CONNECTIONS AND CONFORMAL MAPPING

BY

M. SCHIFFER and N. S. HAWLEY

Stanford University, California, U.S.A. (1)

Introduction

1. In the theory of conformal mapping and of Riemann surfaces, the concepts of invariance and covariance under change of variable play an important role. They allow the study of functions and differentials on abstractly given domains and have been extensively utilized from the early days of the theory. In this development the complete analogy to the tensor calculus, in general differential geometry of surfaces, has been helpful and has motivated and guided the investigations. A differential in the theory of Riemann surfaces is the analogue of a tensor in differential geometry in so far as both entities are transformed by a linear homogeneous operation under change of the coordinate system. However, it is well known that differential geometers were soon led to introduce entities with more complicated transformation laws than those of tensors. In particular, the Christoffel symbols and connections of a surface became an important tool in the study of the geodesics and the curvature of a surface. A connection is an entity which transforms under a linear but non-homogeneous law if the coordinates are changed. It is natural to inquire whether the analogous concept of a connection should be applied likewise in the theory of conformal mapping and Riemann surfaces. The present paper is devoted to an exposition of the role of connections in various applications of this kind. Before entering into a systematic development of the theory of connections, we wish to give in this introduction a brief preview of our results. This will enable the reader to judge at one glance the usefulness and significance of the concept of connection in a systematic study of conformal transformations.

(1) This work was supported in part by the Office of Naval Research under Task Nr 041–086 and by the National Science Foundation, Grant 8199, at Stanford University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

2. In Chapter I we deal with the theory of conformal mappings of planar domains of finite connectivity. In this case, much of the information about the domain is contained in the geometry of the boundary curves and it is shown that the curvature of the boundary curve transforms by an affine law (which is similar to the behavior of a connection) under conformal mappings of the domain, the coefficients of the transformation formula depending upon the mapping function. The transformation law enables us to study the mapping of a domain onto canonical domains with specified laws for the curvature of the boundary.

The simplest canonical domain of this kind is the circular domain where the curvature on each boundary component has a *constant* value. The requirement that an analytic function $w=f(z)$ maps a domain in the z-plane onto such a canonical domain in the w-plane leads now in a natural way to an integral equation for the unknown mapping function $f(z)$. True, the integral equation is nonlinear and the unknown constant curvatures of the canonical domain enter as accessory parameters into the integral equation. However, the integral equation is near enough in type to the well-studied Hammerstein integral equations to suggest an interesting and simple extremum problem with the same solution. In this way, a new and direct existence proof for canonical mapping on circular domains is obtained, and a new functional is introduced which has a remarkable transformation law under conformal mapping and which plays an important role in the later developments of the theory of connections.

While the mapping on circular domains is certainly particularly simple and interesting, mappings on domains with different laws for the curvature of the boundary can be treated in a similar way. If we start with an arbitrary domain and map it onto a domain bounded by convex curves, we can characterize this particular mapping by an extremum problem for the corresponding mapping function. If we map the original domain onto two different domains bounded by convex curves, we obtain two inequalities for the corresponding mapping functions; namely, the fact that each mapping function gives a lesser value to its own functional than its competitor. From these inequalities, we can derive simple distortion theorems for conformal mappings between domains with convex boundary curves.

Finally, we consider the mapping of a domain by means of an analytic function $f(z; t)$ which depends on the real parameter t. The functionals of f, introduced in connection with the above existence theorems and based on the curvature of the image domain, become now functions of t and satisfy useful differential relations and inequalities.

3. Most canonical domains which have been studied in the theory of conformal mappings are closely related to the theory of the Green's function of the given domain. For example, the functions which map onto parallel slit domains, radial and circular slit domains, spiral slit domains, etc., can be expressed in terms of Green's function. Since the change of the Green's function under a deformation of its domain of definition is given by Hadamard's variational formula, it is easy to compute the change of the corresponding canonical domains and to give simple variational formulas for the moduli which characterize the various canonical domains of this kind.

The case of the circular domain, however, is quite different. No variational formula for the moduli was known here and the main purpose of Chapter II is to provide such formulas. For the sake of simplicity and without loss of generality, we may suppose that the original domain is already a circular domain. We subject the domain to an interior variation and ask for the change of its circular moduli. From the general theory of variations of moduli we know that such a variation can be expressed in terms of a quadratic differential of the domain; the problem is to characterize the proper quadratic differential occurring. We single out these quadratic differentials by extremely simple conditions on the value of the integral of those differentials around the various boundary curves and solve the problem completely. It is also shown that the same characterization is valid if the canonical domain is bounded by n similar convex curves instead of n circumferences and if the moduli sought are the locations and scale factors of these convex curves.

The study of the variation of the moduli of a circular domain was naturally motivated by the fact that we had given a new proof for this canonical mapping in Chapter I. However, there is also an unexpected close relation between the quadratic differentials of a planar domain and the curvature of its boundary curves in which we are interested as a connection. In fact, let $\omega(x,y)$ be harmonic in a domain and satisfy on its boundary the relation $\partial^2\omega/\partial s^2 = \varkappa(\partial\omega/\partial n)$ where $\varkappa$ is the local curvature. Then $q(z) = -i(\partial^2\omega/\partial z^2)$ is a quadratic differential of the domain. Conversely, a quadratic differential $q(z)$ can be derived from such a harmonic function if $\int_{C_\nu} q(z)\,dz = \operatorname{Im}\{\int_{C_\nu} zq(z)\,dz\} = 0$ for each boundary curve C_ν of the domain. We arrive thus at an interesting boundary value problem for harmonic functions. To find harmonic functions in the domain with

$$\frac{\partial^2\omega}{\partial n^2} = \frac{\partial^2\omega}{\partial s^2} - \varkappa\frac{\partial\omega}{\partial n} = 0.$$

It is shown that if the domain is the exterior of n convex curves C_ν, no regular harmonic function of this type can exist. However, many quadratic differentials with singularities can be constructed through this boundary value problem for harmonic functions.

Finally, the quadratic differentials which occur in the variation of the circular moduli are constructed explicitly in terms of proper Robin's functions of the domain. The variation of the moduli is given under interior variation as well as under a Hadamard type variation of the boundary. In the latter formulas the curvature $\varkappa$ plays again the principal role.

4. The only connection considered so far was the curvature $\varkappa$ of the boundary curve of a planar domain. Our main interest is naturally the study of analytic functions in the domain which transform according to a linear non-homogeneous law under conformal mapping. We show in Chapter III that there exist functions $\Gamma(z)$ meromorphic in the domain such that $\operatorname{Im}\{\Gamma \dot{z}\}=\varkappa$ on the boundary of the domain where $\dot{z}$ denotes the tangent vector to the boundary curve at the point considered. $\Gamma(z)$ has the following tranformation law under a *conformal mapping* $z^*(z)$:

$$\Gamma^* dz^* = \Gamma\, dz + d \log\left(\frac{dz^*}{dz}\right),$$

i.e., it transforms according to a linear inhomogeneous law.

The first important application of each connection Γ with the above transformation law is the process of differentiation of differentials of the domain which is analogous to the covariant differentiation of tensors by means of connections in differential geometry. Indeed, let $q(z)$ be meromorphic in the domain and transform under conformal mapping like $q^*(dz^*)^n = q\, dz^n$, i.e., let $q(z)$ be a differential of order n. Then $(dq/dz)+n\Gamma q$ will be again a differential of the domain but of order $n+1$. Thus, connections allow the creation of new differentials by differentiation.

It is easy to see that, in general, no regular analytic connection $\Gamma(z)$ can exist. The simplest connection $\Gamma(z;\zeta)$ is regular analytic in the domain D considered except for the point $\zeta \in D$ where it has a simple pole with residue $-N$ (N = number of boundary curves). The most general connection can be built up by linear superposition of these elementary connections. The question now arises how $\Gamma(z;\zeta)$ depends on its pole ζ. For this purpose, we introduce a generalized Neumann's function $H(z;\zeta)$ of the domain D defined by the following properties:

(a) $H(z;\zeta)$ is harmonic for $z\in D$ except at $z=\zeta$ where $H(z;\zeta)+N\log|z-\zeta|$ is harmonic.

(b) On the boundary C of D we have $\partial H/\partial n = -\varkappa$.

(c) We normalize by $\int_C \varkappa H\, ds = 0$. This new Neumann's function has the same symmetry law as the classical Neumann's function; namely, $H(z;\zeta)=H(\zeta;z)$. On the other hand, we have $\Gamma(z;\zeta)=2(\partial/\partial z)H(z;\zeta)$. This identity shows that $\Gamma(z;\zeta)$ depends harmonically upon its parameter point ζ.

While the classical Neumann's function of D has a very involved transformation law under conformal mapping, the new Neumann's function satisfies the law:

$$H^*(z^*;\zeta^*) = H(z;\zeta) + \log\left|\frac{dz^*}{dz}\right| + \log\left|\frac{d\zeta^*}{d\zeta}\right| + \frac{1}{\pi N}[z^*, z]$$

where $[z^*,z]$ is the functional of the mapping function $z^*(z)$ which occurred first in the extremum problem of Chapter I.

In analogy, we may introduce the generalized Green's function $G_0(z;\zeta)$ of the domain by the requirements:

(a) $G_0(z;\zeta)$ is harmonic for $z \in D$ except at $z=\zeta$ where $G_0(z;\zeta)+\log|z-\zeta|$ is harmonic.

(b) $G_0(z;\zeta)$ is constant on each boundary continuum of D.

(c) $\int_{C_\nu}(\partial/\partial n_z)G_0(z;\zeta)\,ds_z = 2\pi/N$ for each boundary continuum C_ν.

(d) We have $\int_C \varkappa(s_z)G_0(z;\zeta)\,ds_z=0$. By these four requirements the new Green's function $G_0(z;\zeta)$ is uniquely determined and can be shown to be symmetric in z and ζ.

If we complete $H(z;\zeta)$ and $G_0(z;\zeta)$ to analytic functions in z, say $\mathfrak{H}(z;\zeta)$ and $P_0(z;\zeta)$ it is seen that $\mathfrak{H}(z;\zeta)$ and $NP_0(z;\zeta)$ have at ζ a simple logarithmic pole with residue N and that they both have periods $\pm 2\pi i$ when continued around any boundary contour C_ν. Their exponentials are, therefore, single-valued analytic functions except for an Nth order pole at the point ζ.

We construct now in Chapter III two kernels $\mathfrak{L}(z;\zeta)$ and $\mathfrak{K}(z;\bar\zeta)$ with the following properties: $\mathfrak{L}(z;\zeta)$ is symmetric and analytic in both arguments except for $z=\zeta$ where it has a simple logarithmic pole. It is determined only up to integer multiples of $2\pi i$. $\mathfrak{K}(z;\bar\zeta)$ is hermitian symmetric in z and ζ, analytic in z and, therefore, anti-analytic in ζ. It is regular and single-valued in D. We then find:

$$\mathfrak{H}(z;\zeta) = \pi N[\mathfrak{L}(z;\zeta) + \mathfrak{K}(z;\bar\zeta)],\quad P_0(z;\zeta) = \pi[\mathfrak{L}(z;\zeta) = \mathfrak{K}(z;\bar\zeta)].$$

These relations throw light on the interrelation between the new Neumann's and the new Green's function.

We make contact with the well-known theory of the kernel functions by the identities:

$$\frac{\partial^2\mathfrak{K}(z;\bar\zeta)}{\partial z\,\partial\bar\zeta} = K_0(z;\bar\zeta);\ -\frac{\partial^2\mathfrak{L}(z;\zeta)}{\partial z\,\partial\zeta} = L_0(z,\zeta)$$

where $K_0(z;\bar\zeta)$ and $L_0(z;\zeta)$ are the Bergman kernel and its associated kernel for the class of all analytic functions $f'(z)$ in D which have a single-valued integral in D.

On the other hand, $\mathfrak{K}(z;\bar\zeta)$ itself may be defined as a reproducing kernel in a proper class of analytic functions defined in D. In fact, consider all functions $f(z)$ with finite norm $\iint_D |f'(z)|^2 < \infty$ and normalization $\int_C \varkappa f\,ds=0$. Their kernel function is

$$\mathfrak{K}(z; \bar{\zeta}) = \sum_{\nu=1}^{\infty} f_\nu(z)\, \overline{f_\nu(\zeta)}$$

where the $\{f_\nu(z)\}$ are any complete orthonormal set in the class. This leads to an explicit construction for the kernels $\mathfrak{K}$ and $\mathfrak{L}$ and thus of the harmonic functions H and G_0.

While the kernels K_0 and L_0 transform under conformal mapping as double differentials, their integrated kernels $\mathfrak{K}$ and $\mathfrak{L}$ transform according to the non-homogeneous law

$$\mathfrak{K}^*(z^*; \bar{\zeta}^*) = \mathfrak{K}(z; \bar{\zeta}) + \frac{1}{2\pi N} \log \frac{dz^*}{dz} + \frac{1}{2\pi N} \log \left(\frac{d\zeta^*}{d\zeta}\right)^- + \frac{1}{2\pi^2 N^2}[z^*, z]$$

and

$$\mathfrak{L}^*(z^*; \zeta^*) = \mathfrak{L}(z; \zeta) + \frac{1}{2\pi N} \log \frac{dz^*}{dz} + \frac{1}{2\pi N} \log \frac{d\zeta^*}{d\zeta} + \frac{1}{2\pi^2 N^2}[z^*, z].$$

We may describe $2\pi N\mathfrak{K}$ and $2\pi N\mathfrak{L}$ as logarithms of multiplicative double differentials of D.

There appears to exist a close relation between the kernels $\mathfrak{K}$ and $\mathfrak{L}$ on the one hand and the expressions $\log K_0$ and $\log L_0$ on the other. All these expressions are logarithms of double differentials, and in the case of simply-connected domains we have the identity

$$\frac{\partial^2}{\partial z\, \partial\bar{\zeta}} \log K_0(z; \bar{\zeta}) = 2\pi\, K_0(z; \bar{\zeta}) = \frac{\partial^2}{\partial z\, \partial\bar{\zeta}} (2\pi\mathfrak{K}(z; \bar{\zeta})).$$

This analogy can now be easily explained. All properties of the kernels K_0 and L_0 can be developed from the relation on the boundary

$$(K_0(z; \bar{\zeta})\, \dot{z})^- + L_0(z; \zeta)\, \dot{z} = 0 \quad (z \in C, \zeta \in D),$$

by means of the method of contour integration. If s is the arc length along C, we obtain by differentiation with respect to s the identity

$$\left(\frac{d}{ds} \log K_0(z; \bar{\zeta})\right)^- = \frac{d}{ds} \log L_0(z; \zeta) + 2i\varkappa.$$

Thus, within the class of functions $f(z)$ with the normalization $\int \varkappa f\, ds = 0$ the kernels $\log K_0$ and $\log L_0$ have analogous boundary behavior as had the kernels K_0 and L_0 before. Using contour integration, we can derive numerous relations between the various kernels.

The role of non-homogeneous transformation laws under conformal mapping has played a central role in these formal developments. Even the logarithms of differentials and their derivations can be considered only if non-homogeneous transformations are admitted. The importance of the curvature $\varkappa$ in normalization of the various fundamental functions and of the function class with kernel $\mathfrak{K}$ is obvious. It leads to simple transformation laws for the expressions defined.

Finally, we show in Chapter III that all differentials of D can be expressed in an elegant and simple way in terms of the connections $\Gamma(z;\zeta)$ such that this term may be considered as the fundamental building block for all important expressions in the domain.

In Chapter IV we study the dependence of the connections $\Gamma(z;\zeta)$ upon their domain of definition. For this purpose, we derive at first a variational formula for its harmonic potential, that is the Neumann's function $H(z;\zeta)$ defined in Chapter III. While this formula is of considerable interest in itself, its main significance seems to be that it leads in a natural way to the combination $\mathfrak{R}(z;\zeta,\eta)=\Gamma(z;\zeta)\Gamma(z;\eta)+\Gamma'(z;\zeta)+\Gamma'(z;\eta)$ which has the simple transformation rule

$$\mathfrak{R}^*\left(\frac{dz^*}{dz}\right)^2=\mathfrak{R}+2\{z^*;\,z\},$$

where $\{z^*;z\}$ denotes the Schwarz differential parameter. We have also the boundary behavior

$$\operatorname{Im}\{\mathfrak{R}\dot{z}^2\}=2\frac{d\varkappa}{ds}\quad\text{at}\quad z=z(s)\in C.$$

Both properties of the expression $\mathfrak{R}(z;\zeta,\eta)$ are very similar to those of the kernel $l(z;z)$ which plays a role in the theory of the Bergman kernel function and by combination of the new and the old domain function, we obtain real quadratic differentials of the domain.

