LIPSCHITZ APPROXIMATIONS TO SUMMABLE FUNCTIONS

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1. Introduction

In [4] I considered a continuous non-negative function ϕ on R^n with certain special properties and I defined for each Lipschitz function f on the closed unit cube Q of R^n,

$$\Psi(f) = \int_Q \phi(\text{grad } f)\, dx.$$

This non-negative functional Ψ was shown to be lower semi-continuous on the set of Lipschitz functions with the $\mathcal{L}_1$ topology and hence could be extended to a non-negative lower semi-continuous functional on the summable functions. The main result of [4] was the following.

If f is continuous on Q and such that $\Psi(f)$ is finite, and if $\varepsilon > 0$, then there exists a Lipschitz function g on Q such that the set

$$\{x;\, x \in Q \text{ and } f(x) \neq g(x)\}$$

has measure less than ε and $\Psi(g) < \Psi(f) + \varepsilon$. This problem arose from a conjecture of C. Goffman concerning the approximation of non-parametric surfaces with finite area, by Lipschitz surfaces. See [2] and [3].

In the present paper this theorem is proved without the continuity restriction, i.e., it is shown that any function f, summable on Q and with $\Psi(f) < \infty$, can be approximated by a Lipschitz function in the manner described above. Also, in the new theorem, a more general ϕ is taken, hence a more general functional Ψ.

Throughout the present paper ϕ denotes a non-negative continuous, real-valued function on R^n with the following properties:

(i) $\phi(\xi) \geq \phi(\xi')$, when $|\xi_1| \geq |\xi_1'|, \ldots, |\xi_n| \geq |\xi_n'|$;

(ii) there exist constants A and B such that

$$\|\xi\| \leqslant A + B\phi(\xi)$$

for all $\xi \in R^n$;

(iii) there exists a continuous non-negative real-valued function θ on R^n such that

$$\phi(\zeta_1 \xi_1, \zeta_2 \xi_2, \ldots, \zeta_n \xi_n) \leqslant \theta(\zeta) \cdot \phi(\xi)$$

for all $\zeta \in R^n$, $\xi \in R^n$;

(iv) $$(s+t)\,\phi\left(\frac{\xi+\zeta}{s+t}\right) \leqslant s\,\phi\left(\frac{\xi}{s}\right) + t\,\phi\left(\frac{\zeta}{t}\right),$$

for all $\xi, \zeta \in R^n$ and $s > 0$, $t > 0$.

It is shown in 2.7 that, if λ is a non-negative continuous function on R^n such that, for each bounded open set U the functional

$$L(f) = \int_U \lambda\,(\operatorname{grad} f)\,dx$$

is lower semi-continuous on the Lipschitz functions with respect to uniform convergence, then

$$(s+t)\,\lambda\left(\frac{\xi+\zeta}{s+t}\right) \leqslant s\,\lambda\left(\frac{\xi}{s}\right) + t\,\lambda\left(\frac{\zeta}{t}\right).$$

Since the function ϕ considered in [4] yielded a lower semi-continuous functional, it must satisfy (iv) above. The ϕ considered in [4] was postulated to satisfy (i) and (ii) above and one of the other conditions on it implies (iii) above. Thus the ϕ considered in the precent paper is at least as general as that of [4]. Example 1 shows that it is more general.

Example 1. Let $n = 2$ and

$$\phi(\xi_1, \xi_2) = \xi_1^2 + \xi_2^2 + |\xi_2|.$$

Then ϕ satisfies (i), (ii) and (iii). It can be shown to satisfy (iv) by means of the inequality

$$\frac{(\mu+t)^{p-1}}{(s+t)^{p-1}} \leqslant \frac{\mu^p}{s^{p-1}} + \frac{\nu^p}{t^{p-1}}, \tag{1}$$

where μ, ν, s, t are positive numbers and $p \geqslant 1$. This inequality can be derived from $(x^{-1}+\alpha)^{p-1}(x^{p-1}+\alpha) \geqslant (1+\alpha)^p$, in which $\alpha > 0$, $x > 0$, by the substitution $\alpha = \mu/\nu$, $x = s\nu/t\mu$. This latter inequality is easily verified by differentiation with respect to x.

Now suppose that ϕ satisfies 1 (iv) of [4]. Then, for each $\xi \in R^n$ with $\xi \neq 0$, there exists a constant $C_\xi > 0$ and such that $\phi(t\xi) = \eta(C_\xi t)$ for all $t > 0$.

But putting $\xi = (1, 0)$, one obtains

$$t^2 = \eta(C' t), \quad t > 0,$$

hence

$$\eta(s) = E s^2, \quad s > 0. \tag{2}$$

By putting $\xi = (0, 1)$, one obtains $t^2 + t = \eta(C'' t)$ for $t > 0$, hence

$$\eta(s) = F^2 s^2 + F s, \quad s > 0,$$

contradicting (2). Thus ϕ does not satisfy 1 (iv) of [4].

It can also be shown that ϕ does not satisfy 1 (iii) of [4], but this is not important, because [4] could have been written with 1 (iii) replaced by (iii) of the present paper.

The following is an example that was not given in [4], the reason being that I could not prove that the postulates were satisfied.

Example 2. Let

$$\phi(\xi) = \| \xi \|^p,$$

where $p \geq 1$. Then ϕ satisfies (i), (ii) and (iii) and by using the inequality (1), one can show that it satisfies (iv).

It is believed that the approximation theorem of the present paper can be applied to the theory of discontinuous parametric surfaces.

2. Preliminaries

Let U be an open set of R^n. As in [4], $\mathcal{L}(U)$ denotes the set of all locally summable real-valued functions on U and $\mathcal{K}(U)$ denotes the subset of $\mathcal{L}(U)$ consisting of all locally Lipschitz functions

For each $f \in \mathcal{K}(U)$ and each Borel subset B of U define

$$\Phi(f, B) = \int_B \phi(\operatorname{grad} f)\, dx.$$

$\mathcal{M}(U)$ denotes the subset of $\mathcal{L}(U)$ consisting of all f with the property that, for each compact subset C of U there exists a sequence $\{f^{(r)}\}$ of functions of $\mathcal{K}(U)$ with

$$\int_C |f^{(r)} - f| \, dx \to 0$$

as $r \to \infty$ and with $$\liminf_{r\to\infty} \Phi(f^{(r)}, C) < \infty.$$

My main aim in 2 is to define the Borel measures $\Phi(f, B)$ for functions f of $\mathcal{M}(U)$.

THEOREM 2.1. *If $f \in \mathcal{K}(U)$ and B is a bounded Borel set with $\bar{B} \subseteq U$, then*

$$\phi\left\{\frac{1}{m(B)}\int_B \text{grad } f\, dx\right\} \leq \frac{1}{m(B)}\int_B \phi(\text{grad } f)\, dx.$$

Proof. f is Lipschitz on B. Take $\varepsilon > 0$. Divide B into a finite number of mutually disjoint measurable subsets $B_1, B_2, \ldots, B_p$ such that, if

$$M_{ri} = \sup_{x \in B_r}\left|\frac{\partial f}{\partial x_i}\right|,$$

then $$\sum_{r=1}^{p} \phi(M_{r1}, \ldots, M_{rn})\, m(B_r) < \int_B \phi(\text{grad } f)\, dx + \varepsilon\, m(B). \tag{1}$$

Now $$\phi\left\{\frac{1}{m(B)}\int_B \text{grad } f\, dx\right\} \leq \phi\left\{\frac{\phi}{m(B)}\int_B \left|\frac{\partial f}{\partial x_1}\right| dx, \ldots, \frac{1}{m(B)}\int_B \left|\frac{\partial f}{\partial x_n}\right| dx\right\}$$

$$\leq \phi\left\{\frac{\sum_{r=1}^{p} M_{r1}\, m(B_r), \ldots, \sum_{r=1}^{p} M_{rn}\, m(B_r)}{\sum_{r=1}^{p} m(B_r)}\right\}$$

and by 1 (iv). $$\leq \frac{1}{m(B)}\sum_{r=1}^{p} m(B_r)\, \phi(M_{r1}, \ldots, M_{rn}),$$

which by (1), $$< \frac{1}{m(B)}\int_B \phi(\text{grad } f)\, dx + \varepsilon.$$

