|^2 + O(3)\\ &= 2\,\mathrm{Re}\,\xi_0 + \mathcal{L}(\varphi_s)_0 + |g|^2 + O(3).\end{aligned}$$

Thus if W is sufficiently small, we have

$$\{z \in \mathbf{C}^{n-q+1} \cap W \,|\, \mathrm{Re}\,\xi_0 \geqslant 0\} \subset \{z \in W \,|\, \tilde{\Phi}(z) \geqslant 0\}.$$

Note that ξ_0, being one of the coordinates in $\mathbf{C}^{n+1}$, has $(d\xi_0)_0 \neq 0$. We therefore have in $\mathbf{C}^{n+1}$ the situation already discussed in Case (a).

There is therefore a sequence $\tilde{Q}_\nu$ of neighborhoods of $0 \in \mathbf{C}^{n+1}$, contained in W, such that letting

$$\tilde{Y} = \{z \in W \,|\, \tilde{\Phi}(z) > 0\},$$

we have isomorphic restrictions:

$$H^0(\tilde{Q}_\nu, \mathcal{O}) \simeq H^0(\tilde{Q}_\nu \cap \tilde{Y}, \mathcal{O}).$$

Let $\mathfrak{J} = \mathcal{O}g$; this is the sheaf of germs of holomorphic functions vanishing on $\tilde{P} = \tau(P)$. Since $\mathfrak{J} \approx \mathcal{O}$, we thus get also isomorphic restrictions:

$$H^0(\tilde{Q}_\nu, \mathfrak{J}) \simeq H^0(\tilde{Q}_\nu \cap \tilde{Y}, \mathfrak{J}).$$

We set $Q_\nu = \tau^{-1}(\tilde{Q}_\nu)$, $Y = \tau^{-1}(\tilde{Y})$. We have a commutative diagram:

$$\begin{array}{ccccccc} 0 \to H^0(\tilde{Q}_\nu, \mathfrak{J}) & \to & H^0(\tilde{Q}_\nu, \mathcal{O}) & \to & H^0(Q_\nu, \mathcal{O}) & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 \to H^0(\tilde{Q}_\nu \cap \tilde{Y}, \mathfrak{J}) & \to & H^0(\tilde{Q}_\nu \cap \tilde{Y}, \mathcal{O}) & \to & H^0(Q_\nu \cap Y, \mathcal{O}) & \to . \end{array}$$

If we show that the map $H^0(\tilde{Q}_\nu \cap \tilde{Y}, \mathcal{O}) \to H^0(Q_\nu \cap Y, \mathcal{O})$ is surjective, then by the "five lemma" and the above remarks it follows that, as we wanted,

$$H^0(Q_\nu, \mathcal{O}) \simeq H^0\, Q_\nu \cap Y, \mathcal{O}).$$

Step 3. We want to prove that (omitting the indices ν)

$$H^0(\tilde{Q} \cap \tilde{Y}, \mathcal{O}) \to H^0(Q \cap Y, \mathcal{O})$$

is surjective. Now we know that $H^0(\tilde{Q}, \mathcal{O}) \simeq H^0(\tilde{Q} \cap \tilde{Y}, \mathcal{O})$.

Hence by the same reasoning as in Step 2, Case (a), it will be sufficient to prove that

$$H^0(\tilde{Q} \cap \bar{\bar{Y}}, O) \to H^0(Q \cap \bar{Y}, O)$$

is surjective. This amounts to showing that $H^1(\tilde{Q} \cap \bar{\bar{Y}}, \mathcal{J}) = 0$ or, since $\mathcal{J} \simeq O$, that $H^1(\tilde{Q} \cap \bar{\bar{Y}}, O) = 0$.

But this is a consequence of Lemma 1 since we now work in $\mathbf{C}^{n+1}$ and, from $q \leqslant n-2$, we get $1 < (n+1) - q - 1$.

Remark. The construction given for the neighborhoods Q_ν satisfying conditions (i) and (ii) of Proposition 1, also satisfies the following condition:

(iii) *For every ν there exists $\varepsilon_\nu > 0$ such that if α is a C^∞ function on U satisfying*

$$\sup_{x \in Q_\nu} \sum_{|r| \leqslant 2} |D^r \alpha(x)| < \varepsilon_\nu,$$

then, setting $\Phi' = \Phi + \alpha$, $Y' = \{z \in U \mid \Phi'(z) > \Phi'(0)\}$, we have that

$$H^0(Q_{\nu+1}, O) \to H^0(Q_{\nu+1} \cap Y', O)$$

is an isomorphism.

4. We now conclude the proof of Theorem 1.

(α). We choose $c_3 = \inf(c_1, c_2)$, c_1, c_2 being defined as in Lemmas 1 and 2 of Section 1.

Choose c with $c_3 > c > \inf_X \Phi$ and a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of ∂B_c by open coordinate balls in X.

For every $x_0 \in \partial B_c$ we choose a coordinate ball $U_i \ni x_0$ and, as in Proposition 1, we can find a closed connected neighborhood $Q_{\nu+1} = Q(x_0)$ in U_i satisfying conditions (i), (ii) of Proposition 1 and (iii) of the Remark at the end of Section 3.

Since ∂B_c is compact, we can find a finite number of such neighborhoods $Q_1, \ldots, Q_t$ such that

$$\partial B_c \subset \bigcup_{s=1}^{t} \mathring{Q}_s.$$

Now let α_s be a C^∞ function on X with the properties

$$0 \leqslant \alpha_s \leqslant 1,$$

$$\operatorname{supp} \alpha_s \subset \mathring{Q}_s,$$

$$\sum \alpha_s(x) > 0 \quad \forall\, x \in \partial B_c.$$

We set
$$\Phi_s(x) = \Phi(x) + \varepsilon_1 \alpha_1 + \ldots + \varepsilon_s \alpha_s,$$
$$\Phi_0(x) = \Phi(x).$$

We can select $\varepsilon_i > 0$ one after the other, sufficiently small, such that for each s the function Φ_s is strongly q-pseudoconvex.

We define
$$B_c^s = \{x \in X \mid \Phi_s(x) > c\}.$$

We then have
$$B_c = B_c^0 \subset B_c^1 \subset \ldots \subset B_c^t \tag{a}$$

since
$$\Phi_{s+1} \geqslant \Phi_s.$$

Also, since
$$\Phi_{s+1} - \Phi_s = \varepsilon_{s+1} \alpha_{s+1},$$

we have
$$B_c^{s+1} - B_c^s \subset\subset \mathring{Q}_{s+1}. \tag{b}$$

Furthermore, since
$$\sum \varepsilon_s \alpha_s(x) > 0 \quad \text{on} \quad \partial B_c,$$

we have
$$B_c \subset\subset B_c^t. \tag{c}$$

Now conditions (a), (b), (c) will be satisfied for all choices of $\varepsilon_i > 0$ sufficiently small.

By condition (iii) of the remark quoted before, we may also assume that

$$H^0(Q_{s+1} \cap B_c^{s+1}, \mathcal{O}) \rightarrow H^0(Q_{s+1} \cap B_c^s, \mathcal{O}) \quad \text{for} \quad 0 \leqslant s \leqslant t-1 \tag{d}$$

is an isomorphism (and indeed both groups will be isomorphic with $H^0(Q_{s+1}, \mathcal{O})$).

We finally choose $\sigma > 0$ such that

$$B_{c-\sigma} \subset B_c^t. \tag{e}$$

This is possible since $\bar{B}_c = \{x \in X \mid \Phi(x) \geqslant c\}$.

(β) Now let $\Gamma = \pi_1(X)$ be considered as the group of automorphisms of the universal covering $\pi: \tilde{X} \rightarrow X$ of X.

For $1 \leqslant s \leqslant t$ the set $\pi^{-1}(Q_s)$ has as many connected components as there are elements in Γ. Let $\tilde{Q}_s$ be one of these components; then

$$\pi^{-1}(Q_s) = \Gamma \tilde{Q}_s.$$

The component $\tilde{Q}_s$ and each one of its transforms is mapped isomorphically by π onto Q_s.

Now let
$$f \in H^0(\pi^{-1}(B_c), O)$$
be a holomorphic function on $\pi^{-1}(B_c) = \tilde{B}_c$.

Let $\tilde{B}^s_c = \pi^{-1}(B^s_c)$. By condition (b) we have

$$\tilde{B}^1_c - \tilde{B}^0_c \subset \Gamma \tilde{Q}_s.$$

By condition (d) it then follows that f can be extended to a holomorphic function f_1 on $\tilde{B}^1_c$.

Repeating the argument with $\tilde{B}^2_c$, $\tilde{B}^1_c$ instead of $\tilde{B}^1_c$, $\tilde{B}^0_c$ and so forth, we prove that

$$H^0(\tilde{B}^t_c, O) \to H^0(\tilde{B}_c, O)$$

is a surjective map.

Consequently, by (e), setting $\pi^{-1}(B_{c-\sigma}) = \tilde{B}_{c-\sigma}$,

$$H^0(\tilde{B}_{c-\sigma}, O) \to H^0(\tilde{B}_c, O)$$

is surjective, and therefore an isomorphism, since both $\tilde{B}_c$ and $\tilde{B}_{c-\sigma}$ are connected.

(γ) We have proved the following statement:

Given any c, $c_3 > c > \inf_X \Phi$, we can find $\sigma > 0$ such that the restriction map

$$H^0(\tilde{B}_{c-\sigma}, O) \to H^0(\tilde{B}_c, O) \qquad (^*)$$

is an isomorphism.

Let us now consider the set Λ of real numbers σ, $0 \leqslant \sigma < c - \inf_X \Phi$ such that $(^*)$ is an isomorphism.

If $\sigma_0 \in \Lambda$, then any σ with $0 \leqslant \sigma < \sigma_0$ also belongs to Λ.

If $\sigma_\nu \nearrow \sigma_0$ and $\sigma_\nu \in \Lambda$ for every ν, then $\sigma_0 \in \Lambda$.

If $\sigma_0 \in \Lambda$, there is an $\varepsilon > 0$ such that $\sigma_0 + \varepsilon \in \Lambda$.

These three statements follow from $(^*)$ and from the fact that all B_c's with $c < c_3$ are connected. This implies that Λ is a closed and open subset of $0 \leqslant \sigma < c - \inf_X \Phi$; thus Λ coincides with that interval and we then have that

$$H^0(\tilde{X}, O) \to H^0(\tilde{B}_c, O)$$

is an isomorphism.

§ 2. On deformations of non-compact complex manifolds

5. *Differentiable families of complex manifolds.* a) By a *differentiable family of complex manifolds*, we mean the set of the following data:

a differentiable manifold M,
a differentiable manifold $\mathfrak{V}$,
a differentiable surjective map $\varpi : \mathfrak{V} \to M$

satisfying the following conditions:

(i) ϖ is of maximal rank at each point,
(ii) for every point $x \in \mathfrak{V}$ we can find:

a neighborhood W of x in $\mathfrak{V}$,
a neighborhood U of $\varpi(x)$ in M,
an open set S in some numerical space $\mathbf{C}^n$, and
a diffeomorphism $\varphi : U \times S \to W$ such that

(α) $pr_U = \varpi \circ \varphi$
(β) if $\varphi_i : U_i \times S_i \to W_i$, $i = 1, 2$, are any two such diffeomorphisms, then $\varphi_2^{-1} \circ \varphi_1$ is an isomorphism of $\varphi_1^{-1}(W_1 \cap W_2)$ and $\varphi_2^{-1}(W_1 \cap W_2)$, endowed with the structural sheaves of germs of C^∞ functions holomorphic on the fibers of the projection pr_{U_i} $(i = 1, 2)$.

It follows then that, for every $t \in M$, $\varpi^{-1}(t) = X_t$ has a natural structure of a complex manifold. We will take as structural sheaf on $\mathfrak{V}$ the sheaf of germs of C^∞ functions holomorphic on the fibers of ϖ.

Analogously one defines a complex analytic (or holomorphic) family of complex manifolds, cf. [3].

b) Let X_0 be a complex manifold. By a (differentiable) *deformation* of X_0 we mean the data of

a differentiable family of complex manifolds $(\mathfrak{V}, \varpi, M)$,
a point $m_0 \in M$,
an isomorphism $i : X_0 \to \varpi^{-1}(m_0)$.

Analogously for a complex analytic deformation.

The definitions of equivalent deformations, locally equivalent deformations and classes of local deformations are as in [13].

Given $M, m_0 \in M$, any deformation $(\mathfrak{V}, \varpi, M)$ of X_0, which is equivalent to the deformation $(X_0 \times M, pr_M, M)$ is called a trivial deformation of X_0.

c) We set the following definitions.

DEFINITION. A deformation $(\mathfrak{V}, \varpi, M)$ of X_0 is called *pseudotrivial* if for every relatively compact open set $A \subset\subset X_0$ we can find an isomorphism

$$g_A : A \times M \to \mathfrak{V}$$

onto an open subset of $\mathfrak{V}$ such that $\varpi \circ g_A = pr_M$.

One then derives from this the notion of a *locally pseudotrivial* deformation.

