A MAXIMAL THEOREM FOR SUBADDITIVE FUNCTIONS

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Introduction

The theory of subadditive functions is sufficiently well developed to suggest that it may be a very useful tool of analysis. The present paper, in which we first prove a maximal theorem for subadditive functions and then apply it to a rather wide class of problems, is offered as further evidence of this point of view.

Our maximal theorem does not seem to be included in the category of maximal ergodic theorems. It does have some points of contact with that of Hardy and Littlewood, but the situation is roughly that our theorem gives more precise information about a smaller class of functions. We first consider some variations of the definition of subadditivity of real-valued functions defined over E_n, n-dimensional Euclidean space. For the maximal theorem itself, a kind of evenness of the functions involved is assumed. We then construct the maximal function corresponding to each properly chosen subadditive function; and the maximal theorem, which is a statement about the comparability of some integral norms involving the original function and its corresponding maximal function is given. In the second theorem, some limitations on the maximal theorem are noted. In the next section, applications are presented, first for some well-known subadditive functions to which the maximal theorem applies directly. A minor variation of the theorem is then applied to some integral transforms. Finally, we obtain a kind of local maximal theorem in a result which is related to the differentiability of integrals. Modifications of the original argument are more serious for this result, and we make use of the maximal theorem of Hardy and Littlewood here. In the last section, sums whose terms involve subadditive functions are introduced. The main result of this section is a statement about the

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equivalence of a sum and an integral involving certain subadditive functions. Finally, we relate this result to our maximal theorem.

For a real-valued function ϕ, defined on E_n, the ordinary definition of subadditivity of ϕ consists of the condition

$$\phi(u+v) \leqslant \phi(u) + \phi(v), \quad u, v \text{ in } E_n. \tag{1}$$

In addition, we shall always insist that a subadditive function (in any of the senses given) be non-negative, measurable, and finite everywhere. The finiteness assumption is rather weak under the circumstances (cf. [5. p. 240]). For our maximal theorem, we can be somewhat more general than in (1) and say that ϕ is *subadditive* on E_n if there exists a constant $C > 0$ such that

$$\phi(u+v) \leqslant C[\phi(u) + \phi(v)], \quad u, v \text{ in } E_n. \tag{2}$$

(2) is much more convenient for several reasons, among which is the fact that any positive power of a subadditive function is subadditive. It is also sufficient for most of our results. Where (1) is required, we shall say that ϕ is *strictly subadditive*. It is known [5] that if ϕ is strictly subadditive on E_1, then it is bounded on compact subsets. We shall prove below that something analagous is true in E_n. Together with the measurability condition, this gives sense to the following definition. The non-negative measurable function ϕ is *generalized subadditive* if there exist constants $C > 0$ and ϱ, $0 < \varrho < 1$, such that

$$\phi(u) \leqslant \frac{C}{|u|^n} \int_{|u-v| < \varrho |u|} \phi(v)\, dv, \quad u \text{ in } E_n, \ u \neq 0. \tag{3}$$

It is indicated below how (2) implies (3). The notion of generalized subadditivity is too broad for most of our results, but it may be used occasionally. We also discuss subadditivity for functions defined on subsets of E_n, e.g. spheres and the interval $(0, \infty)$. The only restriction required in the above definitions is that the points involved be in the appropriate sets.

The following two theorems are essentially known ([2], [3]) and will be useful in what follows. The notation S_R refers to the solid sphere of radius R about the origin in E_n.

THEOREM A. *Let ϕ be subadditive on S_R in E_n, $n > 1$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be linearly independent unit vectors, $0 < p \leqslant \infty$, and $\alpha p > -1$. There exist constants A and B, depending only on α, p, n, and the vectors $\{\mu_i\}$ such that*

$$\int_{S_R} \frac{\phi^p(u)}{|u|^{n+p\alpha}}\,du \leqslant A \sum_{i=1}^{n} \int_{-R}^{R} \frac{\phi^p(r\mu_i)}{|r|^{1+p\alpha}}\,dr \leqslant B \int_{S_R} \frac{\phi^p(u)}{|u|^{n+p\alpha}}\,du.$$

The restriction that $\alpha p > -1$ may be omitted for the second inequality. In [3], the theorem was proved only for ϕ strictly subadditive, $R = \infty$, and $0 < p < \infty$. There is no difficulty in extending that proof to cover the above theorem. For $p = \infty$, the norm is to be interpreted as $\sup \phi(x)/|x|^\alpha$, and the condition $\alpha p > -1$ becomes $\alpha \geqslant 0$.

THEOREM B. *Let ϕ be generalized subadditive on S_R in E_n. Let α be real, and* $1 \leqslant p < q \leqslant \infty$. *Then*

$$\left(\int_{S_R} \frac{\phi^q(u)}{|u|^{n+q\alpha}}\,du\right)^{1/q} \leqslant C \left(\int_{S_R} \frac{\phi^p(u)}{|u|^{n+p\alpha}}\,du\right)^{1/p}.$$

By *generalized subadditive* in S_R, we mean there exist a C and a ϱ, $0 < \varrho < 1$, such that for x in S_R, $x \neq 0$,

$$\phi(x) \leqslant \frac{C}{|x|^n} \int_{N(x)} \phi(u)\,du, \quad N(x) = (x + S_{\varrho|x|}) \cap S_R.$$

In [3], the theorem was stated only for ϕ strictly subadditive and proved only for ϕ subbadditive. We sketch a proof of the above theorem for the sake of completeness. Let $x \neq 0$. By Hölder's inequality

$$\phi(x) \leqslant \frac{C}{|x|^{n/p}} \left(\int_{N(x)} \phi^p(u)\,du\right)^{1/p} \leqslant C \left(\int_{N(x)} \frac{\phi^p(u)}{|u|^n}\,du\right)^{1/p}.$$