The following application of these formal considerations is made. Let C be a smooth closed curve in the complex plane and consider the Poincaré–Fredholm integral equation connected with it. In particular, let $D(\lambda)$ be the Fredholm determinant of this problem. It is known that $D(\lambda)$ is not conformally invariant but changes in a rather unforeseeable way under mappings. However, we show that $D(1)$ can be computed if we can map the interior and the exterior of C onto circular domains and give an explicit formula for this expression. The reason for the identity derived is the variational formula for $D(1)$ which happens to coincide with the variational formula of a certain functional which occurs in the theory of the connections.

It should be observed that the existence proof for the canonical mapping of a multiply-connected domain onto a circular domain can be derived from an extremum problem concerning the Fredholm determinant $D(1)$ [25]. The interrelation between the expression $D(1)$ and the connection theory is, therefore, somehow to be expected.

From the variational formula for $H(z;\zeta)$ we derive corresponding formulas for various other domain functions which arise in the theory of connections.

The concept of "connection" achieves its full significance when we proceed from planar domains to Riemann surfaces. Chapter V deals with the study and classification of connections on closed Riemann surfaces. We show that the sum of residues of every connection is $2-2p$ if p is the genus of the surface. While it is easy to see that the logarithmic differentiation of each Abelian differential leads to a connection, only a subclass of connections (the canonical connections) can be obtained in this way. We call a connection normal if it has at all singularities integer residues. Clearly, each canonical connection is normal. We study now the integrals of normal connections on the Riemann surface; in order to eliminate the multivaluedness due to the various poles, we consider the exponential function of these integrals $\exp\{-\int \Gamma_\alpha dz_\alpha\}$ and study the cohomology classes connected with these expressions. We show that the cohomology class conversely determines the connection up to an additive term $d \log f$ where f is a function on the surface. The canonical connections can be characterized as the normal connections with the trivial cohomology class. Finally, we show that to each cohomology class of the surface a corresponding connection can be found.

The main application of connections on Riemann surfaces is the operation of covariant differentiation with respect to the local uniformizing parameter [8]. We study also the concept of integration by means of connections; this process is well-defined in the small but leads to new problems when considered globally. This question is studied in detail for the case of quadratic differentials when integrated by means of canonical connections. The extension of the theory to normal connections as integrators is briefly indicated.

A connection is called elementary if it possesses only a single pole on the surface. We can normalize the elementary connection $\Gamma(\mathfrak{p};\mathfrak{q})$ in such a way that it is uniquely determined by its pole $\mathfrak{p}$ and its cohomology class on the surface. It will then depend analytically on $\mathfrak{q}$. Every connection can be built up from these normalized elementary connections and differentials of the surface. In Chapter VI we study the dependence of these connections upon the surface and give variational formulas for the normalized elementary connections under infinitesimal deformations of the surface.

The formalism of the variational technique suggests the introduction of the functional $E(\mathfrak{p};\mathfrak{q})$ of the surface with the following properties: $E(\mathfrak{p};\mathfrak{q})$ is a multiplicative differential in $\mathfrak{p}$ and in $\mathfrak{q}$ and is symmetric in both variables. Its factors with respect to the set A_ν of a conjugate cross-cut system (A_ν, B_ν) are unity. It has no poles on the surface and for $\mathfrak{p}=\mathfrak{q}$ vanishes of order $2p-2$. Then $\Gamma(\mathfrak{p};\mathfrak{q}) = -(\partial/\partial p)\log E(\mathfrak{p};\mathfrak{q})$ is the normalized elementary connection. The functional $E(\mathfrak{p};\mathfrak{q})$ is constructed explicitly in terms of the Abelian differentials of the surface.

I. The integral equation for circular mapping

1. We wish to establish in this chapter a new proof for the fact that every finitely connected domain in the complex z-plane can be mapped conformally onto a domain bounded by circumferences. This theorem was proved first by Schottky [27] and has been discussed by many authors [3, 12, 15, 17, 25]. The new feature of our proof is the reduction of the canonical mapping problem to an extremum problem of the Dirichlet type.

Since the consideration of punctured domains does not lead to any significant modification and since all proper boundary continua can be mapped into analytic curves by elementary preparatory mappings, there is no restriction of generality if we assume that the domain D considered is bounded by N closed analytic curves $C_\nu(\nu=1,\dots,N)$.

Let $z=z(s)$ be the parametric representation of the boundary curves in terms of the arc length s. If L_ν is the length of C_ν, the variable s will run from 0 to $L=\sum L_\nu$ and $z(s)$ will be an analytic function of s, except at $L_1, L_1+L_2, \dots$, where it will be discontinuous. We have the Frenet formula for the curvature

$$\varkappa(s)=\frac{1}{i}\frac{\ddot{z}}{\dot{z}}, \quad \dot{z}=\frac{dz}{ds}. \tag{1}$$

The univalent conformal mapping $w=f(z)$ of D carries the curve system C_ν into a system of curves Γ_ν in the w-plane. We assume that $f(z)$ is analytic in $D+C(C=\sum C_\nu)$ and have, therefore, again an analytic curve system $\Gamma=\sum\Gamma_\nu$ bounding the domain Δ. Let σ be the arc length parameter on Γ. Since

$$\frac{d\sigma}{ds}=|f'(z)| \quad (z=z(s)), \tag{2}$$

we find

$$\frac{dw}{d\sigma}=f'(z)\,\dot{z}\,|f'(z)|^{-1} \tag{3}$$

and hence by logarithmic differentiation with respect to σ:

$$\frac{d^2w}{d\sigma^2}\left(\frac{dw}{d\sigma}\right)^{-1}=\left[\frac{f''}{f'}\dot{z}+\frac{\ddot{z}}{\dot{z}}-\frac{d}{ds}\log|f'(z)|\right]\frac{ds}{d\sigma}. \tag{4}$$

Comparing imaginary parts on both sides and using (1) and its analogue for the domain Δ, we obtain

$$\varkappa^*(\sigma)\,d\sigma=\varkappa(s)\,ds+\operatorname{Im}\{d\log f'(z)\}. \tag{5}$$

Here $\varkappa^*(\sigma)$ denotes the curvature of Γ at $w(\sigma)$.

Using (2) and the Cauchy-Riemann equations, we may bring (5) into the form

$$\frac{\partial}{\partial n} \log |f'(z)| = \varkappa(s) - \varkappa^*(\sigma) |f'(z)|, \tag{6}$$

where the operator $\partial/\partial n$ denotes differentiation with respect to the interior normal on C.

2. The identity (6) allows us to deal with problems of conformal mapping in which the curvature of the image curves is prescribed. We shall deal with the interesting special case, namely, the problem of circular mappings. Here $\varkappa^*(\sigma)$ must be constant on each Γ_ν, say, have the value $-r_\nu^{-1}$. Hence, we find the condition on the mapping function

$$\frac{\partial}{\partial n} \log |f'(z)| = \varkappa(s) + \frac{1}{r_\nu} |f'(z)|. \tag{7}$$

Clearly, $f(z)$ will not be uniquely determined by the requirement of a circular mapping. Every linear transformation following the mapping by $f(z)$ will preserve the circular boundary. We have thus still the freedom to normalize $f(z)$. We assume that D contains the point at infinity and that $f(\infty) = \infty$. One advantage of this assumption is that all circles in Δ will have a negative curvature relative to the domain and, hence, the numbers r_ν introduced in (7) will all be the positive radii of the circles.

The condition (7) represents a boundary value problem of the second kind for the harmonic function $\log|f'(z)|$. It can be solved by means of the Neumann's function $N(z;\zeta)$ of the domain D in the form

$$\log |f'(\zeta)| = -\frac{1}{2\pi} \sum_{\nu=1}^{N} \int_{C_\nu} N(z;\zeta) \left[\frac{1}{r_\nu} |f'(z)| + \varkappa(s_z)\right] ds_z. \tag{8}$$

This equation can be considered as a non-linear integral equation for the unknown function $\log|f'(z)|$ on C.

The identity (7) would allow still an arbitrary additive constant on the right side of (8). By setting it equal to zero, we imply

$$\int_C \log |f'(\zeta)|\, ds = 0, \tag{9}$$

a normalization which we have the right to make and which will be most convenient in the sequel.

3. We introduce as new unknown function

$$u(z) = \log |f'(z)|, \quad \left(\int_C u\, ds = 0\right) \tag{10}$$

and consider the non-linear integral equation on the curve system C

$$u(\zeta)=F(\zeta)-\sum_{\nu=1}^{N}\frac{1}{2\pi r_\nu}\int_{C_\nu}N(z;\zeta)\,e^{u(z)}\,ds_z \tag{11}$$

with
$$F(\zeta)=-\frac{1}{2\pi}\int_C N(z;\zeta)\,\varkappa(s_z)\,ds_z. \tag{12}$$

We observe that $F(\zeta)$ is harmonic in D, satisfies the condition

$$\int_C F(\zeta)\,ds_\zeta=0 \tag{13}$$

and has on C the normal derivative

$$\frac{\partial F}{\partial n}=\varkappa(s)+\frac{2\pi}{L}N. \tag{13'}$$

We have in the case of the circular mapping

$$f(z(s))=r_\nu\,e^{i\varphi(s)}\quad\text{on}\quad C_\nu. \tag{14}$$

Hence, by differentiation with respect to arc length,

$$f'(z)\dot{z}(s)=i\dot{\varphi}(s)f(z). \tag{15}$$

We can apply the argument principle to this equation; if we run through the closed curve C_ν in the positive sense with respect to D, we obtain

$$\Delta\arg f'(z)-2\pi=\Delta\arg f(z)=-2\pi. \tag{16}$$

Hence, $\Delta\arg f'(z)=0$, that is, $\log f'(z)$ is a single-valued function in D. We can now integrate (7) over each C_ν and find

$$\int_{C_\nu}|f'(z)|\,ds=\int_{C_\nu}e^u\,ds=2\pi r_\nu. \tag{17}$$

Let $h(\zeta)$ be an arbitrary harmonic function in D with a finite Dirichlet integral; we also suppose that it is continuously differentiable in $D+C$. We multiply the identity (11) by $-(\partial h/\partial n)$, integrate over C and apply Green's identity. Also we may assume

$$\int_C h(\zeta)\,ds_\zeta=0 \tag{18}$$

and obtain
$$\iint_D[\nabla(u-F)\nabla h]\,d\tau+\sum_{\nu=1}^{N}\frac{1}{r_\nu}\int_{C_\nu}e^{u(z)}h(z)\,ds_z=0. \tag{19}$$

This condition which is equivalent to the integral equation (11) suggests the following minimum problem:

Let Ω be the class of all functions $u(z)$ with the following properties:

(a) $u(z)$ is harmonic in D and has a finite Dirichlet integral

(b) $u(z)$ belongs to the class $\mathfrak{L}^2$ on C

(c) $\int_C u\,ds = 0$

(d) e^u belongs to the class $\mathfrak{L}^1$ on C.

We define the functional

$$\Phi[u] = \tfrac{1}{2}\iint_D [\nabla(u-F)]^2\,d\tau + 2\pi \sum_{\nu=1}^{N} \log \int_{C_\nu} e^u\,ds, \tag{20}$$

which has a finite value for every $u \in \Omega$. We pose the problem to find a function $u(z) \in \Omega$ which minimizes the functional. Such an extremum function $u(z)$ would satisfy the minimum condition (19).

4. By the inequality between the geometric and the arithmetic mean [7], [18], we have for every real-valued function $f(z)$ defined on C, of class $\mathfrak{L}^2$ and such that $e^f \in \mathfrak{L}^1$ the following inequality:

$$\log \frac{1}{L_\nu}\int_{C_\nu} e^{f(z)}\,ds \geqslant \frac{1}{L_\nu}\int_{C_\nu} f(z)\,ds. \tag{21}$$

On the other hand, every function $u(z) \in \Omega$ satisfies the estimate

$$\int_C u^2\,ds \leqslant \lambda_1^{-1} \iint_D (\nabla u)^2\,d\tau \tag{22}$$

where λ_1 is the lowest non-trivial Stekloff eigen value of D [2]. That is, λ_1 is the lowest eigen value for which a non-constant harmonic function $S(z)$ in D exists such that

$$\frac{\partial S(z)}{\partial n} = -\lambda S(z) \quad \text{on} \quad C. \tag{23}$$

Applying the Schwarz inequality and (22), we conclude

$$\left(\sum_{\nu=1}^{N} \frac{1}{L_\nu}\int_{C_\nu} u\,ds\right)^2 \leqslant \left(\sum_{\nu=1}^{N}\frac{1}{L_\nu}\right)\int_C u^2\,ds \leqslant k \iint_D (\nabla u)^2\,d\tau, \tag{24}$$

with a properly chosen constant k which depends only on D. Finally, we may combine (21), (24) and the fact that F is regular in $D+C$ to obtain the following estimate for the functional (20)

$$\Phi[u] \geqslant \tfrac{1}{2}\iint_D (\nabla u)^2 \, d\tau + k_1\left[\iint_D (\nabla u)^2 \, d\tau\right]^{\frac{1}{2}} + k_2$$

$$\geqslant \tfrac{1}{2}\left(\left[\iint_D (\nabla u)^2 \, d\tau\right]^{\frac{1}{2}} - k_1\right)^2 + k_3 \geqslant k_3. \tag{25}$$

Thus, $\Phi[u]$ is bounded from below for all $u \in \Omega$. Let

$$\mu = \text{g.l.b. } \Phi[u] \quad \text{for all} \quad u \in \Omega \tag{26}$$

and consider a minimum sequence $v_\varrho(z)$ in Ω such that

$$\mu_\varrho = \Phi[v_\varrho] \to \mu \quad \text{as} \quad \varrho \to \infty. \tag{26'}$$

Since all $\Phi[u_\varrho]$ are bounded, we conclude from (25) and (26′)

$$\iint_D (\nabla u_\varrho)^2 \, d\tau \leqslant B \tag{27}$$

and, consequently, by virtue of (22)

$$\int_C u_\varrho^2 \, ds \leqslant \lambda_1^{-1} B. \tag{27'}$$

We may assume without loss of generality that the minimum sequence $u_\varrho(z)$ converges in D to a harmonic function $u(z)$ and that the sequence converges almost everywhere to the boundary value $u(z)$ on C. Clearly, $u(z) \in \mathfrak{L}^2$ on C and

$$\int_C u \, ds = 0. \tag{28}$$

Consider now

$$\log \frac{2\pi r_\nu^{(\varrho)}}{L_\nu} = \log \frac{1}{L_\nu}\int_{C_\nu} e^{u_\varrho} \, ds \geqslant \frac{1}{L_\nu}\int_{C_\nu} u_\varrho \, ds \geqslant -\left(\frac{1}{L_\nu}\int_{C_\nu} u_\varrho^2 \, ds\right)^{\frac{1}{2}} \geqslant -\left(\frac{B}{L_\nu \lambda_1}\right)^{\frac{1}{2}}. \tag{29}$$

We see that all $r_\nu^{(\varrho)}$ are bounded from below; hence, because of the boundedness of the $\Phi[u_\varrho]$ we can also assert that the $r_\nu^{(\varrho)}$ are bounded from above. Thus, there exists a constant A such that

$$\int_{C_\nu} e^{u_\varrho} \, ds \leqslant A \quad \text{for all } \nu \text{ and all } \varrho. \tag{30}$$

The functions $e^{u_\varrho} \in \mathfrak{L}^1$ are positive, converge almost everywhere to the limit e^u and have bounded integrals. Hence, by Fatou's theorem [19], we know that $e^u \in \mathfrak{L}^1$ and that

$$\int_{C_\nu} e^u ds \leqslant \lim \int_{C_\nu} e^{u_\nu} ds. \tag{31}$$

Inserting this inequality into (26′), we find

$$\Phi[u] \leqslant \lim \mu_\varrho = \mu. \tag{32}$$

Since we have just shown that the limit function $u(z)$ belongs to the class Ω, the very definition of μ implies

$$\Phi[u] = \mu. \tag{33}$$

Thus, the existence of a minimum function $u(z) \in \Omega$ with respect to the functional (20) is proved. It must necessarily satisfy the variational condition (19).

