THE WIENER–HOPF EQUATION IN AN ALGEBRA OF BEURLING

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This paper treats the Wiener–Hopf equation in the framework of a Banach con-volution algebra A^2 of functions on the real line that has been constructed and in-vestigated by Beurling, by whose permission we here reproduce part of that theory. The author also acknowledges with much pleasure his great indebtedness to Professor Beurling for his guidance in this research.

We use few specialized notations and conventions. Our function spaces are all spaces of complex valued functions on the real line, hence we denote by L^1 (L^2) the space of all functions that are integrable (square integrable) over the whole real line. When an integral sign carries no limits, the integral is understood to be taken over the whole real line. The elements of the two principal function spaces A^2 and B^2 (the Banach space dual of A^2) are also distinguished by letting Latin minuscules denote elements of A^2 and Greek minuscules elements of B^2. The norm sign $\|f\|$, $\|\varphi\|$ is only used for the norms of A^2 and B^2. By the above convention no ambiguity may arise therefrom. All infinite sequences are indexed by the natural numbers and sum and product signs with no limits mean sum and product respectively over the whole set of natural numbers (we use only absolutely convergent sums and products).

Since the original paper of Wiener and Hopf [6] many authors have worked to remove its growth restrictions on the kernel of the equation. The case when the Fourier transform of the kernel has no real zeros is treated in a recent big expository paper by Krein [4]. The case when the transform of the kernel is real and has real zeros has been treated by Widom [5] and is also the object of the present paper. The essential feature of this study is that it yields, for kernels $f \in A^2$ that satisfy certain additional restrictions on its Fourier transform $\hat{f}$, all solutions in B^2 of the Wiener–Hopf equation. A similar study was made earlier by Beurling [2], using still

more restrictive conditions of f, namely, that $f-1$ have only a finite number of zeros and near each zero be bounded from below in absolute value by the absolute value of some non-constant linear function.

1. Properties of the Beurling algebra A^2 and its dual space

The algebra A^2

All results in this section (together with many generalizations) are due to Beurling [1]. In order to make our paper self-contained, we give here the part of the theory in the cited paper that we will use in the sequel.

Let Ω be the convex cone in L^1 that consists of those bounded positive functions that are symmetric about the origin and non-increasing on the positive half-axis. For $\omega \in \Omega$, put $N(\omega) = \omega(0) + \int \omega\, dx$. We call the function $N: \Omega \to R^+$ the norm of Ω.

PROPOSITION 1. *The cone Ω is closed under addition and convolution (i.e., Ω is a semiring) and the norm is additive and submultiplicative with respect to convolution. Moreover, an increasing (in the order determined by Ω) sequence of elements of Ω whose norms are bounded converges a.e. to an element of Ω with norm bounded by the same bound.*

The verification of these statements is easy.

We now define a (convolution) subalgebra A^2 of L^1 as follows

$$A^2 = \left\{ f \mid f \in L^1, \ \inf\left\{ N(\omega) \int \frac{|f|^2}{\omega}\, dx \mid \omega \in \Omega \right\} = \|f\|^2 < \infty \right\}. \tag{1}$$

It follows directly from the Cauchy–Schwarz inequality that $\int |f|\, dx \leqslant \|f\|$ so that we could *a priori* have taken f to be only measurable. Another expression for the function $f \to \|f\|$ (which we will soon show is a norm) is obtained by minimizing along half-rays in Ω

$$\|f\| = \tfrac{1}{2} \inf\left\{ N(\omega) + \int \frac{|f|^2}{\omega}\, dx \mid \omega \in \Omega \right\}. \tag{2}$$

So far we do not know if A^2 is a linear space, let alone a Banach space. However, suppose that $f_1, f_2 \in A^2$ and take $\omega_1, \omega_2 \in \Omega$ so that they determine $\|f_1\|$ and $\|f_2\|$ respectively within an arbitrary $\varepsilon > 0$ by equation (2). Then

$$N(\omega_1 + \omega_2) + \int \frac{|f_1 + f_2|^2}{\omega_1 + \omega_2}\, dx \leqslant N(\omega_1) + \int \frac{|f_1|^2}{\omega_1}\, dx + N(\omega_2) + \int \frac{|f_2|^2}{\omega_2}\, dx$$

by an elementary inequality $((a-b)^2 \geqslant 0$, put $a = \omega_1 |f_2|$, $b = \omega_2 |f_1|)$. Thus $f_1 + f_2 \in A$ and $\|f_1 + f_2\| \leqslant \|f_1\| + \|f_2\| + 2\varepsilon$, but ε was arbitrary, hence we have the triangle inequality and we now know that $f \to \|f\|$ is a norm, since it is obviously homogeneous and vanishes only for null functions. Suppose we had a Cauchy sequence $\{f_n\}_1^\infty$ in this norm. By passing to a subsequence we may assume that $\|f_n - f_m\| < 2^{-n}\varepsilon$ for an arbitrary $\varepsilon > 0$, $m \geqslant n$. Since the L^1 norm is dominated by the A^2 norm, the theorem of Beppo Levi shows that the sequence converges almost everywhere to a limit function f. Put $F = \lim_{n \to \infty} \|f_n\|$, obviously $|F - \|f_n\|| < 2^{-n}\varepsilon$. Now choose to each f_n an $\omega_n \in \Omega$ subject to the conditions $N(\omega_n) = F$, and $\int |f_n|^2 \omega_n^{-1} dx < F + 2^{-n+2}\varepsilon$. This can be done, if necessary at the cost of passing again to a subsequence. Now, with the hypotheses made,

$$\left\| \frac{f_n + f_m}{2} \right\| \geqslant F - 2^{-n}\varepsilon$$

if $m \geqslant n$. Since $\quad \tfrac{1}{2} \left(N(\omega_n + \omega_m) + \int \frac{|f_n + f_m|^2}{\omega_n + \omega_m} dx \right) \geqslant \|f_n + f_m\|$

by equation (1) we have then

$$\int \left(\frac{|f_n|^2}{\omega_n} + \frac{|f_m|^2}{\omega_m} - \frac{|f_n + f_m|^2}{\omega_n + \omega_m} \right) dx < 2^{-n+4}\varepsilon,$$

or equivalently, by an elementary identity

$$\int \frac{\omega_n \omega_m}{\omega_n + \omega_m} \left| \frac{f_n}{\omega_n} - \frac{f_m}{\omega_m} \right|^2 dx < 2^{-n+4}\varepsilon.$$

This, however, proves in the standard fashion using the Beppo Levi theorem that $\lim_{n \to \infty} f_n / \omega_n = g$ exist a.e. on the support of f. Namely, if I is an open interval symmetric about the origin and I' its complement, then ω_n is bounded from below on I by $(F + 4\varepsilon)^{-1} \int_{I'} |f_n|^2 \, dx$ and since $\int |f|^2 dx \leqslant \|f\|^2$ for any $f \in A^2$ implying that $f = \lim f_n$ is also an L^2 limit, this bound is uniform for large enough n, provided f does not vanish a.e. on I'. Let J be the union of all symmetric open intervals I such that f does not vanish a.e. on I', then we have actually proved that the limit function g exists a.e. on J and we define $g = 0$ on J'. Put $\underline{\lim} \, \omega_n = \omega$, then $f = \omega g$ almost everywhere since $f = 0$ a.e. on $g^{-1}(0)$ and on the complement of $g^{-1}(0)$ the sequence ω_n converges a.e. to f/g. Now Fatou's lemma yields $N(\omega) \leqslant F$, $\int |f|^2 \omega^{-1} dx = \int |fg| \, dx \leqslant F$ which shows that $\omega \in \Omega$ and that $f \in A^2$ with $\|f\| \leqslant F$. We then apply the whole procedure over again to the sequence $f_n - f_1$ and find that $\|f - f_1\| < \varepsilon$. But $\varepsilon > 0$ was arbitrary, hence we have proved that $\lim \|f - f_n\| = 0$ and that A^2 is a Banach space.

As a byproduct of the proof we have also found that for any $f \in A^2$ the infimum in equation (2) is actually attained at some $\omega_f \in \Omega$ and it is straightforward to check in the manner of the preceding proof that $f \to \omega_f$ is a one-valued application. This is a seemingly non-trivial result that is not in [1]. From the Cauchy–Schwarz type inequality

$$\int |f \! \ast \! g|^2 (\omega_f \! \ast \! \omega_g)^{-1} dx \leqslant \int |f|^2 \omega_f^{-1} dx \int |g|^2 \omega_g^{-1} dx = \|f\| \, \|g\|$$

and

$$N(\omega_f \! \ast \! \omega_g) \leqslant N(\omega_f) \, N(\omega_g) = \|f\| \, \|g\|$$

then follows $\|f \! \ast \! g\| \leqslant \|f\| \, \|g\|$, hence A^2 is a Banach algebra under convolution.