The two C's that occur in the preceding inequality are, of course, different. Normally, throughout the paper, the dependence of constants on parameters, etc. will not be indicated. Thus

$$\frac{\phi(x)}{|x|^\alpha} \leqslant \frac{C}{|x|^\alpha} \left(\int_{N(x)} \frac{\phi^p(u)}{|u|^n}\,du\right)^{1/p} \leqslant C \left(\int_{N(x)} \frac{\phi^p(u)}{|u|^{n+p\alpha}}\,du\right)^{1/p}.$$

Replacing $N(x)$ by S_R in the last integral does not invalidate the inequality, and this proves the theorem for $q = \infty$. If $q < \infty$, denote the right-hand integral of the theorem by M. Since $q - p > 0$,

$$\frac{\phi^q(x)}{|x|^{n+q\alpha}} \leqslant \frac{C\,\phi^p(x)}{|x|^{n+p\alpha}} M^{q-p}, \quad x \neq 0.$$

An integration completes the proof. It is not hard to see that if we take $\alpha \geqslant 0$, then the restriction that $\varrho < 1$ may be omitted in the definition of generalized subadditivity.

1. The maximal theorem

As part of the hypothesis for the maximal theorem, we shall require another condition. If ϕ is subadditive and is also an even function on S_R, then

$$\phi(u) \leqslant C[\phi(u+v) + \phi(v)], \quad u, v \text{ in } S_R. \tag{2'}$$

We shall say that ϕ is *subadditive-even* in S_R if there exists a constant C such that (2) and (2′) are both satisfied. Standard examples of functions, subadditive in the strict sense, are subadditive-even. It is to be noted for future reference that if ϕ satisfies (2) and (2′) on $(0, \infty)$, then its even extension to $(-\infty, \infty)$, with $\phi(0) = 0$, does too.

We now introduce the maximal function corresponding to any subadditive-even function ϕ on S_R. Let

$$\omega(t) = \sup_{|v| \leqslant t} \phi(v), \quad t \text{ in } (0, R). \tag{4}$$

$\omega(t)$ is finite everywhere (cf. Lemma 1), increasing, and subadditive; for

$$\sup_{|v| \leqslant s+t} \phi(v) \leqslant \sup_{\substack{|u| \leqslant s \\ |v| \leqslant t}} \phi(u+v) \leqslant C \sup_{|u| \leqslant s} \phi(u) + C \sup_{|v| \leqslant t} \phi(v).$$

It also satisfies (2′) since it is increasing.

THEOREM 1. *Let ϕ be subadditive-even on S_R in E_n. Let α be real, and $0 < p \leqslant \infty$. If ω is defined by* (3), *then*

$$\int_0^R \frac{\omega^p(t)}{t^{1+p\alpha}}\, dt \leqslant C \int_{S_R} \frac{\phi^p(u)}{|u|^{n+p\alpha}}\, du.$$

The constant C on the right side of the above inequality depends on α, p, n, and the constants occurring in (2) and (2′). It is clear that the integral on the left side of the inequality dominates the one on the right so that the theorem is a statement about the equivalence of the two integrals. For the proof, we shall consider only the case $R = \infty$, i.e. functions ϕ subadditive-even on E_n. The adjustments in the proof for the case $R < \infty$ are quite minor. Also, since any positive power af a subadditive-even function is also subadditive-even, then it is necessary to consider only two values of p, i.e. $p=1$ and $p=\infty$.

Our first lemma states more than is necessary for the proof of the theorem, for which it is implicitly assumed anyway that ϕ is locally integrable, at least away from the origin. However, the extra information is included very cheaply and includes a proof of the fact that a subadditive function is generalized subadditive.

LEMMA 1. *Let ϕ be subadditive on E_n. Then it is bounded on compact subsets of E_n, and there exists a constant C such that*

$$\phi(u) \leqslant \frac{C}{|u|^n} \int_{|v-u/2|<|u|/6} \phi(v)\, dv, \quad u \neq 0.$$

The proof of the first statement is an adaptation of an argument in [5, p. 240]. We first restrict ϕ to an open hyperquadrant, say E_n^+, defined as those u such that each coordinate is strictly positive. Let u belong to E_n^+, and let $\phi(u)=A$. Let $H(u)$ denote the hyperrectangle in E_n^+ with 0 and u as opposite vertices. Let $|H(u)|$ denote its volume. Let $F(u)$ denote the subset of $H(u)$ such that $\phi(v) \geqslant A/2\,C$ for v in $F(u)$ where C is given by (2). Then, $H(u)=F(u) \cup (u-F(u))$ so that $|F(u)|$, the measure of $F(u)$, is at least $|H(u)|/2$. If ϕ were unbounded in R, a hyperrectangle of E_n^+, defined by $\bar{u}$ and $\bar{v}$ as opposite vertices, there exists a sequence u_m in R such that $\phi(u_m) \geqslant 2\,Cm$. For u in R, $|H(u)| \geqslant |H(\bar{u})|$, and the set of points v in $H(\bar{v})$ such that $\phi(v) \geqslant m$ has measure at least $|H(\bar{u})|/2$. This implies that ϕ is ∞ on a set of positive measure, a contradiction. We have thus established that ϕ is bounded on compact sets of E_n^+. Let S_δ denote the sphere of radius δ about the origin. Let 1 denote the point all of whose coordinates are unity. Then

$$\phi(u) \leqslant C\,[\phi(-1)+\phi(u+1)].$$

If u belongs to S_δ with $\delta<1$, then $u+1$ belongs to a compact subset of E_n^+. Thus ϕ is bounded on S_δ and so on any sphere about the origin.