DEFINITION. A deformation $(\mathfrak{V}, \varpi, M)$ of X_0 is called *rigid at infinity* if we can find a compact set $K_0 \subset X_0$ and an isomorphism

$$g : (X_0 - K_0) \times M \to \mathfrak{V}$$

onto an open subset of $\mathfrak{V}$ such that

(i) $\varpi \circ g = pr_M$,

(ii) $\varpi \mid \mathfrak{V} - \mathrm{Im}\,(g)$ is a proper map.

One then defines the notion of deformation *locally rigid at infinity*.

6. a) Let $(\mathfrak{V}, \varpi, M)$ be a differentiable family of complex structures. The bundle Θ of holomorphic tangent vectors to the fibers of ϖ is defined. It is a differentiable bundle on $\mathfrak{V}$ whose restriction to any fiber of ϖ is a holomorphic bundle.

Let $\mathcal{A}(\Theta)$ be the sheaf of germs of C^∞ sections of Θ which are holomorphic on the fibers of ϖ, and lat $\mathcal{A}^{r,s}(\Theta)$ be the sheaf of germs of C^∞ sections of the bundle $\Theta \otimes \Theta^{*r} \otimes \bar{\Theta}^{*s}$ where Θ^* denotes the dual bundle of Θ, Θ^{*r} its exterior power r-times, and the bar denotes the passage to the complex conjugate bundle.

The operator of exterior differentiation along the fibers with respect to anti-holomorphic local coordinates defines a sheaf homomorphism

$$\bar{\partial} : \mathcal{A}^{r,s}(\Theta) \to \mathcal{A}^{rs+1}(\Theta),$$

and one obtains a fine resolution of the sheaf $\mathcal{A}(\Theta)$:

$$0 \to \mathcal{A}(\Theta) \xrightarrow{i} \mathcal{A}^{00}(\Theta) \xrightarrow{\bar{\partial}} \mathcal{A}^{01}(\Theta) \xrightarrow{\bar{\partial}} \dots .$$

In particular $H^q(\mathfrak{V}, \mathcal{A}(\Theta))$ is isomorphic to the qth cohomology group of the complex $\{\oplus_{s\geq 0} \Gamma(\mathfrak{V}, \mathcal{A}^{0s}), \bar{\partial}\}$.

Let ψ be the family of closed subsets F of $\mathcal{V}$ such that $\varpi | F$ is a proper map. Substituting the functor Γ with Γ_ψ we also obtain that $H^q_\psi(\mathcal{V}, \mathcal{A}(\Theta))$ is isomorphic to the qth cohomology group of the complex $\{\bigoplus_{s\geqslant 0}\Gamma_\psi(\mathcal{V}, \mathcal{A}^{0s}), \bar\partial\}$.

b) Let T denote the differentiable tangent bundle to M and let $\mathcal{E}$ be the sheaf of germs of C^∞ sections of T. Then the map

$$\tilde\varrho_M : H^0(M, \mathcal{E}) \to H^1(\mathcal{V}, \mathcal{A}(\Theta))$$

is defined (cf. [13]). If the family is rigid at infinity, this map factors in a natural way through

$$\tilde\sigma_M : H^0(M, \mathcal{E}) \to H^1_\psi(\mathcal{V}, \mathcal{A}(\Theta))$$

so that one has the commutative diagram

$$\begin{array}{ccc} H^0(M,\mathcal{E}) & \xrightarrow{\tilde\varrho_M} & H^1(\mathcal{V}, \mathcal{A}(\Theta)) \\ \tilde\sigma_M \searrow & & \nearrow i \\ & H^1_\psi(\mathcal{V}, \mathcal{A}(\Theta)), & \end{array}$$

i being the natural homomorphism.

c) Let us now suppose that $(\mathcal{V}, \varpi, M)$ is a differentiable family of deformations of $X_0 = \varpi^{-1}(m_0)$. By localization around the point m_0 of the previous considerations one obtains first the mapping of Kodaira-Spencer

$$\tilde\varrho : \mathcal{E}_{m_0} \to \mathcal{R}^1\varpi(\Theta),$$

where $\mathcal{R}^1\varpi(\Theta)$ is the first direct image of the sheaf $\mathcal{A}(\Theta)$ for the projection ϖ.

If we have a deformation which is locally rigid at infinity, we obtain then a mapping

$$\tilde\sigma : \mathcal{E}_{m_0} \to \mathcal{R}^1_\psi\varpi(\Theta),$$

where $\mathcal{R}^1_\psi\ \varpi(\Theta)$ is the first direct image of $\mathcal{A}(\Theta)$ with supports ψ.

One has then the commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{m_0} & \xrightarrow{\tilde\varrho} & \mathcal{R}^1\varpi(\Theta) \\ \tilde\sigma \searrow & & \nearrow i \\ & \mathcal{R}^1_\psi\varpi(\Theta). & \end{array}$$

PROPOSITION 2. *Let $(\mathcal{V}, \varpi, M)$ be a deformation of the complex manifold $X_0 = \varpi^{-1}(m_0), m_0 \in M$.*

If $\tilde{\varrho} = 0$, *then* $(\mathfrak{V}, \varpi, M)$ *is a locally pseudotrivial deformation of* X_0; *actually one can find a neighborhood of* X_0 *in* $\mathfrak{V}$ *which can be isomorphically imbedded in the product* $X_0 \times M$ *with a fiber-preserving map.*

This proposition was proved in [3] for a complex analytic family. The proof holds without any change for differentiable families. In particular, from the same argument one deduces for deformations rigid at infinity the following:

PROPOSITION 3. *Let* $(\mathfrak{V}, \varpi, M)$ *be a deformation of* $X_0 = \varpi^{-1}(m_0)$, $m_0 \in M$ *which is rigid at infinity.*

The necessary and sufficient condition for $(\mathfrak{V}, \varpi, M)$ *to define a locally trivial deformation is that* $\tilde{\sigma} = 0$.

d) Analogous considerations could be repeated for differentiable families of differentiable manifolds. In this case the sheaf $\mathcal{A}(\Theta)$ would be replaced by a fine sheaf and analogue of the previous propositions would lead to the following conclusion ([3]):

PROPOSITION 4. *Let* $(\mathfrak{V}, \varpi, M)$ *be a differentiable deformation of a differentiable manifold* $X_0 = \varpi^{-1}(m_0)$, $m_0 \in M$.

(a) *one can find a neighborhood of* X_0 *in* $\mathfrak{V}$ *which can be imbedded in the product* $X_0 \times M$ *with a fiber-preserving map.*

(b) *if* $(\mathfrak{V}, \varpi, M)$ *is rigid at infinity, then* $(\mathfrak{V}, \varpi, M)$ *is locally trivial.*

7. *Families of uniformizable structures.* a) Let $(\mathfrak{V}, \varpi, M)$ be a differentiable family of complex manifolds parametrized by a connected and simply connected manifold M.

Let $\pi: \tilde{\mathfrak{V}} \to \mathfrak{V}$ be the universal covering manifold of $\mathfrak{V}$; $(\tilde{\mathfrak{V}}, \varpi \circ \pi, M)$ can be considered as a new family of complex manifolds.

Let D be a complex manifold; we will say that $(\mathfrak{V}, \varpi, M)$ is a *family of complex manifolds uniformizable on the manifold* D if we can give an isomorphism

$$\sigma: \tilde{\mathfrak{V}} \to D \times M$$

(with respect to the sheaves of C^∞ functions holomorphic respectively on the fibers of $\varpi \circ \pi$ and pr_M) so that the following is a commutative diagram:

$$\begin{array}{ccc} \tilde{\mathfrak{V}} & \xrightarrow{\sigma} & D \times M \\ \pi \downarrow & & \downarrow pr_M \\ \mathfrak{V} & \xrightarrow{\varpi} & M. \end{array}$$

We will always assume in the sequel that $\mathfrak{V}$ is connected. Hence $\tilde{\mathfrak{V}}$ will be connected and simply connected. If follows that D must be connected and simply connected and that, for each $t \in M$,

$$D \times \{t\} \overset{\pi \circ \sigma^{-1}}{\rightarrow} X_t = \varpi^{-1}(t)$$

is the universal convering of X_t.

Let $\Gamma = \pi_1(\mathfrak{V})$; this can be viewed as the group of automorphisms of the universal covering $\pi: \tilde{\mathfrak{V}} \to \mathfrak{V}$. From the previous remark it then follows that, for each t, we have

$$\pi_1(X_t) \approx \Gamma.$$

b) Let Aut(D) be the group of all complex analytic automorphisms of the manifold D.

By means of σ we identify Γ with $\sigma^{-1}\Gamma\sigma$ as a group of automorphisms of $D \times M$. Every element $\gamma \in \Gamma$ represents then a map $D \times M \to D \times M$ given by equations of the type

$$\begin{cases} z \to \gamma(z, t) & z \in D, t \in M, \\ t \to t \end{cases}$$

where, for every $t \in M, \gamma(z, t) \in \mathrm{Aut}(D)$.

We will assume that Aut(D) (with the compact open topology) *has the structure of a Lie group.*(1) This is the case for instance for a bounded domain D in $\mathbf{C}^n$ (cf. [9]).

One sees then that to give a family of complex manifolds uniformizable on D is the same as to give for every $t \in M$ a representation

$$\varrho_t : \Gamma \to \mathrm{Aut}(D)$$

which is discrete, acts freely on D (i.e., without fixed points), and depends differentiably on t.

c) As in Section 5 we define then, for a manifold X_0 whose universal covering is isomorphic to D, the notion of a deformation of X_0 in the class of uniformizable structures on D. This notion, due to A. Weil [21], is the most natural for the investigation of properly discontinuous groups.

8. *Holomorphic deformations of uniformizable structures.* In the definitions given before, we can replace differentiable families by complex analytic families of uniformizable manifolds.

(1) For a complex manifold D this may not be the case. For instance, if $D = \mathbf{C}^2$, the transformations of type $x \to x + P(y)$, $y \to y$ where $P(y)$ is *any* polynomial in x cannot belong all to the same Lie group.

The condition of complex analyticity for the family is a very restrictive one, as appears from the following theorem:

THEOREM 2. *Let D be a bounded domain in $\mathbf{C}^n$,. Any complex analytic family of complex manifolds uniformizable on D is locally trivial.*

Proof. Let $(\mathfrak{V}, \varpi, M)$ be a complex analytic family of complex manifolds uniformizable on D.

Since the theorem is of local nature, we may assume that the parameter space M is the unit ball in $\mathbf{C}^m$:

$$M = \{t \in \mathbf{C}^m \mid \sum |t^\alpha|^2 < 1\}.$$

On the product $D \times M$ we have then given a group Γ of complex analytic automorphisms of type

$$\gamma = \begin{cases} z^\alpha \rightarrow \gamma^\alpha(z, t) & 1 \leqslant \alpha \leqslant n \\ t^\beta \rightarrow t^\beta & 1 \leqslant \beta \leqslant m, \end{cases}$$

where now, by assumption, the functions $\gamma^\alpha(z, t)$ are holomorphic in z *and* t.

All we have to prove is that actually the functions $\gamma^\alpha(z, t)$ for every $\gamma \in \Gamma$ and $1 \leqslant \alpha \leqslant n$, do not depend on t.

Now M and D are bounded domains; so we can consider the Bergmann metrics ds_M^2 and ds_D^2 of M and D respectively. One easily verifies that for the product of two bounded domains the Bergmann metric is the sum of the metrics of the factors. Thus

$$ds^2_{D \times M} = ds_D^2 + ds_M^2.$$

Any complex analytic automorphism of $D \times M$ is an isometry with respect to the Bergmann metric.

Let $(z_0, t_0) \in D \times M$ and let γ be given. Then

$$\gamma_0 = \begin{cases} z^\alpha \rightarrow \gamma(z^\alpha, t_0) & 1 \leqslant \alpha \leqslant n \\ t^\beta \rightarrow t^\beta & 1 \leqslant \beta \leqslant m \end{cases}$$

is also an automorphism of $D \times M$. We consider the automorphism $\gamma \circ \gamma_0^{-1}$. This will be the identity on $D \times \{t_0\}$.

We take (z_0, t_0) in the origin of the coordinates in $\mathbf{C}^n \times \mathbf{C}^m$; then the equations of $\gamma \circ \gamma_0^{-1}$ will have expressions of this type:

$$\gamma \circ \gamma_0^{-1} = \begin{cases} z' = A\,z + B\,t + O(2) \\ t' = t, \end{cases}$$

where A and B are constant matrices.

Since the origin is a fixed point and $\gamma\circ\gamma_0^{-1}$ is an isometry for the Bergmann metric, it follows then that we must have $B=0$. Moreover, since for $t=0$ the mapping is the identity, then $A=I$. Thus the linear part of $\gamma\circ\gamma_0^{-1}$ is the identity. By a theorem of H. Cartan [8, 9], since $D\times M$ is bounded, it follows then that

$$\gamma\circ\gamma_0^{-1}=\text{identity}\quad \text{i.e., } \gamma=\gamma_0.$$

This proves the theorem.

Remark 1. We have actually proved the triviality of the family on any coordinate ball of the base space M.

Remark 2. There are no restrictions on the dimension of D. Thus for instance if $(\mathfrak{V},\varpi,M)$ is the family of curves of genus $p>1$ over the Teichmüller space M, then the uniformizing parameter on $X_t=\varpi^{-1}(t)$ on the Poincaré unit circle cannot depend analytically on t (or $\tilde{\mathfrak{V}}$ is not analytically isomorphic to the product of M and the unit circle). This fact was first pointed out to us by L. Bers.