The conjugate space B^2

Any given continuous linear functional $f \to \Phi(f)$ on A^2 is also continuous in the stronger topology defined by the pseudo norm $\int |f|^2 \omega^{-1} dx$ for any $\omega \in \Omega$, hence representable where this norm is finite by $\Phi(f) = \int f \bar{\varphi} \, dx$ with some measurable function φ satisfying $\int |\varphi|^2 \omega \, dx < \infty$. Actually, from the definition of norm on A^2 in (1) one readily deduces, putting $f = \varphi \omega$ that $\int |\varphi|^2 \omega \, dx \leqslant N(\omega) \|\Phi\|^2$, where $\|\Phi\|$ denotes the norm of Φ as an element of the strong dual Banach space B^2 of A^2. The representation φ is only defined as a measurable function on the support of ω, however, different representations are equal a.e. on the common part of their domains since if $\omega_1, \omega_2 \in \Omega$ and φ_1, φ_2 are the respective representations, then $\int f(\bar{\varphi}_1 - \bar{\varphi}_2) \, dx$ represents the zero functional on $\{f \mid \int |f|^2 \omega^{-1} dx < \infty, \ \omega = \mathrm{Min}\,(\omega_1, \omega_2)\}$ hence $\varphi_1 = \varphi_2$ a.e. on $\mathrm{supp}\,\omega = \mathrm{supp}\,\omega_1 \cap \mathrm{supp}\,\omega_2$. Thus we may think of $\Phi(f)$ as represented by a measurable function φ on the whole line and the previous representatives as the restrictions of this function to the supports of the respective corresponding ω. We also drop Φ as a notation for the element of B^2 in favor of φ and we then have

$$\|\varphi\|^2 \geqslant \sup \left\{ (N(\omega))^{-1} \int |\varphi|^2 \omega \, dx \mid \omega \in \Omega \right\}$$

$$= \sup \left\{ \sup \left\{ \left| \int f \bar{\varphi} \, dx \right|^2 \mid N(\omega) \int \frac{|f|^2}{\omega} \, dx \leqslant 1 \right\} \mid \omega \in \Omega \right\}$$

$$= \sup \left\{ \left| \int f \bar{\varphi} \, dx \right|^2 \mid N(\omega_f) \int \frac{|f|^2}{\omega_f} \, dx \leqslant 1 \right\} = \|\varphi\|^2.$$

We collect our results in a theorem.

THEOREM 1 (Beurling). *The set A^2 defined by* (1) *is a Banach algebra under convolution with norm given alternatively by* (1) *or* (2). *The strong dual B^2 of A^2 (as a Banach space) can be represented by* $\{\varphi \mid \sup\{(N(w))^{-1}\int|\varphi^2|\,\omega\,dx \mid \omega\in\Omega\} = \|\varphi\|^2 < \infty\}$ *and this norm is the correct dual norm under the duality* $\int f\bar\varphi\,dx : A^2\times B^2 \to \mathbf{C}$.

In this paper, we will always use latin letters for the elements of A^2 and greek letters for the elements of B^2. Hence we take the liberty of using the same sign for the norms in A^2 and B^2.

When computing the norm in B^2 the following proposition is useful. It says roughly that one need only consider those functions ω that are characteristic functions of intervals.

PROPOSITION 2. *The norm of an element $\varphi\in B^2$ is alternatively given by* $\|\varphi\| = \sup\{p(r,\varphi)\mid r>0\}$ *with* $p(r,\varphi) = [(1+2r)^{-1}\int_{-r}^{r}|\varphi|^2\,dx]^{\frac{1}{2}}$.

Proof. If $\omega\in\Omega$ is a step-function, it may be thought of as a *positive* linear combination of characteristic functions of symmetric intervals, whence $\int|\varphi|^2\,\omega\,dx = \sum a_\nu(1+2r_\nu)(p(r_\nu,\varphi))^2 \leqslant N(\omega)(P(\varphi))^2$, putting for the moment $\sup\{p(r,\varphi)\mid r>0\} = P(\varphi)$. An arbitrary function $\omega\in\Omega$ is the limit a.e. of a decreasing sequence of step-functions so that $\int|\varphi|^2\,\omega\,dx \leqslant N(\omega)(P(\varphi))^2$ for all $\omega\in\Omega$, implying $P(\varphi)\geqslant\|\varphi\|$. But the opposite inequality is trivial, hence $P(\varphi) = \|\varphi\|$ as claimed.

An immediate consequence of Proposition 2 is that A^2 and B^2 are not reflexive Banach spaces since the strongly closed subspace $B_0^2 = \{\varphi\mid\lim_{r=\infty} p(r,\varphi) = 0\}$ of B^2 is not orthogonal to any non-vanishing element of A^2. We will show now that A^2 is actually the dual space of B_0^2 (this particular result is not in [1]). We first need a lemma.

LEMMA 1. *For every $f\in A^2$, $f\omega_f^{-1}\in B^2$ and has norm 1.*

Proof. By the Hölder inequality, $\|f\omega_f^{-1}\| \geqslant \|f\|^{-1}\int \bar f f\omega_f^{-1}\,dx = 1$. Conversely, if $\|f\omega_f^{-1}\| > 1$ then by Proposition 2 for some r, $\int_{-r}^{r}|f|^2\,\omega_f^{-2}\,dx > 1+2r$. Put $\omega(y) = \omega_f + y$ if $|x|\leqslant r$, $\omega(y) = \omega_f$ if $|x|>r$, then

$$\left(\frac{d}{dy}N(\omega(y))\int\frac{|f|^2}{\omega(y)}\,dx\right)_{y=0} = \left(1+2r-\int_{-r}^{r}|f|^2\,\omega_f^{-2}\,dx\right)\|f\| < 0,$$

contradicting the definition of ω_f. Hence $\|f\omega_f^{-1}\| = 1$, as claimed.

Now suppose that $\varphi\to F(\varphi): B_0^2\to\mathbf{C}$ is a continuous antilinear functional on B_0^2 and let B_r^2 be the closed subspace of B_0^2 consisting of those functions that vanish

outside of $[-r, r]$. Then if $\varphi \in B_r^2$, $\|\varphi\|^2 \leqslant \int_{-r}^r |\varphi|^2 dx \leqslant (1 + 2r) \|\varphi\|^2$ so that F is representable on B_r^2 by

$$F(\varphi) = \int_{-r}^r f\bar\varphi \, dx \quad (f \in L^2(-r, r)). \tag{3}$$

If f is continued as the zero function outside $[-r, r]$ then $f \in A^2$. Substitution of $\varphi = f\omega_f^{-1}$ in (3) yields, by Lemma 1,

$$\|f\| \leqslant \|F\|.$$

For $r = 1, 2, \ldots$ we construct the corresponding functions $f_1, f_2, \ldots$. On the set $[-n, n] \cap [-m, m]$ the two functions f_n and f_m agree a.e., hence $\lim_{n=\infty} f_n = f$ exists a.e. and by the lemma of Fatou and the Lebesgue dominated convergence theorem $\int f\bar\varphi \, dx$ exists for every $\varphi \in B^2$ and satisfies $|\int f\bar\varphi \, dx| \leqslant \|F\| \|\varphi\|$. Thus $F(\varphi)$ is represented by $\int f\bar\varphi \, dx$ on the closed hull of the union of all B_r^2, which is B_0^2. We have proved Proposition 3.

PROPOSITION 3. A^2 *is the strong dual of* B_0^2.

Our next remark concerns the convolution of functions in A^2 with functions in B^2. Given $f, g \in A^2$ and $\varphi \in B^2$ we have by Theorem 1 the repeated integral inequality

$$\int \left(\int |f(x - y)| \, |g(-y)| \, dy \right) |\varphi(-x)| \, dx \leqslant \|f\| \, \|g\| \, \|\varphi\|.$$

By Fubini's theorem, it is permitted to interchange the order of integration, hence also

$$\int \left(\int f(y - x) \, \varphi(x) \, dx \right) g(y) \, dy \leqslant \|f\| \, \|g\| \, \|\varphi\|.$$

This together with Theorem 1 proves Proposition 4.

PROPOSITION 4. *If* $f \in A^2$, $\varphi \in B^2$ *then* $f * \varphi \in B^2$ *and* $\|f * \varphi\| \leqslant \|f\| \, \|\varphi\|$.

The subspace B_0^2 is stable under the operation of convolution with a function in A^2. It is practically evident that $\varphi \in B_r^2$ implies $f * \varphi \in B_0^2$ for any $f \in A$, since $|\int_{-r}^r f(x - y) \varphi(y) \, dy|^2 \leqslant \int_{-r}^r |f(x - y)|^2 \, dy \int_{-r}^r |\varphi(y)|^2 \, dy$ that tends to zero as x tends to infinity. The continuity of the convolution as expressed by Proposition 4 and the previously used fact $B_0^2 = \overline{\bigcup B_r^2}$ then shows that $f * \varphi \in B_0^2$ for any $\varphi \in B_0^2$. However, a stronger "one-sided" stability holds:

PROPOSITION 5. *If* $f \in A^2$, $\varphi \in B^2$, $\varphi \chi(-\infty, 0) \in B_0^2$ *then* $(f * \varphi) \cdot \chi(-\infty, 0) \in B_0^2$.

Proof. In view of the above remark we need only show that

$$\chi(-\infty, 0) \int_0^\infty f(x-y)\, \varphi(y)\, dy = \psi(x) \in B_0^2.$$

Indeed, $\psi(x)$ tends to zero as $x \to -\infty$, since $|\psi(x)|^2 \leqslant \|\varphi\|^2 \|f\| \int_{-\infty}^x |f|^2 \omega_f^{-1} dx$. Thus Proposition 5 is proved.