THEOREM 2.2. *If V is a bounded open set with $\bar{V} \subseteq U$, then $\Phi(f, V)$ is lower semi-continuous with respect to $\mathfrak{L}_1$ convergence; i.e. if $f, f^{(r)} \in \mathcal{K}(U)$ and*

$$\int_V |f^{(r)} - f|\, dx \to 0$$

as $r \to \infty$, then $$\liminf_{r\to\infty} \Phi(f^{(r)}, V) \geq \Phi(f, V).$$

Proof. For each $g \in \mathcal{K}(U)$ and each bounded Borel set with $\bar{B} \subseteq U$, define

$$\bar{\mu}_i(g, B) = \int_B \left|\frac{\partial g}{\partial x_i}\right| dx.$$

Then, because of 1 (iv),

$$\Phi(g, V) = \sup \sum_{j=1}^{p} \phi\left(\frac{\bar{\mu}_1(g, V_j)}{m(V_j)}, \ldots, \frac{\bar{\mu}_n(g, V_j)}{m(V_j)}\right) m(V_j),$$

where the supremum is taken over all finite collections $V_1, \ldots, V_p$ of mutually disjoint open subsets of V. It is known (see [2], Theorem 3, p. 214) that for each open subset W of V,

$$\liminf_{r\to\infty} \bar{\mu}_i(f^{(r)}, W) \geqslant \bar{\mu}_i(f, W). \tag{1}$$

Take $\varepsilon > 0$ and let $V_1, \ldots, V_p$ be mutually disjoint open subsets of V such that

$$\sum_{j=1}^{p} \phi\left(\frac{\bar{\mu}_1(f, V_j)}{m(V_j)}, \ldots, \frac{\bar{\mu}_n(f, V_j)}{m(V_j)}\right) m(V_j) > \Phi(f, V) - \varepsilon. \tag{2}$$

Then

$$\liminf \phi(f^{(r)}, V) \geqslant \liminf \sum_{j=1}^{p} \phi\left(\frac{\bar{\mu}_1(f^{(r)}, V_j)}{m(V_j)}, \ldots, \frac{\bar{\mu}_n(f^{(r)}, V_j)}{m(V_j)}\right) m(V_j),$$

which, by (1), $\geqslant \sum_{j=1}^{p} \phi\left(\frac{\bar{\mu}_1(f, V_j)}{m(V_j)}, \ldots, \frac{\bar{\mu}_n(f V_j)}{m(V_j)}\right) m(V_j)$

and by (2) $> \Phi(f, V) - \varepsilon.$

When $f \in \mathfrak{L}(U)$, A is a subset of U with $d(A, \sim U) > 0$ and r is a positive integer with $(\sqrt{n}) \cdot r^{-1} < d(A, \sim U)$, the symbol $\mathfrak{I}_r(f)$ will be used (as in [3] and [4]) to denote the well-known integral mean

$$\{\mathfrak{I}_r(f)\}(x) = r^n \int_0^{1/r} \ldots \int_0^{1/r} f(x+\xi)\, d\xi_1 \ldots d\xi_n,$$

which is defined for $x \in A$.

THEOREM 2.3. *If A and B are Borel subsets of U such that $A \subseteq B$ and $d(A, \sim B) > 0$, if r is a positive integer such that $(\sqrt{n}) \cdot r^{-1} < d(A, \sim B)$ and if $f \in \mathfrak{K}(U)$, then*

$$\Phi\{\mathfrak{I}_r(f), A\} \leqslant \Phi(f, B).$$

Proof. $\Phi\{\mathfrak{I}_r(f), A\} = \int_A \phi\{\text{grad } \mathfrak{I}_r(f)\}\, dx$

and since grad $\mathfrak{I}_r(f) = \mathfrak{I}_r$ (grad f),

$$\Phi\{\mathfrak{I}_r(f), A\} = \int_A \phi\{\mathfrak{I}_r(\text{grad } f)\}\, dx.$$

Hence, by 2.1,

$$\Phi\{\mathfrak{J}_r(f), A\} \leqslant \int_A r^n \left[\int_0^{1/r} \cdots \int_0^{1/r} \phi\{(\operatorname{grad} f)(x+\xi)\}\, d\xi\right] dx$$

$$= r^n \int_0^{1/r} \cdots \int_0^{1/r} \left[\int_A \phi\{(\operatorname{grad} f)(x+\xi)\}\, dx\right] d\xi$$

and making the substitution $y = x + \xi$

$$= r^n \int_0^{1/r} \cdots \int_0^{1/r} \left[\int_{A'_\xi} \phi\{(\operatorname{grad} f)(y)\}\, dy\right] d\xi$$

$$\leqslant r^n \int_0^{1/r} \cdots \int_0^{1/r} \left[\int_B \phi\{(\operatorname{grad} f)(y)\}\, dy\right] d\xi = \Phi(f, B).$$

COROLLARY 2.4. *If A and B are Borel subsets of U such that $A \subseteq B$ and $d(A, \sim B) > 0$, if r is a positive integer such that $(\sqrt{n}) \cdot r^{-1} < \frac{1}{2} d(A, \sim B)$ and if $f \in \mathcal{K}(U)$, then*

$$\Phi\{\mathfrak{J}_r^2(f), A\} \leqslant \Phi(f, B).$$

THEOREM 2.5. *If $f \in \mathfrak{M}(U)$, then there exists a subset F of R^1 such that $R^1 \sim F$ is countable and for every open interval I with $\bar{I} \subseteq U$ and the coordinates of its vertices all in F,*

$$\sup \limsup_{r\to\infty} \Phi\{\mathfrak{J}_r^2(f), C\} = \inf \liminf_{r\to\infty} \Phi\{g^{(r)}, W\},$$

where the supremum is taken over all compact subsets C of I and the infimum over all open sets W such that $\bar{I} \subseteq W \subseteq U$ and all sequences $\{g^{(r)}\}$ of functions of $\mathcal{K}(U)$ that converge $\mathfrak{L}_1$ to f on W.

Proof. For each bounded open set V with $\bar{V} \subseteq U$, define

$$\mu(V) = \sup \limsup_{r\to\infty} \Phi\{\mathfrak{J}_r^2(f), C\},$$

where the supremum is taken over all compact subsets C of V and define

$$\nu(V) = \inf \liminf_{r\to\infty} \Phi\{g^{(r)}, W\},$$

where the infimum is taken over all open sets W such that $\bar{V} \subseteq W \subseteq U$ and all sequences $\{g^{(r)}\}$ of functions of $\mathcal{K}(U)$ that converge $\mathfrak{L}_1$ to f on W.

I now prove

$$\mu(V) \leqslant \nu(V) \tag{1}$$

for every bounded open set V, with $\bar{V} \subseteq U$. To show this, let C be a compact subset of V, W be an open set such that $\bar{V} \subseteq W \subseteq U$ and $\{g^{(s)}\}$ be a sequence of functions of $\mathcal{K}(U)$ that converges $\mathfrak{L}_1$ to f on W. Let D be a compact subset of V with $C \subseteq \operatorname{Int}(D)$. Then, for all $r > 2\sqrt{n} \cdot \{d(D, \sim W)\}^{-1}$, one has by 2.4,

$$\Phi\{\mathfrak{J}_r^2(g^{(s)}), D\} \leq \Phi\{g^{(s)}, W\}. \tag{2}$$

But
$$\int_D |\mathfrak{J}_r^2(g^{(s)}) - \mathfrak{J}_r^2(f)|\, dx \leq \int_W |g^{(s)} - f|\, dx \to 0$$

as $s \to \infty$, hence by 2.2,

$$\liminf_{s\to\infty} \Phi\{\mathfrak{J}_r^2(g^{(s)}), \operatorname{Int}(D)\} \geq \Phi\{\mathfrak{J}_r^2(f), \operatorname{Int}(D)\},$$

so that by (2),
$$\Phi\{\mathfrak{J}_r^2(f), C\} \leq \liminf_{s\to\infty} \Phi\{g^{(s)}, W\}$$

for all $r > 2 \cdot \sqrt{n}\{d(D, \sim W)\}^{-1}$. Then

$$\limsup_{r\to\infty} \Phi\{\mathfrak{J}_r^2(f), C\} \leq \liminf_{s\to\infty} \Phi\{g^{(s)}, W\}$$

so that (1) is true.