For the proof of the second inequality, we note that if v belongs to the sphere of radius $|u|/6$ about $u/2$, then so does $u-v$. Thus, integrating the inequality

$$\phi(u) \leqslant C\,[\phi(v)+\phi(u-v)]$$

over this sphere gives

$$|u|^n\,\phi(u) \leqslant C \int_{|v-u/2|\leqslant|u|/6} \phi(v)\, dv$$

as desired.

The following result is the key step in the proof of the theorem.

LEMMA 2. *Let ϕ be subadditive-even on $(0, \infty)$. Then for $u>0$,*

$$\omega(u) = \sup_{0<v\leqslant u} \phi(v) \leqslant \frac{C}{u} \int_{u/30}^{2u/3} \phi(t)\,dt.$$

By Lemma 1,
$$\phi(u) \leqslant \frac{C}{u} \int_{u/3}^{2u/3} \phi(t)\,dt = C\,\psi(u). \tag{5}$$

The second equality is simply a definition of ψ. We have

$$\psi(v) \leqslant \frac{3}{u} \int_{u/9}^{2u/3} \phi(t)\,dt, \quad u/3 \leqslant v \leqslant u. \tag{6}$$

We temporarily fix v in the interval $(0, u/3)$ and consider

$$\chi(s) = \int_{s+v/3}^{s+2v/3} \phi(t)\,dt, \quad \frac{u-v}{3} \leqslant s \leqslant \frac{2(u-v)}{3}.$$

χ, being continuous, has a minimum at s_0, say. Then

$$\frac{u}{9}\,\chi(s_0) \leqslant \frac{u-v}{3}\,\chi(s_0) \leqslant \int_{(u-v)/3}^{2(u-v)/3} \chi(s)\,ds \leqslant \frac{v}{3} \int_{u/3}^{2u/3} \phi(t)\,dt.$$

That is
$$\frac{1}{v}\,\chi(s_0) \leqslant 3\,\psi(u). \tag{7}$$

By the evenness property, $\phi(t) \leqslant C[\phi(s_0) + \phi(s_0+t)]$. An integration shows

$$\psi(v) \leqslant C\left[\phi(s_0) + 3\,\frac{\chi(s_0)}{v}\right] \leqslant C\,[\phi(s_0) + 9\,\psi(u)] \leqslant \frac{C}{s_0} \int_{s_0/3}^{2\,s_0/3} \phi(t)\,dt + 9\,C\,\psi(u)$$

by (5) and (7). Since $u/9 \leqslant (u-v)/3 \leqslant s_0$, the integral on the right does not exceed

$$\frac{C}{u} \int_{u/30}^{2u/3} \phi(t)\,dt.$$

Along with (6), this shows that

$$\psi(v) \leqslant \frac{C}{u} \int_{u/30}^{2u/3} \phi(t)\,dt, \quad v \leqslant u.$$

But $\phi(v) \leqslant C\,\psi(v)$, and so $\omega(u)$ does not exceed a constant multiple of the integral on the right.

To complete the proof of the theorem, we let

$$\phi_i\,(u_i) = \phi(0, \ldots, 0, u_i\, 0, \ldots, 0), \quad i = 1, 2, \ldots, n.$$

Each ϕ_i is subadditive-even in its single variable. Let ω_i be the maximal function corresponding to ϕ_i. By an inductive process,

$$\phi(u) \leqslant C \sum_{i=1}^{n} \phi_i(u_i).$$

Thus,

$$\omega(t) \leqslant C \sum_{i=1}^{n} \omega_i(t).$$

Letting $\omega_i^{(1)}(t) = \sup_{0<u_i\leqslant t} \phi_i(u_i)$ and $\omega_i^{(2)}(t) = \sup_{-t\leqslant u_i<0} \phi_i(u_i)$, we have

$$\omega_i(t) \leqslant \omega_i^{(1)}(t) + \omega_i^{(2)}(t)$$

and

$$\omega_i^{(1)}(t) \leqslant \frac{C}{t} \int_{t/30}^{2t/3} \phi_i(u_i)\, du_i \leqslant C\, t^\alpha \int_{t/30}^{2t/3} \frac{\phi_i(u_i)}{u_i^{1+\alpha}}\, du_i$$

by Lemma 2. Hence

$$\int_0^\infty \frac{\omega_i^{(1)}(t)}{t^{1+\alpha}}\, dt \leqslant C \int_0^\infty \frac{dt}{t} \int_{t/30}^{2t/3} \frac{\phi_i(u_i)}{u_i^{1+\alpha}}\, du_i \leqslant C \int_0^\infty \frac{\phi_i(u_i)}{u_i^{1+\alpha}}\, du_i.$$

A similar inequality applies to $\omega_i^{(2)}(t)$ so that

$$\int_0^\infty \frac{\omega_i(t)}{t^{1+\alpha}}\, dt \leqslant C \int_{-\infty}^\infty \frac{\phi_i(u_i)}{|u_i|^{1+\alpha}}\, du_i.$$

Now an application of the second inequality in Theorem A (for which there is no restriction on α) suffices for the proof of the case $p=1$. Since any positive power of a subadditive-even function is also subadditive-even, we may replace ϕ by ϕ^p, ω by $C\omega^p$, and α by αp for the case $0<p<\infty$ and $p \neq 1$. For the case $p=\infty$, we obtain from Lemma 2 that, if ϕ is subadditive-even on $(0, \infty)$, then

$$\frac{\omega(t)}{t^\alpha} \leqslant C \sup_{0<u\leqslant t} \frac{\phi(u)}{u^\alpha}.$$