The Fourier transform classes of A^2 and B^2

We will now investigate the properties of the Fourier transforms of elements in A^2 and B^2 and we start with the class B^2, which is the easier. However, the Fourier transforms of elements of B^2 do not always exist in the classical sense and we will here represent them by functions that are harmonic in the upper half-plane of a (ξ, η)-coordinate system, in the following way

$$\hat{\varphi}(\xi, \eta) = \widehat{\varphi e^{-\eta |x|}} = \int \varphi(x)\, e^{-i\xi x - \eta |x|}\, dx.$$

For any given $\eta > 0$ the Fourier transform exists and determines the function φ in the classical way. This, of course, holds true in much wider classes of functions (temperate distributions), however, among these the functions in B^2 are characterized by a precise condition on the growth of $\hat{\varphi}(\xi, \eta)$ as η tends to zero. Let $2\pi M\varphi(\eta)$ denote the square of the norm of $\hat{\varphi}(\xi, \eta)$ in $L_\xi^2(-\infty, \infty)$,

$$M\varphi(\eta) = \frac{1}{2\pi} \int |\hat{\varphi}(\xi, \eta)|^2\, d\xi = \int |\varphi(x)|^2\, e^{-2\eta |x|}\, dx,$$

then Theorem 2 gives the precise characterization of B^2 on the Fourier transform side.

THEOREM 2. (Beurling). *If $\varphi \in B^2$, then $M\varphi(\eta) \leqslant \|\varphi\|^2 (1 + \eta^{-1})$ for all $\eta > 0$. Conversely, if $\sup\{(1 + \eta^{-1})^{-1} M\varphi(\eta) \mid \eta > 0\} = M < \infty$, then $\varphi \in B^2$ and $\|\varphi\|^2 \leqslant e^2 M$.*

Proof. Suppose that $\varphi \in B^2$, then by the Parseval formula and Theorem 1,

$$M\varphi(\eta) = \int |\varphi(x)|^2\, e^{-2\eta |x|}\, dx \leqslant \|\varphi\|^2 N(e^{-2\eta |x|}) = \|\varphi\|^2 (1 + \eta^{-1}).$$

Conversely, if $M\varphi(\eta) \leqslant M(1 + \eta^{-1})$, then Proposition 2, the Parseval formula again and some computation shows that

$$\|\varphi\|^2 = \sup_\eta (1 + 2\eta^{-1})^{-1} \int_{-\eta^{-1}}^{\eta^{-1}} |\varphi(x)|^2\, dx \leqslant e^2 \sup_\eta (1 + 2\eta^{-1})^{-1} \int |\varphi(x)|^2\, e^{-2\eta |x|}\, dx \leqslant e^2 M.$$

Hence Theorem 2 has been proved. The subspace B_0^2 also has a characterization by Fourier transforms, as expressed by Proposition 6.

PROPOSITION 6. *A necessary and sufficient condition for an element φ of B^2 to belong to the subspace B_0^2 is $\lim_{\eta=0} \eta M\varphi(\eta) = 0$.*

Proof. We have for all $\varphi \in B^2$ and $\eta > 0$

$$e^2 \eta M\varphi(\eta) = e^2 \eta \int |\varphi(x)|^2 e^{-2\eta|x|} \, dx \geqslant (\eta+2)(1+2\eta^{-1})^{-1} \int_{-\eta^{-1}}^{\eta^{-1}} |\varphi(x)|^2 \, dx.$$

This shows that the condition is sufficient. On the other hand, if $\varphi \in B_0^2$ we get by partial integration putting $\int_{-r}^{r} |\varphi(x)|^2 \, dx = \varepsilon(r)(1+2r)$

$$\eta M\varphi(\eta) = \eta \int |\varphi(x)|^2 e^{-2\eta|x|} \, dx = 2 \int_0^\infty \varepsilon\left(\frac{x}{\eta}\right)(\eta+2x) e^{-2x} \, dx$$

and the last expression tends to zero with η by Lebesgue's bounded convergence theorem since $\varepsilon(x/\eta)$ tends to zero a.e. while bounded by $\|\varphi\|^2$. Hence the condition is necessary too, as claimed.

We conclude the list of properties of $\hat{\varphi}(\xi, \eta)$ with a proposition on the pointwise growth of its absolute value near the ξ-axis.

PROPOSITION 7. *If $\varphi \in B^2$ then $|\hat{\varphi}(\xi, \eta)| \leqslant \{2\eta^{-1} M\varphi(\eta/2)\}^{\frac{1}{2}}$ which is $O(\eta^{-1})$ for small η and $o(\eta^{-1})$ if $\varphi \in B_0^2$.*

Proof. By the Cauchy–Schwarz inequality and Parseval's formula

$$|\hat{\varphi}(\xi, \eta)| \leqslant \int |\varphi(x)| e^{-\eta|x|} \, dx \leqslant \left\{ \int e^{-\eta|x|} \, dx \int |\varphi(x)|^2 e^{-\eta|x|} \, dx \right\}^{\frac{1}{2}} = \{2\eta^{-1} M\varphi(\eta/2)\}^{\frac{1}{2}}.$$

and the rest of Proposition 7 follows by Theorem 2 and Proposition 6.

The Fourier transforms of the elements of A^2 all exist in the classical sense. We collect in Proposition 8 the immediately obvious properties of this class of Fourier transforms.

PROPOSITION 8. *The class $\widehat{A^2}$ of Fourier transforms, of functions in A^2 as a Banach algebra (with norm transferred form A^2) under pointwise multiplication. If $f \in \widehat{A^2}$, then f is a continuous function, $\lim_{\xi=\infty} f(\xi) = 0$ and $f \in L^2(-\infty, \infty)$. The Schwarz class S of infinitely differentiable functions tending to zero at infinity together with all their derivatives more rapidly than any negative power is contained in $\widehat{A^2}$.*

A contractor is a function $z \to K(z) : \mathbf{C} \to \mathbf{C}$ such that $K(0) = 0$ and $|K(z) - K(w)| \leqslant |z - w|$ for all $z, w \in \mathbf{C}$. In [1], Beurling has shown that every contractor operates on $\widehat{A^2}$, in fact, $\|K(\hat{f})\| \leqslant 6 \|\hat{f}\|$ for all $\hat{f} \in \widehat{A^2}$. This, of course, implies that if $F(z)$ is analytic on the set of values of $\hat{f} \in \widehat{A^2}$ and if $F(0) = 0$, then $F(\hat{f}) \in \widehat{A^2}$. On the other hand, there are plenty of non-analytic contractors. This shows how the restriction from L^1 to A^2 has altered the properties of the transform space, since by a theorem of Helson, Kahane, Katznelson and Rudin [3], the space $\hat{L}^1$ of Fourier transforms of functions in L^1 admits no non-analytic operators.

The cited theorem of Beurling has a rather lengthy proof depending on the computation of an equivalent norm in terms of the transformed function. In this paper we need only the weaker property that analytic functions operate on $\widehat{A^2}$, and this will follow from the Wiener–Levy theorem of general Banach algebra theory if we can identify the regular maximal ideals of $\widehat{A^2}$ with the sets $\hat{I}_\alpha = \{\hat{f} \mid \hat{f} \in \widehat{A^2}, \hat{f}(\alpha) = 0\}$, or, equivalently, the regular maximal ideals of A^2 with the sets $I_\alpha = \{f \mid f \in A^2, \int f(x) e^{-i\alpha x} dx = 0\}$. It is trivial to check that these sets are indeed regular maximal ideals, and the essential idea is to prove that every maximal ideal has this form. From the general theory we know that any such ideal is expressible in the form $I_\varphi = \{f \mid f \in A^2, \int f(x) \bar{\varphi}(x) dx = 0\}$ with some non-null function $\varphi \in B^2$ such that the functional $f \to \int f \bar{\varphi} dx$ is multiplicative. Obviously $\int_I \bar{\varphi} dx \neq 0$ for some bounded interval I and its characteristic function may then be approximated from below by a continuously differentiable positive function g so that $\int g \bar{\varphi} dx \neq 0$. The relation $\int g_{x+z} \bar{\varphi} dy \int g \bar{\varphi} dy = \int g_x \bar{\varphi} dy \int g_z \bar{\varphi} dy$ follows from the multiplicativity property with the usual notation $g_a(x) = g(x + a)$; if we differentiate it with respect to z at $z = 0$ we find that the function φ_1 defined by $\varphi_1(x) = \int g_x \bar{\varphi} dy$ $(\int g \bar{\varphi} dy)^{-1}$ satisfies the differential equation $\varphi_1'(x) = \varphi_1'(0) \varphi_1(x)$. Moreover, as a consequence of Proposition 2, $\|\varphi_x\| \leqslant (1 + 2|x|)^{\frac{1}{2}} \|\varphi\|$, hence φ_1 is $O(|x|^{\frac{1}{2}})$ at infinity and consequently is of the form $\varphi_1(x) = e^{i\alpha x}$ for some real α. Now we use the multiplicativity once more, this time together with Fubini's theorem, to deduce that $[\int f(x) \overline{\varphi(x)} dx - \int f(x) e^{-i\alpha x} dx] \int g(y) \overline{\varphi(y)} dy = \int f(-x) [\int g_x \bar{\varphi} dy - e^{i\alpha x} \int g \bar{\varphi} dy] dx = 0$, i.e., $I_\varphi = I_\alpha$. This proves the following proposition.