Now let D' be a compact subset of U. For each rational number α and each $i = 1, \ldots, n$, let

$$L_{\alpha i} = \{x;\ x \in R^n \quad \text{and} \quad x_i < \alpha\}.$$

There exists a strictly increasing sequence $\{r_s\}$ of positive integers such that

$$\psi_i(\alpha) = \lim_{s\to\infty} \Phi\{\mathfrak{J}_{r_s}^2(f),\ L_{\alpha i} \cap \operatorname{Int}(D')\} \tag{3}$$

exists for all rational α and all i. For each i, ψ_i is increasing and bounded, hence it can be extended uniquely to an increasing bounded function on R^1 that is continuous on the left at each irrational point. Let E_i denote the set of points where ψ_i is continuous and put

$$E = \bigcap_{i=1}^{n} E_i.$$

Then $R^1 \sim E$ is countable. I now show that

$$\mu(I) = \nu(I), \tag{4}$$

for every open interval I such that $\bar{I} \subseteq \operatorname{Int}(D)$ and the coordinates of its vertices are all in E. To prove (4) let

$$I = (a_1, b_1) \times \ldots \times (a_n, b_n)$$

be such an interval. Take $\varepsilon > 0$, and let α_i', α_i'', β_i', β_i'', be such that $\alpha_i' < a_i \leqslant \alpha_i'' \leqslant \beta_i' < b_i < \beta_i''$,

$$\psi_i(\alpha_i'') - \psi_i(\alpha_i') + \psi_i(\beta_i'') - \psi_i(\beta_i') < n^{-1} \cdot \varepsilon$$

and the interval
$$K = (\alpha_1', \beta_1'') \times \ldots \times (\alpha_n', \beta_n'')$$

is contained in Int (D). Put

$$J = (\alpha_1'', \beta_1') \times \ldots \times (\alpha_n'', \beta_n').$$

Let C' be a compact subset of I containing J. Then

$$\mu(I) \geqslant \limsup_{r \to \infty} \Phi\{\mathfrak{J}_r^2(f), C'\} \geqslant \limsup_{s \to \infty} \Phi\{\mathfrak{J}_{r_s}^2(f), J\};$$

but, since $\mathfrak{J}_r^2(f)$ converges $\mathfrak{L}_1$ to f on K,

$$\nu(I) \leqslant \liminf_{s \to \infty} \Phi\{\mathfrak{J}_{r_s}^2(f), K\};$$

hence
$$\nu(I) - \mu(I) \leqslant \limsup_{s \to \infty} \Phi\{\mathfrak{J}_{r_s}^2(f), K \sim J\}$$

$$\leqslant \lim_{s \to \infty} \sum_{i=1}^{n} [\Phi\{\mathfrak{J}_{r_s}^2(f), L_{\alpha_i'' i} \sim L_{\alpha_i' i}) \cap \text{Int}(D')\} + \Phi\{\mathfrak{J}_{r_s}^2(f), L_{\beta_i'' i} \sim L_{\beta_i' i}) \cap \text{Int}(D')\}]$$

$$= \sum_{i=1}^{n} [\psi_i(\alpha_i'') - \psi_i(\alpha_i') + \psi_i(\beta_i'') - \psi_i(\beta_i')] < \varepsilon.$$

Thus (4) is true.

Now let $\{D_t\}$ be an increasing sequence of compact subsets of U such that

$$\lim_{t \to \infty} \text{Int}(D_t) = U.$$

By (4), there exists for each t a subset F_t of R^1 such that $R^1 \sim F_t$ is countable and $\mu(I) = \nu(I)$ for every open interval I with $\bar{I} \subseteq \text{Int}(D_t)$ and the coordinates of its vertices all in F_t. Put

$$F = \bigcap_{t=1}^{\infty} F_t.$$

Then $R^1 \sim F$ is countable. Let I be an open interval with $\bar{I} \subseteq U$ and the coordinates of its vertices all in F. Then, for some t, $\bar{I} \subseteq \text{Int}(D_t)$, hence $\mu(I) = \nu(I)$.

THEOREM 2.6. *Let $f \in \mathfrak{M}(U)$. There exists a unique Borel measure η on U such that:*

(i) *for each bounded open set V with $\bar{V} \subseteq U$ and every sequence $\{f^{(r)}\}$ of functions of $\mathfrak{K}(U)$ with*

$$\int_V |f^{(r)} - f| \, dx \to 0$$

as $r \to \infty$, it is true that

$$\liminf_{r \to \infty} \Phi(f^{(r)}, V) \geq \eta(V);$$

(ii) *for each compact subset C of U,*

$$\limsup_{r \to \infty} \Phi\{\mathfrak{J}_r^2(f), C\} \leq \eta(C).$$

The measure η is regular and when $f \in \mathfrak{K}(U)$,

$$\eta(B) = \Phi(f, B)$$

for every Borel subset B of U.

Hence, we define for each $f \in \mathfrak{M}(U)$ and each Borel subset B of U

$$\Phi(f, B) = \eta(B).$$

Proof. Let μ, ν be as in the proof of 2.5. Let F be a subset of R^1 such that $R^1 \sim F$ is countable and if $\mathfrak{J}$ denotes the collection of all open intervals I with $\bar{I} \subseteq U$ and the coordinates of the vertices of I all in F, then $\mu(I) = \nu(I)$ for every $I \in \mathfrak{J}$.

If $I \in \mathfrak{J}$ and $I_1, I_2, \ldots$ are members of $\mathfrak{J}$ countable in number and such that $I \subseteq \bigcup_i I_i$, then

$$\mu(I) \leq \sum_i \mu(I_i); \tag{1}$$

because, given $\varepsilon > 0$, we can choose a compact subset C of I with

$$\limsup_{r \to \infty} \Phi\{\mathfrak{J}_r^2(f), C\} > \mu(I) - \varepsilon;$$

now C can be covered by a finite collection $\mathfrak{J}'$ of the I_i's; to each $J \in \mathfrak{J}'$, we can assign a compact subset C_J of J such that $C \subseteq \bigcup_{J \in \mathfrak{J}'} C_J$. Then

$$\Phi\{\mathfrak{J}_r^2(f), C\} \leq \sum_{J \in \mathfrak{J}'} \Phi\{\mathfrak{J}_r^2(f), C_J\},$$

hence
$$\limsup_{r \to \infty} \Phi(\mathfrak{J}_r^2(f), C\} \leq \sum_{J \in \mathfrak{J}'} \limsup_{r \to \infty} \Phi\{\mathfrak{J}_r^2(f), C_J\}.$$

so that
$$\mu(I) - \varepsilon < \sum_{J \in \mathfrak{J}'} \mu(J) \leq \sum_i \mu(I_i);$$

thus (1) is true.