§ 3. Deformations of structures uniformizable on bounded symmetric domains

9. *W-ellipticity.* We gather here some known facts about the $\bar\partial$-cohomology that we will need later.

Let X be a complex manifold of pure dimension n. Let $\pi:E\to X$ be a holomorphic vector bundle on X. We denote by $C^{pq}(X,E)$ the vector space of C^∞ forms of type (p,q) with values in E; by $\mathfrak{D}^{pq}(X,E)$ we denote the subspaces of those forms with compact support.

Let $h(u,v), u, v\in\pi^{-1}(x)$ be a positive definite hermitian scalar product on the fibers of E depending differentiably on the point $x\in X$. If on the coordinate neighborhood U, $E|_U\simeq U\times\mathbf{C}^r$ ($r=\text{rank } E$) and if $u={}^t(\xi_1,\dots,\xi_r)$, $v={}^t(\eta_1,\dots,\eta_r)$, then on U

$$h(u,v)={}^t\bar\eta\, h_U\,\xi,$$

where h_U is a positive definite hermitian matrix of class C^∞ on U. The local forms

$$l_U=h_U^{-1}\,\partial\, h_U$$

define a ∂-connection on E and the curvature form of this connection is

$$s_U=\bar\partial\, l_U=\bar\partial(h_U^{-1}\,\partial\, h_U)$$

locally given by a $r\times r$ matrix of (1, 1) forms.

Given $\varphi \in C^{pq}(X, E)$, then $\varphi_U = \varphi|_U$ is given by a column vector with r-components each of which is a scalar form of type (p, q). The exterior multiplication by s_U gives a new column vector $s_U \wedge \varphi_U$ of type $(p+1,\ q+1)$. One verifies that we thus obtain a linear mapping

$$e(s): C^{pq}(X, E) \to C^{p+1,\, q+1}(X, E).$$

If E^* is the dual bundle of E, then one can define an anti-isomorphism

$$\#: C^{pq}(X, E) \to C^{qp}(X, E^*)$$

locally given by $\# \varphi_U = \bar{h}_U \bar{\varphi}_U$.

We now consider on X a hermitian metric ds^2. We can then define the $*$ "operator"

$$*: C^{pq}(X, E) \to C^{n-q,\, n-p}(X, E)$$

normalized so that $** = (-1)^{p+q}$.

Given $\varphi, \psi \in C^{pq}(X, E)$, then

$$^t\varphi \wedge *\# \psi = A(\varphi, \psi)\, dX$$

is a (n, n) scalar form that we can write as $A(\varphi, \psi)\, dX, dX$ being the volume element of the metric ds^2.

If $\varphi, \psi \in \mathfrak{D}^{pq}(X, E)$, then

$$(\varphi, \psi) = \int_X A(\varphi, \psi)\, dX$$

is finite and defines on $\mathfrak{D}^{pq}(X, E)$ a complex pre-hilbert structure. We denote by $\mathfrak{L}^{pq}(X, E)$ the completion of $\mathfrak{D}^{pq}(X, E)$ with respect to the norm $\|\varphi\| = (\varphi, \varphi)^{1/2}$.

Since E is a holomorphic vector bundle, exterior differentiation with respect to complex conjugate coordinates defines a linear map

$$\bar{\partial}: C^{pq}(X, E) \to C^{pq+1}(X, E)$$

with $\bar{\partial}\bar{\partial} = 0$. Its formal adjoint is the linear map

$$\mathfrak{b}: C^{pq+1}(X, E) \to C^{pq}(X, E)$$

given by $$\mathfrak{b} = - * \# \bar{\partial} \# *.$$

The Laplace–Beltrami operator is defined as

$$\square = \bar{\partial}\mathfrak{b} + \mathfrak{b}\bar{\partial}: C^{pq}(X, E) \to C^{pq}(X, E).$$

For compactly supported forms one has therefore

$$(\square\,\varphi, \psi) = (\varphi, \square\,\psi) = (\bar\partial\,\varphi, \bar\partial\,\psi) + (\mathfrak{d}\,\varphi, \mathfrak{d}\,\psi).$$

The completion of $\mathcal{D}^{pq}(X, E)$ with respect to the norm

$$N(\varphi) = (\|\varphi\|^2 + \|\bar\partial\,\varphi\|^2 + \|\mathfrak{d}\,\varphi\|^2)^{1/2}$$

is denoted by $W^{pq}(X, E)$.

Definition. We say that E *is* W^{pq} *elliptic* if hermitian metrics on the fibers of E and on X are given, so that, with a positive constant $c > 0$, one has for every $\varphi \in \mathcal{D}^{pq}(X, E)$ the inequality:

$$\|\varphi\|^2 \leqslant c\,(\|\bar\partial\,\varphi\|^2 + \|\mathfrak{d}\,\varphi\|^2).$$

Proposition 5 (cf. [4] or [5]). *If E is W^{pq}-elliptic, then for every $f \in \mathcal{L}^{pq}(X, E) \cap C^{pq}(X, E)$ there exists a unique element $x \in W^{pq}(X, E) \cap C^{pq}(X, E)$ such that*

$$\square\, x = f.$$

We now assume that the hermitian metric ds^2 on X is a *complete metric*. In these conditions for any $\sigma > 0$ one has the following inequality for all forms $\varphi \in C^{pq}(X, E)$ (any p and q) ([4] or [5]):

$$\|\bar\partial\,\varphi\|^2 + \|\mathfrak{d}\,\varphi\|^2 \leqslant \frac{1}{\sigma}\,\|\varphi\|^2 + \sigma\,\|\square\,\varphi\|^2. \tag{1}$$

Obviously the interest of this inequality is for those forms such that

$$\|\varphi\|^2 < \infty,\ \|\square\,\varphi\|^2 < \infty.$$

Also in the case of a complete metric the space $W^{pq}(X, E)$ can be identified with the subspace of $\mathcal{L}^{pq}(X, E)$ of those elements φ for which the distributions $\bar\partial\,\varphi$ and $\mathfrak{d}\,\varphi$ can be represented by elements in $\mathcal{L}^{pq+1}(X, E)$ and $\mathcal{L}^{pq-1}(X, E)$ respectively [5].

From inequality (1), if follows in particular that if $f \in \mathcal{L}^{pq}(X, E) \cap C^{pq}(X, E)$, any $x \in \mathcal{L}^{pq}(X, E) \cap C^{pq}(X, E)$ satisfying the equation $\square x = f$ must be an element of $W^{pq}(X, E) \cap C^{pq}(X, E)$. If E is W^{pq}-elliptic, then it follows that

$$\|x\| \leqslant c\,\|f\|.$$

Theorem 3 ([4], [5]). *If E is W^{pq}-elliptic with respect to a complete metric on X, then for every $f \in \mathcal{L}^{pq}(X, E) \cap C^{pq}(X, E)$ such that $\bar\partial f = 0$ there exists a unique $x \in W^{pq}(X, E) \cap C^{pq}(X, E)$ such that*

$$f = \bar\partial\,\mathfrak{d}\,x, \quad \mathfrak{d}\,\bar\partial\,x = 0.$$

10. *Kähler and Einstein metrics.* a) If the metric ds^2 is a Kähler metric and if

$$\omega = \sqrt{-1} \sum g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$$

is the associated exterior form, one can define the linear mappings

$$L : C^{pq}(X, E) \to C^{p+1, q+1}(X, E)$$

$$\Lambda : C^{p+1, q+1}(X, E) \to C^{pq}(X, E),$$

where $L\varphi = \omega \wedge \varphi$ and $\Lambda = *^{-1} L *$ is the formal adjoint of L.

One has the following identity (cf. [7]):

$$\square - *^{-1} \square * = \sqrt{-1}\,(e(s)\Lambda - \Lambda e(s)) = \sqrt{-1}\,[e(s), \Lambda].$$

Now since for $\varphi \in \mathfrak{D}^{pq}(X, E)$ one has

$$(*^{-1} \square * \varphi, \varphi) = (\square * \varphi, * \varphi) = \|\bar\partial * \varphi\|^2 + \|\mathfrak{d} * \varphi\|^2 \geqslant 0,$$

one obtains the following inequality for any $\varphi \in \mathfrak{D}^{pq}(X, E)$:

$$(\sqrt{-1}\,(e(s)\Lambda - \Lambda e(s))\varphi, \varphi) \leqslant \|\bar\partial\varphi\|^2 + \|\mathfrak{d}\varphi\|^2.$$

Therefore whenever we are able to establish an inequality of the type

$$(\sqrt{-1}\,(e(s)\Lambda - \Lambda e(s))\varphi, \varphi) \geqslant c\,\|\varphi\|^2$$

with $c > 0$, we get a criterion of W-ellipticity,

b) We consider now the special case where

E is the holomorphic tangent bundle Θ to X,
φ is a form of type $(0, q)$ with values in Θ.

Then the Kähler metric on X can be assumed as a metric on the fibers of Θ. We can compute the connection and curvature forms. One has for the ∂-connection:

$$l_\alpha^\varrho = \sum_\beta \Gamma^\varrho_{\alpha\beta}\, dz^\beta \text{ where } \Gamma^\varrho_{\alpha\beta} = \sum g^{\varrho\bar\gamma} \frac{\partial g_{\bar\gamma\alpha}}{\partial z^\beta};$$

for the curvature:

$$s_\alpha^\varrho = \sum R^\varrho{}_{\alpha\beta\bar\gamma}\, dz^\beta \wedge d\bar z^\gamma \text{ where } R^\varrho{}_{\alpha\beta\bar\gamma} = -\frac{\partial \Gamma^\varrho_{\alpha\beta}}{\partial \bar z^\gamma}$$

(the Γ's are the Christoffel symbols and $R^\varrho{}_{\alpha\beta\bar\gamma}$ is the Riemann tensor).

Since φ is of type $(0,q)$, $\Lambda\varphi=0$. so that

$$\sqrt{-1}\,(e(s)\Lambda-\Lambda e(s))\,\varphi=-\sqrt{-1}\,\Lambda e(s)\,\varphi.$$

Let $B=(\beta_1,\dots,\beta_q)$, $B_i'=(\beta_1\dots,\hat{\beta}_i,\dots,\beta_q)$; then for

$$\varphi=\sum_{\beta_1<\dots<\beta_q}\varphi^{\alpha}_{\bar\beta_1\dots\bar\beta_q}\,d\bar z^{\beta_1}\wedge\dots\wedge d\bar z^{\beta_q}=\sum\varphi^{\alpha}_{\bar B}\,\overline{dz^{B}},$$

one obtains:

$$(\sqrt{-1}\,\Lambda\,e(s)\,\varphi)^{\alpha}_{\bar B}=-\sum_{\gamma}R^{\alpha}_{\gamma}\varphi^{\gamma}_{\bar B}-\sum_{i=1}^{q}\sum_{\beta,\gamma}(-1)^{i}R^{\alpha}{}_{\gamma}{}^{\bar\beta}{}_{\bar\beta_i}\,\varphi^{\gamma}_{\bar\beta\bar B_i'},$$

where
$$R_{\alpha\bar\beta}=\sum_{\gamma}R^{\gamma}{}_{\alpha\gamma\bar\beta}$$

is the Ricci tensor.

c) We now introduce the assumption that the metric is a Kähler–Einstein metric, i.e. that one has the relation:

$$R_{\alpha\bar\beta}=\frac{R}{2n}\,g_{\alpha\bar\beta},$$

where $R=2\sum_{\alpha}R^{\alpha}_{\alpha}$ is the scalar curvature of the metric and is constant on X. We then obtain:

$$(-\sqrt{-1}\,\Lambda e(s)\,\varphi)^{\alpha}_{\bar B}=-\frac{R}{2n}\varphi^{\alpha}_{\bar B}-\sum_{i=1}^{q}\sum_{\beta,\gamma}(-1)^{i}R^{\alpha}{}_{\gamma}{}^{\bar\beta}{}_{\bar\beta_i}\,\varphi^{\gamma}_{\bar\beta\bar B_i'}$$

and therefore:

$$-\sqrt{-1}\,A\,(\Lambda\,e(s)\,\varphi,\varphi)=-\frac{R}{2n}\,A\,(\varphi,\varphi)+\frac{1}{(q-1)!}\sum_{\alpha,\beta,\gamma,\delta,B}R_{\alpha}^{\bar\beta\bar\gamma}{}_{\delta}\,\varphi_{\bar\beta,\bar\gamma\bar B'}\,\overline{\varphi^{\alpha,\delta B'}}. \qquad (1)$$

Let $\xi=\{\xi_{\alpha\beta}\}$, $\alpha,\beta=1,\dots,n$, $\xi_{\alpha\beta}=\xi_{\beta\alpha}$ represent a point of $\mathbf{C}^{\frac{1}{2}n(n+1)}$. We consider for every $x_0\in X$ the linear map $L(x_0)$ of $\mathbf{C}^{\frac{1}{2}n(n+1)}$ into itself given by

$$L(x_0):\xi_{\alpha\beta}\to\sum_{\alpha,\beta}R_{\tau}{}^{\alpha\beta}{}_{\gamma}\,(x_0)\,\xi_{\alpha\beta}.$$