It is then an easy step to show, again by the use of Theorem A, that for ϕ subadditive-even on E_n,

$$\sup_{t>0} \omega \frac{(t)}{t^\alpha} \leqslant C \sup_u \frac{\phi(u)}{|u|^\alpha}.$$

We are unable to decide how much the evenness hypothesis can be weakened, but it cannot be omitted entirely from Lemma 2, as can be seen by considering monotone decreasing functions. Lemma 2 provides a direct connection between our maximal theorem and that of Hardy and Littlewood. Thus, if ϕ is any integrable and positive function on $(0, R)$, let

$$\theta(u; \phi) = \sup_{0<\xi<u} \frac{1}{\xi} \int_{u-\xi}^{u} \phi(v)\, dv.$$

This is the maximal function of Hardy and Littlewood. Lemma 2 shows that if ϕ is subadditive-even, then $\omega(u) \leqslant C\,\theta(u; \phi)$. Thus, if ϕ is in L^p, $p>1$, then so is ω. For our theorem, this corresponds to the case $n=1$ and $\alpha p = -1$.

The theorem has been proved for all real α, but for several important cases, there are essential restrictions on α.

THEOREM 2. (i). *Let ϕ be positive and measurable on* $(0, \infty)$ *and satisfy* $\phi(u) \leqslant C \times [\phi(u+v) + \phi(v)]$ *there for some positive* C. *Let*

$$\int_0^\infty \frac{\phi(u)}{u^{1+\alpha}}\, du < \infty$$

for some $\alpha < 0$. *Then ϕ is identically* 0.

(ii). *Let ϕ be strictly subadditive on S_R in E_n. Let*

$$\int_{S_R} \frac{\phi^p(u)}{|u|^{n+p\alpha}}\, du < \infty$$

for some $\alpha \geqslant 1$ *and some* p, $0<p<\infty$. *Then ϕ is identically* 0.

Let m be a large positive integer. By some obvious substitutions, we have

$$2\int_0^\infty \frac{\phi(u)}{u^{1+\alpha}}\, du = \frac{1}{m^\alpha}\int_0^\infty \frac{\phi(mu)}{u^{1+\alpha}}\, du + \frac{1}{(m-1)^\alpha}\int_0^\infty \frac{\phi((m-1)u)}{u^{1+\alpha}}\, du.$$

Using the above inequality shows that this exceeds

$$\frac{1}{C(m-1)^\alpha}\int_0^\infty \frac{\phi(u)}{u^{1+\alpha}}\, du.$$

Letting m go to ∞, we see that the integral is 0 so that ϕ is equivalent to 0. But given $u>0$, there exists $v>0$ such that $\phi(u+v) = \phi(v) = 0$ so that $\phi(u) = 0$.

For the proof of (ii), we may reduce it to the one-dimensional case by Theorem A and use a known theorem for that [2].

Part (i) shows in particular that a non-trivial subadditive-even function cannot be in any L^p class on E^n.

2. Some applications of the maximal theorem

Let f belong to $L^r(E_n)$, $1 \leqslant r \leqslant \infty$, and let

$$\phi_r(u;f) = \left(\int_{E_n} |f(x+u) - f(x)|^r \, dx \right)^{1/r}.$$

This function is subadditive-even on E_n, and a direct application of Theorem 1 to it gives a result due to Taibleson [6], presumably with a quite different proof. If f is in class $L^r(T_n)$, where T_n is the n-dimensional torus, then substitution of T_n for E_n in the above integral gives a subadditive-even function for which the appropriate domain of integration is again T_n. By minor changes in the proof, it can be shown that the statement of the maximal theorem holds in this case.

For the definition of $\phi_r(u;f)$ for $r = \infty$, we mean, as usual, ess $\sup_x |f(x+u) - f(x)|$. If f is bounded in the ordinary sense over E_n, the function defined by

$$\phi(u;f) = \sup_x |f(x+u) - f(x)|$$

is also subadditive-even. This is sometimes called the ordinary modulus of continuity. Let

$$\omega(t;f) = \sup_{|u| \leqslant t} \sup_x |f(x+u) - f(x)|$$

denote the rectified modulus of continuity (cf. [5, p. 249] for the terminology). $\omega(t;f)$ is also the maximal function associated with $\phi(u;f)$.

Other examples of subadditive-even functions to which the maximal theorem applies directly are constructed by use of mixed norms [1] rather than ordinary norms. For convenience, this will be done in only two dimensions. Let f be a measurable function in E_2, and $1 \leqslant p_1, p_2 < \infty$. The mixed norm of f is then given by

$$\|f\|_{p_1, p_2} = \left\{ \int_{E_1} dy \left(\int_{E_1} |f(x,y)|^{p_1} \, dx \right)^{p_2/p_1} \right\}^{1/p_2}.$$

Let $u = (u_1, u_2)$ be a point of E_2, and define $g_u(x,y) = f(x+u_1, y+u_2) - f(x,y)$. Let

$$\phi_{p_1, p_2}(u;f) = \|g_u\|_{p_1, p_2}.$$

Since there is a triangle inequality for mixed norms [1], it is not hard to show that this function is subadditive, in fact strictly subadditive. It is also even, and Theorem 1 applies.