If $I \in \mathfrak{J}$ and $\delta > 0$, $\varepsilon > 0$, then there exist intervals $I_1, \ldots, I_p$ of $\mathfrak{J}$, with diameters less than δ, with $I \subseteq \bigcup_{i=1}^p I_i$ and with

$$\mu(I) > \sum_{i=1}^{p} \mu(I_i) - \varepsilon; \tag{2}$$

because, we can certainly choose mutually disjoint open intervals $J_1, \ldots, J_p \in \mathfrak{J}$ with diameters less than $\frac{1}{2}\delta$ and with $\bar{I} = \bigcup_{j=1}^p \bar{J}_i$; let I_i, K_i be intervals of I such that $\bar{K}_i \subseteq J_i$, $\bar{J}_i \subseteq I_i$,

$$\nu(I_i) - \mu(K_i) < \frac{\varepsilon}{2p}$$

and each I_i has diameter less than δ; now let W be an open set containing $\bar{I}$ and $g^{(r)}$ a sequence of functions of $\mathcal{K}(U)$ converging $\mathfrak{L}_1$ to f on W and such that

$$\liminf_{r \to \infty} \Phi\{g^{(r)}, W\} < \nu(I) + \tfrac{1}{2}\varepsilon = \mu(I) + \tfrac{1}{2}\varepsilon;$$

then
$$\mu(I) + \tfrac{1}{2}\varepsilon > \liminf_{r \to \infty} \sum_{i=1}^{p} \Phi(g^{(r)}, J_i)$$

$$\geqslant \sum_{i=1}^{p} \liminf_{r \to \infty} \Phi(g^{(r)}, J_i) \geqslant \sum_{i=1}^{p} \nu(K_i)$$

$$= \sum_{i=1}^{p} \mu(K_i) > \sum_{i=1}^{p} \nu(I_i) - \tfrac{1}{2}\varepsilon = \sum_{i=1}^{p} \mu(I_i) - \tfrac{1}{2}\varepsilon;$$

thus (2) is true.

It follows immediately from (1) and (2) that there exists a unique Borel measure η on U with

$$\eta(I) = \mu(I) = \nu(I)$$

for every $I \in \mathfrak{J}$. It follows from (1) that η is regular.

Let V be a bounded open set with $\bar{V} \subseteq U$ and let $\{f^{(r)}\}$ be a sequence of functions of $\mathcal{K}(U)$ such that

$$\int_V |f^{(r)} - f| \, dx \to 0$$

as $r \to \infty$. Take $\varepsilon > 0$. There exists a finite number $I_1, \ldots, I_p$ of mutually disjoint open intervals of $\mathfrak{J}$ with each $I_j \subseteq V$ and with

$$\sum_{i=1}^{p} \eta(I_j) > \eta(V) - \varepsilon.$$

Now
$$\liminf_{r\to\infty} \Phi\{f^{(r)}, V\} \geqslant \liminf_{r\to\infty} \sum_{j=1}^{p} \Phi\{f^{(r)}, I_j\}$$
$$\geqslant \sum_{j=1}^{p} \liminf_{r\to\infty} \Phi\{f^{(r)}, I_j\} \geqslant \sum_{j=1}^{p} \eta(J_j)$$

(where each J_j is an interval of $\mathfrak{J}$ with $\bar{J}_j \subseteq I_j$). Hence

$$\liminf_{r\to\infty} \Phi\{f^{(r)}, V\} \geqslant \sum_{i=1}^{p} \eta(I_j) > \eta(V) - \varepsilon.$$

Thus (i) holds.

Now take a compact subset C of U. Let $\varepsilon > 0$. There exists a finite number $I_1, \ldots, I_p$ of open cubes of I such that $C \subseteq \bigcup_{j=1}^{p} I_j$ and

$$\sum_{j=1}^{p} \mu(I_j) < \eta(C) + \varepsilon.$$

One can now choose compact sets $C_1, \ldots, C_p$ such that $C_j \subseteq I_j$ for each j and $C \subseteq \bigcup_{j=1}^{p} C_j$. Then

$$\limsup_{r\to\infty} \Phi\{\mathfrak{J}_r^2(f), C\} \leqslant \limsup_{r\to\infty} \sum_{j=1}^{p} \Phi\{\mathfrak{J}_r^2(f), C_j\}$$
$$\leqslant \sum_{j=1}^{p} \limsup_{r\to\infty} \Phi\{\mathfrak{J}_r^2(f), C_j\} \leqslant \sum_{j=1}^{p} \mu(I_j) < \eta(C) + \varepsilon.$$

Thus (ii) holds.

We have prowed the existence of the measure η. If there existed a second measure η_1, it follows immediately from (i) and (ii) that $\eta_1(I) = \mu(I) = \nu(I)$ for every $I \in \mathfrak{J}$, hence η_1 and η would be identical.

When $f \in \mathcal{K}(U)$ and $I \in \mathfrak{J}$ we obtain from (i), by putting $f^{(r)} = f$,

$$\eta(I) \leqslant \Phi(f, I).$$

By (ii)
$$\eta(\bar{I}) \geqslant \limsup_{r\to\infty} \Phi\{\mathfrak{J}_r^2(f), \bar{I}\}$$

and, since $\mathfrak{J}_r^2(f)$ converges $\mathfrak{L}_1$ to f on I it follows from 2.2 that

$$\eta(\bar{I}) \geqslant \Phi(f, I).$$

But since η is regular, $\eta(\bar{I}) = \nu(I) = \eta(I)$. Thus $\eta(I) = \Phi(f, I)$ for all $I \in \mathfrak{J}$. Hence

$$\eta(B) = \Phi(f, B)$$

for every Borel subset B of U.

The following theorem, mentioned in the introduction, shows that 1 (iv) is a necessary condition for ϕ to yield a functional that is lower-semicontinuous with respect to $\mathfrak{L}_1$ convergence on the open sets.

THEOREM 2.7. *Let λ be a non-negative continuous function on R^n such that for every bounded open set U of R^n, the functional*

$$L(f)=\int_U \lambda(\operatorname{grad} f)\,dx,$$

is lower semi-continuous on the Lipschitz functions with respect to uniform convergence. Then

$$(s+t)\,\lambda\left(\frac{\xi+\zeta}{s+t}\right)\leqslant s\,\lambda\left(\frac{\xi}{s}\right)+t\,\lambda\left(\frac{\xi}{t}\right)$$

for all $\xi,\zeta\in R^n$ and $s>0$, $t>0$.

Proof. Let $\xi,\zeta\in R^n$ and $s>0$, $t>0$. If one defines $\lambda_1(x)=\lambda(x\cdot A)$, where A is an orthogonal matrix, it is easily verified that λ_1 yields a lower semi-continuous functional. Hence it can be assumed that $\xi/s-\zeta/t=\omega$ is of the form $(\varrho,0,\dots,0)$, where $\varrho\geqslant 0$.

Define
$$f(x)=\frac{\xi+\zeta}{s+t}\cdot x=\frac{\xi}{s}\cdot x-\frac{t x_1\varrho}{s+t}=\frac{\zeta}{t}\cdot x+\frac{s x_1 \varrho}{s+t}.$$

Put
$$U=(0,s+t)\times(0,1)\times(0,1)\times\dots\times(0,1).$$

Define
$$\begin{aligned} f^{(r)}(x)&=\frac{\xi\cdot x}{s}-\frac{(k-1)t\varrho}{r} \quad\text{if}\quad \frac{k-1}{r}(s+t)\leqslant x_1\leqslant\frac{k-1}{r}(s+t)+\frac{s}{r},\\ &=\frac{\zeta\cdot x}{t}+\frac{ks\varrho}{r}\quad\text{if}\quad \frac{k-1}{r}(s+t)+\frac{s}{r}\leqslant x_1\leqslant\frac{k}{r}(s+t).\end{aligned}$$

Then, each $f^{(r)}$ is Lipschitz and $f^{(r)}\to f$ uniformly on U.

But
$$\operatorname{grad} f=\frac{\xi+\zeta}{s+t}.$$

Now
$$\begin{aligned}\operatorname{grad} f^{(r)}&=\frac{\xi}{s}\quad\text{if}\quad\frac{k-1}{r}(s+t)\leqslant x_1\leqslant\frac{k-1}{r}(s+t)+\frac{s}{r},\\ &=\frac{\zeta}{t}\quad\text{if}\quad\frac{k-1}{r}(s+t)+\frac{s}{r}\leqslant x_1\leqslant\frac{k}{r}(s+t),\end{aligned}$$

hence
$$L(f^{(r)}) = s\lambda\left(\frac{\xi}{s}\right) + t\lambda\left(\frac{\zeta}{t}\right).$$

But, by hypothesis
$$\liminf_{r\to\infty} L(f^{(r)}) \geqslant L(f),$$

so that
$$s\lambda\left(\frac{\xi}{s}\right) + t\lambda\left(\frac{\zeta}{t}\right) \geqslant (s+t)\,\lambda\left(\frac{\xi+\zeta}{s+t}\right).$$

3. Some approximation theorems

THEOREM 3.1. *Let C be a compact subset of an open set U and let $f \in \mathcal{M}(U)$. Let $\varepsilon > 0$. There exists a Lipschitz function g on R^n with compact support and such that the set*
$$\{x;\ x \in C \ \text{ and } \ f(x) \neq g(x)\}$$
has measure less than ε.