Identifying $\mathbf{C}^{\frac{1}{2}n(n+1)}$ with the fiber of the fiber bundle of symmetric tensors of the described type and using on it the metric induced by the metric on X, we have a scalar product

$$\langle\xi,\eta\rangle=\sum\xi_{\alpha\beta}\,\overline{\eta^{\alpha\beta}}\ \text{ for }\ \xi,\eta\in\mathbf{C}^{\frac{1}{2}n(n+1)}.$$

Now let $\delta(x_0)$ be the smallest eigenvalue of $L(x_0)$ so that

$$\langle L(x_0)\,\xi,\xi\rangle\geqslant\delta(x_0)\,\langle\xi,\xi\rangle\quad\forall\,\xi\in\mathbf{C}^{\frac{1}{2}n(n+1)}.$$

By the symmetries of the Riemann tensor $(R_\alpha{}^{\beta\gamma}{}_\delta = R_\alpha{}^{\gamma\beta}{}_\delta = R_\delta{}^{\beta\gamma}{}_\alpha)$ we then obtain:

$$\sum R_{\bar\alpha}{}^{\bar\beta\bar\gamma}{}_{\bar\delta}\, \varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\alpha,\delta B'}} = \tfrac{1}{4} \sum R_{\bar\alpha}{}^{\bar\beta\bar\gamma}{}_{\bar\delta}\, (\varphi_{\bar\beta,\bar\gamma\bar B'} + \varphi_{\bar\gamma,\bar\beta\bar B'})\, (\overline{\varphi^{\alpha,\delta B'} + \varphi^{\delta,\alpha B'}})$$
$$\geqslant \tfrac{1}{2}\, \delta(x_0) \sum (\varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\beta,\gamma B'}} + \varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\gamma,\beta B'}}). \tag{2}$$

Now we have
$$\sum \varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\beta,\gamma B'}} = q\,!\, A(\varphi, \varphi). \tag{3}$$

$$\sum \varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\gamma,\beta B'}} = \frac{1}{q} \sum \varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\beta,\gamma B'}} - \frac{q+1}{q} \sum \varphi_{[\bar\beta\bar\gamma\bar B]}\, \overline{\varphi^{[\beta\gamma B]}},$$

where
$$\varphi_{[\bar\alpha,\bar\beta_1\ldots\bar\beta_q]} = \frac{1}{q+1} \{\varphi_{\bar\alpha,\bar\beta_1\ldots\bar\beta_q} - \varphi_{\bar\beta_1,\bar\alpha\bar\beta_2\ldots\bar\beta_q} + \ldots \pm \varphi_{\bar\beta_q,\bar\beta_1\ldots\bar\beta_{q-1}\bar\alpha}\}.$$

Therefore
$$\sum \varphi_{\bar\beta,\bar\gamma\bar B'}\, \overline{\varphi^{\gamma,\beta B'}} \leqslant (q-1)\,!\, A(\varphi, \varphi). \tag{4}$$

From (1), (2), (3) and (4) we deduce the following

LEMMA 5 (cf. [7]). *Let X be a complex n-dimensional Einstein–Kähler manifold with scalar curvature R and Riemann tensor $R^\alpha{}_{\beta\gamma\bar\delta}$. Let, for $x_0 \in X$, $\delta(x_0)$ be the smallest eigenvalue of the linear transformation*

$$L(x_0): \xi_{\alpha\beta} \to \sum R_\tau{}^{\alpha\beta}{}_\gamma (x_0)\, \xi_{\alpha\beta} \quad (\xi_{\alpha\beta} = \xi_{\beta\alpha}).$$

If $\delta(x_0) \leqslant 0$, then for any $\varphi \in C^{0q}(X, \Theta))$ one has at x_0:

$$-\sqrt{-1}\, A_{x_0} (\Lambda\, e(s)\, \varphi, \varphi) \geqslant \left\{\delta(x_0)\, \frac{q+1}{2} - \frac{R}{2n}\right\} A_{x_0} (\varphi, \varphi).$$

Now we remark that
$$R = 2 \sum R_\alpha^\alpha = 2 \text{ trace } L(x_0) \geqslant n(n+1)\, \delta(x_0)$$

so that if $R \leqslant 0$, then certainly $\delta(x_0) \leqslant 0$ at each point of X.

COROLLARY. *If X is n-dimensional Einstein–Kähler with $R < 0$, then*

$$\delta(x_0) \leqslant R/n(n+1) < 0,$$

and if
$$\operatorname*{Inf}_{x_0 \in X} \left\{\frac{R}{n\,\delta(x_0)} - (q+1)\right\} > 0,$$

then the tangent bundle Θ is W^{0q}-elliptic.

11. *Bounded domains and their quotients.* a) Let D be a bounded domain in $\mathbf{C}^n$; let

$$\Xi = K(z)\, dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n$$

be the Bergmann kernel; let

$$\sum g_{\alpha\bar{\beta}}(z)\, dz^\alpha\, d\bar{z}^\beta = \partial \cdot \bar{\partial} \log K(z)$$

be the Bergmann metric; and let

$$dX = \left(\frac{\sqrt{-1}}{2}\right)^n \det\,(g_{\alpha\bar{\beta}}(z))\, dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n$$

be the corresponding volume element. Clearly Ξ/dX is a function on D invariant by analytic automorphisms; in particular it is a constant if D is homogeneous. Now since the Ricci tensor is given by

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2 \log \det\,(g_{\alpha\bar{\beta}})}{\partial z^\alpha\, \partial \bar{z}^\beta}$$

we obtain the following (cf. e.g. [12]):

LEMMA 6. *If D is a bounded homogeneous domain in $\mathbf{C}^n$, then the Bergmann metric on D is Kähler and Einstein; the scalar curvature is $R = -2n$.*

b) To a bounded homogeneous domain we can therefore apply the criterion of W-ellipticity given in the Corollary at the end of Section 10. We remark that $\delta(x_0)$ is now a constant.

In particular for the *irreducible bounded symmetric domains*, following the classification of C. L. Siegel [17] and the table, p. 499 of [7], we obtain the following data:

Type of D		$\dim_{\mathbf{C}} D$	Θ is W^{0q}-elliptic for
$\mathrm{I}_{m,m'}$	$(1 \leqslant m \leqslant m')$	mm'	$0 \leqslant q < m + m' - 1$
II_m	$(m \geqslant 2)$	$\frac{1}{2} m(m-1)$	$0 \leqslant q < 2m - 3$
III_m	$(m \geqslant 1)$	$\frac{1}{2} m(m+1)$	$0 \leqslant q < m$
IV_m	$(m \geqslant 3)$	m	$0 \leqslant q < m - 1$
V		16	$0 \leqslant q < 11$
VI		27	$0 \leqslant q < 17$

We note that $R/n\delta$ is not changed if we replace the given metric by a proportional one.

c) Let $D = D_1 \times \ldots \times D_r$ be a product of r bounded homogeneous domains D_i, $1 \leqslant i \leqslant r$. Let $R/n\delta$ and $R_i/n_i\delta_i$ be the corresponding invariants for D and D_i, $1 \leqslant i \leqslant r$.

Then since the Bergmann metric on D is the sum of the Bergmann metrics on the factors, one obtains that

$$\frac{R}{n\delta} = \operatorname{Inf} \frac{R_i}{n_i\,\delta_i};$$

hence from the above table and the corollary of Section 10 we deduce [7] in particular the following

THEOREM 4. *Let X be a complex manifold whose universal covering space is isomorphic to a product $D = D_1 \times \ldots \times D_r$ of bounded irreducible symmetric domains D_i $(1 \leqslant i \leqslant r)$.*

Let Θ be the holomorphic tangent bundle of X and let us consider on X the Kähler–Einstein metric which on the universal covering reduces to the Bergmann metric.

If $\dim_{\mathbf{C}} D_i \geqslant 2$ *for all i $(1 \leqslant i \leqslant r)$, then Θ is W^{01}-elliptic,* i.e., *there is a constant $c = c(D_1, \ldots, D_r)$ such that for any $\varphi \in \mathcal{D}^{01}(X, \Theta)$ we have*

$$\|\varphi\|^2 \leqslant c\,(\|\bar\partial\varphi\|^2 + \|\mathfrak{d}\,\varphi\|^2).$$

In particular, since the Kähler–Einstein metric considered on X is a complete metric, then for every $\varphi \in \mathcal{L}^{01}(X, \Theta) \cap C^{01}(X, \Theta)$ such that $\bar\partial\varphi = 0$, we can find

$$\psi \in \mathcal{L}^{00}(X, \Theta) \cap C^{00}(X, \Theta)$$

such that

$$\varphi = \bar\partial\,\psi.$$

12. *Hermitian metrics close to the Bergmann metric.* In the sequel we have to consider hermitian metrics not necessarily Kähler but close to the Einstein and Kähler metric of the previous section. We need to prove that if the metric is perturbed only on a compact set and if the perturbation is sufficiently small, the contention of Theorem 4 is still valid.

a) *General hermitian metrics.* We adopt the notations introduced in Section 9 and let

$$ds^2 = 2\,{}^t\overline{dz}\,g\,dz = 2\sum g_{\alpha\bar\beta}\,dz^\alpha\,\overline{dz^\beta}$$

be the hermitian metric on X. Let Θ^* denote the dual of the holomorphic tangent bundle on X.

We consider on $E \otimes \Theta^{*p} \otimes \overline{\Theta^{*q}}$ (where $\Theta^{*p} = \wedge_1^p \Theta^*$) the connection defined by $l = h^{-1}\partial h$ on E, the connection defined by $g^{-1}\partial g$ on Θ, the riemannian connection on $\overline{\Theta}$. The covariant differential operators ∇ and $\overline{\nabla}$ will be computed with respect to the chosen connections:

$$\nabla : C^{pq}(X, E) \to C^{pq}(X, E \otimes \Theta^*),$$

$$\bar{\nabla} : C^{pq}(X, E) \to C^{pq}(X, E \otimes \bar{\Theta}^*).$$

In terms of the curvature of the chosen connections one defines the operator (cf. [7] n. 13)

$$\varkappa : C^{pq}(X, E) \to C^{pq}(X, E)$$

(note that $\varkappa = 0$ if $q = 0$).

The identity we used in the Kähler case was

$$\Box - *^{-1} \Box * = \sqrt{-1}\,(e(s)\Lambda - \Lambda e(s)) \quad \text{(Kähler)}. \tag{1}$$

This is now replaced by an identity of the type (cf. [5] n. 13).

$$(\Box - *^{-1} \Box *)\varphi = (\varkappa - *^{-1} \varkappa *)\,\varphi + F_1 \varphi + F_2 \nabla \varphi + F_3 \bar{\nabla} \varphi, \tag{2}$$

where $F_1 F_2, F_3$ are linear mappings:

$$F_1 : C^{pq}(X, E) \to C^{pq}(X, E),$$

$$F_2 : C^{pq}(X, E \otimes \Theta^*) \to C^{pq}(X, E),$$

$$F_3 : C^{pq}(X, E \otimes \bar{\Theta}^*) \to C^{pq}(X, E),$$

which are identically zero if the metric is Kähler, and in that case

$$\varkappa - *^{-1} \varkappa * = \sqrt{-1}\,(e(s)\Lambda - \Lambda e(s)) \quad \text{(Kähler)}. \tag{3}$$

The local expressions of F_1, F_2, F_3 are given by linear functions in the components of φ, $\nabla \varphi$, $\bar{\nabla} \varphi$, respectively, with coefficients which are polynomials in the metric tensor $g_{\alpha\bar{\beta}}$ and $g^{\bar{\alpha}\beta}$, in the torsion tensor $S^{\alpha}_{\beta\gamma}$, in the components of the connection $\Gamma^{\alpha}_{\beta\bar{\gamma}}$, and in the covariant derivatives of these components.

The relation between the operators $\varkappa$ and $\bar{\nabla}$ and the "Dirichlet norm" is given by the following equality:

$$\| \bar{\nabla} \varphi \|^2 + (\varkappa\varphi, \varphi) = \| \bar{\partial}\varphi \|^2 + \| \mathfrak{d}\varphi \|^2 \tag{4}$$

for any $\varphi \in \mathfrak{D}^{pq}(X, E)$ ([5] n. 15).

LEMMA 7. *For any compact set $K \subset X$ we can find a constant $C(K) > 0$ such that for any $\varphi \in \mathfrak{D}^{pq}(X, E)$ we have*

$$\| \nabla \varphi \|^2_K + \| \bar{\nabla} \varphi \|^2_K \leqslant C(K)\,\{\| \varphi \|^2 + \| \bar{\partial}\varphi \|^2 + \| \mathfrak{d}\varphi \|^2\}.$$

Proof. α) First we establish the desired inequality for forms $\varphi \in \mathcal{D}^{pq}(X, E)$ with support in a given compact set $K_1 \subset X$:

(i) $$\operatorname{supp} \varphi \subset K_1.$$

From (4) we deduce first that we get an inequality

$$\|\bar{\nabla}\varphi\|^2 \leqslant \|\bar{\partial}\varphi\|^2 + \|\mathfrak{d}\varphi\|^2 + C_1(K_1)\|\varphi\|^2$$

if we choose $C_1(K_1)$ so that

$$|(\varkappa\varphi, \varphi)| \leqslant C_1(K_1)\|\varphi\|^2.$$

This is certainly possible in view of assumption (i).