PROPOSITION 9. *If $\hat{f} \in \widehat{A^2}$ and $F(z)$ is a function satisfying $F(0) = 0$ and analytic on the set of values of $\hat{f}$, then $F(\hat{f}) \in \widehat{A^2}$.*

2. Formal solutions

The Wiener–Hopf equation

We will study the Wiener–Hopf equation

$$\int_0^\infty f(x-y)\,\varphi(y)\,dy = \varphi(x) \quad (x \geqslant 0) \tag{4}$$

for a kernel f in the algebra A^2. Under certain additional assumptions on f we will find all functions φ in B^2 that satisfy equation (4). We use the standard reformulation of that equation; with the understanding that $\varphi(x) = 0$ for $x < 0$, $\psi(x) = 0$ for $x \geqslant 0$ we write

$$\int f(x-y)\,\varphi(y)\,dy = \varphi(x) + \psi(x). \tag{5}$$

Equation (5) contains equation (4) together with a relation that defines $\psi \in B^2$ once a solution $\varphi \in B^2$ has been found. Proposition 5 shows that in such case one actually has $\psi \in B_0^2$. The harmonic transform of ψ, $\hat{\psi}(\xi, \eta) = \int_{-\infty}^0 \psi(x)\,e^{-i\xi x + \eta x}\,dx = \int_{-\infty}^0 \psi(x)\,e^{-i(\xi + i\eta)x}\,dx$ is evidently an analytic function in the upper $\xi\eta$-plane and will be denoted by $\hat{\psi}(\xi + i\eta)$. Correspondingly, the harmonic transform of φ is anti-analytic, hence we make the notation $\hat{\varphi}(\xi, \eta) = \hat{\varphi}(\xi - i\eta)$ with $\hat{\varphi}$ analytic in the lower half-plane.

Since $\sigma \in B^2$, $|\tau(x)| \leqslant |\sigma(x)|$ a.e. implies $\tau \in B^2$ and $\|\tau\| \leqslant \|\sigma\|$ the function

$$\delta_\eta(x) = \int f(x-y)\,\varphi(y)\,[e^{-\eta|y|} - e^{-\eta|x|}]\,dy$$

is in B^2 for all η. Moreover, $|\delta_\eta(x)| \leqslant |f| * |\varphi|$, hence by the Lebesgue dominated convergence theorem $\delta_\eta(x) \to 0$ a.e. as $\eta \to 0$ (in fact everywhere, as the family $\delta_\eta(x)$ is equicontinuous). By the same theorem, $\int \delta_\eta(x)\,g(x)\,dx \to 0$ as $\eta \to 0$ for all $g \in A^2$. This condition is, by the Parseval relation, equivalent to the following where we have introduced our new notations for the Fourier transforms of φ and ψ and made use of equation (5),

$$\lim_{\eta = +0} \int [(\hat{f}(\xi) - 1)\,\hat{\varphi}(\xi - i\eta) - \hat{\psi}(\xi + i\eta)]\,\hat{g}(\xi)\,d\xi = 0 \quad \text{for all} \quad \hat{g} \in \widehat{A^2}. \tag{6}$$

Equation (6) is the starting point for the computation of the solution of the Wiener–Hopf equation. The remainder of the work on this solution will be carried out in three stages. In the first we derive a formal solution, in the second we derive necessary conditions on the parameters of this solution and in the third we prove that

these conditions are also sufficient. In the first and second stages we make additional restrictions on the nature of the kernel, but the third stage contains no additional assumptions in this direction. We thus obtain in the end a sharp result that is summarized in the last section of the paper (Theorem 3).

The factorization lemma

We now restrict ourselves to the case where f is real-valued (i.e., the kernel is Hermitian symmetric). Furthermore, we suppose that $\log|\hat{f}(\xi) - 1|$ is locally integrable, hence f is not identically equal to one on any interval. We also introduce the notation (a_ν, b_ν) for those maximal disjoint open intervals on which $\hat{f}(\xi) \geq 1$, so that $\{\xi \mid \hat{f}(\xi) \geq 1\}^0 = \bigcup_1^\infty (a_\nu, b_\nu)$. Since f tends to zero at infinity, this union is a bounded set. Put $\zeta = \xi + i\eta$ and let $\log([\zeta - b_\nu]/[\zeta - a_\nu])$ denote the' branch of the logarithm function (with a cut from a_ν to b_ν along the real axis) that takes real values on the real axis outside $[a_\nu, b_\nu]$. Then the relations

$$\left.\begin{array}{c} h_1(\zeta) \\ h_2(\zeta) \end{array}\right\} = \frac{1}{2\pi i} \int \frac{\log|\hat{f}(t) - 1|}{t - \zeta}\, dt + \frac{1}{2} \sum \log \frac{\zeta - b_\nu}{\zeta - a_\nu} \mp i\frac{\pi}{2} \qquad (7)$$

define $h_1(\zeta)$ as an analytic function in the upper half-plane and $h_2(\zeta)$ as an analytic function in the lower half-plane, the upper sign of the last term taken in the former case and the lower sign in the latter. Indeed, the infinite parts of the integral in (7) converge since both $\log|\hat{f} - 1|$ and $(t - \zeta)^{-1}$ are square integrable outside some bounded interval. Also, the sum

$$\sum \log\left(\frac{\zeta - b_\nu}{\zeta - a_\nu}\right) = \sum \log\left(1 - \frac{b_\nu - a_\nu}{\zeta - a_\nu}\right)$$

converges uniformly outside each neighbourhood of $\overline{\{a_\nu\} \cup \{b_\nu\}}$. The two functions have limiting values $h_1(\xi)$ and $h_2(\xi)$ a.e. on the real axis, given by the expressions

$$\left.\begin{array}{c} h_1(\xi) \\ h_2(\xi) \end{array}\right\} = \frac{1}{2}(\pm h(\xi) - ig(\xi)) + \frac{1}{2}\left(\sum \log\left|\frac{\xi - b_\nu}{\xi - a_\nu}\right| \mp \log \operatorname{sgn}(\hat{f}(\xi) - 1)\right),$$

where $\qquad\qquad h(\xi) = \log|\hat{f}(\xi) - 1| \quad$ and $\quad g(\xi) = \frac{1}{\pi}\int \frac{h(t)}{t - \xi}\, dt$.

The Hilbert transform g of h exists since h can be expressed as the sum of one function in L^1 and one in L^2. Also, we have put $\log(+1) = 0$, $\log(-1) = i\pi$. Hence the following factorization holds a.e. on the real axis

$$e^{h_1(\xi)} e^{-h_2(\xi)} = \hat{f}(\xi) - 1.$$

Thus $\hat{f}-1$ can be written as the product of two factors, one that is the limit of an analytic (and zero free) function on the upper half-plane and another that satisfies these conditions for the lower half-plane. The limits of these functions on the real line, i.e., the factors themselves, are not so well-behaved; however, in a certain sense, they are in $\widehat{A^2}$ on every finite closed interval on which $\hat{f}-1$ does not vanish. The following lemma expresses exactly what we will need for our purpose.

LEMMA 3. *If* $\hat{k}\in\widehat{A^2}$, $\hat{k}(\xi)=0$ *for* $\xi\notin[a,b]$ *and* $\hat{f}(\xi)\neq1$ *for* $\xi\in[a,b]$, *then* $\hat{k}(\xi)\,e^{-h_1(\xi)}\in\widehat{A^2}$.

Proof. It is sufficient to find for each of the four factors $e^{-\frac{1}{2}h(\xi)}$, $e^{-\frac{1}{2}ig(\xi)}$, $\prod|\xi-a_\nu/\xi-b_\nu|^{\frac{1}{2}}$, and $e^{\frac{1}{2}\log\mathrm{sgn}(\hat{f}(\xi)-1)}$ in

$$e^{-h_1(\xi)}=e^{-\frac{1}{2}h(\xi)}e^{-\frac{i}{2}g(\xi)}\prod\left|\frac{\xi-a_\nu}{\xi-b_\nu}\right|^{\frac{1}{2}}e^{\frac{1}{2}\log\mathrm{sgn}(\hat{f}(\xi)-1)} \tag{8}$$

a function in $\widehat{A^2}$ that coincides with the respective factor on $[a,b]$. As the Schwarz class S of infinitely differentiable rapidly decreasing functions is contained in A^2, it is contained in $\widehat{A^2}$ too. For any real numbers $a'''<a''<a'<a$, $b<b'<b''<b'''$ such that $\hat{f}(\xi)\neq1$ on $[a''',b''']$ there exist functions r, $s\in S$ with values in $[0,1]$ such that $s(\xi)=1$ for $\xi\in[a,b]$ and $s(\xi)=0$ for $\xi\notin(a',b')$, $r(\xi)=1$ for $\xi\in[a'',b'']$ and $r(\xi)=0$ for $\xi\notin(a''',b''')$. The third and fourth factors become functions in S after multiplication with r or s, and the factors thus modified coincide with the original ones on $[a,b]$. Suppose $\hat{f}<1$ on $[a,b]$. Then $h(\xi)=\log|\hat{f}(\xi)-1|=\log(1-\hat{f}(\xi))$ and if we put $h_m(\xi)=\log(1-r(\xi)\hat{f}(\xi))$ then $h_m\in\widehat{A^2}$ by Proposition 9 since $F(z)=\log(1-z)$ is analytic on the set of values of $r\hat{f}$. If we have $\hat{f}>1$ we may put $h_m(\xi)=\log(1+r(\xi)(\hat{f}(\xi)-2))$ since the function $r(\hat{f}-2)$ has values in $(-1,\infty)$ and $F(z)=\log(1+z)$ is analytic on this set. In both cases $h_m=h$ on $[a'',b'']$. Hence $e^{-\frac{1}{2}h_m(\xi)}-1+r(\xi)$ is a function in $\widehat{A^2}$ that coincides with the first factor on $[a,b]$, as desired.