Proof. Let D be a compact subset of U such that $C \subseteq \operatorname{Int}(D)$. There exists a sequence $\{f^{(r)}\}$ of functions of $\mathcal{K}(U)$ with
$$\int_D |f^{(r)} - f|\, dx \to 0$$
as $r \to \infty$ and with
$$\liminf_{r\to\infty} \Phi(f^{(r)}, D) < \infty,$$
hence by 1 (ii),
$$\liminf_{r\to\infty} \int_D [1 + \|\operatorname{grad} f^{(r)}\|^2]^{\frac{1}{2}}\, dx < \infty. \tag{1}$$

Let $J_1, J_2, \ldots, J_p$ be a finite number of mutually non-overlapping closed cubes such that $C \subseteq \bigcup_{j=1}^{p} J_j \subseteq D$. By (1) and [3] 4.3, each of the functions,
$$f_j(x) = f(x) \quad \text{if} \quad x \in J,$$
$$= 0 \quad \text{if} \quad x \notin J,$$
belongs to the class $\mathcal{B}$ of [3]. Hence, the function
$$f^* = \sum_{j=1}^{p} f_j$$
belongs to $\mathcal{B}$, so that by [3] 3.1, there exists a Lipschitz function g on R^n with compact support and agreeing with f^* except on a set of measure less than ε. Since f^* agrees with f almost everywhere on C, g is the required function.

LEMMA 3.2. *Let A be a bounded measurable subset of R^1 and U an open set. Let $\varepsilon > 0$. There exist a finite number $I_1, I_2, \ldots, I_k$ of non-overlapping closed intervals such that:*

(i) *each I_j is contained in U, has its endpoints in A and has length less than ε;*

(ii) *$(A \cap U) \sim \bigcup_{j=1}^{k} I_j$ has measure less than ε.*

Proof. Let A' be the set of all those points x of $A \cap U$ such that every open interval of R^1 containing x contains points of $A \cap U$ to the left of x and points of $A \cap U$ to the right of x. Then $(A \cap U) \sim A'$ is countable, hence has zero measure. Let A_1 be a compact subset of A' such that $A' \sim A_1$ has measure less than ε. Let $\mathfrak{J}$ be the collection consisting of all closed intervals that are contained in U, have both endpoints in A and have length less than ε. The interiors of the intervals of $\mathfrak{J}$ cover A_1, hence there is a finite subcollection $\{J_1, \ldots, J_p\}$ of $\mathfrak{J}$ with

$$A_1 \subseteq \bigcup_{r=1}^{p} \operatorname{Int}(J_r).$$

Now let $I_1, \ldots, I_k$ be mutually non-overlapping closed intervals with

$$\bigcup_{j=1}^{k} I_j = \bigcup_{r=1}^{p} J_r,$$

with the endpoints of every I_j occurring among the endpoints of the J_r's and with each I_j contained in a J_r. The I_j's have the required properties.

THEOREM 3.3. *Let f be Lipschitz on R^n, A be a measurable subset of R^n and $\{f_r\}$ be a sequence of Lipschitz functions on R^n such that $\lim_{r\to\infty} f_r(x) = f(x)$ for almost all $x \in A$. Let $\eta > 0$ and U be an open subset of R^n. Put*

$$A_r = \{x;\ x \in R^n \text{ and } |f_r(x) - f(x)| \leqslant \eta\}.$$

Then

$$\text{(a)} \qquad \liminf_{r\to\infty} \int_{A_r \cap U} \left|\frac{\partial f_r}{\partial x_i}\right| dx \geqslant \int_{A \cap U} \left|\frac{\partial f}{\partial x_i}\right| dx, \quad \textit{and}$$

$$\text{(b)} \qquad \liminf_{r\to\infty} \Phi\{f_r, A_r \cap U\} \geqslant \Phi(f, A \cap U).$$

Proof. (a) In proving (a) we can assume A and U are bounded, because if not we could approximate the integral on the right-hand side of (a) with an integral over $A_1 \cap U_1$, where A_1 and U_1 are bounded.

(i) When $n=1$. Suppose (a) is not true. Choose a subsequence $\{f_{r_s}\}$ such that

$$\lim_{s\to\infty}\int_{A_{r_s}\cap U}\left|\frac{df_{r_s}}{dx}\right|dx=\int_{A\cap U}\left|\frac{df}{dx}\right|dx-\alpha,\quad \alpha>0$$

and

$$\lim_{s\to\infty} f_{r_s}(x)=f(x)$$

for all x in a subset B of A with $m(A\sim B)=0$.

Let $\delta>0$ be such that for any finite set $[a_1,b_1],\ldots,[a_p,b_p]$ of mutually non-overlapping closed intervals, contained in U and with lengths $<\delta$, one has

$$\sum_{j=1}^{p}|f(b_j)-f(a_j)|>\sum_{j=1}^{p}\int_{a_j}^{b_j}\left|\frac{df}{dx}\right|dx-\tfrac{1}{2}\alpha.$$

Let $K>0$ be a Lipschitz constant for f.

By 3.2, there exist non-overlapping closed intervals $I_1,\ldots,I_k$ such that:

(A) each I_j is contained in U, has its endpoints in B and has length less than $\min(\delta,\eta/2K)$;

(B)
$$\sum_{j=1}^{k}\int_{I_j}\left|\frac{df}{dx}\right|dx>\int_{A\cap U}\left|\frac{df}{dx}\right|dx-\tfrac{1}{2}\alpha.$$

Let $I_j=[\alpha_j,\beta_j]$. Then for each j and each s,

$$\int_{I_j\cap A_{r_s}}\left|\frac{df_{r_s}}{dx}\right|dx\geqslant|f(\beta_j)-f(\alpha_j)|-|f(\alpha_j)-f_{r_s}(\alpha_j)|-|f(\beta_j)-f_{r_s}(\beta_j)|;\tag{1}$$

because, if $I_j\subseteq A_{r_s}$, one has

$$\begin{aligned}\int_{I_j\cap A_{r_s}}\left|\frac{df_{r_s}}{dx}\right|dx&\geqslant|f_{r_s}(\beta_j)-f_{r_s}(\alpha_j)|\\&=|\{f(\beta_j)-f(\alpha_j)\}+\{f(\alpha_j)+f_{r_s}(\alpha_j)\}-\{f(\beta_j)-f_{r_s}(\beta_j)\}|\\&\geqslant|f(\beta_j)-f(\alpha_j)|-|f(\alpha_j)-f_{r_s}(\alpha_j)|-|f(\beta_j)-f_{r_s}(\beta_j)|,\end{aligned}$$

and, if I_j is not contained in A_{r_s}, one can proceed as follows. Since $|f(\beta_j)-f(\alpha_j)|<\tfrac{1}{2}\eta$, it can be assumed that each of $|f(\alpha_j)-f_{r_s}(\alpha_j)|$, $|f(\beta_j)-f_{r_s}(\beta_j)|$ is less than η, hence $\alpha_j,\beta_j\in A_{r_s}$. Let $\xi,\zeta\in I_j$ be such that $\xi<\zeta$, $[\alpha_j,\xi]\subseteq A_{r_s}$, $|f(\xi)-f_{r_s}(\xi)|=\eta$, $[\zeta,\beta_j]\subseteq A_{r_s}$ and $|f(\zeta)-f_{r_s}(\zeta)|=\eta$. Then

$$\int_{I_j\cap A_{r_s}}\left|\frac{df_{r_s}}{dx}\right|dx\geqslant|f_{r_s}(\xi)-f_{r_s}(\alpha_j)|+|f_{r_s}(\beta_j)-f_{r_s}(\zeta)|.$$