In order to utilize the extremum condition (19), we introduce the Green's function of the domain D, considered, $G(z,\zeta)$ and the reproducing kernel [2]

$$K(z,\zeta) = N(z,\zeta) - G(z,\zeta). \tag{34}$$

It is well-known that $K(z,\zeta)$ is harmonic as a function of z in the closure $D+C$ as long as ζ is kept fixed in D. We may use $K(z,\zeta)=h(z)$ in (19). Observe that for $z \in C$ we have $K(z,\zeta) = N(z,\zeta)$ and that

$$\iint_D [\nabla(u-F)\nabla K(z,\zeta)]\, d\tau_z = u(\zeta) - F(\zeta). \tag{35}$$

Hence, we obtain for arbitrary $\zeta \in D$ the identity

$$u(\zeta) = F(\zeta) - \sum_{\nu=1}^{N} \frac{1}{2\pi r_\nu} \int_{C_\nu} N(z,\zeta)\, e^{u(z)}\, ds_z \tag{36}$$

which goes over into our initial integral equation (11) as $\zeta \to C$. The existence of a solution $u(z)$ has thus been established for this non-linear integral equation.

The integral equation (11) is closely related to the non-linear integral equations considered by Hammerstein [6]. It does not entirely fit into the Hammerstein theory since the kernel $N(z,\zeta)$ is not bounded on C. However, the same solution method has been followed as in the classical case, namely the reduction to an extremum problem in function space.

5. For the sake of completeness we shall now show that $u(z)$ is the real part of a complex-valued function which is analytic in $D+C$. We bring condition (19) into the form (cf. (18)).

$$\int_C u \frac{\partial h}{\partial h} ds - \sum_{\nu=1}^{N} \int_{C_\nu} \left[\frac{1}{r_\nu} e^{u(z)} + \frac{\partial F}{\partial n} - \frac{2\pi}{L} N \right] h\, ds = 0. \tag{19'}$$

We define on each boundary curve C_ν the absolutely continuous function

$$\varphi(s) = \int \left(\frac{e^u}{r_\nu} + \frac{\partial F}{\partial n} - \frac{2\pi}{L} N \right) ds = \int \left(\frac{e^u}{r_\nu} + \varkappa(s) \right) ds = \int \frac{e^u}{r_\nu} ds + \frac{1}{i} \log (\dot{z}\, i). \tag{37}$$

By virtue of (17), we conclude that $\varphi(s)$ is single-valued on each C_ν and we can, therefore, find a single-valued harmonic function $v(z)$ in D which takes the boundary values (37). Identity (19′) asserts that

$$W(z) = u(z) - iv(z) \tag{38}$$

is an analytic function in D. So is also $V(z) = e^{W(z)}$ with the boundary values in C_ν

$$V(z) = e^{W(z)} = \dot{z}^{-1} e^u \frac{1}{i} \exp \left\{ -i \int \frac{e^u}{r_\nu} ds \right\}. \tag{39}$$

We can state that
$$\frac{d}{ds} \int V\, dz = r_\nu \frac{d}{ds} \exp \left\{ -i \int \frac{e^u}{r_\nu} ds \right\}. \tag{40}$$

Hence, the analytic function $\int V\, dz$ satisfies on each C_ν

$$\int V\, dz = \text{const.} + r_\nu \exp \left\{ -i \int \frac{e^u}{r_\nu} ds \right\}. \tag{41}$$

It maps the analytic arc C_ν onto a circumference of radius r_ν. Hence, $V(z)$ is analytic on C_ν and so is, consequently, $W(z)$. This proves our assertion. Instead of the assertion that the integral equation (11) has a solution of class $\mathfrak{L}^2$ on C we have now the theorem that $u(z)$ may be differentiated any number of times with respect to the arc length on the boundary.

6. It is equally simple to prove the uniqueness of the circular mapping. We observe that

$$\frac{d^2}{d\varepsilon^2} \Phi [u + \varepsilon v] = \iint_D (\nabla v)^2 d\tau + 2\pi \sum_{\nu+1}^{N} \frac{1}{2\pi r_\nu} \int_{C_\nu} e^{u+\varepsilon v} v^2 ds - 2\pi \sum_{\nu=1}^{N} \frac{1}{(2\pi r_\nu)^2} \left(\int_{C_\nu} e^{u+\varepsilon v} v\, ds \right)^2, \tag{42}$$

with
$$2\pi r_\nu = \int_{C_\nu} e^{u+\varepsilon v} ds. \tag{42'}$$

But by the Schwarz inequality we have for non-constant v

$$2\pi r_\nu \int_{C_\nu} e^{u+\varepsilon v} v^2 ds > \left(\int_{C_\nu} e^{u+\varepsilon v} v\, ds \right)^2. \tag{43}$$

We recognize that $\Phi[u+\varepsilon v]$ is convex in dependence on ε:

$$\frac{d^2}{d\varepsilon^2}\Phi[u+\varepsilon v]>0. \tag{44}$$

Since for the minimum function $u(z)$ holds

$$\frac{d}{d\varepsilon}\Phi[u+\varepsilon v]=0 \quad \text{for} \quad \varepsilon=0 \tag{45}$$

we have

$$\frac{d}{d\varepsilon}\Phi[u+\varepsilon v]>0 \quad \text{for} \quad \varepsilon>0. \tag{45'}$$

Hence, $\Phi[u+v]>\Phi[u]$ for every choice of v. Furthermore, no other local minimum can occur since the functional $\Phi[u]$ must surely decrease in the linear family which connects the function considered with the minimum function $u(z)$.

7. It should be observed that the functional (20) can be written in the form

$$\Phi[u]=\tfrac{1}{2}\iint_D(\nabla u)^2\,d\tau+\int_C u\varkappa(s)\,ds+2\pi\sum_{\nu=1}^{N}\log\int_{C_\nu}e^u\,ds+\tfrac{1}{2}\iint_D(\nabla F)^2\,d\tau. \tag{46}$$

Since the harmonic function $F(z)$ is a fixed function for a given domain D, we may characterize $u(z)$ as that function of the class Ω which minimizes the functional

$$\hat{\Phi}[u]=\tfrac{1}{2}\iint_D(\nabla u)^2\,d\tau+\int_C u\varkappa\,ds+2\pi\sum_{\nu=1}^{N}\log\int_{C_\nu}e^u\,ds. \tag{47}$$

The advantage of this formulation of the minimum principle for circular mapping is that the function $F(z)$ has been eliminated. Thus, we do not have to solve an auxiliary boundary value problem in the domain D in order to determine $F(z)$ and to set up the original extremum problem.

We have the identity, valid for arbitrary constant c,

$$\hat{\Phi}[u+c]=\hat{\Phi}[u]. \tag{48}$$

Thus, the side condition that the contour integral of u over C should vanish can now be dropped. We introduce the wider function class Ω^* which consists of all functions $u(z)$ harmonic in D such that $u\in\mathfrak{L}^2$ and $e^u\in\mathfrak{L}^1$ on C. The extremum problem for $\hat{\Phi}[u]$ may be framed best within this class Ω^*.

8. The preceding reasoning can be applied in order to prove more general existence theorems for harmonic functions with non-linear boundary conditions. Let, for example,

$p(s)$ be a positive and continuous function on the system C of boundary curves C_ν. Consider the functional

$$\Psi[u] = \tfrac{1}{2}\iint_D [\nabla(u-F)]^2 \, d\tau + 2\pi \sum_{\nu=1}^{N} \log \int_{C_\nu} e^u p(s)\, ds \tag{49}$$

which is well-defined in the class Ω. In view of definition (20), we can assert that

$$\Psi[u] \geqslant \Phi[u] + 2\pi N \log(\min p(s)). \tag{50}$$

Thus, the new functional has a finite lower bound within Ω. We can, therefore, repeat the above arguments to show that there exists a unique function $u(z) \in \Omega$ for which the minimum value of $\psi[u]$ with respect to this class is attained. It is immediately seen that the minimum function satisfies on C_ν the boundary condition

$$\frac{\partial u}{\partial n} = \varkappa(s) + \frac{1}{m_\nu} p(s)\, e^u \tag{51}$$

with

$$2\pi m_\nu = \int_{C_\nu} e^u p(s)\, ds. \tag{51'}$$

We conclude from (51) and (51′) that the harmonic function $u(z)$ can be completed to an analytic function, say $\log f'(z)$, which is single valued in the domain D. We can then determine the analytic function $f(z)$ which maps the curve C_ν onto curves Γ_ν such that the curvature of Γ_ν at the image point of $z(s)$ has the value $-p(s)/m_\nu$. This is an immediate consequence of (6) and (51).

There arises now the question of the behavior of the analytic function $f(z)$ in the large. We know that $f'(z)$ is non-zero, regular and singlevalued in D; but its integral $f(z)$ might have a logarithmic singularity at infinity and have further periods due to the multiple connectivity of the domain. Thus, we cannot assert, in general, that the curves Γ_ν are closed.

It is a fortunate accident that we can assert for $p(s)$ = constant that $f(z)$ does not possess any additive periods. This follows from the fact that a circle does not allow a translation into itself and can also be seen from the following argument. We have

$$f'(z) = e^{u-iv}, \quad v = \int \frac{\partial u}{\partial n}\, ds \tag{52}$$

and, hence, are led to the condition for single-valuedness

$$\int_{C_\nu} e^{u-iv}\, dz = 0. \tag{53}$$

Observe now that by (51), (1) and (6) we have on C_ν

$$v = \int \frac{\partial u}{\partial n} ds = -i \log \dot{z} + \frac{1}{m_\nu} \int e^u p(s)\, ds; \tag{52'}$$

hence (53) takes the form

$$\int_{C_\nu} e^u \exp \left\{ \frac{1}{i m_\nu} \int^s e^u p(\sigma)\, d\sigma \right\} ds = 0, \tag{53'}$$

that is

$$\int_{C_\nu} \frac{1}{p(s)} \frac{d}{ds} \exp \left\{ \frac{1}{i m_\nu} \int^s e^u p(\sigma)\, d\sigma \right\} ds = 0. \tag{53''}$$

In case that $p(s)$ is constant we can infer from (51′) that the left-hand integral in (53″) does indeed vanish and that $f(z)$ is single-valued and hence univalent in D. In every other case, the condition for $p(s)$ seems to be rather involved.

9. We return to the interesting functional $\hat{\Phi}[u]$ defined by (47). Until now we have considered this functional for a fixed domain D and all admissible functions $u(z) \in \Omega$. It is of interest to study how this functional varies if we change our original domain D by a conformal mapping or equivalently, under change of the uniformizer z. We refer the domain D to another domain D^* by the univalent conformal mapping

$$z = k(z^*) \tag{54}$$

which we assume to be continuously differentiable in $D + C$ and to have a non-vanishing derivative at infinity. We translate functions $u(z)$ in D into functions $u^*(z^*)$ in D^* by the correspondence rule

$$u(z) = \log \left| \frac{df}{dz} \right| \leftrightarrow u^*(z^*) = \log \left| \frac{df^*}{dz^*} \right| \tag{55}$$

i.e., we transplant the generating function $f(z)$ of $u(z)$ into $f(k(z^*)) = f^*(z^*)$ for $u^*(z^*)$. We have therefore

$$u^*(z^*) = u(z) + \log |k'(z^*)|. \tag{55'}$$

Since we have on the corresponding curve system C_ν^*

$$e^{u^*} ds^* = e^u ds \tag{56}$$

we see that the integrals $\int e^u ds$ are unchanged in the transition.

Next we apply the identity (6) to our particular mapping and find

$$\frac{\partial}{\partial n^*} \log |k'(z^*)| = \varkappa^*(s^*) - \varkappa(s) |k'(z^*)|. \tag{57}$$

Hence, an easy calculation yields

$$\frac{1}{2}\iint_D (\nabla u)^2 d\tau + \int_C u\varkappa\, ds = \frac{1}{2}\iint_{D^*} (\nabla u^*)^2 d\tau + \int_C u^* \varkappa^* ds^*$$
$$- \int_C \log|k'(z^*)|\varkappa\, ds + \frac{1}{2}\iint_{D^*} (\nabla \log|k'(z^*)|)^2 d\tau^*. \quad (58)$$

As was to be expected, the difference between the functionals $\hat{\Phi}^*[u^*]$ and $\hat{\Phi}[u]$ is independent of u and depends only on the domains D and D^*. Hence, the minimum function in D is referred by our rule (55) into the minimum function of D^*.

The transformation rule (58) can be written in very elegant form if we define the expression

$$[w, z] = \frac{1}{2}\iint_D \left(\nabla \log\left|\frac{dw}{dz}\right|\right)^2 d\tau + \int_C \log\left|\frac{dw}{dz}\right| \cdot \varkappa\, ds \quad (59)$$

connecting the two variables $w(z)$ and z with respect to the domain of the variable z. Putting $w = f(z)$ where $f(z)$ is the generating function of $u(z)$, we can formulate the identity (58) as follows:

$$[w, z] = [w, z^*] + [z^*, z]. \quad (60)$$

We can now state
$$\hat{\Phi}[u] = \hat{\Phi}^*[u^*] + [z^*, z]. \quad (61)$$

It is easily seen that this transformation law is valid for arbitrary differentiable functions $u(z)$ provided that they transform according to the law (55′).

The formal expression (59) will play an essential role in the general theory of connections which will be developed in the later chapters. It is clear from the very definition that

$$[z, z] = 0. \quad (59')$$

We shall use later the same expression for finite and simply-connected domains and show that $[w, z] = 0$ if $w(z)$ is a mapping of such a domain onto itself.

10. We shall now combine the results of the two preceding sections in order to derive an interesting inequality for conformal mappings between domains with convex boundary curves.

Let $f_1(z)$ and $f_2(z)$ be two univalent analytic functions in the original domain D of the z-plane which map the point at infinity into itself and carry D into domains Δ_1 and Δ_2, respectively, with boundaries Γ_1 and Γ_2 which are composed of convex curves. We denote by $-p_1(s)$ and $-p_2(s)$ the curvature at the image points $f_1(z(s))$ and $f_2(z(s))$ and, by our assumption, the $p_i(s)$ are positive.

If we denote $$\Psi_i[u] = \frac{1}{2}\iint_D (\nabla u)^2 d\tau + \int_C \varkappa u\, ds + 2\pi \sum_{\nu=1}^{N} \log \int_{C_\nu} e^u p_i(s)\, ds \tag{62}$$

and $$u_i(z) = \log |f_i'(z)|, \tag{62'}$$

our results in Section 8 allow us the following conclusions. The function $u_i(z)$ belongs to the class Ω^* and yields within this class the minimum value for the functional $\Psi_i[u]$. Hence, we have

$$\Psi_1[u_1] \leqslant \Psi_1[u_2], \quad \Psi_2[u_2] \leqslant \Psi_2[u_1]. \tag{63}$$

Adding these two inequalities and using the definition (62) of the functionals, we find

$$\sum_{\nu=1}^{N} \log \int_{C_\nu} e^{u_1} p_1\, ds \cdot \int_{C_\nu} e^{u_2} p_2\, ds \leqslant \sum_{\nu=1}^{N} \log \int_{C_\nu} e^{u_2} p_1\, ds \cdot \int_{C_\nu} e^{u_1} p_2\, ds. \tag{64}$$

Observe that $$e^{u_i} ds = d\sigma_i, \quad -p_i(s) = \varkappa_i(\sigma_i). \tag{65}$$

Hence $$\int_{C_\nu} e^{u_i} p_i\, ds = 2\pi \tag{65'}$$

and (64) can be expressed in the simple form

$$\prod_{\nu=1}^{N} \left(\frac{1}{2\pi}\int_{C_\nu} p_1\, d\sigma_2\right)\left(\frac{1}{2\pi}\int_{C_\nu} p_2\, d\sigma_1\right) \geqslant 1. \tag{66}$$

This is a very interesting inequality connecting the two conformally equivalent curve systems Γ_1 and Γ_2. The intermediate curve system C is quite unimportant and might, for example, be chosen as either Γ_i.