Finally, the function $g(\xi)$ that enters into the second factor of (8) is modified according to the formula

$$g_m(\xi)=\frac{1}{\pi}\int\frac{h_m(t)}{t-\xi}\,dt+\frac{s(\xi)}{\pi}\int\frac{h(t)-h_m(t)}{t-\xi}\,dt.$$

Clearly $g_m=g$ on $[a,b]$. The first term in the expression for g_m is in $\widehat{A^2}$ since the Hilbert transform is an isometry of $\widehat{A^2}$ (it corresponds to multiplication by $-i\,\mathrm{sgn}\,x$ in A^2 which is an obvious isometry). The second term is even in S, because it is the

product of one infinitely differentiable factor that vanishes outside (a', b') and one factor which is analytic on (a'', b''). Hence $g_m \in \widehat{A^2}$ and the desired modification of the second factor is given e.g. by $e^{-\frac{1}{2} i g_m(\xi)} - 1 + s(\xi)$. Lemma 3 is now proved.

We remark that, by a classical theorem of Szegö, the condition that $\log |f - 1|$ is locally integrable is not only sufficient but also necessary for the existence of a factorization of the desired type. Thus this condition stands apart from the more ad hoc conditions that will be imposed on f later. The author has not been able to prove or disprove that the mere existence of a non-trivial solution to equation (4) would imply $\log |f-1| \in L^1_{\mathrm{loc}}$. However, it is true that if $f - 1 = 0$ on some interval then φ und ψ must vanish. To outline the proof, let I be such an interval, then equation (6) implies that $\lim_{\eta = +0} \int \hat{\psi}(\xi + i\eta) \, \hat{g}(\xi) \, d\xi = 0$ for all $\hat{g} \in \widehat{A^2}$ with support in I. Then by the same method that will be used in the proof of Proposition 10 below (the continuation principle) it follows that $\hat{\psi}$ may be continued analytically across I to the zero function in the lower half-plane, hence $\psi = 0$. Now we repeat the same argument on some interval on which $f - 1$ does not vanish and obtain $\varphi = 0$.

The continuation principle

According to the preceding section the function $e^{-h_1(\zeta)} \hat{\psi}(\zeta)$ is analytic in the upper half-plane and the function $e^{-h_2(\zeta)} \hat{\varphi}(\zeta)$ is analytic in the lower half-plane. We will show in this section that these two functions continue each other across the real axis at every point ξ for which $f(\xi) \neq 1$. Since $\log |f - 1|$ is now supposed to be locally integrable, this in fact means on a set that is open and dense in the real axis and contains a neigbourhood of infinity. The resulting function $E(\zeta)$ that is analytic and single-valued in the whole plane except at the set $f^{-1}(1)$ on the real axis will be called the formal solution of the Wiener-Hopf equation, since its values on any line $\mathrm{Im}\, \zeta = \mathrm{const} < 0$ yield the function φ by multiplication with $e^{h_2(\zeta)}$ and inverse Fourier transformation.

We will need the following elementary lemma in the proof.

LEMMA 4. *Put* $w(\zeta) = (\zeta - a)(b - \zeta)$, a, b *real. Then* $w^+(\xi) \in \widehat{A^2}$ *and vanishes outside* (a, b) *and the same holds true for* $w^+(\xi) u(\xi)$ *if* u *is infinitely differentiable on some open set containing* $[a, b]$.

Proof. Since w is negative on the real axis outside $[a, b]$, we could prove the rest of the lemma by the same methods as those of the proof of Lemma 3 if we only knew that $w^+(\xi) \in \widehat{A^2}$. Now this clearly follows for arbitrary a, b when proved

for $a = -1$, $b = 1$. But $(1 - \xi^2)^+$ is the transform of $g(x) = \pi^{-1}(1 + d^2/dx^2) x^{-1} \sin x$ and an elementary computation shows that $\int |g|^2/\omega \, dx$ is finite if $\omega(x) = (1 + x^2)^{-1}$. Hence Lemma 4 is proved.

We formulate the principal result of this section in a proposition.

PROPOSITION 10. *The function* $E(\zeta)$ *defined by* $E(\zeta) = e^{-h_1(\zeta)} \hat{\psi}(\zeta)$ *for* $\mathrm{Im}\, \zeta > 0$ *and* $E(\zeta) = e^{-h_2(\zeta)} \hat{\varphi}(\zeta)$ *for* $\mathrm{Im}\, \zeta < 0$ *can be continued to a function that is single-valued and analytic in the whole plane except on the subset* $\hat{f}^{-1}(1)$ *of the real axis.*

Proof. Let the auxiliary function $E_\theta(\zeta)$, $\theta > 0$, be defined on the complement of the real line by $E_\theta(\zeta) = e^{-h_1(\zeta)} \hat{\psi}(\zeta + i\theta)$ for $\mathrm{Im}\, \zeta > 0$, $E_\theta(\zeta) = e^{-h_2(\zeta)} \hat{\varphi}(\zeta - i\theta)$ for $\mathrm{Im}\, \zeta < 0$. Suppose $\hat{f} \neq 1$ on $[a, b]$ and let C be the rectangle with vertices $a \pm i$, $b \pm i$. For $\zeta \in \mathrm{Int}\, C - (a, b)$, consider the identity (note that $(\hat{f} - 1) e^{-h_1} = e^{-h_2}$ on the real axis)

$$E_\theta(\zeta) = \frac{1}{2\pi i w(\zeta)} \left\{ \int_C \frac{E_\theta(z) w(z)}{z - \zeta} \, dz - \int_{-\infty}^{\infty} [(\hat{f}(\xi) - 1) \hat{\varphi}(\xi - i\theta) - \hat{\psi}(\xi + i\theta)] \cdot \frac{w^+(\xi) e^{-h_1(\xi)}}{\xi - \zeta} \, d\xi \right\}, \quad (9)$$

where $w(\zeta) = (\zeta - a)(b - \zeta)$ (cf. Lemma 4). The function w serves a convergence factor for the first integral of the right hand side of equation (9). Indeed, the pointwise growth estimates of Proposition 7 show that the integrand in the contour integral is uniformly bounded in z as θ tends to zero. The limit of the integral therefore defines a function that is analytic in $\mathrm{Int}\, C$ including the interval (a, b) of the real axis. The second integral on the right in equation (9) tends to zero for every non-real ζ as shown by Lemma 3, Lemma 4 and equation (6). Since $E(\zeta) = \lim_{\theta=0} E_\theta(\zeta)$ on the complement of the real line, $E(\zeta)$ has a continuation that is analytic in (a, b). But since every point ξ such that $\hat{f}(\xi) \neq 1$ is interior to some interval $[a, b]$ that is contained in the complement of $\hat{f}^{-1}(1)$ this proves the continuation property claimed by Proposition 10. The single-valuedness is evident.

In the sequel $E(\zeta)$ will denote the continued function.

3. Necessary conditions

The pole nature of certain singularities

We have shown that to every solution of the Wiener–Hopf equation (equation (4) or (5)) with kernel f satisfying $\log |\hat{f} - 1| \in L^1_{\mathrm{loc}}$ there corresponds a function E, called the formal solution, which is analytic in the whole plane except at $\hat{f}^{-1}(1)$ on the real axis. The problem is now to determine what conditions must be imposed on a formal solution, i.e. a function E that is analytic outside the subset $\hat{f}^{-1}(1)$ of the

real axis, in order that it determines a solution of the Wiener–Hopf equation by the appropriate backward transformation.

To do this, we need first two propositions that declare singularities at zero and infinity respectively to be poles if certain growth requirements are satisfied.

PROPOSITION 11. *If* $f(z)$ *is a non-constant function, analytic in the extended plane except at zero (i.e.* $f(z^{-1})$ *is entire), that satisfies* $|f(z)| \leqslant C |\operatorname{Im} z|^{-\alpha}$ *in some neighbourhood of zero for some* α, $0 < \alpha < \infty$, *then* f *has a pole at* 0.

PROPOSITION 12. *If* $f(z)$ *is an entire function that satisfies* $|f(z)| \leqslant \operatorname{Max} (C |\operatorname{Im} z|^{-\alpha},$ $D |\operatorname{Im} z|^{\beta})$ *for some positive and finite constants* C, D, α, *and* β, *then* f *is a polynomial.*

Proof. We prove here only Proposition 11, since the proof of Proposition 12 is entirely similar. The function $\log^+ |f|$ is subharmonic in the extended plane outside $\{0\}$. Hence we may use Green's function $G(z, e^{i\theta}) = (2\pi)^{-1} \operatorname{Re} (z + e^{-i\theta}) (z - e^{-i\theta})^{-1}$ for the exterior of the unit circle to obtain the following estimate

$$\log^+ |f(z)| \leqslant \int_0^{2\pi} G\left(2 \frac{z}{|z|}, e^{i\theta}\right) \log^+ \left| f\left(\frac{|z|}{2} e^{i\theta}\right) \right| d\theta \leqslant 3\alpha \log^+ \frac{1}{|z|} + 3 \log 4^\alpha C,$$

where we have also used the elementary relations

$$2\pi \left| G\left(2 \frac{z}{|z|}, e^{i\theta}\right) \right| \leqslant 3 \quad \text{and} \quad \int_0^{2\pi} \log |\sin \theta| d\theta = -2\pi \log 2.$$

This shows that the function $z^n f(z)$ is continuous at zero for large enough n, as claimed.