We need now an analogous estimate for $\|\nabla\varphi\|^2$. First we remark that it is enough to establish such an inequality for forms of type $(0, q)$ only. In fact, any $\varphi \in \mathcal{D}^{pq}(X, E)$ can be considered as an element $\tilde{\varphi} \in \mathcal{D}^{0q}(X, E \otimes \dot{\Theta}^{*p})$, and one has, by the choice made of the connections:

$$\|\nabla\varphi\| = \|\nabla\tilde{\varphi}\|, \quad \|\bar{\nabla}\varphi\| = \|\bar{\nabla}\tilde{\varphi}\|.$$

Now if $\varphi \in \mathcal{D}^{0q}(X, E)$, then $\#\varphi \in \mathcal{D}^{q0}(X, E^*)$ and

$$\bar{\nabla}\#\varphi = \bar{\partial}\#\varphi, \quad \mathfrak{d}\#\varphi = 0,$$

so that $$\|\bar{\nabla}\#\varphi\|^2 = (\square_{E^*}\#\varphi, \#\varphi).$$

Moreover, $\overline{\nabla\varphi}$ and $\bar{\nabla}\#\varphi$ differ only by terms not involving derivatives of the components of φ and analogously for $\bar{\nabla}\varphi$ and $\overline{\nabla\#\varphi}$. We then have inequalities of the type:

$$\|\nabla\#\varphi\|^2 \leqslant C_2(K_1)\|\varphi\|^2 + \|\bar{\nabla}\ \varphi\|^2,$$
$$\|\bar{\nabla}\#\varphi\|^2 \leqslant C_2(K_1)\|\varphi\|^2 + \|\nabla\ \varphi\|^2,$$
$$\|\nabla\ \varphi\|^2 \leqslant C_2(K_1)\|\varphi\|^2 + \|\bar{\nabla}\#\varphi\|^2.$$

From this last inequality we get an inequality:

$$\|\nabla\varphi\|^2 + \|\bar{\nabla}\varphi\|^2 \leqslant \|\bar{\partial}\varphi\|^2 + \|\mathfrak{d}\varphi\|^2 + (C_1 + C_2)\|\varphi\|^2 + (\square_{E^*}\#\varphi, \#\varphi).$$

Also from (2) we get (since $\varkappa_{E^*}\#\varphi = 0$)

$$(\square_{E^*}\#\varphi, \#\varphi) = (\square_{E^*} * \#\varphi, * \#\varphi) + (\varkappa * \#\varphi, * \#\varphi) + (F_1\#\varphi, \#\varphi)$$
$$+ (F_2\nabla\#\varphi, \#\varphi) + (F_3\bar{\nabla}\#\varphi, \#\varphi).$$

Again by assumption (i) we get estimates:

$$|(\varkappa * \# \varphi, \# \varphi)| \leqslant C_3(K_1) \|\varphi\|^2,$$

$$|(F_1 \# \varphi, \# \varphi)| \leqslant C_3(K_1) \|\varphi\|^2,$$

$$|(F_2 \nabla \# \varphi, \# \varphi)| \leqslant C_3(K_1) \left\{\sigma \|\nabla \# \varphi\|^2 + \frac{1}{\sigma} \|\varphi\|^2\right\},$$

$$|(F_2 \overline{\nabla} \# \varphi, \# \varphi)| \leqslant C_3(K_1) \left\{\sigma \|\overline{\nabla} \# \varphi\|^2 + \frac{1}{\sigma} \|\varphi\|^2\right\},$$

for any $\sigma > 0$, while we have for any $\varphi \in \mathfrak{D}^{0q}(X, E)$

$$(\Box_{E^*} * \# \varphi, * \# \varphi) = \|\bar{\partial} \varphi\|^2 + \|\mathfrak{d} \varphi\|^2.$$

We then derive, for sufficiently small σ, an inequality

$$\|\nabla \varphi\|^2 + \|\overline{\nabla} \varphi\|^2 \leqslant C_4(K_1) \{\|\varphi\|^2 + \|\bar{\partial} \varphi\|^2 + \|\mathfrak{d} \varphi\|^2\}$$

β) Let $\varphi \in \mathfrak{D}^{pq}(X, E)$ and let $K \subset \mathring{K}_1$ where K_1 is a fixed compact neighborhood of K in X. Let μ be a C^∞ function, $0 \leqslant \mu \leqslant 1$, with support in K_1 and $= 1$ on K. We apply the previous inequality to $\mu \varphi$ (which satisfies assumption (i)) and we get

$$\|\nabla \varphi\|_K^2 + \|\overline{\nabla} \varphi\|_K^2 \leqslant C_4(K_1) \{\|\mu \varphi\|^2 + \|\bar{\partial} \mu \varphi\|^2 + \|\mathfrak{d} \mu \varphi\|^2\}.$$

But now we can find a constant $C_5(K_1)$ such that

$$\|\bar{\partial} \mu \varphi\|^2 \leqslant \|\bar{\partial} \varphi\|^2 + C_5(K_1) \|\varphi\|^2,$$

$$\|\mathfrak{d} \mu \varphi\|^2 \leqslant \|\mathfrak{d} \varphi\|^2 + C_5(K_1) \|\varphi\|^2,$$

while

$$\|\mu \varphi\|^2 \leqslant \|\varphi\|^2.$$

From these inequalities the lemma follows.

b) We consider now a deformation $(\mathfrak{V}, \varpi, M)$ of the complex manifold X; we assume that

$$M = \{t \in \mathbf{R}^m \mid \sum t_i^2 < 1\}.$$

Let $ds_t^2 = 2 \sum g_{\alpha \bar{\beta}}(z, t)\, dz^\alpha\, d\bar{z}^\beta$ be a hermitian metric on the fibers of $\mathfrak{V}$ which depends differentiably on the parameters t.

Let F be a closed subset of $\mathfrak{V}$ such that $\varpi | F$ is a proper map. From the previous Lemma 7 we derive the following:

Corollary. *Let $K_t = F \cap X_t$ where $X_t = \varpi^{-1}(t)$. We can find a continuous function $c(t)$ such that for any $\varphi \in \mathfrak{D}^{pq}(\mathfrak{V}, \Theta)$ we have*

$$\| \nabla_t \varphi_t \|^2_{K_t} + \| \bar{\nabla}_t \varphi_t \|^2_{K_t} \leqslant c(t) \{ \| \varphi_t \|^2 + \| \bar{\partial} \varphi_t \|^2 + \| \mathfrak{d} \varphi_t \|^2 \}.$$

Proof. Inspecting the proof of the previous lemma one sees that the constants $c_i(K_t)$, $1 \leqslant i \leqslant 5$, are locally bounded functions of t (i.e., given $t_0 \in M$ there is a neighborhood $U(t_0)$ in M and a constant $c_0 > 0$ such that for $t \in U(t_0), 0 < c_i(K_t) < c_0$. Hence the constant $c(K_t)$ of the lemma is also locally bounded. Thus there exists a continuous function $c(t) > c(K_t)$ $\forall\, t \in M$.

We can now prove the following proposition that we will need later:

PROPOSITION 6. *We assume that the hermitian metric ds_t^2 has the following properties:*

a) *on $X_0 \cup \bigcup_{t \in M} (X_t - K_t)$ the metric is a Kähler metric;*

b) *at each point $x \in X_0 \cup \bigcup_{t \in M} (X_t - K_t)$ we have for any $\varphi \in C^{pq}(X, \Theta)$*

$$A_t(\sqrt{-1}\,[e(s), \Lambda]\varphi, \varphi) \geqslant c_0\, A_t(\varphi, \varphi)$$

(where $c_0 > 0$ is a constant independent of t and φ).

Then we can find $\varepsilon > 0$ such that for any $\varphi \in \mathfrak{D}^{pq}(X, \Theta)$ and any $t \in M$ with $\sum t_i^2 < \varepsilon$ we have

$$\tfrac{1}{4} c_0 \| \varphi_t \|_t^2 \leqslant \| \bar{\partial} \varphi_t \|_t^2 + \| \mathfrak{d} \varphi_t \|_t^2.$$

Proof. From identity (2) we get

$$((\varkappa - \ast^{-1} \varkappa \ast)\varphi, \varphi) \leqslant \| \bar{\partial} \varphi \|^2 + \| \mathfrak{d} \varphi \|^2 + |(F_{1t} \varphi, \varphi)| + |F_{2t} \nabla \varphi, \varphi)| + |(F_{3t} \bar{\nabla} \varphi, \varphi)|. \quad (\alpha)$$

Let $\mathfrak{W} = X_0 \cup (\mathfrak{V} - F)$; at each point of $\mathfrak{W}$ we have by the Kähler assumption a)

$$\varkappa - \ast^{-1} \varkappa \ast = \sqrt{-1}\,[e(s), \Lambda].$$

From b) and the fact that $\varkappa_t$ and $\ast_t$ depend continuously on t we get for any $\varphi \in \mathfrak{D}^{pq}(X, \Theta)$:

$$\begin{aligned} ((\varkappa - \ast^{-1} \varkappa \ast)\varphi_t, \varphi_t) &= ((\varkappa - \ast^{-1} \varkappa \ast)\varphi_t, \varphi_t)_{X_t - K_t} + ((\varkappa - \ast^{-1} \varkappa \ast)\varphi_t, \varphi_t)_{K_t} \\ &\geqslant c_0 \| \varphi_t \|^2_{X_t - K_t} + \tfrac{2}{3} c_0 \| \varphi_t \|^2_{K_t}, \end{aligned} \quad (\beta)$$

provided t is sufficiently small, say $\sum t_i^2 < \varepsilon$.

Also the operators F_{it} vanish on $\mathfrak{W}$, hence their support is in K_t and we thus can find a constant $c_1(\varepsilon) > 0$ such that

$$\lim_{\varepsilon \to 0} c_1(\varepsilon) = 0,$$

and such that for any t with $\sum t_i^2 < \varepsilon$ we have:

$$A_t(F_{1t}\varphi, F_{1t}\varphi) \leqslant c_1(\varepsilon)\, A_t(\varphi,\varphi),$$

$$A_t(F_{2t}\nabla\varphi, F_{2t}\nabla\varphi) \leqslant c_1(\varepsilon)\, A_t(\nabla\varphi,\nabla\varphi),$$

$$A_t(F_{3t}\overline{\nabla}\varphi, F_{3t}\overline{\nabla}\varphi) \leqslant c_1(\varepsilon)\, A_t(\overline{\nabla}\varphi,\overline{\nabla}\varphi).$$

We then get from Schwarz's inequality for any $\sigma > 0$

$$|(F_{1t}\varphi,\varphi)| \leqslant \tfrac{1}{2}\left(\sigma + \frac{c_1(\varepsilon)}{\sigma}\right)\|\varphi\|^2,$$

$$|(F_{2t}\nabla\varphi,\varphi)| + |(F_{3t}\overline{\nabla}\varphi,\varphi)| \leqslant \sigma\|\varphi\|^2 + \frac{c_1(\varepsilon)}{2\sigma}\left(\|\nabla\varphi\|^2_{K_t} + \|\overline{\nabla}\varphi\|^2_{K_t}\right).$$

Moreover, by the previous lemma

$$\|\nabla\varphi\|^3_{K_t} + \|\overline{\nabla}\varphi\|^2_{K_t} \leqslant c(t)\left\{\|\varphi\|^2 + \|\bar\partial\varphi\|^2 + \|\mathfrak{d}\varphi\|^2\right\}.$$

From (α), (β) and these estimates we thus get for $\sum t_i^2 < \varepsilon$:

$$\tfrac{2}{3}c_0\|\varphi\|^2 \leqslant \tfrac{1}{2}\left\{3\sigma + \frac{c_1(\varepsilon)}{\sigma}(1+c(t))\right\}\|\varphi\|^2 + \left(\frac{c_1(\varepsilon)\,c(t)}{2\sigma} + 1\right)\left(\|\bar\partial\varphi\|^2 + \|\mathfrak{d}\varphi\|^2\right).$$

Now we choose $\sigma = \frac{1}{18}c_0$. When $\varepsilon \to 0$, the coefficient of $\|\varphi\|^2$ on the right-hand side tends to $c_0/12$, while the coefficient of $\|\bar\partial\varphi\|^2 + \|\mathfrak{d}\varphi\|^2$ tends to 1. Hence if ε is sufficiently small they are less than $c_0/6$ and 2. From this we get the result.

13. *The rigidity theorem.* Let $M = \{t \in \mathbf{R}^m \mid \sum t_\mu^2 < 1\}$ be the unit ball in $\mathbf{R}^m$. We want to prove the following

THEOREM 5. *Let* $(\mathfrak{V}, \varpi, M)$ *be a differentiable family of deformations of a complex manifold* $X_0 = \varpi^{-1}(0)$. *We assume that*

a) *the family is a family of uniformizable structures on a bounded symmetric domain* D *none of whose irreducible components is of* $\dim_{\mathbf{C}} = 1$,

b) *the deformation is rigid at infinity,*

c) X_0 *is a* q*-pseudoconcave manifold with* $0 \leqslant q \leqslant \dim_{\mathbf{C}} X - 2$,

d) *the fundamental group* $\pi_1(X_0)$ *is finitely generated.*

Then the whole deformation $(\mathfrak{V}, \varpi, M)$ *is trivial.*

Proof. α) From assumption b) and Proposition 4 we deduce the existence of a diffeomorphism

$$g: X_0 \times M \to \mathfrak{V}$$

with the following properties:

(i) g is fiber-preserving i.e. $pr_M = \varpi \circ g$,

(ii) for a convenient compact set $K_0 \subset X_0$ the map

$$g : (X_0 - K_0) \times M \to g((X_0 - K_0) \times M),$$

is an isomorphism.