If, in the definition of $\phi_r(u;f)$, the first difference of f is replaced by the second symmetric difference, a function is obtained which is also important in applications. Thus, let

$$\sigma_r(u;f) = \left(\int_{E_n} |f(x+u)+f(x-u)-2f(x)|^r\,dx\right)^{1/r}.$$

Although this function is not subadditive, it has certain features which allow analysis of the above type in obtaining a maximal theorem, at least in dimension one. It is easy to show (cf. [2, p. 381]) that

$$\sigma_r(u+v;f) \leqslant 2\,\sigma_r(u;f) + 2\,\sigma_r(v;f) + \sigma_r(u-v;f);$$

$$\sigma_r(v;f) \leqslant 2\,\sigma_r(u;f) + 2\,\sigma_r(u-v;f) + \sigma_r(2u-v;f).$$

The first inequality is the analogue of the subadditivity property, and the second follows from it by the evenness of σ_r. We indicate briefly how it is possible to prove a maximal theorem for σ_r in dimension one from these two properties. From the first inequality, if follows that σ_r is generalized subadditive. Thus, if $u>0$,

$$\sigma_r(u;f) \leqslant \frac{20}{u}\int_{u/8}^{7u/8} \sigma_r(v;f)\;dv.$$

Now fix v, $0<v<u/8$, and let

$$\chi(s) = \int_{s-7v/8}^{s-v/8} \sigma_r(t;f)\;dt + \int_{2s-7v/8}^{2s-v/8} \sigma_r(t;f)\;dt,\quad \frac{u-v}{8} \leqslant s \leqslant \frac{7(u-v)}{8}.$$

Taking $\chi(s_0)$ as the minimum functional value, we may show, as in the proof of Lemma 2, that

$$\frac{\chi(s_0)}{v} \leqslant \frac{C}{u}\int_{u/16}^{7u/4} \sigma_r(t;f)\;dt.$$

From the second of the above inequalities for σ_r, we have for $0<t<s_0$ that

$$\sigma_r(t;f) \leqslant 2\,\sigma_r(s_0;f) + 2\,\sigma_r(s_0-t;f) + \sigma_r(2s_0-t;f).$$

Integration of this over $(v/8,\ 7v/8)$ when $0<v<u/8$ shows that

$$\sigma_r(v;f) \leqslant 2\,\sigma_r(s_0;f) + C\,\frac{\chi(s_0)}{v} \leqslant 2\,\sigma_r(s_0;f) + \frac{C}{u}\int_{u/16}^{7u/4} \sigma_r(t;f)\,dt.$$

A similar inequality holds if $u/8 \leqslant v \leqslant u$ so that

$$\sup_{0<v\leqslant u} \sigma_r(v;f) \leqslant \frac{C}{u}\int_{u/16}^{7u/4} \sigma_r(t;f)\,dt.$$

The proof may be completed as in Theorem 1.

Certain integral transforms of positive functions will be generalized subadditive if the kernel satisfies a kind of uniform generalized subadditivity condition in one of the variables. The situation is well illustrated in the following transform, which is a kind of Riesz fractional derivative. Let

$$\phi(u) = \int_{E_n} |u-v|^{\beta} f(v)\, dv, \quad \beta > 0. \tag{8}$$

We shall assume that f is non-negative, integrable, and with compact support so that there is no question about the definition of ϕ. It is not hard to show that ϕ is generalized subadditive. Furthermore, a maximal theorem holds for ϕ. $\omega(t)$ will denote, as before, $\sup_{0<|u|\leqslant t} \phi(u)$.

THEOREM 3. *Let ϕ be defined by* (8) *with f non-negative, integrable, and with compact support. Let $0<\beta$. Then ϕ is generalized subadditive, and if $n=1$, and α is real, $0<p<\infty$, then*

$$\int_0^{\infty} \frac{\omega^p(t)}{t^{1+p\alpha}}\, dt \leqslant C \int_{E_1} \frac{\phi^p(u)}{|u|^{1+p\alpha}}\, du.$$

To prove the first statement of the theorem, it is enough to establish the existence of C, independent of u and w, such that

$$|u-w|^{\beta} \leqslant \frac{C}{|u|^n} \int_{|u-v|<|u|/2} |w-v|^{\beta}\, dv, \quad u \neq w.$$

But there is a hyperplane through u which is orthogonal to the line segment joining u and w, and which divides E_n into two half spaces. Let H^- denote the half space not containing w. If v belongs to H^- and satisfies $|u-v|<|u|/2$, then

$$|u-w| \leqslant |w-v| \text{ and } |u-w|^{\beta} \leqslant |w-v|^{\beta}.$$

Since the measure of the set of such points is one-half the volume of the solid sphere defined by $|u-v|<|u|/2$, the above inequality is established.

From the obvious inequalities

$$|w-u-v|^{\beta} \leqslant C\,[|w-2u|^{\beta} + |w-2v|^{\beta}],$$

$$|w-u|^{\beta} \leqslant C\,[|w-\tfrac{1}{2}(u+v)|^{\beta} + |w-v|^{\beta}]$$

it follows that

$$\phi(u+v) \leqslant C\,[\phi(2u)+\phi(2v)], \quad \phi(u) \leqslant C\,[\phi(\tfrac{1}{2}(u+v)) + \phi(v)]. \tag{9}$$

Now it is enough to confine attention to the interval $(0, \infty)$. Rewriting the first inequality of (9) as $\phi(u) \leqslant C[\phi(2u-2v)+\phi(2v)]$ and integrating over $(u/4, 3u/4)$, we obtain

$$\phi(u) \leqslant \frac{C}{u}\int_{u/2}^{3u/2} \phi(v)\,dv = C\,\psi(u).$$

The last expression is a definition of $\psi(u)$. We have

$$\psi(v) \leqslant \frac{C}{u}\int_{u/6}^{3u/2} \phi(w)\,dw, \quad \frac{u}{3} \leqslant v \leqslant u.$$

Fix v in the interval $(0, u/3)$, and let

$$\chi(s) = \int_{s/2+v/4}^{s/2+3v/4} \phi(t)\,dt, \quad \frac{u-v}{2} \leqslant s \leqslant \frac{3(u-v)}{2}.$$

Let $\chi(s_0)$ be a minimum for χ. It can be shown, as in the proof of Lemma 2, that

$$\frac{\chi(s_0)}{v} \leqslant \frac{C}{u}\int_{u/4}^{3u/4} \phi(t)\,dt.$$

The second inequality of (9) may be written as

$$\phi(t) \leqslant C[\phi(\tfrac{1}{2}(s_0+t)) + \phi(s_0)].$$

Substitution of this into the integral defining ψ and using the above estimate of $\chi(s_0)/v$ shows that

$$\psi(v) \leqslant \frac{C}{u}\int_{u/6}^{3u/2} \phi(t)\,dt.$$

Thus,

$$\sup_{0<v\leqslant u} \phi(v) \leqslant \frac{C}{u}\int_{u/6}^{3u/2} \phi(t)\,dt.$$

Now the proof can be completed as in Theorem 1.