Analysis of the square of the formal solution

We now turn our attention to the function $B(\zeta)$ defined in the domain of regularity of $E(\zeta)$ by $B(\zeta) = E(\zeta) \overline{E(\bar\zeta)}$. From equation (7) and the relations defining $E(\zeta)$ it follows that

$$B(\zeta) = -\overline{\hat\phi(\bar\zeta)} \hat\psi(\zeta) \prod \frac{\zeta - a_\nu}{\zeta - b_\nu} \quad \text{for} \quad \operatorname{Im} \zeta > 0$$

$$\left(\text{and, of course,} \quad B(\zeta) = -\hat\phi(\zeta) \overline{\hat\psi(\bar\zeta)} \prod \frac{\zeta - a_\nu}{\zeta - b_\nu} \quad \text{for} \quad \operatorname{Im} \zeta < 0\right).$$

Suppose that ξ is an isolated singularity of $B(\zeta)$, i.e., some isolated point of $\hat f^{-1}(1)$. Then, if γ is a small circle with center at ξ whose exterior contains the rest of $\hat f^{-1}(1)$, the function $B_\xi(\zeta)$ defined outside γ by $B_\xi(\zeta) = -(2\pi i)^{-1} \int_\gamma B(z) (z - \zeta)^{-1} dz$ has as its

only singularity a singularity at ξ of the same nature as the corresponding singularity of $B(\zeta)$. Proposition 7 shows that $B(\zeta)$ satisfies an inequality of the type $|B(\zeta)| \leqslant C |\operatorname{Im} \zeta|^{-\alpha}$ for $|\operatorname{Im} \zeta|$ small enough. This obviously implies that the same type of inequality holds for $B_\xi(\zeta)$, since the singularity at ξ was isolated. Hence Proposition 11 applied to the function $f(z) = B_\xi(z + \xi)$ shows that the singularity of B_ξ at ξ is a pole. Thus we now know that every isolated singularity of B is a pole; an inspection of Proposition 7 together with the fact that the pole must have even order now reveals that if $\xi \neq b_\nu$ for all ν, the "pole" has order zero whereas if $\xi = b_\nu$ the order may be 2 or 0. A similar argument applied to the function $B_\infty(\zeta)$ defined inside a circle Γ surrounding all singularities of B by $B_\infty(\zeta) = (2\pi i)^{-1} \int_\Gamma B(z) (z - \zeta)^{-1} dz$ and employing Proposition 12 instead of Proposition 11 shows that $B(\zeta)$ tends to zero at infinity.

At this stage we introduce a new restricting hypothesis on the kernel function f that can be stated as $\sum |b_\nu| < \infty$. This, in particular, means that zero is the only point of accumulation for the set $\{b_\nu\}$. It will be clear from the following work that we could carry out the same computations for kernels with an arbitrary finite set of accumulation points of this type. In order to utilize the above reductions to conclude that B is regular in the extended plane except at $\overline{\{b_\nu\}} = \{b_\nu\} \cup 0$ we need now precisely suppose that $f^{-1}(1)$ is countable. Namely, since this set is closed, removing recursively the isolated points of the remainder of $f^{-1}(1)$ one after the other transfinitely we arrive after a countable number of steps (Cantor–Bendixson decomposition) at a perfect set which is then either void or uncountable.

The condition $\sum |b_\nu| < \infty$ makes it possible to remove all singularities of B except the one at zero by multiplication with the appropriate infinite product. Indeed, we may choose an integer n and a positive constant $|c|^2$ so that the function $B_1(\zeta) = \zeta^{2n} |c|^{-2} \prod (1 - b_\nu/\zeta)^2 B(\zeta)$ has value 1 at infinity and is analytic except at zero. Since B_1 is also the "square" of an analytic function $E_1(\zeta) = \zeta^n c^{-1} \prod (1 - b_\nu/\zeta) E(\zeta)$, i.e., $B_1(\zeta) = E_1(\zeta) \overline{E_1(\bar{\zeta})}$ the zeros of B_1 must be either real and of even order or non-real and occurring in conjugate pairs. We may thus denote the set of zeros of B_1 by $\{\beta_\nu\} \cup \{\bar{\beta}_\nu\}$, and we order the sequence $\{\beta_\nu\}$ so that $|\beta_\nu| \geqslant |\beta_{\nu+1}|$ for all ν.

LEMMA 5. *The series* $\sum |\beta_\nu|$ *converges.*

Proof. We introduce the counting function $n(t) = 2 \operatorname{Max} \{\nu \mid |\beta_\nu| \geqslant t\}$ and apply Jensen's formula to the entire function $B_1(z^{-1})$;

$$N(r) = \int_r^\infty t^{-1} n(t) \, dt = (2\pi)^{-1} \int_0^{2\pi} \log |B_1(re^{i\theta})| \, d\theta.$$

The sum of the series $\sum |\beta_\nu|$, whether finite or infinite, equals the value of the integral $-\frac{1}{2} \int_0^\infty t\,dn(t)$. This integral may be estimated by partial integration.

$$-\int_r^\infty t\,dn(t) = rn(r) + \int_r^\infty n(t)\,dt \leqslant 2\int_{\frac{1}{2}r}^r n(t)\,dt + \int_r^\infty n(t)\,dt \leqslant 2\int_{\frac{1}{2}r}^\infty n(t)\,dt$$

$$= -2\int_{\frac{1}{2}r}^\infty t\,dN(t) \leqslant 4\int_{\frac{1}{4}r}^\infty N(t)\,dt \leqslant 4\int_{\frac{1}{4}r}^\infty n(t)\,dt \leqslant -4\int_{\frac{1}{4}r}^\infty t\,dn(t).$$

Hence the series $\sum |\beta_\nu|$ converges or diverges together with the integral

$$2\pi \int_0^\infty N(t)\,dt = \int_0^\infty \left\{ \int_0^{2\pi} \log \left| \prod (1 - a_\nu/t\,e^{i\theta}) \right| d\theta \right\} dt$$

$$+ \int_0^\infty \left\{ \int_0^{2\pi} \log \left| \prod (1 - b_\nu/t\,e^{i\theta}) \right| d\theta \right\} dt + \int_0^\infty \left\{ 2\int_0^\pi \log \left| (t\,e^{i\theta})^{2n}\,c^{-2}\,\hat\varphi(t\,e^{-i\theta})\,\hat\psi(t\,e^{i\theta}) \right| d\theta \right\} dt.$$

The convergence of the first integral to the left is equivalent with the convergence of $\sum |a_\nu|$. This becomes clear if one substitutes a_ν for β_ν in the definition of $n(t)$ and copies the above argument showing the equivalence of the convergence of $\sum |\beta_\nu|$ and $\int_0^\infty N(t)\,dt$. Similarly the second integral converges or diverges together with $\sum |b_\nu|$. The third integral converges like $\int_0^1 \log t\,dt$ by the bounds given in Proposition 7. Since the convergence of $\sum |a_\nu|$ is obviously equivalent to that of $\sum |b_\nu|$, we have proved Lemma 5. Note that the integrals within $\{\ \}$-brackets above vanish for large t by Jensen's formula and by the condition $B_1(\infty) = 1$.

Because of Lemma 5 we may now divide out the zeros of B_1 by simple infinite products. Consider the function $B_2(\zeta) = \log B_1(\zeta) \prod (1 - \beta_\nu/\zeta)^{-1}(1 - \bar\beta_\nu/\zeta)^{-1}$; it is single-valued and analytic except at zero, and $B_2(\infty) = 0$. The infinite products $\prod (1 - \beta_\nu/\zeta)$, $\prod (1 - \bar\beta_\nu/\zeta)$, $\prod (1 - a_\nu/\zeta)$ and $\prod (1 - b_\nu/\zeta)$ are entire functions of ζ^{-1} with growth at most of order one, minimal type, whereas $\log \hat\varphi$ and $\log \hat\psi$ grow along vertical lines like $\log |\operatorname{Im} \zeta|^{-1}$, uniformly in $\operatorname{Re} \zeta$. Hence B_2 satisfies the requirements of Proposition 11 (with $\alpha = 1$) and so either has a simple pole at zero or else vanishes indentically. However, the growth of B_2 as one approaches zero along the imaginary axis is $o(|\zeta|^{-1})$, thus $B_2 \equiv 0$. We sum up the result of this section in a lemma.