Let Φ be a continuous function on X_0 which is strongly q-pseudoconvex on the complement of a compact set $K \subset X_0$ and such that

$$B_c = \{x \in X_0 \mid \Phi(x) > c\} \subset \subset X \quad \forall\, c > \inf_{X_0} \Phi.$$

We may assume that $K = K_0 \subset B_{c_0}$ where $c_0 > \inf_{X_0} \Phi$.

Let $\mathcal{A} = g((X - B_{c_0}) \times M)$ and set on $\mathcal{V}$

$$\Psi(v) = \begin{cases} c_0 \text{ if } v \in \mathcal{V} - \mathcal{A}, \\ \Phi \circ pr_{X_0} \circ g^{-1} \text{ if } v \in \mathcal{A}. \end{cases}$$

Then Ψ is continuous on $\mathcal{V}$ and its restriction to $X_t = \varpi^{-1}(t)$, $(t \in M)$ will be strongly q-pseudoconvex outside the set $K_t = g(\bar{B}_{c_0} \times \{t\})$. Moreover the sets

$$B_c(t) = \{x \in X_t \mid \Psi(x) > c\}$$

will be relatively compact in X_t for $c > \inf_{X_0} \Phi = \inf_{X_t} \Psi|_{X_t}$.

It follows then that for any $t \in M, X_t = \varpi^{-1}(t)$ satisfies the same assumptions required for X_0.

β) We now remark that the group of analytic automorphisms of X_0 is a Lie group. This follows from the

LEMMA. *Let D be a connected, simply connected, bounded domain in $\mathbf{C}^n$. Then for any manifold X whose universal covering is isomorphic to D the group* $\mathrm{Aut}(X)$ *of all complex analytic automorphisms of X is a Lie group.*

Proof of Lemma. We know that $\mathrm{Aut}(D)$ is a Lie group by a theorem of H. Cartan [9]. Let $\Gamma = \pi_1(X)$ be considered as a discrete subgroup of $\mathrm{Aut}(D)$. Let $\pi : D \to D/\Gamma = X$ be the natural projection. Since every holomorphic map $\tau : Y \to X$ of a simply connected manifold Y into X can be factored through the universal covering map $\pi : D \to X$, it follows that for any $\alpha \in \mathrm{Aut}(X)$ we can find $\eta \in \mathrm{Aut}(D)$ such that

$$\alpha \circ \pi(x) = \pi \circ \eta(x) \quad \forall x \in D.$$

This means that $\eta \in N(\Gamma) = \{\sigma \in \mathrm{Aut}\ (D) \mid \sigma\Gamma = \Gamma\sigma\}.$

We thus hawe the exact sequence

$$e \to \Gamma \to N(\Gamma) \to \mathrm{Aut}\,(X) \to e.$$

Since $N(\Gamma)$ is a closed subgroup of $\mathrm{Aut}(D)$, we see that $N(\Gamma)$ and hence $\mathrm{Aut}(X) = N(\Gamma)/\Gamma$ is a Lie group.

γ) We now remark that it will be sufficient to prove the following

PROPOSITION 7. *Let* $(\mathfrak{V}, \varpi, M)$ *satisfy the assumptions of Theorem 5. Then* $(\mathfrak{V}, \varpi, M)$ *represents a locally trivial deformation of* X_0.

In fact, if follows first that all fibers $X_t = \varpi^{-1}(t)$ of $\mathfrak{V}$ $(\forall t \in M)$ are isomorphic because the local triviality entails that $\{t \in M \mid X_t \approx X_0\}$ is open and closed in M.

Then by the previous remark and the local triviality if follows that $(\mathfrak{V}, \varpi, M)$ is a fiber bundle over M, with typical fiber X_0 and with structure group $\mathrm{Aut}\,(X_0)$ which is a Lie group.

Since M is contractible, $(\mathfrak{V}, \varpi, M)$ is topologically trivial, hence also differentiably trivial (cf. [19], p. 25). This implies Theorem 5.

14. *Proof of Proposition 7.* α) Let $\mathfrak{U} = \{U_i\}$ be a locally finite coordinate covering of $\mathfrak{V}$ with coordinate patches (z_i, t) and coordinate transformations

$$\begin{cases} z_i^\alpha = h_{ij}^\alpha\,(z_j, t) & 1 \leqslant \alpha \leqslant n = \dim_{\mathbf{C}} X_0 \\ t_\mu = t_\mu & 1 \leqslant \mu \leqslant m. \end{cases}$$

Let $F = g(K_0 \times M)$. We may assume without loss of generality that the functions $h_{ij}^\alpha\,(z_j, t)$ are independent of t whenever $U_i \cap U_j \cap F = \emptyset$. Let $v = \sum_1^m v^\mu\,(t)\,\partial/\partial t_\mu$ be a C^∞ vector field on M and let us consider at a point $p \in U_i \cap U_j$, $p = \{(z_i, t) = (h_{ij}\,(z_j, t), t)\}$,

$$\theta_{ij}^\alpha\,(z_i, t) = \sum v^\mu\,(t)\,\frac{\partial h_{ij}^\alpha\,(z_j, t)}{\partial t_\mu}.$$

Then $\{\theta_{ij}\} \in Z^1(\mathfrak{U}, \Theta)$ is the deformation cocycle corresponding to the vector field v and the coordinate covering $\mathfrak{U}$.

By the previous assumption we have

$$\mathrm{supp}\;\{\theta_{ij}\} \subset F_1,$$

where $F_1 = g(K_1 \times M)$, K_1 being a convenient compact neighborhood of K_0.

Let $\{\varrho_i\}$ be a C^∞ partition of unity, subordinate to $\mathcal{U}$, and set

$$\psi_i^\alpha(z_i, t) = \sum \varrho_j \theta_{ij}^\alpha(z_i, t) \quad \text{on } U_i.$$

We get

$$\sum_\beta \psi_j^\beta \frac{\partial z_i^\alpha}{\partial z_j^\beta} - \psi_i^\alpha = \theta_{ij}^\alpha,$$

so that

$$\varphi = \bar\partial \psi_i^\alpha = \bar\partial \psi_j^\beta \cdot \frac{\partial z_i^\alpha}{\partial z_j^\beta} = \sum \varphi^\nu{}_{\bar\beta}\, d\bar z^\beta$$

is a $(0,1)$ $\bar\partial$-closed form with values in Θ, and

$$\operatorname{supp} \varphi \subset F_1.$$

The form φ is the element corresponding to the class of $\{\theta_{ij}\}$ by the Dolbeault isomorphism.

If $\varphi_t = \varphi \mid X_t$, then for every $t \in M$ we have

$$\varphi_t \in \mathcal{D}^{01}(X_t, \Theta_t)$$

where $\Theta_t = \Theta \mid X_t$.

β) We consider

$$\pi: D \times M \to \mathfrak{V}$$

the universal covering of $\mathfrak{V}$, as assumed in the hypothesis a) of Theorem 5.

We also consider the mapping

$$\sigma: D \times M \to X_0 \times M$$

defined by $\sigma(\xi, t) = (\pi(\xi, 0), t)$ obtained by trivial extension of the universal covering map $\pi: D \times \{0\} \to X_0$ induced by π over X_0.

The diffeomorphism

$$g: X_0 \times M \to \mathfrak{V}$$

can be lifted to a fiber-preserving diffeomorphism $\tilde g$ of the universal coverings, i.e., we will have a commutative diagram

$$\begin{array}{ccc} D \times M & \xrightarrow{\tilde g} & D \times M \\ \sigma \downarrow & & \pi \downarrow \\ X_0 \times M & \xrightarrow{g} & \mathfrak{V}. \end{array}$$

Here g and $\tilde g$ are only fiber-preserving diffeomorphisms but not morphisms for the structures of analytic fiberings over M.

Let $A = X_0 - K_0, D_A = \pi^{-1}|_{X_0}(A)$; then by restriction to $A \times M$ and $D_A \times M$ in the above diagram we get a diagram in which also $g|A \times M$ and $\tilde{g}|D_A \times M$ are morphisms for the structures of analytic fiberings over M.

The holomorphic tangent bundle along the fibers of $D \times M$ can be identified with $(D \times \mathbf{C}^n) \times M$. The Bergmann metric on D is a function

$$\beta : D \times \mathbf{C}^n \to \mathbf{R}$$

which is invariant by the operations of Aut (D) on the tangent bundle $D \times \mathbf{C}^n$. By trivial extension on $D \times M$ we then have a hermitian (Kähler and Einstein) metric along the fibers of $D \times M$

$$\tilde{\beta} : (D \times \mathbf{C}^n) \times M \to \mathbf{R}$$

invariant by the operations of $\mathrm{Aut}\,(D \times M) =$ differentiable maps of M into $\mathrm{Aut}\,(D)$.

Then if Γ is the automorphism group of the universal covering $(D \times M, \pi, \mathfrak{V})$, we have $\Theta \approx \{(D \times \mathbf{C}^n) \times M\}/\Gamma$ and $\tilde{\beta}$ defines a hermitian (Kähler–Einstein) metric

$$ds_\pi^2 \text{ along the fibers of } (\mathfrak{V}, \varpi, M);$$

analogously if Γ_0 is the automorphism group of $(D, \pi|_{D \times \{0\}}, X_0)$. We have $\Theta_0 = (D \times \mathbf{C}^n)/\Gamma_0$ and β defines a Kähler-Einstein metric ds_0^2 on X_0. By trivial extension of Γ_0 to $D \times M$ we get from $\tilde{\beta}$ a Kähler–Einstein metric

$$ds_\sigma^2 \text{ along the fibers of } X_0 \times M \text{ such that}$$

$$ds_\sigma^2 = pr_{X_0}^* ds_0^2.$$

On $g(A \times M)$, $(g^{-1})^* ds_\sigma^2$ will remain Kähler–Einstein and "independent of $t \in M$".

Now let ϱ be a C^∞ function on X_0 with the properties

$$0 \leqslant \varrho \leqslant 1, \qquad \varrho(x) = \begin{cases} 1 & \text{if } x \in K_0, \\ 0 & \text{if } x \in X_0 - K_1. \end{cases}$$

Let
$$\mu = \varrho \circ g^{-1}$$

and set on $\mathfrak{V}$
$$ds^2 = \mu\, ds_\pi^2 + (1 - \mu)\,(g^{-1})^*\, ds_\sigma^2.$$

This metric will have the following properties:

(i) on $g(K_0 \times M)$ and $g((X_0 - K_1) \times M)$ it is a Kähler–Einstein metric,

(ii) on X_0 $ds^2 = ds_0^2$ is the natural Kähler–Einstein metric.

γ) Let $\bar{\partial}_t$ be the $\bar{\partial}$-operator on the fiber X_t and let us denote, with respect to the metric ds_t^2 just defined, by $\mathfrak{d}_t$ and $\square_t$, the operators $\mathfrak{d}$ and $\square$, for the bundle Θ_t.

By the assumption a), Theorem 4, Proposition 6, and Theorem 3, we deduce that there exists a $\sigma>0$ such that, for $\sum t_i^2<\sigma$, we can find one and only one element $x_t \in W^{01}(X_t, \Theta_t) \cap C^{01}(X_t, \Theta_t)$ such that

$$\varphi_t=\bar{\partial}_t \mathfrak{d}_t x_t \quad (\text{and } \mathfrak{d}_t \bar{\partial} x_t=0).$$

Let $U\{z, t\}$ be a coordinate patch on $\mathfrak{V}$; then in that coordinate patch x_t is represented by a differential form

$$x_t=\sum a_{\bar{\beta}}^{\alpha}(x, t)\, d\bar{z}^{\beta}.$$

We want to prove the following regularity theorem for x_t:

(A) *The coefficients $a_{\bar{\beta}}^{\alpha}(z, t)$ and all their partial derivatives with respect to the fiber coordinates z are continuous functions of t.*

Let $z^{\alpha}=x_{\alpha}+\sqrt{-1}\, x_{n+\alpha}$, $1 \leqslant \alpha \leqslant n$, and $h=(h_1, \ldots, h_{2n}) \in \mathrm{N}^{2n}$. We set

$$D^h=\frac{\partial^{|h|}}{\partial x_1^{h_1} \ldots \partial x_{2n}^{h_{2n}}}, \quad |h|=\sum_1^{2n} h_i.$$

We choose $\varepsilon'>0$ and $\delta>0$ (with $\delta<\sigma$) such that the ball

$$B_r=\{(x, t) \in U \mid t=t_0, \sum x_i^2<r\}$$

is contained in $U \cap X_{t_0}$ for any $r \leqslant \varepsilon'$ and any choice of t_0 with $\sum (t_0^j)^2<\delta$.