The result (66) can be better understood and even be generalized by the following consideration. Let us denote

$$\mathfrak{L}(w, p) = 2\pi \sum_{\nu=1}^{N} \log \frac{1}{2\pi}\int_{C_\nu} \left|\frac{dw}{dz}\right| p(s)\, ds. \tag{67}$$

With $w_i = f_i(z)$ and the notation (59) and (67), we may express the first inequality (63) in the form

$$[w_1, z] + \mathfrak{L}(w_1, p_1) \leqslant [w_2, z] + \mathfrak{L}(w_2, p_1). \tag{68}$$

But by virtue of (62′), (65′) and (67), we have

$$\mathfrak{L}(w_1, p_1) = 0. \tag{68'}$$

Finally, using the transformation law (60), we can bring (68) into the form:

$$[w_1, w_2] \leqslant \mathfrak{L}(w_2, p_1). \tag{69}$$

Let now $w_\varrho(z)$ be a sequence of conformal mappings of our initial domain D onto domains Δ_ϱ with convex boundary $\Gamma_\varrho(\varrho=1,\dots,m)$. Since by extension of (60)

$$[w_1, w_m] = \sum_{\varrho=1}^{m-1} [w_\varrho, w_{\varrho+1}], \tag{70}$$

we obtain

$$[w_1, w_m] \leqslant \sum_{\varrho=1}^{m-1} \mathfrak{L}(w_{\varrho+1}, p_\varrho). \tag{71}$$

In particular, if $w_m = w_1$ we arrive at the inequality

$$0 \leqslant \sum_{\varrho=1}^{m-1} \mathfrak{L}(w_{\varrho+1}, p_\varrho), \tag{72}$$

which reduces for $m=3$ to (66).

11. In order to understand the close relation between the functionals $[w, z]$ and $\mathfrak{L}(w, p)$, let us consider the one-parameter family of univalent conformal mappings $w=w(z, t)$ which depend differentiably upon the parameter t and which map infinity into infinity. Let

$$\Phi(t) = [w(z,t), w(z, 0)] \tag{73}$$

and denote

$$\frac{d\Phi}{dt} = \dot{\Phi}(t), \quad \frac{\partial}{\partial t} \log \left| \frac{\partial w(z,t)}{\partial z} \right| = \dot{u}(z,t). \tag{74}$$

From definition (59), we calculate

$$\dot{\Phi}(t) = \iint_D \nabla \log \left| \frac{\partial w}{\partial z} \right| \cdot \nabla \dot{u}(z,t)\, d\tau + \int_C \varkappa\, \dot{u}\, ds. \tag{75}$$

By the Green's identity, this expression may be reduced to a line integral over C:

$$\dot{\Phi}(t) = \int_C \dot{u}(z,t) \left[\varkappa(s) - \frac{\partial}{\partial n} \log \left| \frac{\partial w}{\partial z} \right| \right] ds. \tag{76}$$

Next we apply the transformation law (6) for the curvature. Let

$$\varkappa_t = -p(s, t) \tag{77}$$

be the curvature of the image Γ_t at the point $w(z(s),t)$. Then (76) can be brought into the simple form:

$$\dot{\Phi}(t) = \int_C \dot{u}(z,t)\, \varkappa_t \left| \frac{\partial w(z,t)}{\partial z} \right| ds = \int_C \varkappa_t \frac{\partial}{\partial t} \left| \frac{\partial w(z,t)}{\partial z} \right| ds. \tag{78}$$

We have, on the other hand,

$$\mathfrak{L}(w(z,t+\Delta t),\ p(s,t)) = 2\pi \sum_{\nu=1}^{N} \log \left\{ \frac{1}{2\pi} \int_{C_\nu} \left| \frac{\partial w(z,t+\Delta t)}{\partial z} \right| p(s,t)\, ds \right\}. \tag{79}$$

We observe that in analogy to (68′) we have

$$\mathfrak{L}(w(z,\ t),\ p(s,\ t)) = 0. \tag{79′}$$

Hence, developing the right hand side into a Taylor series in powers of Δt and using (77) and (78), we find

$$\mathfrak{L}(w(z,\ t+\Delta t),\ p(s,\ t)) = -\dot{\Phi}(t)\Delta t + 0(\Delta t^2). \tag{80}$$

We have thus proved the interesting identity

$$\frac{d}{dt}[w(z,t),\ w(z,0)] = -\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathfrak{L}(w(z,t+\Delta t),\ p(s,t)) \tag{81}$$

which is by no means restricted to convex mappings only. We have, indeed, for every one-parameter family of univalent mappings

$$[w(z,0),\ w(z,t)] = \lim_{n\to\infty} \sum_{\varrho=0}^{n} \mathfrak{L}(w(z,t_{\varrho+1})\ p(s,t_\varrho)) \tag{82}$$

provided that $\max(t_{\varrho+1} - t_\varrho)$ tends to zero with $n \to \infty$. The interesting fact in the case of convex mappings is expressed by the inequality (71), which states that in this special case each finite approximate sum is always larger than the limit approximated. We learn, on the other hand, that the inequality (71) is sharp in the sense that the difference between both sides can be made arbitrarily small for large enough values of m.

II. Variational theory for Moduli

1. Given a domain D bounded by N curves C_ν and containing the point at infinity, we introduce the class $\mathfrak{C}$ of all domains Δ, which can be obtained from D by conformal mapping with a univalent mapping function $\zeta = f(z)$, which has at infinity a Laurent development

$$\zeta = f(z) = z + \frac{k_1}{z} + \frac{k_2}{z^2} + \cdots. \tag{1}$$

From the results of the preceding chapter it follows that there exists in $\mathfrak{C}$ a unique domain Δ which is bounded by N circumferences. Let $c_\nu = a_\nu + ib_\nu$ denote the coordinates of the centers of these circles and let $r_\nu > 0$ denote their radii. The $3N$ real numbers a_ν, b_ν and r_ν are characteristic for the equivalence class $\mathfrak{C}$; they are called a set of moduli for the class $\mathfrak{C}$.

The importance of moduli in the theory of conformal mapping is well known. The above definition of moduli is a special case of the following general method for defining moduli. Let $\varphi(z)$ be an analytic single-valued function defined on the unit circle $|z| = 1$, which maps the circumference in a one-to-one manner onto the closed curve Γ_0. The functions

$$\zeta = c_\nu + r_\nu \varphi(e^{i\tau}), \quad r_\nu > 0, \quad 0 \leqslant \tau \leqslant 2\pi \tag{2}$$

will then lead to curves Γ_ν, which might be called curves similar to Γ_0. For many curves Γ_0 it can be shown that there exists in each equivalence class $\mathfrak{C}$ precisely one domain Δ whose boundary curves Γ_ν are similar to Γ_0. This domain Δ may be called the canonical domain in $\mathfrak{C}$ with respect to the curve Γ_0. It is well known that there exists a canonical domain in $\mathfrak{C}$ with respect to every convex curve Γ_0 [4]. The numbers c_ν and r_ν are then the moduli of the class $\mathfrak{C}$ with respect to Γ_0.

An important question arises now: how do the moduli of $\mathfrak{C}$ depend on the original domain D? That is, if the boundary curves C_ν are subjected to a specified infinitesimal deformation to find the corresponding variation of the moduli under the resulting change of equivalence class. This problem has been solved in some special cases, e.g., the parallel-slit mapping. This is due to the fact that many important canonical mappings can be expressed in terms of the Green's function of the domain D considered and that the variation of the Green's function with the domain is well known. Previously, no analogous formula had been established for the moduli of the circular mapping. Thus, we propose to study in this chapter the variational formulas for these moduli. The significance of such variational theory in extremum problems of conformal mapping and in the study of conformal equivalence is evident.

2. For the sake of simplicity, we start with a domain D which is already in canonical form with respect to a given convex curve Γ_0. That is, each boundary curve C_ν of D admits the representation (2) in terms of a parameter τ. Let then z_0 be an arbitrary point in D and consider the analytic function

$$z^* = z + \frac{\varrho^2 e^{i\alpha}}{z - z_0} \quad (\varrho > 0). \tag{3}$$

This function is univalent in the circular region $|z - z_0| > \varrho$ and hence, for small enough ϱ, we can assume that all curves C_ν lie in the region of univalence of z^* and are mapped into new simple curves C_ν^*. These curves will determine a new domain D^* which will be considered as the variation of D. In general, the new domain D^* will not belong to the equivalence class

$\mathfrak{C}$ of D but will admit new moduli c_ν^* and r_ν^*. Our aim is to express these new moduli in terms of domain functions and moduli of the original domain D. Specifically, we shall derive asymptotic formulas for c_ν^* and r_ν^* in powers of ϱ^2. That such a development is possible in principle follows from the non-linear integral equation (1.11) in the case of circular canonical domains. We shall restrict ourselves here to a purely formal derivation of the variational formulas; the results will then apply to any canonical domain for which an asymptotic formula can be obtained at all.

We observe that the varied domain D^* will, in general, not be in canonical form. There exists, therefore, in D^* a univalent function $\zeta = h(z)$ with the normalization (1) which carries D^* into canonical form. We have for $h(z)$ the asymptotic development

$$\zeta = h(z) = z + \varrho^2 l(z) + o(\varrho^2) \tag{4}$$

since it reduces to the identity mapping for $\varrho = 0$. The function $l(z)$ is regular analytic in D^* and continuous in $D^* + C^*$.

Setting
$$z^*(\tau) = z^*(c_\nu + r_\nu \varphi(e^{i\tau})) \tag{5}$$

we can use for the curves C_ν^* the same parameter τ as on the curves C_ν. On the other hand, we have by the definition of $h(z)$ the representation

$$\zeta = h(z^*(\tau)) = c_\nu^* + r_\nu^* \varphi(e^{i\tau^*}). \tag{6}$$

We set up the asymptotic developments

$$c_\nu^* = c_\nu + \varrho^2\gamma_\nu + o(\varrho^2); \quad r_\nu^* = r_\nu + \varrho^2 R_\nu + o(\varrho^2); \quad \tau = \tau^* + \varrho^2 T(\tau^*) + o(\varrho^2). \tag{7}$$

It is clear from these definitions that the γ_ν are complex numbers while R_ν and $T(\tau^*)$ are real-valued.

Inserting (3), (4) and (5) into (6), we obtain

$$c_\nu^* + r_\nu^* \varphi(e^{i\tau^*}) = z + \frac{e^{i\alpha}\varrho^2}{z^* - z_0} + \varrho^2 l(z^*) + o(\varrho^2) \tag{8}$$

and thus by (7)

$$c_\nu + \varrho^2\gamma_\nu + (r_\nu + \varrho^2 R_\nu)\varphi(e^{i\tau^*}) + o(\varrho^2) = c_\nu + r_\nu\varphi(e^{i\tau^*} + ie^{i\tau^*}\varrho^2 T(\tau^*)) + \frac{e^{i\alpha}\varrho^2}{z^* - z_0} + \varrho^2 l(z^*). \tag{8'}$$

Comparing the coefficients of ϱ^2 on both sides of this asymptotic identity, we find

$$\gamma_\nu + R_\nu\varphi(e^{i\tau^*}) - r_\nu\varphi'(e^{i\tau^*})\, ie^{i\tau^*}T(\tau^*) = \frac{e^{i\alpha}}{z^* - z_0} + l(z^*). \tag{9}$$

We have on the left side of this equation a function determined only on the boundary curves C_ν^*, but on the right side a meromorphic function defined in D^*. Except for errors of higher order in ϱ^2, $l(z^*)$ will coincide in the common interior of D and D^* with the analytic function in D defined by the boundary condition

$$\frac{e^{i\alpha}}{z-z_0}+l(z)=\gamma_\nu+R_\nu\varphi(e^{i\tau})-r_\nu\varphi'(e^{i\tau})\,ie^{i\tau}T(\tau) \tag{9'}$$

on C. Hence, we are permitted to use this function $l(z)$ in the asymptotic development for $h(z)$ in the interior of D.

We make use of the fact that by virtue of (2) the unit tangent vector at $z \in C_\nu$ has the form

$$\dot{z}=i\,r_\nu e^{i\tau}\varphi'(e^{i\tau})\cdot\frac{d\tau}{ds}\cdot \tag{10}$$

Hence, we can bring condition (9′) into the form

$$\frac{e^{i\alpha}}{z-z_0}+l(z)=\gamma_\nu+\frac{R_\nu}{r_\nu}(z-c_\nu)-T(\tau)\frac{d\tau}{ds}\cdot\dot{z}, \tag{11}$$

on each C_ν.

We shall show in the next section that this boundary value problem for analytic functions in D can be solved only for a unique choice of the parameters γ_ν and R_ν. We shall give the necessary values for these parameters which allow a solution and determine in this way the first term in the asymptotic formulas (7) for the c_ν^* and r_ν^*.

3. We bring the boundary condition (11) for $l(z)$ into the form

$$\operatorname{Im}\left\{\bar{\dot{z}}\left[l(z)+\frac{e^{i\alpha}}{z-z_0}-\gamma_\nu-\frac{R_\nu}{r_\nu}(z-c_\nu)\right]\right\}=0, \tag{12}$$

which is a simplification since the unknown function $T(\tau)$ has been eliminated. The potential theoretic character of the boundary value problem (12) can be understood best if we introduce the harmonic potential $\omega(x, y)$ for the analytic function $l(z)$ in D:

$$l(z)=\frac{\partial\omega}{\partial x}-i\frac{\partial\omega}{\partial y}\cdot \tag{13}$$

Then (12) can be written as

$$\frac{\partial\omega}{\partial x}\dot{y}+\frac{\partial\omega}{\partial y}\dot{x}=\operatorname{Im}\left\{\bar{\dot{z}}\left[\frac{e^{i\alpha}}{z-z_0}-\gamma_\nu-\frac{R_\nu}{r_\nu}(z-c_\nu)\right]\right\}\cdot \tag{14}$$

This is an oblique Neumann problem for the domain D considered and the γ_ν, R_ν are the accessory parameters which must be chosen in a compatible way.

In order to determine the admissible values for the γ_ν and R_ν we proceed as follows. We introduce the class $\mathfrak{q}$ of quadratic differentials $Q(z)$ in D. A function $Q(z)$ belongs to $\mathfrak{q}$ if it is regular analytic in D, vanishes at infinity and satisfies on the boundary C of D the condition

$$Q(z)\,\dot{z}^2 = \text{real}. \tag{15}$$

Let us denote, on the other hand,

$$q_\nu(z) = l(z) + \frac{e^{i\alpha}}{z - z_0} - \gamma_\nu - \frac{R_\nu}{r_\nu}(z - c_\nu). \tag{16}$$

Clearly, we have by (12) and (15) on each C_ν

$$q_\nu(z)\,Q(z)\,dz = \bar{\dot{z}}\,q_\nu(z)\,\dot{z}^2 Q(z)\,ds = \text{real}. \tag{17}$$

Hence, we can assert the equation

$$\operatorname{Im}\left\{\sum_{\nu=1}^{N}\int_{C_\nu} q_\nu(z)\,Q(z)\,dz\right\} = 0 \tag{17'}$$

for each $Q(z) \in \mathfrak{q}$.

Observe further that the function (4) has the normalisation (1) which implies $l(\infty) = 0$. Hence, we can apply the residue theorem as follows:

$$\sum_{\nu=1}^{N}\int_{C_\nu} l(z)\,Q(z)\,dz = 0, \quad \sum_{\nu=1}^{N}\int_{C_\nu}\frac{Q(z)}{z - z_0}\,dz = 2\pi i\,Q(z_0). \tag{18}$$

These two equations allow us to calculate explicitly a large part of the terms in (17′) by use of (16). There remains the equivalent equation

$$\operatorname{Im}\left\{2\pi i\,e^{i\alpha}Q(z_0) - \sum_{\nu=1}^{N}\left[\gamma_\nu\int_{C_\nu} Q(z)\,dz + \frac{R_\nu}{r_\nu}\int_{C_\nu}(z - c_\nu)\,Q(z)\,dz\right]\right\} = 0. \tag{17''}$$

Thus, every quadratic differential $Q(z) \in \mathfrak{q}$ provides one linear equation for the $3N$ unknowns $\alpha_\nu = \operatorname{Re}\{\gamma_\nu\}$, $\beta_\nu = \operatorname{Im}\{\gamma_\nu\}$ and R_ν

$$\operatorname{Re}\{e^{i\alpha}Q(z_0)\} = \operatorname{Im}\left\{\sum_{\nu=1}^{N}\left[\gamma_\nu\cdot\frac{1}{2\pi}\int_{C_\nu} Q(z)\,dz + \frac{R_\nu}{r_\nu}\cdot\frac{1}{2\pi}\int_{C_\nu}(z - c_\nu)\,Q(z)\,dz\right]\right\}. \tag{19}$$

Therefore, the question arises how many linearly independent quadratic differentials are in the class $\mathfrak{q}$ and whether the equations (19) determine the unknowns in a unique manner.