LEMMA 6. *If $\sum |b_\nu| < \infty$, and $\hat f^{-1}(1)$ is countable, then there exists a positive integer n, a positive real number $|c|^2$ and a complex sequence $\{\beta_\nu\}$ satisfying $\sum |\beta_\nu| < \infty$ so that*

$$E(\zeta)\,\overline{E(\bar\zeta)} = |c|^2\,\zeta^{-2n} \prod (1 - \beta_\nu/\zeta)\,(1 - \bar\beta_\nu/\zeta)\,(1 - b_\nu/\zeta)^{-2}.$$

The parameters of the formal solution

Our object is now to "take the square root" of the above expression for B. First we rearrange (if necessary) the set $\{\beta_\nu\} \cup \{\bar\beta_\nu\}$ of zeros of B so that $\{\beta_\nu\}$ is the set of zeros of E. Then the function E_2 defined by $E_2(\zeta) = E(\zeta) c^{-1} \zeta^n \prod (1 - \beta_\nu/\zeta)^{-1} (1 - b_\nu/\zeta)$ is defined analytically outside the subset $\hat f^{-1}(1)$ of the real axis. Also, since $E_2(\zeta)\overline{E_2(\bar\zeta)} = 1$, it is zero free and we may adjust the argument of the constant c so that $E_2(\infty) = 1$. The function $E_3(\zeta) = \log E_2(\zeta)$ is then defined as an analytic and single-valued function outside $\hat f^{-1}(1)$ by the requirement $E_3(\infty) = 0$. If ξ is an isolated point of $\hat f^{-1}(1)$ and γ is a circle with center at ξ, so small that the rest of $\hat f^{-1}(1)$ is outside, then as before the function E_ξ defined outside γ by $E_\xi(\zeta) = - (2\pi i)^{-1} \int_\gamma E_3(z) (z - \zeta) dz$ is such that $E_\xi(z - \xi)$ satisfies the conditions of Proposition 11 with $\alpha = 1$. Namely, of the logarithms of the factors in E_2 the only one that has not already been analyzed is $\int (t - \zeta)^{-1} \log |\hat f(t) - 1| dt$ (c.f. equation (7)), however, since $\log |\hat f - 1|$ is the sum of one L^1 and one L^2 function, this term of E_3 is $O(|\mathrm{Im}\,\zeta|^{-1})$, uniformly in $\mathrm{Re}\,\zeta$, and pointwise $o(|\mathrm{Im}\,\zeta|^{-1})$. Hence the singularity of E_ξ at ξ is at most a pole and since the growth of E_ξ at ξ along the line $\mathrm{Re}\,\zeta = \xi$ is $o(|\zeta|^{-1})$, the "pole" is in fact a regular point. Thus E_3 is analytic in the extended plane by the transfinite induction argument used earlier on B. Hence $E_3 = 0$ as desired and we have proved Lemma 7.

LEMMA 7. *Under the hypotheses* $\log |\hat f - 1| \in L^1_{\mathrm{loc}}$, $\hat f^{-1}(1)$ *countable and* $\sum |b_\nu| < \infty$ *the parameters of the formal solution* $E(\zeta) = c \zeta^{-n} \prod (1 - \beta_\nu/\zeta) (1 - b_\nu/\zeta)^{-1}$ *are the complex number* c, *the positive integer* n *and the complex sequence* $\{\beta_\nu\}$ *satisfying* $\sum |\beta_\nu| < \infty$.

4. Sufficient conditions

The solution of the Wiener–Hopf equation

The main theorem of this paper, that we prove in this section, is the following

THEOREM 3. *Under the hypotheses so far made on the transform* $\hat f$ *of the kernel, a formal solution* E *yields a solution of the Wiener–Hopf equation* (5) *by means of the formulae* $\hat\varphi(\xi - i\eta) = e^{h_2(\xi - i\eta)} E(\xi - i\eta)$, $\hat\psi(\xi + i\eta) = e^{h_1(\xi + i\eta)} E(\xi + i\eta)$ $(\eta > 0)$ *if and only if, for sufficiently small* $\eta > 0$, $\int |\hat\varphi(\xi - i\eta)|^2 d\xi \leqslant M\eta^{-1}$ *for some constant* M *and* $\int |\hat\psi(\xi + i\eta)|^2 d\xi \leqslant \varepsilon(\eta) \eta^{-1}$ *for some bounded function* ε *that tends to zero with* η.

In terms of the parameters of the formal solution the conditions of the theorem are as follows $(\zeta = \xi + i\eta)$

$$\int (r(\xi,\eta))^{-1}\,|\zeta|^{-2n}\,\prod |1-\beta_\nu/\zeta|^2\,|1-a_\nu/\zeta|^{-1}\,|1-b_\nu/\zeta|^{-1}\,d\xi \leqslant M\eta^{-1}$$

$$\int r(\xi,\eta)\,|\zeta|^{-2n}\,\prod |1-\beta_\nu/\zeta|^2\,|1-a_\nu/\zeta|^{-1}\,|1-b_\nu/\zeta|^{-1}\,d\xi \leqslant \varepsilon(\eta)\,\eta^{-1}$$

with $r(\xi,\eta) = \exp\left(\pi^{-1}\int \eta[(t-\xi)^2+\eta^2]^{-1}\log|\hat f(t)-1|\,dt\right)$. Note that $r(\xi,\eta)$ tends to $|\hat f(\xi)-1|$ as $\eta \to 0$.

Suppose for the moment that the set $\{b_\nu\}$ were finite, then going back to the construction of B an easy inspection reveals that this function is then rational, hence in particular the set $\{b_\nu\}$ is finite too. Then the second condition is a consequence of the first, since as η tends to zero the proportion of the first integral that comes from an arbitrary neighbourhood of $\hat f^{-1}(1)$ tends to one whereas on precisely these neighbourhoods $r(\xi,\eta)$ becomes small. It is easy to make this argument precise.

Also, in this case, the growth of the integrals depends on the local behaviour at a countable number of points. For each point $\xi_\mu \in \hat f^{-1}(1)$ we then introduce an order d_μ defined by the relation

$$d_\mu = \{\min d \mid d \text{ non-negative integer, } \exists\, \varepsilon,\, M > 0$$

such that
$$\int_{\xi_\mu-\varepsilon}^{\xi_\mu+\varepsilon} (r(\xi,\eta))^{-1}[(\xi-\xi_\mu)^2+\eta^2]^{d-\frac12 e}\,d\xi \leqslant M\eta^{-1},$$

where $\qquad e = 1$ if $\xi_\mu \in \{a_\nu\} \cup \{b_\nu\}$ and $e = 0$ otherwise$\}$

and our conditions may be reformulated: the Wiener–Hopf equation has a solution if and only if $\sum d_\mu < m$, where m is the number of points of $\{b_\nu\}$ and the formal solution can be written $E(\zeta) = P(\zeta)\prod (\zeta-\xi_\mu)^{d_\mu}\prod_1^m(\zeta-b_\nu)^{-1}$, where $P(\zeta)$ is an arbitrary polynomial of degree less than $m-\sum d_\mu$. Hence the space of solutions of equation (5) is spanned by a certain minimal solution (corresponding to a constant $P(\zeta)$) and its derivatives of order less than $m-\sum d_\mu$.

Another immediate remark is that for $\eta > \eta_0 > 0$, both $r(\xi,\eta)$ and $(r(\xi,\eta))^{-1}$ are bounded by constants depending only on η_0. Hence we may replace the left hand side of the inequalities by $M(1+\eta^{-1})$ and $\varepsilon(\eta)(1+\eta^{-1})$ respectively.

We now begin the proof of Theorem 3. Suppose that E were given and satisfied the conditions of the theorem. We then know by these conditions, the above remark, Theorem 2 and Proposition 6 that $\hat\varphi$ and $\hat\psi$ are the transforms of functions $\varphi \in B^2$ and $\psi \in B_0^2$ respectively and also that $\varphi(x)$ vanishes for $x < 0$ and $\psi(x)$ for $x > 0$. Hence $\delta \in B^2$, where $\delta(x) = \int f(x-y)\varphi(y)\,dy - \varphi(x) - \psi(x)$ and we consider that the formal solution yields a solution in the ordinary sense if $\delta(x) = 0$ for almost all x.

In order to show this we define the function V as an analytic function in the complement of the real line by $V(\zeta) = \int_{-\infty}^{0} \delta(x)\,e^{-i\zeta x}\,dx$ for $\operatorname{Im} \zeta > 0$ and $V(\zeta) = -\int_{0}^{\infty} \delta(x)\,e^{-i\zeta x}$ for $\operatorname{Im} \zeta < 0$. Our first goal is to prove the following lemma.

LEMMA 8. *The function V can be continued to a function analytic outside the closed subset $\hat{f}^{-1}(1)$ of the real axis.*

Proof. This proof follows closely the proof of Proposition 10. Suppose, as we did there, that $[a, b]$ is a closed interval on which $\hat{f} \neq 1$ and let C be the rectangle with vertices $a \pm i$, $b \pm i$. For $\theta > 0$, define V_θ by $V_\theta(\zeta) = V(\zeta + i\theta)$ for $\operatorname{Im} \zeta > 0$, $V_\theta(\zeta) = V(\zeta - i\theta)$ for $\operatorname{Im} \zeta < 0$. The counterpart of equation 9 is the identity

$$V_\theta(\zeta) = \frac{1}{2\pi i w(\zeta)} \left(\int_C \frac{V_\theta(z)\,w(z)}{z - \zeta}\,dz + \int \left\{ \int \delta(x)\,e^{-\theta|x|}\,e^{-i\xi x}\,dx \right\} \frac{w^+(\xi)}{\xi - \zeta}\,d\xi \right) \tag{10}$$

valid for $\zeta \in \operatorname{Int} C - (a, b)$. Because of its definition and of Proposition 7, $V(\zeta)$ grows as $O(|\operatorname{Im} \zeta|^{-1})$ when approaching the real axis along vertical lines, hence following the reasoning of the proof of Proposition 10, equation (10) will prove Lemma 8 provided we can show that the second integral on the left hand side tends to zero with θ for fixed ζ with $\operatorname{Im} \zeta \neq 0$.