For any form $\psi \in \bigoplus_{p+q=r} C^{pq}(X_t, \Theta_t)$ we consider its expression in the local coordinates x_α in $U \cap X_t$:

$$\psi=\left\{\sum_S \psi_S^{\alpha}(x)\, dx^S\right\} \quad 1 \leqslant \alpha \leqslant n,$$

S being a block of $r=p+q$ indices between 1 and $2n$. We then define for a given $\varepsilon, 0<\varepsilon<\varepsilon'$ and for any integer $k \geqslant 0$:

$$(\|\psi\|_k^{B_\varepsilon})^2=\sum_{\alpha=1}^{n} \sum_S \sum_{|h| \leqslant k} \int_{B_\varepsilon}|D^h \psi_S^{\alpha}(x)|\, dx_1 \wedge \ldots \wedge dx_{2n}.$$

Now ∂B_ε has the "cone property" so that we can apply Sobolev's inequality (cf. e.g. [10], pp. 232–233), i.e., we can find a positive constant $c=c(\varepsilon, k, h)$ such that

$$|D^h \psi_S^{\alpha}(y)| \leqslant c(\varepsilon, k, h)\, \|\psi\|_{|h|+k}^{B_\varepsilon}$$

for any $y \in B$, any h and $k>n$.

Also one has for any $k>0$ the following Friedrichs inequality [11]:

$$(\|\psi\|_{k+2}^{B\varepsilon})^2 \leqslant c_1(\varepsilon, \varepsilon', k)\{(\|\square_t \psi\|_k^{B\varepsilon'})^2 + (\|\psi\|_0^{B\varepsilon'})^2\},$$

where $c_1 = c_1(\varepsilon, \varepsilon' k)$ is a positive constant.

To prove contention (A) it is enough to show, by Sobolev's inequality, that for s, t in the coordinate ball $\{\sum t_j^2 < \delta\}$ on M we have

$$\lim_{t\to s} \|x_t - x_s\|_k^{B\varepsilon} = 0$$

for a sufficiently large k. Now by the Friedrichs inequality we have:

$$(\|x_t - x_s\|_{k+2}^{B\varepsilon})^2 \leqslant c_1\{(\|\square_t(x_t - x_s)\|_k^{B\varepsilon'})^2 + (\|x_t - x_s\|_0^{B\varepsilon'})^2\}. \tag{1}$$

Thus it is enough to show that both terms on the right-hand side tend to zero as $t \to s$.

(α) For the first term we write

$$\square_t(x_t - x_s) = \square_t x_t - \square_s x_s - (\square_t - \square_s) x_s = \varphi_t - \varphi_s - (\square_t - \square_s) x_s.$$

Thus
$$\|\square_t(x_t - x_s)\|_k^{B\varepsilon'} \leqslant \|\varphi_t - \varphi_s\|_k^{B\varepsilon'} + \|(\square_t - \square_s) x_0\|_k^{B\varepsilon'}.$$

Now since φ is C^∞ in the parameters t,

$$\lim_{t\to s} \|\varphi_t - \varphi_s\|_k^{B\varepsilon'} = 0.$$

Also $\square_t$ has C^∞ coefficients in t, hence

$$\lim_{t\to s} \|(\square_t - \square_s) x_s\|_k^{B\varepsilon'} = 0.$$

Hence the first term on the right-hand side of (1) tends to zero for $t \to s$.

(β) It remains to show that

$$\lim_{t\to s} \|x_t - x_s\|_0^{B\varepsilon'} = 0.$$

Let $g_t: X_t \to X_0$ be the diffeomorphism defined by

$$g_t = pr_{X_0} \circ g^{-1} | X$$

(cf. Section 12 point α) of the proof). If we set

$$g_{ts} = g_s^{-1} \circ g_t : X_t \to X_s,$$

we get a diffeomorphism which is also a complex analytic isomorphism outside $K_t = g_t^{-1}(K_0)$, $K_s = g_s^{-1}(K_0)$ in X_t, X_s, respectively.

By transposition the diffeomorphism g_{ts} defines an isomorphism g_{ts}^* of the space of C^∞ forms with values in the real tangent bundle of X_s onto the analogous space for X_t.

We will denote by

$$\prod_{p,q}{}_t^r \quad (r=0,1)$$

the projection operator on X_t which associates to each C^∞ form of degree $p+q$, with values in the real tangent bundle to X_t, the part of type (p,q) with values in Θ_t (if $r=0$) or $\bar{\Theta}_t$ (if $r=1$).

In the coordinate neighborhood U, g_{ts} will be represented by a coordinate transformation depending differentiably upon t and s and reducing to the identity for $t=s$. From this it follows that

$$\lim_{t\to s} \|x_s - g_{ts}^* x_s\|_0^{B\varepsilon'} = 0.$$

Also setting

$$\check{\prod}_t = \prod_{10}{}_t^0 + \prod_{01}{}_t^1 + \prod_{10}{}_t^1,$$

we have

$$g_{ts}^* x_s = \prod_{01}{}_t^0 g_{ts}^* x_s + \check{\prod}_t g_{ts}^* x_s$$

and

$$\lim_{t\to s} \|\check{\prod}_t g_{ts}^* x_s\|_0^{B\varepsilon'} = 0,$$

since

$$\prod_{01}{}_t^0 x_t = x_t.$$

It is therefore enough to prove that

$$\lim_{t\to s} \|x_t - \prod_{01}{}_t^0 g_{ts}^* x_s\|_0^{B\varepsilon'} = 0.$$

Now with respect to the chosen metric on the fibers of $\mathfrak{V}$ we can consider the global norms defined in Section 9. The above condition will certainly be satisfied if we can prove that $\|x_t - \prod_{01}{}_t^0 g_{ts}^* x_s\|$ is finite and tends to zero as t tends to s.

Now we have by virtue of the triviality at infinity and the special choice of the metric on the fibers:

(a) $\prod_{01}{}_t^0 g_{ts}^* \mathcal{L}^{01}(X_s, \Theta_s) \subset \mathcal{L}^{01}(X_t, \Theta_t)$,

(b) $\Box_t \prod_{01}{}_t^0 g_{ts}^* - \prod_{01}{}_t^0 g_{ts}^* \Box_s$ is a compactly supported operator on X_t.

Therefore from

$$\Box_t (x_t - \prod_{01}{}^0_t \, g^*_{ts} x_s) = \varphi_t - \prod_{01}{}^0_t \, g^*_{ts} \varphi_s - (\Box_t \prod_{01}{}^0_t \, g^*_{ts} - \prod_{01}{}^0_t \, g^*_{ts} \Box_s) \, x_s$$

we see that the right-hand side is compactly supported while $x_t - \prod_{01}{}^0_t \, g^*_{ts} x_s$ is square integrable.

From a previous remark (at the end of Section 9) it follows then that

$$x_t - \prod_{01}{}^0_t \, g^*_{ts} x_s \in W^{01} (X_t \, \Theta_t),$$

and therefore by virtue of Proposition 6, for $\sum t_i^2 < \sigma$, we have

$$\| x_t - \prod_{01}{}^0_t \, g^*_{ts} x_s \| \leqslant c \, \{ \| \varphi_s - \prod_{01}{}^0_t \, g^*_{ts} \varphi_s \| + \| (\Box_t \prod_{01}{}^0_t \, g^*_{ts} - \prod_{01}{}^0_t \, g^*_{ts} \Box_s) \, x_s \| \}.$$

It is now clear that for $t \to s$ the right-hand side of this inequality tends to zero.

Contention (A) is therefore proved.

15. *Continuation of the proof.* α) Our next step is to show that $\mathfrak{V}$ as a *continuous* family of deformations of X_0 is locally pseudotrivial.

We set $$M_r = \{ t \in M \mid \sum t_\mu^2 < r \}$$

for $0 < r < 1$.

Let $$y_t = \mathfrak{d}_t \, x_t$$

so that $$\varphi_t = \bar{\partial} \, y_t.$$

We have seen that y_t represents a section of Θ on $\pi^{-1}(M_\sigma)$ which is C^∞ along the fibers and continuous on $\mathfrak{V}$.

Let this section y_t be locally represented on the coordinate covering

$$\mathfrak{U} \cap \pi^{-1}(M_\sigma) = \{ U_i \cap \pi^{-1}(M_\sigma) \}_{i \in I}$$

by the vector fields along the fibers

$$\{ y^\alpha (z_i, t) \} \quad 1 \leqslant \alpha \leqslant n.$$

We then have on each U_i:

$$\bar{\partial} \, (\psi_i^\alpha - y^\alpha) = 0,$$

i.e., $$\theta_i^\alpha (z_i, t) = \psi_i^\alpha (z_i \, t) - y^\alpha (z_i, t)$$

is a holomorphic vector field along the fibers (continuous in t) and one has

$$\theta^{\alpha}_{ij}(z_i, t) = \sum_{\beta} \theta^{\beta}_{j} \frac{\partial z^{\alpha}_{i}}{\partial z^{\beta}_{j}} - \theta^{\alpha}_{i}.$$

β) Let us choose in particular $v(t) = \partial/\partial t_{\mu}$; correspondingly one has a deformation cocycle $\{\theta^{\alpha}_{\mu ij}(z_i, t)\}$ and vector fields $\{\theta^{\alpha}_{\mu i}(z_i, t)\}$ such that

$$\theta^{\alpha}_{\mu ij}(z_i\, t) = \sum_{\beta} \theta^{\beta}_{\mu j}(z_j, t) \frac{\partial z^{\alpha}_{i}}{\partial z^{\beta}_{j}} - \theta^{\alpha}_{\mu i}(z_i, t).$$

One then verifies that

$$X_{\mu}(z_i, t) = \frac{\partial}{\partial t_{\mu}} - \theta^{\alpha}_{\mu i}(z_i, t) \frac{\partial}{\partial z^{\alpha}_{i}}$$

is a global (continuous) vector field on $\pi^{-1}(M_{\sigma})$ whose projection on M is the vector field $\partial/\partial t_{\mu}$.

This can be done for $\mu = 1, 2, \ldots, m$.

γ) We now introduce the following notations:

$$M_r(s) = \{(t_1, \ldots, t_s) \in \mathbf{C}^s \mid |t_{\mu}| < r \quad 1 \leqslant \mu \leqslant s\}$$

$$I_{\varepsilon}(h) \ = \{t_h \in \mathbf{C} \mid |t_h| < \varepsilon\}.$$

Let $\mathfrak{V}_r(s) = \varpi^{-1}(M_r(s))$, and let $\mathfrak{U}_0 = \{U_i\}_{i \in I_0}$ be the set of those $U_i \in \mathfrak{U}$ such that $U_i \cap X_0 \neq \emptyset$.

We can choose $\mathfrak{U}'_0 = \{U'_i\}_{i \in I_0}$ and $\mathfrak{U}^*_0 = \{U^*_i\}_{i \in I_0}$ both coverings of X_0 in $\mathfrak{V}$, such that

$$U'_i \subset\subset U^*_i \subset\subset U_i \quad \forall\, i \in I_0.$$

For every $i \in I_0$ we can find $\varepsilon_i > 0$ such that the system of ordinary differential equations

$$\begin{cases} \dfrac{\partial g^{\alpha}_{i}(\xi_i, t)}{\partial t_m} + \theta^{\alpha}_{mi}(g_i(\xi, t), t) = 0 \\ 1 \leqslant \alpha \leqslant n \end{cases}$$

has a solution $g_i(\xi, t)$ of class C^1 defined for

$$t \in M_{r_1}(m-1) \times I_{\varepsilon_i}(m), \quad r_1 = \frac{1}{2} \frac{\sigma}{\sqrt{m}}$$

with initial values:

$$\begin{cases} g^{\alpha}_{i}(\xi_i, t_1, \ldots, t_{m-1}, 0) = \xi^{a}_{i} \\ 1 \leqslant \alpha \leqslant m, \end{cases}$$

$\xi_i \in U^*_i \cap X_0$ and such that $g_i(\xi, t) \in U_i$.

Moreover, we can assume that the functions $g_i^\alpha(\xi, t)$, t will define a system of holomorphic coordinates on

$$U_i' \cap \varpi^{-1}(M_{r_1}(m-1) \times I_{\varepsilon_i}(m)) = U_i''.$$

Making the change of coordinates $z_i^\alpha = g_i^\alpha(\xi_i, t)$, we then see that if $\xi_i^\alpha = k_{ij}^\alpha(\xi_i, t), t = t$, are the new coordinate transformations on $U_i'' \cap U_j''$, we must have

$$\sum \frac{\partial g_i^\alpha}{\partial \xi_i^\beta} \frac{\partial k_{ij}^\beta}{\partial t^m} = 0 \quad \text{or} \quad \frac{\partial k_{ij}^\beta}{\partial t_m} = 0.$$

This shows that in the covering $\bigcup_{i \in I_0} U_i''$ of X_0 in $\mathfrak{V}$ the change of coordinates is independent of the variable t^m or that there is a neighborhood $\mathcal{A}$ of X_0 in $\mathfrak{V}$ which can be imbedded by an isomorphism of class C^1 in the product $\mathfrak{V}_{r_1}(m-1) \times \mathbf{C}$, the isomorphism being the identity on $\mathfrak{V}_{r_1}(m-1)$.