4. A quadratic differential in the class $\mathfrak{q}$ is called a regular quadratic differential if if vanishes at infinity at least in the fourth order. It is well known that a domain D with N boundary continua C_ν $(N > 2)$ has precisely $3N - 6$ regular quadratic differentials which are linearely independent over the field of real numbers [26]. Consider now the following analytic functions defined in D:

(a) $p(z)$ is the analytic completion of the Green's function $g(z)$ with the logarithmic pole at infinity; $p'(z) \sim 1/z$ at infinity and $p'(z)\dot{z}$ = imaginary for $z \in C$.

(b) $f_0(z)$ and $f_1(z)$ are univalent in D and map this domain onto parallel-slit domains in the directoin of the real and imaginary axis, respectively, such that $f_0'(\infty) = f_1'(\infty) = 1$.

Then $f_0'(z)\dot{z}$ and $if_1'(z)\dot{z}$ are real for $z \in C$. We construct then $if_0'(z)p'(z)$ and $f_1'(z)p'(z)$ which are two quadratic differentials of class $\mathfrak{q}$ which vanish at infinity only in the first order and which are evidently independent. Next, given two independent real differentials $w_1'(z)$ and $w_2'(z)$ of D, that is analytic functions of z which satisfy $w_\nu'(z)\dot{z}$ = real for $z \in C$ and which vanish at infinity to the second order, we construct $f_0(z)w_1'(z)$ and $f_0(z)w_2'(z)$ which lie both in the class $\mathfrak{q}$ and vanish at infinity in second order. Finally, $ip'(z)w_1'(z)$, $ip'(z)w_2'(z)$ lie in $\mathfrak{q}$ and vanish at infinity in the third order. Clearly, from the six quadratic differentials which were constructed explicitly and from the $3N-6$ regular quadratic differentials all elements of the class $\mathfrak{q}$ can be obtained. We have thus shown that for $N>2$ the class $\mathfrak{q}$ has precisely $3N$ linearly independent quadratic differentials.

The same assertion can be made in the cases $N=1$ and $N=2$. For N = 1 the elements $if_0'(z)p'(z)$, $f_1'(z)p'(z)$ and $p'(z)^2$ are the basis for the class $\mathfrak{q}$, and for $N=2$ we have the basis $if_0'(z)p'(z)$, $f_1'(z)p'(z)$, $f_0'(z)w'(z)$, $if_1'(z)w'(z)$, $ip'(z)w'(z)$, $w'(z)^2$ where $w'(z)$ is the one real differential of D. This shows that $\mathfrak{q}$ has in every case $3N$ independent elements.

We select a basis of $3N$ quadratic differentials in $\mathfrak{q}$ and denote them by $Q_\mu(z)\,(\mu = 1,\ldots,3N)$. We shall have to solve sets of linear equations with respect to various periods of the Q_μ with respect to the curves C_ν and the following determinant will play a central role:

$$\Delta = \left\| \operatorname{Im}\left\{\frac{1}{2\pi}\int_{C_\nu} Q_\mu dz\right\};\ \operatorname{Re}\left\{\frac{1}{2\pi}\int_{C_\nu} Q_\mu dz\right\};\ \operatorname{Im}\left\{\frac{1}{2\pi}\int_{C_\nu} z\,Q_\mu dz\right\} \right\| \tag{20}$$

where $\mu = 1,2,\ldots,3N$ is the row index, while $\nu = 1,\ldots,N$ determines the respective columns in the three vertical sections of Δ.

We wish to prove the fundamental inequality

$$\Delta \neq 0. \tag{21}$$

Indeed, suppose that for the domain D considered the determinant Δ did vanish. We could then obviously solve an appropriate homogeneous equation system with this determinant and construct an element $q(z) \in \mathfrak{q}$ which does not vanish identically and satisfies the conditions

$$\int_{C_\nu} q(z)\,dz = 0, \quad \operatorname{Im}\left\{\int_{C_\nu} z\,q(z)\,dz\right\} = 0. \tag{22}$$

This particular element of $\mathfrak{q}$ must also satisfy the equation (19), but now the right side of this equation would vanish because of (22). Since $e^{i\alpha}$ and z_0 are arbitrary, it follows that $q(z)$ must be identically zero, contrary to our assumption. Thus, we derived a contradiction from the hypothesis $\Delta = 0$ and the inequality (21) is proved.

The preceding proof that no quadratic differential $q(z) \in \mathfrak{q}$ with the properties (22) can exist holds only for canonical domains D for which the preceding variational theory is valid. It is, therefore, of interest to prove the following more general theorem:

Let D be a domain containing the point at infinity and bounded by N convex analytic curves C_ν. There does not exist a quadratic differential $q(z) \in \mathfrak{q}$ which satisfies the equations (22).

Indeed, let us assume there were such a $q(z)$. From the first set of equations (22) we could conclude that $q(z)$ would have a single-valued integral $F(z)$ in D. We may assume without loss of generality that $F(\infty) = 0$. The second set of equations shows next that

$$\int_{C_\nu} F'(z)\, z\, dz = -\int_{C_\nu} F(z)\, dz = \text{real}. \tag{23}$$

The Cauchy-Riemann equations between the real and imaginary parts of $F(z)$ can be interpreted as integrability conditions for the existence of a harmonic potential $\omega(x, y)$ such that

$$iF(z) = \frac{1}{2}\left(\frac{\partial \omega}{\partial x} - i\frac{\partial \omega}{\partial y}\right) = \frac{\partial \omega}{\partial z}. \tag{24}$$

Since D is not simply-connected, $\omega(x, y)$ would not need to be single-valued in D. However, (23) takes now the form

$$\int_{C_\nu} \left(\frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy\right) = \int_{C_\nu} d\omega = 0, \tag{25}$$

which shows that $\omega(x, y)$ actually is single-valued in D.

It is possible that $F(z)$ has a residue at infinity; if so, we can infer from (23) that it must be pure imaginary. Hence, $\omega(x, y)$ may have a logarithmic pole at infinity. In this case we may assume without loss of generality that $\omega \to +\infty$ as we approach the point at infinity and that $\omega(x, y)$ is bounded from below in D. Since D is analytically bounded, $q(z)$ is analytic in the closure of D and $\omega(x, y)$ is regular harmonic even on C. (As we shall see, this remark is important!)

We start with the identities

$$\frac{\partial \omega}{dz} = iF(z), \quad \frac{\partial^2 \omega}{\partial z^2} = iq(z), \tag{26}$$

where $q(z)$ is the hypothetical quadratic differential. We have the formal relations

$$\frac{\partial\omega}{\partial s}=\frac{\partial\omega}{\partial z}\dot{z}+\frac{\partial\omega}{\partial\bar{z}}\dot{\bar{z}}=2\ \mathrm{Re}\,\{iF(z)\,\dot{z}\} \tag{27}$$

and

$$\frac{\partial^2\omega}{\partial s^2}=2\ \mathrm{Re}\,\{iF'(z)\,\dot{z}^2+iF(z)\,\ddot{z}\}=2\ \mathrm{Re}\,\{iF(z)\,\ddot{z}\}, \tag{27'}$$

since $F'(z)=q(z)$ is a quadratic differential. By the Frenet formula we have

$$\ddot{z}=i\varkappa\dot{z}; \tag{28}$$

hence

$$\frac{\partial^2\omega}{\partial s^2}=-2\varkappa\ \mathrm{Re}\,\{F(z)\,\dot{z}\}. \tag{29}$$

On the other hand,

$$\frac{\partial\omega}{\partial n}=-\frac{\partial\omega}{\partial x}\dot{y}+\frac{\partial\omega}{\partial y}\dot{x}=-2\ \mathrm{Im}\left\{\frac{\partial\omega}{\partial z}\dot{z}\right\}=-2\ \mathrm{Re}\,\{F(z)\,\dot{z}\}. \tag{30}$$

Thus, $\omega(x, y)$ is harmonic in D with a possible logarithmic pole at infinity and satisfies the boundary condition

$$\frac{\partial^2\omega}{\partial s^2}=\varkappa\frac{\partial\omega}{\partial n}. \tag{31}$$

Having thus cast the problem of the quadratic differential $q(z)$ in the form of a boundary value problem for harmonic functions, we can now show that under our assumptions on ω we must have $\omega\equiv$ const. in D.

In fact, if ω is not constant in D it must have its minimum value on C since it becomes positive infinite at its logarithmic pole in D. We can assume without loss of generality that the minimum point lies at the origin and that the boundary curve C has there the x-axis as tangent and that the positive y-axis issues into D. We may also assume that $\omega(0,0)=0$ since ω is only determined up to an additive constant. Since $\omega(x, y)$ is regular harmonic on C we can develop it near the origin in a power series in x and y

$$\omega(x, y)=a_1x+a_2y+a_3(x^2-y^2)+a_4xy+\dots, \tag{32}$$

where the following terms are harmonic polynomials of the third and higher order. Since C is convex, we have $\varkappa<0$ and know that in a sufficiently small neighborhood of the origin all points with $y>0$ lie in D. Hence, the fact that $\omega=0$ is the minimum value of ω implies

$$a_1=0,\quad a_2\geqslant 0,\quad a_3\geqslant 0. \tag{33}$$

Clearly, at the origin

$$\frac{\partial\omega}{\partial n}=a_2\geqslant 0 \tag{34}$$

and
$$\frac{\partial^2 \omega}{\partial s^2} \geqslant 0 \tag{35}$$

because of the minimum property. But since $\varkappa < 0$, the boundary condition (31) implies an equality between a non-negative and a non-positive quantity; whence

$$a_2 = 0, \quad a_3 = 0. \tag{34'}$$

If $a_4 \neq 0$, we could find points near the origin with $y > 0$, but $\omega < 0$ in contradiction to the minimum property of $\omega(x, y)$ at the origin. Let then $P_l(x, y)$ be the first harmonic polynomial in the development (32) of degree l which does not vanish. Since $l \geqslant 3$ and

$$P_l(x, y) = A_l r^l \cos(l\varphi + \beta_l) \tag{36}$$

there exist surely points in D near the origin where $P_l(x, y) < 0$. Since P_l is the decisive term in the development (32) for a sufficiently small neighborhood of the origin, $\omega(x, y)$ could not be positive throughout D and hence could not attain its minimum at the origin. Thus the development (32) cannot have non-vanishing terms and our assertion is proved.

It might be interesting to point out the well-known identity

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = \frac{\partial^2 \omega}{\partial s^2} - \varkappa \frac{\partial \omega}{\partial n} + \frac{\partial^2 \omega}{\partial n^2} \tag{37}$$

valid for all twice differentiable functions of x and y. Thus, we may also formulate our result in the following form:

Let D be a domain containing the point at infinity and bounded by N convex analytic curves C_ν. The only harmonic functions $\omega(x, y)$ in D which satisfy the boundary condition

$$\frac{\partial^2 \omega}{\partial n^2} = 0 \quad \text{on } C \tag{31'}$$

are the constants. This result remains valid even if we allow a logarithmic pole of ω at infinity.

5. We return now to the canonical domain D and apply the inequality (21) in order to determine the unknowns γ_ν and R_ν from the equations (19). For this purpose we introduce in $\mathfrak{q}$ a particular convenient basis $\{Q_\nu^{(1)}(z), Q_\nu^{(2)}(z), Q_\nu^{(3)}(z)\}$ of $3N$ quadratic differentials, normalized as follows:

$$\operatorname{Re}\left\{\frac{1}{2\pi}\int_{C_\nu} Q_\mu^{(1)}(z)\,dz\right\} = \delta_{\nu\mu}, \quad \operatorname{Im}\left\{\frac{1}{2\pi}\int_{C_\nu} Q_\mu^{(1)}(z)\,dz\right\} = 0, \quad \operatorname{Im}\left\{\frac{1}{2\pi}\int_{C_\nu} z\,Q_\mu^{(1)}(z)\,dz\right\} = 0; \tag{38}$$

$$\operatorname{Re}\left\{\frac{1}{2\pi}\int_{c_\nu} Q_\mu^{(2)}(z)\,dz\right\}=0, \quad \operatorname{Im}\left\{\frac{1}{2\pi}\int_{c_\nu} Q_\mu^{(2)}(z)\,dz\right\}=\delta_{\nu\mu}, \quad \operatorname{Im}\left\{\frac{1}{2\pi}\int_{c_\nu} z\,Q_\mu^{(2)}(z)\,dz\right\}=0; \quad (38')$$

$$\frac{1}{2\pi}\int_{c_\nu} Q_\mu^{(3)}(z)\,dz=0, \quad \operatorname{Im}\left\{\frac{1}{2\pi}\int_{c_\nu} z\,Q_\mu^{(3)}(z)\,dz\right\}=\delta_{\nu\mu}. \quad (38'')$$

Such a basis can always be constructed and is uniquely determined because of $\Delta \neq 0$.

Using these particular quadratic differentials of $\mathfrak{q}$ as test functions in (19), we find immediately

$$\frac{R_\nu}{r_\nu}=\operatorname{Re}\{e^{i\alpha}Q_\nu^{(3)}(z_0)\} \quad (39)$$

$$\operatorname{Re}\{\gamma_\nu\}=\operatorname{Re}\{e^{i\alpha}Q_\nu^{(2)}(z_0)\}+\frac{R_\nu}{r_\nu}\operatorname{Re}\{c_\nu\} \quad (39')$$

$$\operatorname{Im}\{\gamma_\nu\}=\operatorname{Re}\{e^{i\alpha}Q_\nu^{(1)}(z_0)\}+\frac{R_\nu}{r_\nu}\operatorname{Im}\{c_\nu\}. \quad (39'')$$

In the notation of the variational calculus we may express our result as follows. If the canonical domain D is subjected to the particular variation (3), we have the following variation of moduli:

$$\delta\log r_\nu=\operatorname{Re}\{e^{i\alpha}\varrho^2Q_\nu^{(3)}(z_0)\} \quad (40)$$

$$\delta\operatorname{Re}\left\{\frac{c_\nu}{r_\nu}\right\}=\operatorname{Re}\{e^{i\alpha}\varrho^2Q_\nu^{(2)}(z_0)\}\cdot\frac{1}{r_\nu} \quad (40')$$

$$\delta\operatorname{Im}\left\{\frac{c_\nu}{r_\nu}\right\}=\operatorname{Re}\{e^{i\alpha}\varrho^2Q_\nu^{(1)}(z_0)\}\cdot\frac{1}{r_\nu}. \quad (40'')$$

It is to be expected from the general theory of moduli that their variation should be described by means of quadratic differentials. The main result of this section is the simple and general characterization of the quadratic differentials which belong to the various moduli.

Just as we did in Section 4, we can express the conditions on the various quadratic differentials $Q_\nu^{(i)}(z)$ in a potential theoretic form. For example, starting from (38″) we conclude the existence of a regular analytic function $F_\nu(z)$ in D which vanishes at infinity and satisfies

$$\frac{d}{dz}F_\nu(z)=Q_\nu^{(3)}(z). \quad (41)$$

We introduce again a (multivalued) harmonic potential $\omega_\nu(x,y)$ by the definition

$$iF_\nu(z)=\frac{\partial\omega_\nu}{\partial z}. \quad (42)$$

But now we have
$$\frac{1}{2}\int_{C_\mu} d\omega_\nu = \delta_{\nu\mu}. \tag{43}$$

We have again the boundary condition on C:
$$\frac{\partial^2 \omega_\nu}{\partial n^2} = 0. \tag{44}$$

The periodicity condition (43) and the boundary condition (44) determine the harmonic function $\omega_\nu(x, y)$ in a unique way. The corresponding quadratic differential $Q_\nu^{(3)}(z)$ is finally determined by means of
$$Q_\nu^{(3)}(z) = \frac{1}{i}\frac{\partial^2 \omega_\nu}{\partial z^2}. \tag{45}$$

Similar formulas can be established for the other quadratic differentials.