But
$$|f_{r_s}(\xi)-f_{r_s}(\alpha_j)|=|\{f_{r_s}(\xi)-f(\xi)\}+\{f(\alpha_j)-f_{r_s}(\alpha_j)\}+\{f(\xi)-f(\alpha_j)\}|$$
$$\geqslant|f_{r_s}(\xi)-f(\xi)|-|f(\alpha_j)-f_{r_s}(\alpha_j)|-|f(\xi)-f(\alpha_j)|$$
$$>\tfrac{1}{2}\eta-|f(\alpha_j)-f_{r_s}(\alpha_j)|$$
$$>\tfrac{1}{2}|f(\beta_j)-f(\alpha_j)|-|f(\alpha_j)-f_{r_s}(\alpha_j)|$$
and similarly
$$|f_{r_s}(\beta_j)-f_{r_s}(\zeta)|>\tfrac{1}{2}|f(\beta_j)-f(\alpha_j)|-|f(\beta_j)-f_{r_s}(\beta_j)|,$$

so that (1) is also true in this case.

Now it follows from (1) that

$$\int_{A_{r_s}\cap U}\left|\frac{df_{r_s}}{dx}\right|dx\geqslant\sum_{j=1}^{k}|f(\beta_j)-f(\alpha_j)|-\sum_{j=1}^{k}|f(\alpha_j)-f_{r_s}(\alpha_j)|-\sum_{j=1}^{k}|f(\beta_j)-f_{r_s}(\beta_j)|$$

and since each $\alpha_j, \beta_j, \in B$,

$$\liminf_{s\to\infty}\int_{A_{r_s}\cap U}\left|\frac{df_{r_s}}{dx}\right|dx\geqslant\sum_{j=1}^{k}|f(\beta_j)-f(\alpha_j)|>\sum_{j=1}^{k}\int_{I_j}\left|\frac{df}{dx}\right|dx-\tfrac{1}{2}\alpha>\int_{A\cap U}\left|\frac{df}{dx}\right|dx-\alpha.$$

A contradiction.

(ii) When $n>1$. One can assume $i=n$. For each $y\in R^{n-1}$ and each subset W of R^n, let
$$W(y)=\{t;\ t\in R^1 \text{ and } (y,t)\in W\}.$$
Then
$$\liminf_{r\to\infty}\int_{A_r\cap U}\left|\frac{\partial f_r}{\partial x_n}\right|dx=\liminf_{r\to\infty}\int_{R^{n-1}}\left[\int_{A_r(y)\cap U(y)}\left|\frac{\partial f_r}{\partial x_n}\right|dx_n\right]dx_1\dots dx_{n-1}$$

{where $y=(x_1,\dots,x_{n-1})$} and by Fatou's lemma

$$\geqslant\int_{R^{n-1}}\left[\liminf_{r\to\infty}\int_{A_r(y)\cap U(y)}\left|\frac{\partial f_r}{\partial x_n}\right|dx_n\right]dx_1\dots dx_{n-1},$$

and by (i)
$$\geqslant\int_{R^{n-1}}\left[\int_{A(y)\cap U(y)}\left|\frac{\partial f}{\partial x_n}\right|dx_n\right]dx_1\dots dx_{n-1}=\int_{A\cap U}\left|\frac{\partial f}{\partial x_n}\right|dx.$$

(b) We can again assume that A and U are bounded.

For each Lipschitz function g on R^n and each bounded measurable subset B of R^n, put
$$\bar{\mu}(g,B)=\left(\int_B\left|\frac{\partial g}{\partial x_1}\right|dx,\dots,\int_B\left|\frac{\partial g}{\partial x_n}\right|dx\right).$$

Since ϕ is continuous and satisfies 1 (iv), it is easily verified that

$$\Phi(g, B) = \sup \sum_{j=1}^{p} \phi\left\{\frac{\bar{\mu}(g, B \cap U_j)}{m(B \cap U_j)}\right\} m(B \cap U_j),$$

where the supremum is taken over all finite collections $U_1, \ldots, U_p$ of mutually disjoint open subsets of R^n.

Take $\varepsilon > 0$ and let $V_1, \ldots, V_q$ be mutually disjoint open subsets of U such that

$$\sum_{j=1}^{q} \phi\left\{\frac{\bar{\mu}(f, A \cap V_j)}{m(A \cap V_j)}\right\} m(A \cap V_j) > \Phi(f, A \cap U) - \tfrac{1}{2}\varepsilon.$$

For each j, let W_j be an open subset of V_j such that $A \cap V_j \subseteq W_j$ and $m(W_j)$ is sufficiently close to $m(A \cap V_j)$ that

$$\phi\left\{\frac{\bar{\mu}(f, A \cap V_j)}{m(W_j)}\right\} > \phi\left\{\frac{\bar{\mu}(f, A \cap V_j)}{m(A \cap V_j)}\right\} - \frac{\varepsilon}{2 \cdot q \cdot m(A \cap V_j)}.$$

Then $A \cap V_j = A \cap W_j$. By (a)

$$\liminf_{r \to \infty} \int_{A_r \cap W_j} \left|\frac{\partial f_r}{\partial x_i}\right| dx \geqslant \int_{A \cap W_j} \left|\frac{df}{\partial x_i}\right| dx,$$

hence by 1 (i),

$$\liminf_{r \to \infty} \phi\left\{\frac{\bar{\mu}(f_r, A_r \cap W_j)}{m(A_r \cap W_j)}\right\} \geqslant \phi\left\{\frac{\bar{\mu}(f, A \cap W_j)}{m(W_j)}\right\}$$

$$> \phi\left\{\frac{\bar{\mu}(f, A \cap V_j)}{m(A \cap V_j)}\right\} - \frac{\varepsilon}{2 \cdot q \cdot m(A \cap V_j)}.$$

Therefore

$$\liminf_{r \to \infty} \Phi(f_r, A_r \cap U) \geqslant \liminf_{r \to \infty} \sum_{j=1}^{q} \phi\left\{\frac{\bar{\mu}(f_r, A_r \cap W_j)}{m(A_r \cap W_j)}\right\} m(A_r \cap W_j)$$

$$\geqslant \sum_{j=1}^{q} \phi\left\{\frac{\bar{\mu}(f, A \cap V_j)}{m(A \cap V_j)}\right\} m(A \cap V_j) - \tfrac{1}{2}\varepsilon > \Phi(f, A \cap U) - \varepsilon.$$

THEOREM 3.4. *Let C be a compact subset of an open set U and let $f \in \mathfrak{M}(U)$. Let $\varepsilon > 0$. There exists a Lipschitz function f_0 on U such that:*

(i) *the set $\{x;\ x \in C$ and $f(x) \neq f_0(x)\}$ has measure less than ε;*

(ii) $\Phi(f_0, C) < \Phi(f, C) + \varepsilon$.