The considerations of the next example are very much in the spirit of the maximal theorem, but rather more modifications are necessary. It is a kind of local version of the first example, and we shall take advantage of the Hardy–Littlewood maximal theorem. Our motivation for this example is its importance in a convergence theorem for trigonometric polynomials (cf. [4]). Since the result is a kind of local one, it is more appropriate to use bounded regions of integration and to consider periodic functions. For technical reasons, we shall confine attention to the E_1 case. For a function f, locally integrable on E_1, let

$$\omega(t, x) = \sup_{0<|v|\leqslant t} \int_{x-t}^{x+t} |f(s+v) - f(s)|\, ds, \quad |x| \leqslant \pi, 0 < t \leqslant \pi. \tag{10}$$

Let f satisfy the condition

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x+u) - f(x)|^p}{|u|^{1+p\alpha}}\, dx\, du < \infty. \tag{11}$$

THEOREM 4. *Let the periodic function f satisfy* (11) *with $1 < p < \infty$ and $0 < \alpha < 1$. Let $\omega(t, x)$ be defined by* (10). *Then the function*

$$\mu(x) = \sup_{0<t\leqslant\pi} \frac{\omega(t, x)}{t^{1+\alpha}}$$

belongs to $L^p(-\pi, \pi)$.

Let
$$\phi_r(u, x) = \int_{x-r}^{x+r} |f(s+u) - f(s)|\, ds, \quad r > 0.$$

The following inequality, which is quite easy to see, is the analogue of the subadditivity property.

$$\phi_r(u+v, x) \leqslant C[\phi_{2r}(u, x) + \phi_{2r}(v, x)], \quad |v| \leqslant r. \tag{12}$$

It is also easy to see that $\phi_r(-u, x) \leqslant \phi_{2r}(u, x)$ if $|u| \leqslant r$, and so

$$\phi_r(u, x) \leqslant C[\phi_{2r}(u+v, x) + \phi_{2r}(-v, x)] \leqslant C[\phi_{4r}(u+v, x) + \phi_{4r}(v, x)], \quad |u|, \ |v| \leqslant r.$$

The latter inequality is an analogue of the evenness property. (12) leads to

$$\phi_r(u, x) \leqslant C[\phi_{2r}(u-v, x) + \phi_{2r}(v, x)], \quad |v| \leqslant r.$$

Let $0 < u \leqslant r$. Integrate the preceding to obtain

$$\phi_r(u, x) \leqslant \frac{C}{u} \int_{u/3}^{2u/3} \phi_{2r}(v, x)\, dv = C\, \psi_{2r}(u, x).$$

We are now prepared to repeat the proof of Lemma 2 with only minor modifications. This leads to the result that if v belongs to $(0, u)$, then

$$\psi_{2r}(v, x) \leqslant \frac{C}{u} \int_{u/30}^{2u/3} \phi_{16r}(t, x)\, dt.$$

But $\phi_r(v, x) \leqslant C\, \psi_{2r}(v, x)$, and so

$$\sup_{0<v\leqslant u} \phi_r(v, x) \leqslant \frac{C}{u} \int_{u/30}^{2u/3} \phi_{16r}(t, x)\, dt, \quad 0 < u \leqslant r.$$

Since $\phi_r(-u, x) \leqslant \phi_{2r}(u, x)$, then

$$\omega_r(u, x) = \sup_{0<|v|\leqslant u} \phi_r(v, x) \leqslant \frac{C}{u} \int_{u/30}^{2u/3} \phi_{32r}(t, x)\, dt, \quad 0 < u \leqslant r.$$

Let $r = t$. Then $\omega_r(t, x) = \omega(t, x)$ as defined by (10). By Fubini's theorem,

$$\omega(t, x) \leqslant \frac{C}{t} \int_{x-32t}^{x+32t} ds \int_{t/30}^{2t/3} |f(s+v) - f(s)|\, dv \leqslant C\, t^{\alpha-(1/q)} \int_{x-32t}^{x+32t} ds \int_{t/30}^{2t/3} \frac{|f(s+v)-f(s)|}{v^{\alpha+(1/p)}}\, dv$$

where $1/p + 1/q = 1$. Applying Hölder's inequality to the inner integral gives

$$\omega(t, x) \leqslant C\, t^{\alpha} \int_{x-32t}^{x+32t} ds \left(\int_0^{\pi} \frac{|f(s+v)-f(s)|^p}{v^{1+p\alpha}}\, dv \right)^{1/p} = C t^{\alpha} \int_{x-32t}^{x+32t} g(s)\, ds$$

where the last equality simply defines $g(s)$. Thus

$$\sup_{0<t\leqslant\pi} \frac{\omega(t, x)}{t^{1+\alpha}} \leqslant C \sup_{0<t\leqslant\pi} \frac{1}{32\, t} \int_{x-32t}^{x+32t} g(s)\, ds.$$

Since $|x| \leqslant \pi$ and $0 < t \leqslant \pi$, then $|x \pm 32t| \leqslant 33\pi$. Thus, the right side of the preceding inequality is dominated by the Hardy-Littlewood maximal function corresponding to the function g and the interval $(-33\pi, 33\pi)$ (cf. [7, p. 30]). But g is periodic and of class L^p. Hence μ is also of class L^p.