6. We have seen in the preceding sections the close relation between the problem of finding harmonic functions $\omega(x, y)$ in a domain D with the boundary condition
$$\frac{\partial^2 \omega}{\partial n^2} = 0 \tag{46}$$

and the problem of constructing certain quadratic differentials of D. If we know a function $w(x, y)$ which satisfies (46) and is harmonic in D except for specified singular points, we can assert that
$$Q(z) = \frac{1}{i}\frac{\partial^2 \omega}{\partial z^2} \tag{46'}$$

is a quadratic differential of D with known singularities.

It is now quite remarkable that in the case of a circular domain quadratic differentials can be connected with another classical problem of potential theory. We consider the harmonic functions $\omega_\nu(x, y)$ and express the boundary condition (44) by means of (37) as follows:
$$\frac{\partial^2 \omega_\nu}{\partial s^2} + \frac{1}{r_\mu}\frac{\partial \omega_\nu}{\partial n} = 0 \text{ on } C_\mu, \tag{47}$$

where r_μ is the radius of the circle C_μ. Clearly, (47) implies
$$\int_{C_\mu} \frac{\partial \omega_\nu}{\partial n} ds = 0 \tag{47'}$$

since $\omega_\nu(x, y)$ has in D single-valued derivatives. Equation (47') shows that $\omega_\nu(x, y)$ possesses in D a single-valued harmonic conjugate function $\eta_\nu(x, y)$. Clearly, the Cauchy-Riemann equations imply

$$\frac{\partial \omega_\nu}{\partial s} = \frac{\partial \eta_\nu}{\partial n} \quad \frac{\partial \omega_\nu}{\partial n} = -\frac{\partial \eta_\nu}{\partial s}. \tag{48}$$

Hence, (47) leads to

$$\frac{\partial^2 \omega_\nu}{\partial s^2} = \frac{1}{r_\mu}\frac{\partial \eta_\nu}{\partial s} \quad \text{on} \quad C_\mu \tag{49}$$

and by integration

$$\frac{\partial \omega_\nu}{\partial s} - \frac{1}{r_\mu}\eta_\nu = \frac{\partial \eta_\nu}{\partial n} - \frac{1}{r_\mu}\eta_\nu = \frac{1}{r_\mu}k_{\nu\mu} \quad \text{on} \quad C_\mu \tag{49'}$$

where the $k_{\nu\mu}$ are constants. In the case of a circular domain D the determination of quadratic differentials thus has been reduced to the solution of the simple Robin boundary value problem for harmonic functions: To determine harmonic functions $\eta(x, y)$ such that

$$\frac{\partial \eta}{\partial n} - \frac{1}{r_\mu}\eta = k_\mu \quad \text{on} \quad C_\mu \tag{50}$$

with constants k_μ.

At first let us show conversely that every function $\eta(x, y)$ satisfying (50) and being harmonic in D except for specified isolated singularities gives rise to quadratic differentials of D with known singularities. In fact, let us put

$$\eta(x, y) = \operatorname{Re}\{\Phi(z)\}, \quad z = c_\nu + r_\nu e^{-i\varphi}. \tag{51}$$

Then (50) takes the form

$$\operatorname{Re}\{\Phi'(z)e^{-i\varphi} - \frac{1}{r_\mu}\Phi(z)\} = k_\mu \quad \text{on} \quad C_\mu; \tag{51'}$$

we differentiate this identity with respect to φ

$$\operatorname{Re}\{-i\Phi''(z)\,r_\mu e^{-2i\varphi} - i\Phi' e^{-i\varphi} + i\Phi' e^{-i\varphi}\} = 0, \tag{52}$$

i.e.,

$$\Phi''(z)\,\dot{z}^2 = \text{real on } C, \tag{52'}$$

which shows that

$$Q(z) = \frac{\partial^2 \eta(x, y)}{\partial z^2} \tag{53}$$

is a quadratic differential of D.

7. The reduction of the boundary value problem for the harmonic functions $\omega_\nu(x, y)$ to a Robin boundary value problem for their conjugates $\eta_\nu(x, y)$ in the case of a circular domain allows us to express here the variation of the moduli r_ν in a particularly explicit form.

We introduce for this purpose the Robin's function $R(z;\zeta)$ of D with the following properties [2]:

(a) $R(z;\zeta)$ is harmonic in D except at $z=\zeta$ where

$$R(z;\zeta)=\log\frac{1}{|z-\zeta|}+\text{harmonic function}; \tag{54}$$

(b) $R(z;\zeta)$ satisfies on each C_ν the boundary condition

$$\frac{\partial}{\partial n}R(z;\zeta)-\frac{1}{r_\nu}R(z;\zeta)=0. \tag{55}$$

The existence of such a Robin function follows from the general theory of the Green's functions of the domain D; it is also well known that $R(z;\zeta)$ is symmetric in both arguments. If $j(x,y)$ is an arbitrary harmonic function in D, we derive easily from Green's identity the representation

$$j(x,y)=-\frac{1}{2\pi}\sum_{\nu=1}^{N}\int_{C_\nu}R(z;\zeta)\left[\frac{\partial j(\xi,\eta)}{\partial n}-\frac{1}{r_\nu}j(\xi,\eta)\right]ds_\zeta \tag{56}$$

with $z=x+iy$, $\zeta=\xi+i\eta$.

Let us define now N harmonic functions

$$j_\nu(x,y)=-\frac{1}{2\pi r_\nu}\int_{C_\nu}R(z;\zeta)\,ds_\zeta\,. \tag{57}$$

Each $j_\nu(x,y)$ is regular harmonic in D and, in view of (56), it has to satisfy the boundary condition

$$\frac{\partial j_\nu}{\partial n}-\frac{1}{r_\mu}j_\nu=\frac{1}{r_\nu}\delta_{\nu\mu}\quad\text{on}\quad C_\mu. \tag{57'}$$

Thus, the $j_\nu(x,y)$ form a basis for all regular harmonic functions $\eta(x,y)$ which satisfy (50). They lead to a set of quadratic differentials of D by means of formula (53).

Observe that the N harmonic functions $j_\nu(x,y)$ are not independent. In fact, their sum

$$j(x,y)=\sum_{\nu=1}^{N}j_\nu(x,y) \tag{58}$$

satisfies the boundary condition

$$\frac{\partial j}{\partial n}-\frac{1}{r_\mu}j=\frac{1}{r_\mu}\quad\text{on}\quad C_\mu \tag{58'}$$

which is also fulfilled by the harmonic function $j=-1$. Hence, the uniqueness theorem for this boundary value problem implies

$$j(x, y) = \sum_{\nu=1}^{N} j_\nu(x, y) = -1. \tag{58''}$$

Observe also that the harmonic functions $\eta_\nu(x, y)$ needed in our variational theory satisfy by (43) and (48) the equation

$$\int_{C_\mu} \frac{\partial \eta_\nu}{\partial n} ds = \int_{C_\mu} \frac{\partial \omega_\nu}{\partial s} ds = 2\delta_{\nu\mu}. \tag{59}$$

Thus

$$\int_C \frac{\partial \eta_\nu}{\partial n} ds = 2, \tag{59'}$$

that is each $\eta_\nu(x, y)$ has at infinity a logarithmic pole of strength π^{-1}. Hence, the regular harmonic functions $j_\nu(x, y)$ will not be sufficient to construct these $\eta_\nu(x, y)$. We introduce, therefore, the Robin's function $R(z)$ with pole at infinity

$$R(z) = \log|z| + \text{regular harmonic function} \tag{60}$$

and the boundary conditions (55).

From (49'), (57'), (59') and (60) we derive the representation

$$\eta_\nu(x, y) = \frac{1}{\pi} R(z) + \sum_{\mu=1}^{N} k_{\nu\mu} j_\mu(x, y). \tag{61}$$

8. We have now expressed the unknown function $\eta_\nu(x, y)$ in terms of $R(z)$ and the $j_\nu(x, y)$ which depend only on the Robin's function. There remains the final problem to determine the constants $k_{\nu\mu}$. They have to be adjusted to the period conditions (59). We define for this purpose the constants

$$A_\nu = \int_{C_\nu} \frac{\partial R(z)}{\partial n} ds = \frac{1}{r_\nu} \int_{C_\nu} R(z)\, ds$$

and

$$m_{\nu\mu} = \int_{C_\nu} \frac{\partial j_\mu}{\partial n} ds = \frac{1}{r_\nu} \int_{C_\nu} j_\mu\, ds + 2\pi\delta_{\nu\mu} \tag{63}$$

$$= -\frac{1}{2\pi r_\nu r_\mu} \int_{C_\nu} \int_{C_\mu} R(z; \zeta)\, ds_z\, ds_\zeta + 2\pi\delta_{\nu\mu} = m_{\mu\nu}.$$

We may now express the conditions (59) as a system of linear equations for the $k_{\nu\mu}$

$$2\delta_{\nu\sigma} = \frac{1}{\pi} A_\sigma + \sum_{\mu=1}^{N} k_{\nu\mu} m_{\mu\sigma} \quad (\sigma, \nu = 1, 2, \ldots, N). \tag{64}$$

It is easy to show that $((m_{\mu\nu})) = M$ is a positive semi-definite matrix. In fact, we can write the definition (63) of $m_{\mu\nu}$ as follows:

$$m_{\mu\nu} = \sum_{\varrho=1}^{N} \int_{C_\varrho} \frac{\partial j_\mu}{\partial n} \left[r_\varrho \frac{\partial j_\nu}{\partial n} - j_\nu \right] ds$$

$$= \sum_{\varrho=1}^{N} r_\varrho \int_{C_\varrho} \frac{\partial j_\mu}{\partial n} \frac{\partial j_\nu}{\partial n} ds + \iint_D \nabla j_\mu \cdot \nabla j_\nu \, d\tau. \tag{63'}$$

Let the $x_\nu (\nu = 1, \ldots, N)$ be N arbitrary real numbers and let

$$I(x, y) = \sum_{\nu=1}^{N} x_\nu j_\nu(x, y). \tag{65}$$

Clearly
$$\sum_{\mu,\nu=1}^{N} m_{\mu\nu} x_\mu x_\nu = \sum_{\varrho=1}^{N} r_\varrho \int_{C_\varrho} \left(\frac{\partial I}{\partial n}\right)^2 ds + \iint_D (\nabla I)^2 d\tau \geqslant 0. \tag{66}$$

The quadratic form can vanish only for $I = \text{const.}$, i.e., for

$$x_1 = x_2 = \ldots = x_N. \tag{67}$$

Indeed, it follows from the identity (58″) that

$$\sum_{\sigma=1} m_{\mu\sigma} = 0 \quad (\mu = 1, 2, \ldots, N). \tag{68}$$

But except for the constant vector (67), the quadratic form (66) is positive-definite. In particular, we can assert that the principal submatrix $M_{N-1} = ((m_{\nu\mu}))_{\mu,\nu=1,\ldots,N-1}$ is non-degenerate.

We consider now the equation system (64) for fixed ν. We see that the N-th equation is a consequence of the preceding $N-1$ equations because of (68) and

$$\sum_{\sigma=1}^{N} A_\sigma = \int_C \frac{\partial R}{\partial n} ds = 2\pi. \tag{69}$$

On the other hand, the unknowns $k_{\nu\mu}$ are determined only up to an additive constant which may depend on ν. This follows again from (68) and the symmetry of the matrix $m_{\nu\mu}$; it can also be inferred from the definition (49′) of the $k_{\nu\mu}$. In fact, $\eta_\nu(x, y)$ is only defined up to an additive constant which leads to a corresponding indeterminacy for the $k_{\nu\mu}$.

There is, therefore, no loss in generality if we assume

$$k_{\nu N} = 0 \quad \text{for} \quad \nu = 1, 2, \ldots, N. \tag{70}$$

We can now use the first $N-1$ equations (64) to determine the $N-1$ unknowns $k_{\nu\mu} (\mu = 1, \ldots, N-1)$. Since the matrix M_{N-1} of this inhomogeneous system of linear equations is non-degenerate this system has a unique solution.

The rather unusual boundary value problem for the harmonic functions $\omega_\nu(x, y)$ has been reduced in the case of a circular domain to a well determined boundary value problem of the Robin type and has been completely solved. For more general canonical domains the problem of boundary values for the second normal derivative of a harmonic function poses an interesting topic for further research. It is important to remember that the existence of a solution for this unconventional boundary value problem is guaranteed in our special case by the theory of quadratic differentials of a planar domain.

9. Until now we considered the very special set of domain variations whose kinematic is described by (3). The variation of the moduli was expressed by formulas (40), (40′) and (40″), which are all built in an analogous fashion, differing only from each other by different choice of the quadratic differentials used.

We proceed now to transform these variational formulas in such a way that their geometrical significance is clearly displayed, which will enable us to generalize them and to bring them into the conventional form of the calculus of variations. We start with the Cauchy formula

$$\operatorname{Re}\{e^{i\alpha}\varrho^2 Q(z_0)\} = \operatorname{Re}\left\{\frac{1}{2\pi i}\int_C Q(z)\frac{e^{i\alpha}\varrho^2}{z-z_0}\,dz\right\} \tag{71}$$

valid for all $Q(z) \in \mathfrak{q}$. Observe next that by (3) the quantity

$$\delta n = \operatorname{Re}\left\{\frac{e^{i\alpha}\varrho^2}{i(z-z_0)\dot z}\right\} \tag{72}$$

measures the normal shift of each point $z \in C$ under the variation considered. Because of the boundary condition (15) satisfied by every quadratic differential, we can bring (71) into the form

$$\operatorname{Re}\{e^{i\alpha}\varrho^2 Q(z_0)\} = \frac{1}{2\pi}\int_C [Q(z)\dot z^2]\,\delta n\,ds. \tag{73}$$

Thus, the variational equations (40)–(40″) may be expressed in the standard Hadamard variational form

$$\delta \log r_\nu = \frac{1}{2\pi}\int_C [Q_\nu^{(3)}(z)\dot z^2]\,\delta n\,ds, \tag{74}$$

$$\delta \operatorname{Re}\left\{\frac{c_\nu}{r_\nu}\right\} = \frac{1}{2\pi r_\nu}\int_C [Q_\nu^{(2)}(z)\dot z^2]\,\delta n\,ds, \tag{74′}$$

$$\delta \operatorname{Im}\left\{\frac{c_\nu}{r_\nu}\right\} = \frac{1}{2\pi r_\nu}\int_C [Q_\nu^{(1)}(z)\dot z^2]\,\delta n\,ds. \tag{74″}$$

Since an application of Runge's theorem shows that the most general admissible δn-variation can be approximated arbitrarily by a superposition of elementary variations (3), we have thus derived the variation of the moduli under the δn-deformations.

The preceding variational formulas can be brought into completely real form by use of the harmonic potentials $\omega_\nu(x, y)$ of the quadratic differentials. We shall exemplify this for the formulas (74). We introduce the functions $\omega_\nu(x, y)$ which lead by (45) to the $Q_\nu^{(3)}(z)$. From (42), (27) and (30) we deduce

$$\frac{\partial \omega_\nu}{\partial s} = 2 \operatorname{Re}\{iF_\nu(z)\dot{z}\}, \quad \frac{\partial \omega_\nu}{\partial n} = 2 \operatorname{Re}\{F_\nu(z)\dot{z}\} \tag{75}$$

and hence by (41) and (28)

$$\frac{\partial^2 \omega_\nu}{\partial s \partial n} = -2[Q_\nu^{(3)}(z)\dot{z}^2] - 2 \operatorname{Re}\{F_\nu(z)\ddot{z}\} = -2[Q_\nu^{(3)}(z)\dot{z}^2] - \varkappa \frac{\partial \omega_\nu}{\partial s}. \tag{76}$$

Thus, (74) can be expressed as

$$\delta \log r_\nu = -\frac{1}{4\pi} \int_C \left[\frac{\partial^2 \omega_\nu}{\partial s \partial n} + \varkappa \frac{\partial \omega_\nu}{\partial s}\right] \delta n \, ds. \tag{77}$$

Similar potential-theoretic expressions can also be obtained for the formulas (74′) and (74″).