We now replace $\mathfrak{V}$ with $\mathcal{A}$ as subset of $\mathfrak{V}_{r_1}(m-1) \times \mathbf{C}$; then the deformation cocycle for $v = \partial/\partial t^{m-1}$ will be, in the new coordinates, again written as a co-boundary of a co-chain locally given by sections of Θ holomorphic along the fibers and continuous in the parameters t. The same will be true for the restrictions to $\mathfrak{V}_{r_1}(m-1)$.

By the previous argument we can find a neighborhood of X_0 in $\mathfrak{V}_{r_1}(m-1)$ which can be C^1-isomorphically imbedded in the product $\mathfrak{V}_{r_2}(m-2) \times \mathbf{C}$ where $r_2 = \frac{1}{2} r_1$, the isomorphism being the identity on $\mathfrak{V}_{r_2}(m-2)$.

By this procedure we find, after m steps, a neighborhood of X_0 in $\mathfrak{V}$ which can be isomorphically imbedded in the product $X_0 \times \mathbf{C}^m$ by a C^1-isomorphism.

This proves that the local deformation defined by $\mathfrak{V}$ as a continuous (actually of class C^1) deformation is pseudotrivial.

16. *End of the proof; the concavity assumption.* α) Let us resume the notations of Section 13 α) and let us set

$$c_\infty = \inf_{X_0} \Phi = \inf_{\mathfrak{V}} \Psi.$$

From the assumptions c) and d) of Theorem 5 and from Theorem 1 we deduce that:

We can find a constant $c_3 > c_\infty$ such that, for any c, $c_\infty < c \leqslant c_3$, we have:

(i) the sets $B_c(t) = \{x \in X_t \mid \Psi(x) > c\}$ are connected, relatively compact in X_t;

(ii) the envelope of holomorphy of $\pi^{-1}(B_c(t))$ is the domain D.

We fix c_4 with $c_\infty < c_4 < c_3$. From the (continuous) pseudotriviality of $\mathfrak{V}$ (Section 13) we deduce that there exists an $\varepsilon = \varepsilon(c_4) > 0$ and an injective continuous map

$$\chi : B_{c_4}(0) \times M_\varepsilon \to \mathfrak{V},$$

which is open, is a homeomorphism onto its image, is fiber-preserving, and is a complex analytic isomorphic imbedding on each fiber.

If $\varepsilon = \varepsilon(c_4)$ is sufficiently small, we may assume that

$$\text{(iii)} \qquad \chi(B_{c_4}(0) \times \{t\}) \supset B_{c_5}(t) \quad \forall\, t \in M_\varepsilon.$$

β) We claim that we can find a lifting of the map χ

$$\tilde{\chi} : \pi^{-1}(B_{c_4}(0)) \times M_\varepsilon \to D \times M,$$

so that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(B_{c_4}(0)) \times M_\varepsilon & \xrightarrow{\tilde{\chi}} & D \times M \\ \downarrow \pi|_{\tilde{X}_0} \times \mathrm{id.} & & \downarrow \pi \\ B_{c_4}(0) \times M_\varepsilon & \xrightarrow[\chi]{} & \mathfrak{V}. \end{array}$$

Indeed χ is a homotopy of the map

$$\chi_0 : B_{c_4}(0) \to \mathfrak{V},$$

which is given by the inclusions $B_{c_4}(0) \subset X_0 \subset \mathfrak{V}$.

Then $\chi \circ (\pi|_{\tilde{X}_0} \times \mathrm{id})$ is a homotopy of the map $\chi_0 \circ \pi|_{\tilde{X}_0}$. This can be lifted to the inclusion map

$$\tilde{\chi}_0 : \pi^{-1}(B_{c_4}(0)) \to D,$$

D being identified with χ_0, the universal covering of X_0.

The existence of $\tilde{\chi}$ follows then by the lifting homotopy theorem.

Now χ and π being local homeomorphisms, $\tilde{\chi}$ is also a local homeomorphism. Moreover, χ is one to one, hence $\tilde{\chi}$ must also be one to one because it is a homotopy of $\tilde{\chi}_0$ which is one to one.

Finally $\chi_t = \chi \mid B_{c_4}(0) \times \{t\}$ is a holomorphic map. It then follows also that

$$\tilde{\chi}_t = \tilde{\chi} \mid \pi^{-1}(B_{c_4}(0)) \times \{t\}$$

is a holomorphic map.

γ) Now we remark that since D is the envelope of holomorphy of $\pi^{-1}(B_{c_4}(0))$, the map

$$\chi_t : \pi^{-1}(B_{c_4}(0)) \to D$$

extends in a unique way to a holomorphic map (cf. Section 2 Lemma 3)

$$F_t : D \to D$$

having the properties:

(a) if $\varrho_t : \Gamma \to \operatorname{Aut}(D)$ is the family of representations of $\Gamma = \pi_1(X_0)$ associated with the family $(\mathfrak{V}, \varpi, M)$ (cf. Section 7 b)), then

$$F_t(\varrho_0(\gamma) z) = \varrho_t(\gamma) F_t(z) \quad \forall z \in D, \ \forall \gamma \in \Gamma;$$

(b) the map

$$F : D \times M_\varepsilon \to D,$$

defined by the set of mappings $\{F_t\}_{t \in M_\varepsilon}$, is continuous.

This last assertion follows from the fact that the restriction map

$$H^0(D, O) \simeq H^0(\pi^{-1}(B_{c_4}(0)), O)$$

is continuous for the topology of Fréchet spaces of the vector spaces of holomorphic functions on D and $\pi^{-1}(B_{c_4}(0))$. This restriction being an isomorphism, it is also a homeomorphism by the Banach Theorem.

δ) We now show that, for every $t \in M$,

$$F_t \in \operatorname{Aut}(D).$$

For this we have to produce the inverse map of F_t.

By condition (iii), for $\forall t \in M$,

$$\pi^{-1}(\chi(B_{c_4}(0) \times \{t\})) \supset \pi^{-1}(B_{c_3}(t)).$$

Hence $\tilde{\chi}_t^{-1} | \pi^{-1}(B_{c_3}(t))$ extends in a unique way to a holomorphic map

$$G_t : D \to D,$$

because $\pi^{-1}(B_{c_3}(t))$ has D as its envelope of holomorphy (condition (ii)). Now on $\tilde{\chi}_t^{-1}(\pi^{-1}(B_{c_3}(t))$ we have $G_t \circ F_t =$ identity. Hence by analytic continuation

$$G_t \circ F_t = \text{ identity on } D.$$

Analogously $F_t \circ G_t =$ identity on D. This proves our contention.

We have therefore constructed an automorphism of $D, F_t \in \operatorname{Aut}(D)$, such that for any $t \in M$ we have

$$\varrho_t = F_t \circ \varrho_0 \circ F_t^{-1}.$$

Moreover, F_t depends continuously on t.

ε) The proof of the proposition will be completed if we show that we can choose a system of automorphisms of D which depends *differentiably* on the parameters t.

Let us denote by G the group $\mathrm{Aut}(D)$ identified to a closed (algebraic) subgroup of some linear group $GL(N, \mathbf{R})$.

Let
$$Z(\varrho_0) = \{g \in GL(N, \mathbf{R}) \mid g\varrho_0(\gamma)g^{-1} = \varrho_0(\gamma), \forall \gamma \in \Gamma\}$$
$$C(\varrho_t) = \{g \in GL(N, \mathbf{R}) \mid g\varrho_0(\gamma)g^{-1} = \varrho_t(\gamma), \forall \gamma \in \Gamma\}.$$

Clearly $F_t \in C(\varrho_t)$ so that, for $t \in M$, $C(\varrho_t) \neq \emptyset$. Hence
$$C(\varrho_t) = F_t \cdot Z(\varrho_0).$$

Let $\gamma_1, \ldots, \gamma_k$ be a set of generators of Γ and let us set $g_i(t) = \varrho_t(\gamma_i)$ for $1 \leqslant i \leqslant k$. Then $C(\varrho_t)$ is the set of all matrices g with non-vanishing determinants (i.e., elements of $GL(N, \mathbf{R})$) satisfying the linear equations
$$g g_i(0) = g_i(t) g \quad 1 \leqslant i \leqslant k. \tag{1}$$

Now on $\mathbf{R}^{N^2}$ this is a linear space, and since $C(\varrho_t) = F_t Z(\varrho_0)$, its dimension is independent of $t \in M_\varepsilon$. It follows that the rank of the linear system $(1)_t$ is independent of t.

It follows that for any $t^0 \in M_\varepsilon$, we can find in a neighborhood of t^0 a parametric solution $g = g(t)$ of $(1)_t$ with the properties:

(i) $g(t^0) = F_{t_0}$,

(ii) $g(t)$ is a matrix with C^∞ coefficients in t.

This means the following:

Consider the map
$$\tau : M_\varepsilon \to GL(N, \mathbf{R})/Z(\varrho_0)$$

defined by the composition of $F : M_\varepsilon \to G$, the inclusion of G in $GL(N, \mathbf{R})$, and the natural map $GL(N, \mathbf{R}) \to GL(N, \mathbf{R})/Z(\varrho_0)$.

Then τ by the above remark is a differentiable map.

We now set
$$Z_0 = Z(\varrho_0) \cap \mathrm{G},$$

and remark that τ actually, by its very construction, factors through a map
$$\sigma : M \to G/Z_0,$$

so that we have the commutative diagram:

$$\begin{tikzcd} M_\varepsilon \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & GL(N, \mathbf{R})/Z(\varrho_0) \\ & G/Z_0 \arrow[ur] & \end{tikzcd}$$

Now $G/Z_0 \to GL(N, \mathbf{R})/(\sigma_0)$ is an inclusion of G/Z_0 as a closed submanifold of the latter space.

It follows then that also σ is a differentiable map.

We can now consider the fiber space $p: G \to G/Z_0$. Since M_ε is contractible, we can lift $\sigma: M_\varepsilon \to G/Z_0$ to a *differentiable* map

$$\tilde{F}: M_\varepsilon \to G \quad p \circ \tilde{F} = \sigma.$$

For every $t \in M_\varepsilon$ there is a unique $\mu(t) \in Z_0$ such that

$$\tilde{F}(t) = F_t \circ \mu(t).$$

Then $\tilde{F}(t) \in C(\varrho_t) \cap G$. This completes the proof.

References

[1]. ANDREOTTI, A. & GRAUERT, H., Algebraische Körper von automorphen Funktionen. *Nachr. Akad. Wiss. Göttingen*, Math.–Phys. Kl. II, 1961, 39–48.

[2]. —— Théorèmes de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90 (1962), 193–259.

[3]. ANDREOTTI, A. & VESENTINI, E., On the pseudorigidity of Stein magnifolds. *Ann. Scuola Norm. Sup. Pisa*, (3) 16 (1962), 213–223.

[4]. —— Les théorèmes fondamentaux de la théorie des espaces holomorphiquement complets. *Séminaire C. Ehresmann*, Vol. IV, 1962–63, 1–31.

[5]. —— Carleman estimates for the Laplace–Beltrami equation on complex manifolds. *Inst. Hautes Études Sci. Publ. Math.*, 25. To appear.

[6]. BOREL, A., On the curvature tensor of hermitian symmetric manifolds. *Ann. of. Math.*, 71 (1960), 508–521.

[7]. CALABI, E. & VESENTINI, E., On compact, locally symmetric Kähler manifolds. *Ann. of Math.*, 71 (1960), 472–507.

[8]. CARTAN, H., Les fonctions de deux variables complexes et le problème de la représentation conforme. *J. Math. Pures Appl.*, (9) 10 (1931), 1–114.

[9]. CARTAN, H., *Sur les groupes de transformations analytiques*. Hermann, Paris, 1935.

[10]. COURANT, R., & HILBERT, D., *Methods of mathematical physics*, Vol. II, Interscience Publishers, New York, 1962.

[11]. FRIEDRICHS, K. O., On the differentiability of the solutions of partial differential equations. *Comm. Pure Appl. Math.*, 6 (1953), 299–326-

[12]. KOBAYASHI, S., Geometry of bounded domains. *Trans. Amer. Math. Soc.*, 82 (1959), 267–290.

[13]. Kodaira, K. & Spencer, D. C., On deformations of complex analytic structures, I and II. *Ann. of Math.*, 67 (1958), 328–466.
[14]. —— On deformations of complex analytic structures, III. *Ann. of Math.*, 71 (1960), 43–76.
[15]. Selberg, A., On discontinuous groups in higher dimensional symmetric spaces. *Contributions to function theory*, Bombay, 1960, 147–164.
[16]. Serre, J. P., Groupes d'homotopie et classes de groupes abéliens. *Ann. of Math.*, 58 (1953), 258–294.
[17]. Siegel, C. L., *Analytic functions of several complex variables*. Mimeographed notes, Institute for Advanced Study, Princeton, 1949.
[18]. Spilker, J., Algebraische Körper von Automorphen Funktionen. *Math. Ann.*, 149 (1963), 341–360.
[19]. Steenrod, N., *The topology of fibre bundles*. Princeton University Press, Princeton, 1951.
[20]. Weil, A., On discrete subgroups of Lie groups. *Ann. of Math.*, 72 (1960). 369–384.
[21]. —— On discrete subgroups of Lie groups, II. *Ann. of Math.*, 75 (1962), 578–602.
[22]. —— *Introduction à l'étude des variétés kählériennes*. Hermann, Paris, 1958.

Received May 28, 1964.