3. Sums and integrals involving subadditive functions

In the sense that a countable set is of dimension 0, our first result of this section is in the same category as Theorem A: it asserts the equivalence of an integral over a certain domain with an integral over a lower dimensional domain. This theorem generalizes one of Peetre, who is responsible for the L^{∞} case in unpublished work. We may dispense with the evenness requirement here. However, the result is rather delicate, and we shall require both strict subadditivity and a continuity condition. Without the latter, the theorem is false, as shown by simple examples. We use below the notation "$\cong$" to mean the equivalence of two integrals: i.e. both are infinite, both are 0, or both are finite, non-zero with their ratios bounded above and below by constants independent of the functions chosen from a certain function class.

THEOREM 5. *Let ϕ be strictly subadditive and left lower continuous on $(0, \infty)$. Let $0 < \alpha < 1$ and $0 < p \leqslant \infty$. Then*

$$\int_0^\infty \frac{\phi^p(u)}{u^{1+p\alpha}}\,du \cong \sum_{k=-\infty}^{\infty} \frac{\phi^p(2^k)}{2^{kp\alpha}}.$$

There is no particular significance in the use of the sequence $\{2^k\}$ in the sum on the right. We may use any sequence of the form $\{b^k\}$, $b>1$. For $p=\infty$, we mean

$$\sup_{u>0} \frac{\phi(u)}{u^\alpha} \cong \sup_k \frac{\phi(2^k)}{2^{k\alpha}}.$$

The fact that the series does not exceed a constant multiple of the integral needs proof only for the case $p<\infty$, where it is quite elementary, and a considerable weakening of the hypothesis is possible. Let $2^{k-2} \leqslant u \leqslant 2^{k-1}$. Then,

$$\phi^p(2^k) \leqslant C[\phi^p(u) + \phi^p(2^k-u)].$$

Since, under the circumstances, $2^{k-1} \leqslant 2^k - u \leqslant 2^k$, we have

$$\frac{\phi^p(2^k)}{2^{kp\alpha}} \leqslant C\,\frac{\phi^p(2^k)}{2^{kp\alpha}} \int_{2^{k-2}}^{2^{k-1}} \frac{du}{u} \leqslant \frac{C}{2^{kp\alpha}} \int_{2^{k-2}}^{2^k} \frac{\phi^p(u)}{u}\,du \leqslant C \int_{2^{k-2}}^{2^k} \frac{\phi^p(u)}{u^{1+p\alpha}}\,du.$$

Summing over k now gives the result.

For the proof that the integral does not exceed a constant multiple of the series, we begin with a lemma.

Lemma 3. *Let $0<v<\infty$, $0<\alpha<1$, and let ϕ be as above. Then*

$$\sup_{u\leqslant v} \frac{\phi(u)}{u^\alpha} \leqslant C \sup_{2^k\leqslant v} \frac{\phi(2^k)}{2^{k\alpha}}.$$

Since, for u in $(0,v)$, $u=\sum_{2^k\leqslant u} \varepsilon_k 2^k$ with ε_k equal to 1 or 0, it follows from the continuity condition that

$$\phi(u) \leqslant \sum_{2^k\leqslant u} \varepsilon_k \phi(2^k) \leqslant \sum_{2^k\leqslant u} \phi(2^k).$$

Let $\phi(2^k) \leqslant A\,2^{k\alpha}$ for all 2^k in $(0,v)$. We may assume A is finite, for otherwise there is nothing to prove. Then

$$\phi(u) \leqslant A \sum_{2^k\leqslant u} 2^{k\alpha} \leqslant C\,A\,u^\alpha.$$

This completes the proof of the lemma, and by letting v approach ∞, we have the proof of the theorem for the case $p=\infty$. The case $p=1$ is relatively easy to treat; and if $0<p<1$, then ϕ^p is strictly subadditive and satisfies the same continuity condition as ϕ. Since $0<\alpha p<1$, this case can be reduced to that for which $p=1$.

Thus, from now on, we restrict attention to values of p satisfying $1 < p < \infty$. For this, the following easily verified inequality is required. Given p, $1 < p < \infty$, and $\varepsilon > 0$, there is a constant A, depending only on p and ε such that if x and y are complex numbers, then

$$|x+y|^p \leqslant A|x|^p + (1+\varepsilon)\,|y|^p. \tag{13}$$

Let $x = \phi(2^k)$ and $y = \phi(u-2^k)$ in (13). Thus

$$\phi^p(u) \leqslant |\phi(2^k) + \phi(u-2^k)|^p \leqslant A\,\phi^p(2^k) + (1+\varepsilon)\,\phi^p(u-2^k).$$

After multiplication by $2^{-k(1+p\alpha)}$, an integration shows that

$$2^{-k(1+p\alpha)} \int_{2^k}^{2^{k+1}} \phi^p(u)\,du \leqslant A\,2^{-kp\alpha}\,\phi^p(2^k) + (1+\varepsilon)\,2^{-k(1+p\alpha)} \int_0^{2^k} \phi^p(u)\,du.$$

Summing with respect to k, we obtain

$$\sum_{k=-N}^{N} 2^{-k(1+p\alpha)} \int_{2^k}^{2^{k+1}} \phi^p(u)\,du \leqslant A \sum_{k=-N}^{N} 2^{-kp\alpha}\,\phi^p(2^k) + (1+\varepsilon) \sum_{k=-N}^{N} 2^{-k(1+p\alpha)} \int_0^{2^k} \phi^p(u)\,du.$$

Denote the sum on the left by S_N. We shall prove that

$$(1+\varepsilon) \sum_{k=-N}^{N} 2^{-k(1+p\alpha)} \int_0^{2^k} \phi^p(u)\,du \leqslant \frac{1+\varepsilon}{2^{1+p\alpha}-1}\, S_N + T_N, \quad T_N = o(1). \tag{14}$$

Since ε can be chosen so that $1+\varepsilon < 2^{1+p\alpha}-1$, and since $\lim_N S_N$ is clearly equivalent to the integral of the theorem, establishing (14) will complete the proof of the theorem.