In view of the boundary condition (31) satisfied by the $\omega_\nu(x, y)$, we can simplify (77) to

$$\delta \log r_\nu = -\frac{1}{4\pi} \int_C \delta n \left\{\frac{\partial}{\partial s}\left(\frac{1}{\varkappa} \frac{\partial^2 \omega_\nu}{\partial s^2}\right) + \varkappa \frac{\partial \omega_\nu}{\partial s}\right\} ds = -\frac{1}{4\pi} \int_C \frac{\partial \omega_\nu}{\partial s} \left\{\frac{d}{ds}\left(\frac{1}{\varkappa} \frac{d(\delta n)}{ds}\right) + \varkappa \delta n\right\} ds. \tag{77′}$$

III. Connections on plane domains

1. The transformation formula (I.6) for the curvature of the boundary of a domain under conformal mapping suggests the definition of an analytic function $\Gamma(z)$ in D such that

$$\operatorname{Im}\{\Gamma(z)\dot{z}\} = \varkappa(s) \quad \text{for} \quad z = z(s) \in C. \tag{1}$$

We will show that the transformation law of $\varkappa(s)$ induces a simple transformation formula for $\Gamma(z)$ under the mapping. But we have first to discuss in how far $\Gamma(z)$ is determined by the boundary condition (1).

This is, indeed, the case if we make the additional assumption that Γ possesses a single-valued harmonic potential $H(z)$, i.e., a real-valued harmonic function of z such that

$$\Gamma(z) = 2\frac{\partial H(z)}{\partial z} = (H_x - iH_y) \quad (z = x + iy). \tag{2}$$

The boundary condition (1) on $\Gamma(z)$ takes then the form

$$\frac{\partial H(z)}{\partial n} = -\varkappa(s) \quad (z = z(s) \in C). \tag{3}$$

Thus, the quest for $\Gamma(z)$ satisfying (1) becomes a Neumann type boundary value problem for the harmonic potential $H(z)$. On the other hand, the single-valued character of $H(z)$ in D implies the integral condition for $\Gamma(z)$:

$$2\int_{C_\nu} dH = \mathrm{Re}\left\{\int_{C_\nu} \Gamma(z)\,dz\right\} = 0. \tag{2'}$$

We add this requirement to the boundary condition (1) for Γ.

Observe, however, that condition (3) leads to the equation

$$\int_C \frac{\partial H}{\partial n}\,ds = 2\pi N. \tag{4}$$

This shows that $H(z)$ cannot be regular harmonic in D. It must have at least logarithmic poles in D with a total strength N. The simplest assumption we can make on $H(z)$ is that it be regular harmonic everywhere in D with the exception of an arbitrary but fixed point $\zeta \in D$ where

$$H(z) = -N \log|z-\zeta| + \text{regular harmonic}. \tag{5}$$

The conditions (3) and (5) determine the harmonic potential $H(z)$ uniquely up to an additive constant which may depend on ζ. We dispose of this constant by the normalization

$$\int_C \varkappa(s_z)\,H(z)\,ds_z = 0. \tag{6}$$

To stress the dependence of $H(z)$ upon its point of singularity ζ, we denote it from now on by $H(z;\zeta)$. This function is uniquely determined by the conditions (3), (5) and (6). Consequently, its derivative

$$\Gamma(z;\zeta) = 2\frac{\partial}{\partial z}H(z;\zeta) \tag{7}$$

is uniquely characterized by the conditions (1), (2') and its singularity character at ζ:

$$\Gamma(z;\zeta) = -\frac{N}{z-\zeta} + \gamma(z;\zeta) \tag{7'}$$

where $\gamma(z;\zeta)$ is regular analytic for $z \in D$.

In order to study the dependence of the harmonic function $H(z;\zeta)$ upon its source point ζ, we consider the Green's identity

$$H(\zeta;\eta) - H(\eta;\zeta) = \frac{1}{2\pi N}\int_C \left[H(z;\eta)\frac{\partial H(z;\zeta)}{\partial n} - H(z;\zeta)\frac{\partial H(z;\eta)}{\partial n}\right] ds \tag{8}$$

valid for any pair of points ζ, η in D. Using now the conditions (3) and (6), we verify that the right hand integral in (8) vanishes and find the symmetry law:

$$H(z; \zeta) = H(\zeta; z). \tag{9}$$

Thus, we can now assert that $H(z; \zeta)$ and $\Gamma(z; \zeta)$ depend harmonically upon their source point ζ.

2. Having established the existence and uniqueness of the functions $H(z; \zeta)$ and $\Gamma(z; \zeta)$, we shall now derive from (I.6) their transformation law with respect to the conformal mapping $w = f(z)$. Let $\omega = f(\zeta)$ and $H^*(w; \omega)$ be the corresponding harmonic potential of the image domain Δ. Consider the difference function

$$H^*[f(z), f(\zeta)] - H(z; \zeta) = \alpha(z; \zeta). \tag{10}$$

This is a symmetric regular harmonic function in D and it has on C the normal derivative (cf. (3) and (I.6))

$$\frac{\partial}{\partial n_z} \alpha(z; \zeta) = \varkappa - \varkappa^* |f'(z)| = \frac{\partial}{\partial n_z} \log |f'(z)|. \tag{11}$$

In view of the symmetry of $\alpha(z; \zeta)$ in z and ζ, we can conclude

$$H^*(w; \omega) = H(z; \zeta) + \log |f'(z)| + \log |f'(\zeta)| + a, \tag{12}$$

where a is a constant which depends on the mapping function $f(z)$.

We determine a from the equation (6) applied to H^*:

$$\int_C \left(\varkappa(s_z) - \frac{\partial}{\partial n} \log |f'(z)| \right) [H(z; \zeta) + \log |f'(z) f'(\zeta)| + a] \, ds_z = 0. \tag{13}$$

We find after elementary rearrangements using (3) and (6):

$$2\pi N a = 2 \int_C \varkappa(s) \log |f'(z)| ds + \iint_D (\nabla \log |f'(z)|)^2 \, d\tau. \tag{13'}$$

Thus, with the notation (I.59) we obtain:

$$a = \frac{1}{\pi N} [w, z] \tag{14}$$

and (12) may be restated in the form:

$$H^*(w; \omega) = H(z; \zeta) + \log |f'(z) f'(\zeta)| + \frac{1}{\pi N} [w, z]. \tag{15}$$

It is interesting to have the formal expression $[w,z]$ reappear which played such an important role in the theory of circular mappings. The addition law (I.60) is, of course, an immediate consequence of the transformation formula (15).

By differentiating the identity (15) with respect to z and using the definition (2), we obtain the transformation law for the meromorphic domain functions $\Gamma(z;\zeta)$:

$$\Gamma^*(w;\omega)\,dw = \left[\Gamma(z;\zeta)\,dz + d\,\log\left(\frac{dw}{dz}\right)\right]. \tag{16}$$

We shall call domain functions which transform under conformal mapping according to the linear but inhomogeneous law (16) "connections" on the domain considered. This is done in analogy to the corresponding quantities in differential geometry. The use of the concept of a connection for conformal mapping will become quite apparent in the sequel. But we wish to point out, at this stage, an important application of the connections to differentials of analytic domain functions and to their transformation theory.

Let $q_n(z)$ be an analytic function in the domain and defined in such a way in dependence of the domain that it transforms under conformal mapping according to the rule

$$q_n^*(w)\,dw^n = q_n(z)\,dz^n. \tag{17}$$

Such function $q_n(z)$ is called a differential of the domain of order n. It is easily verified that each differential of order n can be transformed into a differential of order $n+1$ by the operation

$$q_{n+1}(z) = \frac{dq_n(z)}{dz} + n\Gamma(z)\,q_n(z) \tag{18}$$

with an arbitrary connection $\Gamma(z)$ on D. Thus, the process of differentiation combined with a simple operation involving a connection leads to an unending sequence of differentials of increasing order. The analogy of (10) to the process of covariant differentiation is obvious.

The preceding statements will be discussed in greater detail in Chapter V. We have anticipated these facts in order to show that the concept of a connection is suggested from very different types of approach, a fact which enhances its significance.

3. The existence of the harmonic function $H(z;\zeta)$ was established by a potential theoretic argument. We shall now derive a constructive procedure to obtain this domain function and shall find in this way a new insight into the nature of $H(z;\zeta)$.

Let $G(z;\zeta)$ be the Green's function of the domain D and consider the difference function

$$d(z;\zeta) = \frac{1}{2\pi N}\,[H(z;\zeta) - NG(z;\zeta)]. \tag{19}$$

It is regular harmonic in D and symmetric in both arguments. We define the class Σ of all functions $\psi(z)$ which are harmonic in D, have a finite Dirichlet integral, are continuous in $D+C$ and have the normalization

$$\int_C \varkappa\psi\, ds = 0. \tag{20}$$

Clearly, $d(z;\zeta)$ lies in Σ as a function of z and ζ. For an arbitrary $\psi(z) \in \Sigma$, we have the identity

$$\iint_D \nabla\psi(z) \cdot \nabla d(z;\zeta)\, d\tau = \frac{1}{2\pi}\int_C \psi \frac{\partial G}{\partial n} ds = \psi(\zeta). \tag{21}$$

Thus, $d(z;\zeta)$ is the reproducing kernel within the class Σ under the Dirichlet multiplication. Let $\{\psi_\nu(z)\}$ be a complete system of harmonic functions in Σ, orthonormalized by the condition

$$\iint_D \nabla\psi_\nu \cdot \nabla\psi_\mu\, d\tau = \delta_{\nu\mu}. \tag{22}$$

Then, $d(z;\zeta)$ can be expressed as the kernel function of this orthonormal system:

$$\frac{1}{2\pi N}[H(z;\zeta) - NG(z;\zeta)] = \sum_{\nu=1}^{\infty} \psi_\nu(z)\,\psi_\nu(\zeta). \tag{23}$$

This formula may serve to construct $H(z;\zeta)$ explicitly. It shows also that the right-hand kernel $d(z;\zeta)$ is positive-definite, that is:

$$\sum_{\nu,\mu=1}^{M} d(\zeta_\nu;\zeta_\mu)\, x_\nu x_\mu \geqslant 0 \quad (x_\nu \text{ real}, \zeta_\nu \in D). \tag{24}$$

In particular,

$$d(\zeta,\zeta) = \lim_{z\to\zeta} \frac{1}{2\pi N}[H(z;\zeta) - NG(z;\zeta)] > 0. \tag{24'}$$

Another simple way to construct the function $H(z;\zeta)$ is the use of the function $F(z)$ defined in Chapter I (see (I.12)). Indeed,

$$\frac{1}{N}[H(z;\zeta) + F(z) + F(\zeta)] = S(z;\zeta) \tag{25}$$

is symmetric in z and ζ, harmonic, except for a pole with strength 1 at $z=\zeta$, and satisfies on C the boundary condition (cf. (I.13′))

$$\frac{\partial S(z;\zeta)}{\partial n_z} = \frac{2\pi}{L}. \tag{26}$$

Hence,

$$S(z;\zeta)=N(z;\zeta)+a, \tag{27}$$

where $N(z;\zeta)$ is the Neumann's function of D and a is an additive constant. We determine the constant by multiplying (27) with $\varkappa$ and integrating over C. From (6) and (I.12) follows:

$$a=-\frac{1}{2\pi N^2}\int_C F(z)\,\varkappa\,ds=-\frac{1}{2\pi N^2}\int_C F\frac{\partial F}{\partial n}ds=\frac{1}{2\pi N^2}\iint_D(\nabla F)^2\,d\tau. \tag{28}$$

Hence, finally: $$\frac{1}{N}[H(z;\zeta)+F(z)+F(\zeta)]=N(z;\zeta)+\frac{1}{2\pi N^2}\iint_D(\nabla F)^2\,d\tau. \tag{29}$$

The calculation of $H(z;\zeta)$ has thus been reduced to the calculation of the ordinary Neumann's function and the one new function $F(\zeta)$ which depends only on one variable.

4. We wish to analyze in greater detail the representation (23) for $H(z;\zeta)$. We define, for this purpose, the N harmonic functions

$$\omega_\nu(\zeta)=\frac{1}{2\pi}\int_{C_\nu}\frac{\partial G(z;\zeta)}{\partial n_z}ds_z, \tag{30}$$

the so-called harmonic measure of C_ν at ζ. Clearly

$$\omega_\nu(\zeta)=\delta_{\nu\mu}\quad\text{for}\quad\zeta\in C_\mu. \tag{31}$$

Consider next the matrix of "induction coefficients":

$$P_{\mu\nu}=-\frac{1}{2\pi}\int_{C_\mu}\frac{\partial\omega_\nu}{\partial n}ds=-\frac{1}{4\pi^2}\int_{C_\nu}\int_{C_\mu}\frac{\partial G(z;\zeta)}{\partial n_z\,\partial n_\zeta}ds_z\,ds_\zeta=P_{\nu\mu}. \tag{32}$$

It is known that if we restrict the indices ν, μ to the values from 1 to $N-1$, the submatrix $((P_{\mu\nu}))$ is positive-definite. Hence, we see that no $\omega_\nu(z)$ can be completed to a single-valued analytic function in D.

Consider, on the other hand, any $\psi(z)\in\Sigma$ which is orthogonal to all harmonic measures $\omega_\nu(z)$. From

$$\iint_D(\nabla\psi\cdot\nabla\omega_\nu)\,d\tau=-\int_C\omega_\nu\frac{\partial\psi}{\partial n}ds=-\int_{C_\nu}\frac{\partial\psi}{\partial n}ds=0, \tag{33}$$

it follows that such a function $\psi(z)$ can be completed to a single-valued analytic function in D. Conversely, the real and imaginary part of every single-valued analytic function in D must be orthogonal to all harmonic measures $\omega_\nu(z)$.

This observation leads to a very useful construction of a complete orthogonal system $\{\psi_\nu(z)\}$ in Σ. We introduce the linear space A of all single-valued analytic functions $f(z)$ in D which have a finite norm

$$\iint_D |f'(z)|^2 d\tau < \infty, \tag{34}$$

which are continuous in $D+C$ and satisfy the normalization condition

$$\int_C \varkappa(s_z) f(z)\, ds_z = 0. \tag{35}$$

Let $\{f_\nu(z)\}$ be a complete orthonormal system in A, that is

$$\iint_D f'_\nu \overline{f'_\mu}\, d\tau = \delta_{\nu\mu}. \tag{36}$$

Putting
$$f_\nu(z) = u_\nu + i v_\nu, \tag{37}$$

we obtain pairs of harmonic functions in the class Σ, and the orthonormalization (36) implies by means of the Cauchy-Riemann equations that all u_ν and v_ν are orthonormal in the Dirichlet sense:

$$\iint_D \nabla u_\nu \cdot \nabla u_\mu\, d\tau = \iint_D \nabla v_\nu \cdot \nabla v_\mu\, d\tau = \delta_{\nu\mu} \tag{38}$$

$$\iint_D \nabla u_\nu \cdot \nabla v_\mu\, d\tau = 0.$$

Thus, a complete orthonormal set in the space A of analytic functions yields a complete orthonormal set of harmonic functions in that subspace of Σ which is orthogonal to the $w_\nu(z)$. We may then write the formula (23) for the reproducing kernel of Σ as follows:

$$\begin{aligned}\frac{1}{2\pi N}[H(z;\zeta) - NG(z;\zeta)] &= \sum_{\nu,\mu=1}^{N-1} c_{\nu\mu}\, \omega_\nu(z)\, \omega_\mu(\zeta) + \sum_{\nu=1}^{\infty} [u_\nu(z)\, u_\nu(\zeta) + v_\nu(z)\, v_\nu(\zeta)] \\ &= \sum_{\nu,\mu=1}^{N-1} c_{\nu\mu}\, \omega_\nu(z)\, \omega_\mu(\zeta) + \mathrm{Re}\left\{\sum_{\nu=1}^{\infty} f_\nu(z)\, \overline{f_\nu(\zeta)}\right\}.\end{aligned} \tag{39}$$

We have thus been led to the kernel function

$$\mathfrak{K}(z;\bar{\zeta}) = \sum_{\nu=1}^{\infty} f_\nu(z)\, \overline{f_\nu(\zeta)} \tag{40}$$

of the linear space A of single-valued analytic functions with normalization (35). This linear space has some remarkable properties which will come out clearly from our considerations in Section 8. We recognize here its role in the construction of the harmonic potential $H(z;\zeta)$.