Proof. Since there exists a Borel measurable function that agrees with f almost everywhere, we can assume that f is Borel measurable. Let C_1 be a compact set contained in U and with C in its interior. Define

$$f_1(x) = f(x) \quad \text{if} \quad x \in C_1,$$
$$= 0 \quad \text{if} \quad x \notin C_1.$$

Then f_1 is Borel measurable and summable on R^n and

$$\Phi(f, C) = \Phi(f, C). \tag{1}$$

By 3.1, there exists a Lipschitz function g on R^n with compact support and such that, if

$$E = \{x;\ x \in C \ \text{ and } \ f_1(x) \neq g(x)\},$$

then
$$m(E) < \tfrac{1}{2}\varepsilon. \tag{2}$$

Let δ be such that $0 < \delta < \frac{1}{2}\varepsilon$ and for every Borel subset G of R^n with $m(G) < \delta$, one has

$$\Phi(g, G) < \tfrac{1}{4}\varepsilon. \tag{3}$$

Choose $\eta > 0$ and such that, if

$$B = \{x;\ x \in R^n \ \text{ and } \ |f_1(x) - g(x)| \leqslant \eta\} \ \text{ and } \ A = \{x;\ x \in R^n \ \text{ and } \ f_1(x) = g(x)\},$$

then
$$m(B \sim A) < \tfrac{1}{2}\delta, \tag{4}$$

and
$$m\{x;\ x \in R^n \ \text{ and } \ |f_1(x) - g(x)| = \eta\} = 0.$$

Let $\{s_r\}$ be an increasing sequence of positive integers such that

$$f^{(r)} = \mathfrak{J}^2_{s_r}(f_1)$$

approaches f_1 almost everywhere. Put

$$A_r = \{x;\ x \in R^n \ \text{ and } \ |f^{(r)}(x) - g(x)| \leqslant \eta\}.$$

Denoting characteristic functions by χ, we have

$$\lim_{r \to \infty} \chi_{A_r}(x) = \chi_B(x)$$

for almost all x, hence by bounded convergence,

$$\lim_{r \to \infty} \int_{R^n} |\chi_{A_r}(x) - \chi_B(x)|\, dx = 0.$$

Thus there exists an r_1 such that $m\{(A_r \sim B) \cup (B \sim A_r)\} < \frac{1}{2}\delta$ for all $r \geqslant r_1$. Hence by (4),

$$m(A_r \sim A) < \delta \tag{5}$$

and $$m(A \sim A_r) < \delta \tag{6}$$

for all $r \geqslant r_1$.

Let V be a bounded open set, containing C (consisting of the union of a finite number of open intervals) and such that

$$\Phi(f_1, \overline{V}) < \Phi(f_1, C) + \tfrac{1}{4}\varepsilon. \tag{7}$$

Let r_2 be such that $$\Phi\{f^{(r)}, \overline{V}\} < \Phi(f_1, \overline{V}) + \tfrac{1}{4}\varepsilon \tag{8}$$

for all $r \geqslant r_2$.

By 3.3 there exists an r_s such that

$$\Phi\{f^{(r)}, V \cap A_r\} > \Phi(g, V \cap A) - \tfrac{1}{4}\varepsilon \tag{9}$$

for all $r \geqslant r_3$.

Let $$r' = \max(r_1, r_2, r_3) \text{ and put } h = f^{(r')}.$$

Define
$$\begin{aligned} f_0(x) &= g(x) \quad \text{if} \quad x \in A_{r'}, \\ &= h(x) - \eta \quad \text{if} \quad h(x) > g(x) + \eta, \\ &= h(x) + \eta \quad \text{if} \quad h(x) < g(x) - \eta. \end{aligned}$$

Then f_0 is Lipschitz. To show that f_0 satisfies condition (i), let $x \in C$ and $f(x) \neq f_0(x)$. Then $f_1(x) \neq f_0(x)$. If $x \in A_{r'}$, we have $f_1(x) \neq g(x)$, so that $x \in E$. If $x \notin A_{r'}$ and $x \notin E$, then $x \in A \sim A_{r'}$. Thus we have shown that

$$\{x;\ x \in C \text{ and } f(x) \neq f_0(x)\} \subseteq E \cup (A \sim A_{r'})$$

and therefore, by (2) and (6), has measure less than ε. To verify that (ii) is satisfied, we observe that

$$\begin{aligned} \Phi(f_0, C) &= \Phi(f_0, C \sim A_{r'}) + \Phi(f_0, C \cap A_{r'}) \\ &= \Phi(h, C \sim A_{r'}) + \Phi(g, C \cap A_{r'}) \\ &\leqslant \Phi(h, C \sim A_{r'}) + \Phi(g, C \cap A) + \Phi(g, A_{r'} \sim A) \end{aligned}$$

and by (3) and (5)

$$< \Phi(h, C \sim A_{r'}) + \Phi(g, V \cap A) + \tfrac{1}{4}\varepsilon,$$

which by (9),

$$< \Phi(h, C \sim A_{r'}) + \Phi(h, V \cap A_{r'}) + \tfrac{1}{2}\varepsilon$$

$$< \Phi(h, V) + \tfrac{1}{2}\varepsilon$$

and by (7), (8) and (1),

$$< \Phi(f, C) + \varepsilon.$$

4. Approximation of functions on Q

In this section, the approximation theorem described in the introduction is proved (4.3).

The functional Ψ was defined in the introduction for Lipschitz functions f on the unit cube Q by

$$\Psi(f) = \int_Q \phi(\text{grad } f)\, dx = \Phi(f, Q) = \Phi\{f, \text{Int}(Q)\}.$$

THEOREM 4.1. *Ψ is lower semi-continuous on the Lipschitz functions with respect to $\mathfrak{L}_1$ convergence.*

Proof. Let $g, g^{(r)}$ be Lipschitz functions on Q such that

$$\int_Q |g^{(r)} - g|\, dx \to 0$$

as $r \to \infty$. Take $\varepsilon > 0$ and let Q_1 be an open cube with $\overline{Q_1} \subseteq \text{Int}(Q)$ and

$$\Phi(g, Q_1) > \Phi(g, Q) - \varepsilon = \Psi(g) - \varepsilon. \tag{1}$$

Then
$$\liminf_{r\to\infty} \Psi(g^{(r)}) \geqslant \liminf_{r\to\infty} \Phi(g^{(r)}, Q_1)$$

and by 2.2 and (1),
$$\geqslant \Psi(g) - \varepsilon.$$

As a result of 4.1, Ψ extends to a lower semi-continuous functional on the set of functions summable on Q. Thus for a function f summable on Q,

$$\Psi(f) = \inf\,[\liminf_{r\to\infty} \Psi(f^{(r)})],$$

where the infimum is taken over all sequences $\{f^{(r)}\}$ of Lipschitz functions that converge $\mathfrak{L}_1$ to f.

If f is summable on Q and $\Psi(f)$ is finite, it follows immediately that $f \in \mathfrak{M}\{\text{Int}(Q)\}$. Also, for every open cube Q_1 with $\overline{Q_1} \subseteq \text{Int}(Q)$, it follows from 2.6 that $\Phi(f, Q_1) \leqslant \Psi(f)$, hence for every compact subset C of $\text{Int}(Q)$, $\Phi(f, C) \leqslant \Psi(f)$. Therefore

$$\Phi\{f, \text{Int}(Q)\} \leqslant \Psi(f).$$

THEOREM 4.2. *Let f be bounded, summable on Q and such that $\Psi(f)$ is finite. Let $\varepsilon > 0$. There exists a Lipschitz function g on Q such that the set*

$$\{x;\ x \in Q \quad \textit{and} \quad f(x) \neq g(x)\}$$

has measure less than ε *and*

$$\Phi(g, Q) < \Phi\{f, \operatorname{Int}(Q)\} + \varepsilon.$$

Proof. Let $|f(x)| \leqslant K$ for all $x \in Q$. Let $a = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and for each $t \in [0, \frac{1}{2}]$, put

$$Q_t = \{2t(x-a) + a;\ x \in Q\}.$$

Let D be the set of all $t \in (0, \frac{1}{2})$ for which $\Phi\{f, \operatorname{Fr}(Q_t)\} = 0$. The complement of D in $(0, \frac{1}{2})$ is countable. Let $t_0 \in D$ be such that $0 < t_0 < \frac{1}{2}$,

$$m(Q \sim Q_{t_0}) < \tfrac{1}{2}\varepsilon \tag{1}$$

and
$$\Phi\{f, \operatorname{Int}(Q) \sim Q_{t_0}\} < \{1 + \theta(1, 1, \ldots, 1)\}^{-1} 2^{-n-2} \varepsilon \tag{2}$$