We apply Abel's transformation to the sum on the left of (14) to obtain

$$\sum_{k=-N}^{N} 2^{-k(1+p\alpha)} \int_0^{2^k} \phi^p(u) \leqslant \frac{1}{2^{1+p\alpha}-1}\, S_N + 2^{2+p\alpha}\, 2^{N(1+p\alpha)} \int_0^{2^{-N}} \phi^p(u)\,du.$$

Denote the second term on the right by T^N. We note, by use of Lemma 3, that

$$T_N \leqslant 2^{2+p\alpha}\, 2^N \int_0^{2^{-N}} \frac{\phi^p(u)}{u^{p\alpha}}\,du \leqslant 2^{2+p\alpha} \sup_{u \leqslant 2^{-N}} \frac{\phi^p(u)}{u^{p\alpha}} \leqslant C \sup_{k \leqslant -N} 2^{-kp\alpha}\,\phi^p(2^k).$$

The last term on the right is $o(1)$ since the series of the theorem is assumed to converge. (Otherwise there is nothing to prove.) This completes the proof of (14) and of the theorem.

If ϕ is strictly subadditive and left lower semi-continuous on a finite interval $(0, a)$, then the equivalence of the theorem holds for this interval since ϕ may be defined as 0 on (a, ∞) and the theorem applied to this extended function.

If ϕ is strictly subadditive on E_n and satisfies the proper continuity condition, then it can be shown by Theorems A and 5 that certain integrals over E_n are equivalent to infinite series, the terms of which involve the values of ϕ on the coordinate axis. An application of Theorem B in the one-dimensional case to the integral of Theorem 5 gives an inequality involving series. Thus, if ϕ is continuous and strictly subadditive on $(0, \infty)$, if $0 < \alpha < 1$, and if $1 \leqslant p < q \leqslant \infty$, then.

$$\left(\sum_{k=-\infty}^{\infty} \frac{\phi^q(2^k)}{2^{kq\alpha}} \right)^{1/q} \leqslant C \left(\sum_{k=-\infty}^{\infty} \frac{\phi^p(2^k)}{2^{kp\alpha}} \right)^{1/p}. \tag{15}$$

It is instructive to compare this inequality with the series analogue of Theorem B. Thus, we shall say that the sequence $\{a_n\}$ of positive reals is subadditive if $a_{m+n} \leqslant a_m + a_n$, $m, n = 1, 2, \ldots$. Let $0 < p < q \leqslant \infty$, and let α be real. Then

$$\left(\sum_{n=1}^{\infty} \frac{a_n^q}{n^{1+q\alpha}} \right)^{1/q} \leqslant C \left(\sum_{n=1}^{\infty} \frac{a_n^p}{n^{1+p\alpha}} \right)^{1/p}.$$

The proof of this is a direct adaptation of the proof of Theorem B. However, the result does not apply directly to (15) since $\{2^k\}$ is not an additive class, and so $\{\phi(2^k)\}$ is not necessarily a subadditive sequence.

Now let ϕ be strictly subadditive-even on $(0, \infty)$; i.e. let it satisfy (2) and (2') with $C = 1$. Also let ϕ be left lower semi-continuous there. Most of the examples we have cited are strictly subadditive, even, and continuous so that we are talking about a large class of functions. Let $\omega(u) = \sup_{0 < v \leqslant u} \phi(v)$. This is the maximal function associated with the even extension to E_1 of ϕ. It is not hard to see that ω is left continuous and strictly subadditive so that Theorem 5 is applicable. (Actually, the conclusion of Theorem 5 is valid for any positive, monotone function.) By use of the maximal theorem, we are led to our final result.

COROLLARY. *Let ϕ be strictly subadditive-even and left lower semi-continuous on $(0, \infty)$. Let $0 < p \leqslant \infty$, and $0 < \alpha < 1$. Then*

$$\sum_{k=-\infty}^{\infty} \frac{\omega^p(2^k)}{2^{kp\alpha}} \cong \sum_{k=-\infty}^{\infty} \frac{\phi^p(2^k)}{2^{kp\alpha}}.$$

References

[1]. Benedek, A. & Panzone, R., The spaces L^p with mixed norm. *Duke Math. J.*, 28 (1961), 301–324.
[2]. Gosselin, R. P., Some integral inequalities. *Proc. Amer. Math. Soc.*, 13 (1962), 378–384.
[3]. —— Integral norms of subadditive functions. *Bull. Amer. Math. Soc.*, 69 (1963), 255–259.
[4]. —— On the approximation of L^p functions by trigonometric polynomials. *Fund. Math.*, 53 (1964), 121–134.
[5]. Hille, E. & Phillips, R. S., *Functional analysis and semi-groups*. Providence, 1957.
[6]. Taibleson, M., Lipschitz classes of functions and distributions in E_n. *Bull. Amer. Math. Soc.*, 69 (1963), 487–493.
[7]. Zygmund, A., *Trigonometrical series*, vol. I. Cambridge, 1959.

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