By Parseval's equation this integrals equals

$$\int \left\{ \int \int f(x - y)\,\varphi(y)\,e^{-\theta|x|}\,dy - (\varphi(x) + \psi(x))\,e^{-\theta|x|} \right\} g(x)\,dx \tag{11}$$

with g denoting the Fourier transform of $w^+(\xi)\,(\xi - \zeta)^{-1}$ (g clearly belongs to A^2). However, $\int \left\{ \int f(x - y)\,\varphi(y)\,[e^{-\theta|y|} - e^{-\theta|x|}]\,dy \right\} g(x)\,dx$ tends to zero with θ by the Lebesgue dominated convergence theorem since $\int f(x - y)\,\varphi(y)\,[e^{-\theta|y|} - e^{-\theta|x|}]\,dy$ does so too for all x by the same theorem (exactly this reasoning has been used once before, in the introductory section on the Wiener–Hopf equation). Hence (11) has the same limit as $\int \{ f(x - y)\,\varphi(y)\,e^{-\theta|y|}\,dy - (\varphi(x) + \psi(x))\,e^{-\theta|x|} \} g(x)\,dx$ and this, by the Parseval relation equals $\int [(\hat{f}(\xi) - 1)\,\hat{\varphi}(\xi - i\theta) - \hat{\psi}(\xi + i\theta)]\,w^+(\xi)\,(\xi - \zeta)^{-1}\,d\xi = \int [e^{h_1(\xi - i\theta) - h_1(\xi)}\,E(\xi - i\theta) - e^{h_2(\xi + i\theta) - h_2(\xi)}\,E(\xi + i\theta)]\,w^+(\xi)\,e^{h_1(\xi)}(\xi - \zeta)^{-1}\,d\xi$ that obviously tends to zero with θ since the integrand is continuous on, say, the rectangle $a \leqslant \xi \leqslant b$, $0 \leqslant \theta \leqslant \frac{1}{2}|\operatorname{Im} \zeta|$. We have now proved Lemma 8.

Exactly as we have done before with the functions B and E_3 we now see that the possible singularities of V at the isolated points of $\hat{f}^{-1}(1)$ are at most poles. However, since $\delta\chi(-\infty, 0) \in B_0^2$ (cf. Proposition 5), Proposition 7 shows that V is of growth $o(|\operatorname{Im} \zeta|)^{-1}$ as one approaches the real axis perpendicularly from above. It

follows that the isolated singularities are removable and we continue V transfinitely to an analytic function in the whole plane. However, by Theorem 2, Proposition 7 and Proposition 12 this entire function must be a polynomial, whereas the Riemann–Lebesgue lemma applied to the definition of V shows that V tends to zero when $\operatorname{Re} \zeta$ tends to infinity for fixed $\operatorname{Im} \zeta \neq 0$. Hence V is the zero function and must be the transform of a function that vanishes a.e. Now Theorem 3 is proved.

Examples

It is not obvious that there exists any non-trivial solution satisfying the conditions of Theorem 3 for any kernel f with $\{b_\nu\}$ infinite. In the final Proposition 13 we give a more stringent set of sufficient conditions and we show afterwards that there are examples that satisfy these conditions.

PROPOSITION 13. *If the analytic function* $\zeta^{-2n} \prod (1 - \beta_\nu/\zeta)^2 (1 - a_\nu/\zeta)^{-1} (1 - b_\nu/\zeta)^{-1}$ *is the sum of a series of partial fractions of the type* $C/\zeta + \sum A_\nu (\zeta - a_\nu)^{-1} + B_\nu (\zeta - b_\nu)^{-1}$, $\sum |A_\nu| + |B_\nu| < \infty$, *and if* $m\{t \mid |\hat{f}(t) - 1| < s\} = O(s^q)$ *for some* $q > 1$, *then the conditions of Theorem 3 are satisfied.*

Proof. In the conditions of Theorem 3 we may replace integration over the whole axis by integration over some bounded interval I that contains the set $\hat{f}^{-1}(1)$ in its interior. Since $\int_I |\zeta - t|^{-1} d\xi < M \log \eta$ with a constant M independent of the real variable t and $\int_I |\zeta - t|^{-q} d\xi < M_q \eta^{1-q}$ we have, by the Hölder inequality and the fact that $r(\xi, \eta) \leq \sup |\hat{f}(\xi) - 1|$ that the conditions of Theorem 3 are satisfied provided that $\int_I (r(\xi, \eta))^{-p} d\xi = O(\eta^{-p/q})$ with $p^{-1} + q^{-1} = 1$. Now we will show that, in fact, the second condition of Proposition 13 implies that $r(\xi, \eta) > \delta \eta^{1/q}$ with a positive δ independent of ξ. For this, we may assume that $|f - 1| \leq 1$. By partial integration we obtain $r(\xi, \eta) = \exp \left(-\pi^{-1} \int_0^1 s^{-1} \alpha(\xi, \eta, s) \, ds \right)$, where $\alpha(\xi, \eta, s)$ denotes the angle subtended at $\xi + i\eta$ by the set $\{t \mid |\hat{f}(t) - 1| \leq s\}$ whose measure is bounded by $M s^q$ for some $M < \infty$, by hypothesis. We suppose that $\eta < 1$ and use the bounds $\eta^{-1} M s^q$ and π for α on $[0, \eta^{1/q}]$ and $[\eta^{1/q}, 1]$ respectively. This yields $r(\xi, \eta) > \delta \eta^{1/q}$ with $\delta = \exp (-M/\pi q) > 0$, as claimed, and we have now proved Proposition 13.

To construct an example that satisfies the conditions of Proposition 13 we need a simple lemma.

LEMMA 9. *If* f *and its derivative are both in* L^2, *then* $f \in A^2$.

Proof. By Parseval's relation, $\int |f|^2 dx < \infty$ and $\int x^2 |f|^2 dx < \infty$, hence $\int |f|^2 \omega^{-1} dx < \infty$ with $\omega = (1 + x^2)^{-1}$.

We will make f continuous and let it vanish outside a compact set, hence it will be obvious that $f \in L^2$. Since it is easy to reconstruct f from $g(\xi) = |\hat{f}(\xi) - 1|$ whose derivative has the same L^2 norm as that of f, we give here a description of g only. Suppose that all a_ν and b_ν are positive and let t_ν be the monotone decreasing sequence whose set of values is $\{a_\nu\} \cup \{b_\nu\}$. Put $g(\xi) = \mathrm{Inf}\,(1, |2\xi|^{1/q}, |\xi - t_1|^{1/q}, ((t_{\nu+1} - \xi)^+ /\gamma_{\nu+1} + (\xi - t_{\nu+1})^+/\gamma_\nu)^{1/q})$, where γ_ν is another monotone decreasing sequence, $0 < \gamma_\nu \leqslant 1$, with $\sum \gamma_\nu < \infty$. Since $m\{t \mid g(t) < s\} \leqslant 2(1 + \sum \gamma_\nu)\,s^q$ the second condition of Proposition 13 is indeed satisfied. If $q < 2$ then $\sum \gamma^{-2/q}(t_\nu - t_{\nu+1})^{-1+2/q} < \infty$ implies that $g' \in L^2$ and hence, by Lemma 9, that $f \in A^2$. One implementation of these conditions is $q = 3/2$, $\gamma_\nu = \nu^{-3/2}$ and $t_\nu \leqslant \nu^{-10}$.

We now turn to the partial fractions condition of Proposition 13. Suppose that $\{b_\nu\}$ is positive and monotone decreasing. All zeros of the sum $\sum b_\nu^2/(\zeta - b_\nu)$ lie in the interval $(0, b_1)$, exactly one in each of the intervals $(b_{\nu+1}, b_\nu)$, and we let a_ν denote the zero that lies in $(b_{\nu+1}, b_\nu)$. Then

$$\prod (1 - b_\nu/\zeta) \sum b_\nu^2/(\zeta - b_\nu) = \zeta^{-1} \sum b_\nu^2 \prod (1 - a_\nu/\zeta)$$

since the left-hand side is an entire function of $1/\zeta$ of at most order one, minimal type. Now

$$\zeta^{-2} \prod (1 - a_\nu/\zeta)(1 - b_\nu/\zeta)^{-1} = (-\zeta^{-1} \sum b_\nu + \sum b_\nu/(\zeta - b_\nu))(\sum b_\nu^2)^{-1}$$

and we have an example of the type required by Proposition 13 if we take $n = 1$ and $\beta_\nu = a_\nu$ for all ν.

If one has one decomposition into a partial fractions series of the type required then an obvious computation shows that one may change a finite number of the non-zero parameters β_ν and still have the same kind of decomposition of the function $\zeta^{-2n} \prod (1 - \beta_\nu/\zeta)^2(1 - a_\nu/\zeta)^{-1}(1 - b_\nu/\zeta)^{-1}$. Furthermore, by linear combinations one may then reach arbitrary high values of the index n. One may also multiply by ζ as long as this doesn't depress the index below one. As an application of a polynomial in the differentiation operator to the solution of the Wiener–Hopf equation corresponds to a multiplication of the above function with a polynomial in ζ having zeros of even order, we see that in all cases covered by Proposition 13 the solution space is infinite dimensional. Also, in contradistinction to the case when $\{b_\nu\}$ is finite, there is no minimal solution from which all others may be obtained by differentiation and linear combination.

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