(θ is the function described in 1 (iii)). Let t_1 be such that $t_0 < t_1 < \frac{1}{2}$, and

$$t_1 - t_0 > \tfrac{1}{2}(\tfrac{1}{2} - t_0), \tag{3}$$

$$\theta(\zeta) \leqslant 1 + \theta(1, 1, \ldots, 1) \tag{4}$$

for all $\zeta \in R^n$ such that $|\zeta_i - 1| \leqslant 1 - (t_1 - t_0)/(\frac{1}{2} - t_0)$ for all i. By 3.4, there exists for each r, a Lipschitz function $g^{(r)}$ on $\operatorname{Int}(Q)$ such that

$$m\{x;\ x \in Q_{t_1} \quad \text{and} \quad f(x) \neq g^{(r)}(x)\} < r^{-1}$$

and
$$\Phi(g^{(r)}, Q_{t_1}) \leqslant \Phi(f, Q_{t_1}) + r^{-1}. \tag{5}$$

We can assume that $|g^{(r)}(x)| \leqslant K$ for all $x \in \operatorname{Int}(Q)$. Then $g^{(r)} \to f$ in the $\mathfrak{L}_1$ topology on Q_{t_1} so that by 2.6

$$\liminf_{r \to \infty} \Phi(g^{(r)}, Q_{t_0}) \geqslant \Phi(f, Q_{t_0}).$$

But by (5)
$$\limsup_{r \to \infty} \Phi(g^{(r)}, Q_{t_1}) \leqslant \Phi(f, Q_{t_1}),$$

hence
$$\limsup_{r \to \infty} \Phi(g^{(r)}, Q_{t_1} \sim Q_{t_0}) \leqslant \Phi(f, Q_{t_1} \sim Q_{t_0}).$$

Hence one can choose a large r, put $h = g^{(r)}$ and obtain

$$m\{x;\ x \in Q_{t_1} \quad \text{and} \quad f(x) \neq h(x)\} < \tfrac{1}{2}\varepsilon, \tag{6}$$

$$\Phi(h, Q_{t_1}) < \Phi(f, Q_{t_1}) + \tfrac{1}{2}\varepsilon \tag{7}$$

$$\Phi(h, Q_{t_1} \sim Q_{t_0}) < \{1 + \theta(1, \ldots, 1)\}^{-1} 2^{-n-1} \varepsilon. \tag{8}$$

Let η be the function on $[0, 1]$ that is linear on each of the intervals $[0, \frac{1}{2}-t_0]$, $[\frac{1}{2}-t_0, \frac{1}{2}+t_0]$, $[\frac{1}{2}+t_0, 1]$ and has $\eta(0)=\frac{1}{2}-t_1$, $\eta(\frac{1}{2}-t_0)=\frac{1}{2}-t_0$, $\eta(\frac{1}{2}+t_0)=\frac{1}{2}+t_0$, $\eta(1)=\frac{1}{2}+t_1$. For each $x \in Q$, define

$$p(x) = (\eta(x_1),\ \eta(x_2), \ldots, \eta(x_n)).$$

Then p is a $1-1$ Lipschitz transformation of Q onto Q_{t_1} such that $p(x)=x$ for all $x \in Q_{t_0}$. Define

$$g(x) = h\{p(x)\}, \quad x \in Q.$$

Then g is Lipschitz on Q and by (1) and (6), the set $\{x;\, x \in Q \text{ and } f(x) \neq g(x)\}$ has measure less than ε. For almost all $x \in Q \sim Q_{t_0}$ we have

$$\frac{\partial g}{\partial x_i} = \left\{\frac{\partial h}{\partial y_i}\right\}_{y=p(x)} \eta'(x_i),$$

so that by 1 (iii)

$$\phi(\text{grad } g) \leqslant \theta\{\eta'(x_1), \ldots, \eta'(x_n)\} \cdot \phi[\{\text{grad } h(y)\}_{y=p(x)}]$$

and, since $\eta'(x_i)=1$ or $(t_1-t_0)/(\frac{1}{2}-t_0)$, it follows from (4) that

$$\phi(\text{grad } g) \leqslant \{1+\theta(1, \ldots, 1)\} \cdot \phi[\{\text{grad } h(y)\}_{y=p(x)}],$$

hence
$$\int_{Q\sim Q_{t_0}} \phi(\text{grad } g)\, dx \leqslant \{1+\theta(1, \ldots, 1)\} \int_{Q\sim Q_{t_0}} \phi[\{\text{grad } h(y)\}_{y=p(x)}]\, dx.$$

But
$$\frac{\partial(p)}{\partial(x)} = \eta'(x_1)\cdot\eta'(x_2) \ldots \eta'(x_n) \geqslant \left(\frac{t_1-t_0}{\frac{1}{2}-t_0}\right)^n \geqslant 2^{-n}$$

so that

$$\int_{Q\sim Q_{t_0}} \phi(\text{grad } g)\, dx \leqslant \{1+\theta(1, \ldots, 1)\}\, 2^n \int_{Q\sim Q_{t_0}} \phi[\{\text{grad } h(y)\}_{y=p(x)}] \frac{\partial(p)}{\partial(x)}\, dx$$

$$= \{1+\theta(1, \ldots, 1)\}\, 2^n \int_{Q_{t_1}\sim Q_{t_0}} \phi(\text{grad } h)\, dy,$$

which by (8), $< \frac{1}{2}\varepsilon$. Then

$$\Phi(g, Q) \leqslant \Phi(h, Q_{t_0}) + \tfrac{1}{2}\varepsilon \leqslant \Phi(h, Q_{t_1}) + \tfrac{1}{2}\varepsilon$$

and by (7),
$$< \Phi(f, \text{Int}(Q)) + \varepsilon.$$

THEOREM 4.3. *If f is summable on Q, $\Psi(f)$ is finite and $\varepsilon > 0$, then there exists a Lipschitz function g on Q such that the set*

$$\{x;\, x \in Q \quad and \quad f(x) \neq g(x)\}$$

has measure less than ε *and* $\qquad \Psi(g) < \Psi(f) + \varepsilon.$

Proof. Let $N > 0$ be sufficiently large that the function

$$\begin{aligned} h(x) &= N \quad \text{if} \quad f(x) \geq N, \\ &= f(x) \quad \text{if} \quad |f(x)| \leq N, \\ &= -N \quad \text{if} \quad f(x) \leq -N. \end{aligned}$$

agrees with f except on a set of measure less than $\frac{1}{2}\varepsilon$. Let $\{f^{(r)}\}$ be a sequence of Lipschitz functions converging $\mathfrak{L}_1$ to f on Q and such that

$$\lim_{r \to \infty} \int_Q \phi(\text{grad } f^{(r)})\, dx = \Psi(f).$$

Define

$$\begin{aligned} h^{(r)}(x) &= N \qquad \text{if} \quad f^{(r)}(x) \geq N, \\ &= f^{(r)}(x) \quad \text{if} \quad |f^{(r)}(x)| \leq N, \\ &= -N \qquad \text{if} \quad f^{(r)}(x) \leq -N. \end{aligned}$$

Then each $h^{(r)}$ is Lipschitz, $h^{(r)} \to h$ in the $\mathfrak{L}_1$ topology and by 1 (i),

$$\int_Q \phi(\text{grad } h^{(r)})\, dx \leq \int_Q \phi(\text{grad } f^{(r)})\, dx,$$

hence $\Psi(h) \leq \Psi(f)$.

By 4.2, there exists a Lipschitz function g on Q agreeing with H except on a set of measure less than $\frac{1}{2}\varepsilon$ and with

$$\Phi(g, Q) < \Phi(h, \text{Int } (Q)) + \varepsilon$$

hence

$$\Psi(g) < \Psi(f) + \varepsilon.$$

g is the required function.

References

[1]. Fleming, W. H., Functions whose partial derivatives are measures. *Illinois J. Math.*, 4 (1960), 452–478.

[2]. Goffman, C., Lower semi-continuity and area functionals I. The non-parametric case. *Rend. Circ. Mat. Palermo* (2), 2 (1953), 203–235.

[3]. Michael, J. H., The equivalence of two areas for non-parametric discontinuous surfaces. *Illinois J. Math.*, 7 (1963), 59–78.

[4]. ——, Approximation of functions by means of Lipschitz functions. *J. Austral. Math. Soc.* (To appear.)

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