We can compute the coefficients $c_{\mu\nu}$ in (39) by taking the normal derivatives on both sides and integrating over C_ϱ. In view of the single-valuedness of the $f_\nu(z)$, and in view of (3), (30) and (32), we obtain

$$\omega_\varrho(\zeta) - \frac{1}{N} = 2\pi \sum_{\nu,\mu=1}^{N} c_{\nu\mu} P_{\nu\varrho}\, \omega_\mu(\zeta). \tag{41}$$

Sending $\zeta \to C_\mu$, we find, because of (31),

$$\delta_{\mu\varrho} - \frac{1}{N} = 2\pi \sum_{\nu=1}^{N} c_{\nu\mu} P_{\nu\varrho}. \tag{41'}$$

Because of the identity

$$\sum_{\varrho=1}^{N} \omega_\varrho(z) = 1 \tag{42}$$

we have the period relation

$$\sum_{\varrho=1}^{N} P_{\nu\varrho} = 0 \tag{42'}$$

which shows that the equations (41′) for the $c_{\nu\mu}$ are not independent. But it is seen that the left-hand side of (41′) is compatible with this linear dependence. The system (41′) can be solved, but the $c_{\nu\mu}$ are only determined up to an additive constant c_μ. These constants must be chosen in such a way that the harmonic function on the left side of (39) lies in Σ. This is the case if

$$\sum_{\nu,\mu=1}^{N} c_{\nu\mu}\, \omega_\mu(\zeta) = 0 \quad (\zeta \in D), \tag{43}$$

i.e., if

$$\sum_{\nu=1}^{N} c_{\nu\mu} = 0 \quad \text{for} \quad \mu = 1, 2, \ldots, N. \tag{43'}$$

These conditions determine the $c_{\nu\mu}$ in a unique way and the harmonic potential $H(z;\zeta)$ is completely determined.

An important aspect of the preceding analysis is the following. Since $G(z;\zeta)$, the harmonic measures $\omega_\nu(z)$, the coefficients $P_{\mu\nu}$ and consequently the $c_{\mu\nu}$ are all conformal invariants, the transformation law of the kernel function $\mathfrak{K}(z;\bar\zeta)$ is entirely determined by the transformation law (15) of $H(z;\zeta)$ and the relation (39). Indeed, let $w = f(z)$ be a conformal mapping of the domain D into the domain Δ with the corresponding kernel $\mathfrak{K}^*(w;\bar\omega)$. If $\omega = f(\zeta)$, we read off from (39) the equation

$$\operatorname{Re}\left\{\mathfrak{K}^*(w;\bar\omega) - \left[\mathfrak{K}(z;\bar\zeta) + \frac{1}{2\pi N}\log f'(z) + \frac{1}{2\pi N}\overline{\log f'(\zeta)} + \frac{1}{2\pi^2 N^2}[w,z]\right]\right\} = 0. \tag{44}$$

The term inside the Re{ }-operation is a hermitian kernel analytic in z and anti-analytic in ζ. It must, therefore, equal an imaginary constant. But since this term becomes real for $z=\zeta$, this imaginary constant must vanish. Thus, we arrive at the transformation law for the kernel function:

$$2\pi N\mathfrak{K}^*(w;\bar{\omega}) = 2\pi N\mathfrak{K}(z;\bar{\zeta}) + \log f'(z) + \log\overline{f'(\zeta)} + \frac{1}{\pi N}[w,z]. \tag{45}$$

5. We define the harmonic function

$$G_0(z;\zeta) = G(z;\zeta) + 2\pi\sum_{\nu,\mu=1}^{N} c_{\nu\mu}\,\omega_\nu(z)\,\omega_\mu(\zeta) \tag{46}$$

where the $c_{\nu\mu}$ are defined by (41′) and (43′). It has a logarithmic pole for $z=\zeta$, that is, near ζ we have

$$G_0(z;\zeta) = \log\frac{1}{|1-\zeta|} + \text{regular harmonic}, \tag{46′}$$

but else it is harmonic in z. In view of (39), we may write

$$G_0(z;\zeta) = \frac{1}{N}H(z;\zeta) - 2\pi Re\{\mathfrak{K}(z;\bar{\zeta})\}. \tag{47}$$

From this representation we can infer that $G_0(z;\zeta)$ is symmetric in z and ζ and satisfies the relations

$$\frac{1}{2\pi}\int_{C_\nu}\frac{\partial}{\partial n_z}G_0(z;\zeta)\,ds_z = \frac{1}{N} \quad (\nu=1,2,\dots,N). \tag{48}$$

On each boundary continuum C_ϱ, $G_0(z;\ \zeta)$ takes a constant value:

$$G_0(z;\zeta) = k_\varrho(\zeta) \quad (z\in C_\varrho). \tag{49}$$

Finally, from (47), (6) and the normalization (35) of the class A follows:

$$\int_C \varkappa(s_z)\,G_0(z;\ \zeta)\,ds_z = 0. \tag{50}$$

It is evident that the conditions (46′), (48), (49) and (50) determine the harmonic function $G_0(z;\ \zeta)$ in a unique way.

Let $f(z)$ be any analytic function of the class A; then

$$\iint_D \frac{\partial G_0(z;\zeta)}{\partial\zeta}f'(\zeta)\,d\tau_\zeta = \frac{1}{2i}\int_C G_0(z;\zeta)\,df(\zeta) = 0 \tag{51}$$

since G_0 is constant on each C_ϱ and $f(z)$ is single-valued in D. Observe that the left-hand integral in (51) is an improper integral because of the logarithmic pole of $G_0(z;\ \zeta)$ at ζ. But the derived function

$$K_0(z; \bar{\zeta}) = -\frac{2}{\pi} \frac{\partial^2 G_0(z; \zeta)}{\partial z \partial \bar{\zeta}} \tag{52}$$

has lost its singularity at $z=\zeta$ and is analytic in z, anti-analytic in ζ and hermitian in dependence on both variables. From (51) we obtain by differentiation with respect to z and by use of the Poisson-Laplace equation

$$\iint_D K_0(z; \bar{\zeta}) f'(\zeta) d\tau_\zeta = f'(z). \tag{53}$$

Let then B be the class of all analytic functions $f'(z)$ in D with single-valued integral $f(z)$ and with a finite norm (34). We see that $K_0(z; \bar{\zeta})$ belongs to this class since (48) implies

$$\int_{C_\nu} \frac{\partial}{\partial n_z} \left(\frac{\partial G_0(z; \zeta)}{\partial \bar{\zeta}} \right) ds_z = 0; \tag{54}$$

this together with (49) and (52) guarantees, indeed, that

$$\int_{C_\nu} K_0(z; \bar{\zeta}) dz = 0. \tag{55}$$

Hence, (53) shows that $K_0(z; \bar{\zeta})$ is the reproducing kernel for the function class B in the sense of the operation (53).

On the other hand, each complete orthonormal system $\{f_\nu(z)\}$ within the class A leads to a complete orthonormal system $\{f'_\nu(z)\}$ for the class B. Thus, we can express the reproducing kernel $K_0(z; \bar{\zeta})$ in the form

$$K_0(z; \bar{\zeta}) = \sum_{\nu=1}^{\infty} f'_\nu(z) \overline{f'_\nu(\zeta)} = \frac{\partial^2}{\partial z \partial \bar{\zeta}} \mathfrak{K}(z; \bar{\zeta}). \tag{56}$$

We find thus the important identity:

$$-\frac{2}{\pi} \frac{\partial^2 G_0(z; \zeta)}{\partial z \partial \bar{\zeta}} = \frac{\partial^2 \mathfrak{K}(z; \bar{\zeta})}{\partial z \partial \bar{\zeta}}. \tag{57}$$

From (47) and (57) follows further

$$\frac{2}{\pi N} \frac{\partial^2 H(z; \zeta)}{\partial z \partial \bar{\zeta}} = \frac{\partial^2 \mathfrak{K}(z; \bar{\zeta})}{\partial z \partial \bar{\zeta}}. \tag{58}$$

In order to study the meaning of the identity (57), it is very helpful to complete the harmonic function $G_0(z; \zeta)$ to an analytic function $P_0(z; \zeta)$ of z in D. Because of (48) this analytic function will have the periods $2\pi i/N$ with respect to each boundary continuum C_ν and will have a logarithmic pole for $z=\zeta$. We can now construct a function $S(z; \zeta)$, which has the same singularity and periodicity as $P_0(z;\zeta)$, but is a "geometric" domain function. That is, $S(z;\zeta)$ can be constructed by simple integrations and does not presuppose the solution of a boundary value problem of Laplace's equation in D as $P_0(z;\zeta)$ does. We define

$$\nu(z)=\frac{1}{2\pi N}\int_C \log\left(\frac{1}{z-t}\right)\cdot \varkappa(s_t)\,ds_t. \tag{59}$$

This function is regular analytic for all finite $z\in D$. At infinity we have

$$\nu(z)=\log z+\text{regular analytic}. \tag{60}$$

$\nu(z)$ has the period $2\pi i/N$ with respect to each boundary component C_ν. Let now

$$S(z;\zeta)=\log\frac{1}{z-\zeta}+\nu(z)+\nu(\zeta)+\frac{1}{2\pi N}\int_C \varkappa(s_t)\,\nu(t)\,ds_t. \tag{61}$$

This function has a logarithmic pole at $z=\zeta$, but it is elsewhere regular analytic in D. It has the period $2\pi i/N$ with respect to each boundary continuum C_ν and the period $2\pi i$ with respect to the pole ζ. We have given $S(z;\zeta)$ such a constant term that

$$Re\left\{\int_C \varkappa(s_z)\,S(z;\zeta)\,ds_z\right\}=0 \tag{62}$$

We can now assert that

$$Q(z;\zeta)=P_0(z;\zeta)-S(z;\zeta) \tag{63}$$

is regular analytic and single-valued in D. Since $P_0(z;\zeta)$ is only defined by the requirement

$$G_0(z;\zeta)=\mathrm{Re}\,\{P_0(z;\zeta)\}, \tag{64}$$

we may add to it an arbitrary imaginary term which depends on ζ. We make use of this arbitrariness in order to fulfill the condition

$$\int_C \varkappa(s_z)\,Q(z;\zeta)\,ds_z=0. \tag{65}$$

This is possible because of (50) and (62); $Q(z;\zeta)$ is now uniquely determined.

We have defined $S(z;\zeta)$ in such a manner that it depends analytically on z and ζ. Hence, (57) and (63) yield

$$-\frac{1}{\pi}\frac{\partial^2 Q(z;\zeta)}{\partial z\,\partial\bar{\zeta}}=\frac{\partial^2\mathfrak{K}(z;\bar{\zeta})}{\partial z\,\partial\bar{\zeta}}. \tag{66}$$

We can conclude, therefore,

$$\frac{1}{\pi}Q(z;\zeta)+\mathfrak{K}(z;\bar{\zeta})=\tilde{l}(z;\zeta)+\overline{\lambda(\zeta)} \tag{67}$$

where $\tilde{l}(z;\zeta)$ and $\lambda(\zeta)$ depend analytically on z and ζ.

From (65) and the fact that $\mathfrak{K}(z;\bar{\zeta})\in A$ follows

$$\int_C \tilde{l}(z;\zeta)\,\varkappa(s_z)\,ds_z=2\pi N\overline{\lambda(\zeta)}. \tag{68}$$

The left side term is analytic in ζ, the right side is anti-analytic. This clearly implies $\lambda(\zeta)=\text{const}$, and we may assume $\lambda\equiv 0$. This yields:

$$\int_C \tilde{l}(z;\zeta)\,\varkappa(s_z)\,ds_z=0, \tag{68'}$$

i.e., $\tilde{l}(z;\zeta)$ belongs to the class A. Now, (67) reduces to

$$\frac{1}{\pi}Q(z;\zeta)+\mathfrak{K}(z;\bar{\zeta})=\tilde{l}(z;\zeta). \tag{69}$$

We also have the symmetry condition

$$\tilde{l}(z;\zeta)=\tilde{l}(\zeta;z) \tag{70}$$

which follows from the fact that the left side of (69) has a real part which is symmetric in z and ζ.

Let finally

$$\mathfrak{L}(z;\zeta)=\frac{1}{\pi}S(z;\zeta)+\tilde{l}(z;\zeta). \tag{71}$$

Then, (63) and (69) lead to

$$P_0(z;\zeta)=\pi[\mathfrak{L}(z;\zeta)-\mathfrak{K}(z;\bar{\zeta})]. \tag{72}$$

We deduce also from (47) the identity

$$\frac{1}{N}H(z;\zeta)=\operatorname{Re}\{\pi[\mathfrak{L}(z;\zeta)+\mathfrak{K}(z;\bar{\zeta})]\} \tag{73}$$

such that the analytic function in z given by $\pi[\mathfrak{L}+\mathfrak{K}]$ represents the analytic completion *of the harmonic potential* $H(z;\zeta)$.

6. Because of the central role which the kernels $\mathfrak{L}(z;\zeta)$ and $\mathfrak{K}(z;\bar{\zeta})$ play in constructing important domain functions, we shall collect in this section their various properties and interrelations.

We start with the characteristic property of $\mathfrak{K}(z;\bar{\zeta})$ as kernel function of the linear function space A:

$$\iint_D \overline{\mathfrak{K}'(z;\bar{\zeta})}\, f'(z)\, d\tau_z = f(\zeta), \tag{74}$$

where the dash denotes differentiation with respect to the first variable. Applying this identity to the particular function $\mathfrak{K}(z;\bar{\eta}) \in A$ we find

$$\iint_D \overline{\mathfrak{K}'(z;\bar{\zeta})}\, \mathfrak{K}'(z;\bar{\eta})\, d\tau_z = \mathfrak{K}(\zeta;\bar{\eta}). \tag{75}$$

This identity shows the positive-definite character of the hermitian kernel $\mathfrak{K}(z;\bar{\zeta})$.

From (74), we deduce by means of (75) and the Schwarz inequality:

$$|f(\zeta)|^2 \leqslant \mathfrak{K}(\zeta;\bar{\zeta}) \cdot \iint_D |f'(z)|^2 d\tau. \tag{76}$$

Equality in this equation holds for $f(z) = a\mathfrak{K}(z;\bar{\zeta})$ and only for these functions. Thus, we can define $\mathfrak{K}(\zeta;\bar{\zeta})$ as follows:

$$\mathfrak{K}(\zeta;\bar{\zeta})^{-1} = \min_{f\in A} \frac{\iint_D |f'(z)|^2 d\tau}{|f(\zeta)|^2}, \tag{77}$$

and the functions which solve this minimum problem are up to a constant factor, precisely the kernel $\mathfrak{K}(z;\bar{\zeta})$. Thus, the determination of this kernel can be reduced to a minimum problem in the function space A.

We observe next that the identity (51) may be transformed by means of (64) into

$$\iint_D \overline{P_0'(z;\zeta)}\, f'(z)\, d\tau_z = 0, \tag{78}$$

valid for every $f(z) \in A$. Hence, (72) leads by means of (74) to

$$\iint_D \overline{\mathfrak{L}'(z;\zeta)}\, f'(z)\, d\tau_z = f(\zeta). \tag{79}$$

Thus, $\mathfrak{L}(z;\zeta)$ has the same reproducing property as $\mathfrak{K}(z;\bar{\zeta})$ under the same integral operation. But $\mathfrak{L}(z;\zeta)$ is neither regular nor single-valued in D.

We apply the identity (79) to the particular function $\mathfrak{K}(z;\bar{\eta}) \in A$ and find:

$$\iint_D \overline{\mathfrak{L}'(z;\zeta)}\, \mathfrak{K}'(z;\bar{\eta})\, d\tau_z = \mathfrak{K}(\zeta;\bar{\eta}). \tag{79'}$$

Inserting now into this identity the representation (71) for $\mathfrak{L}(z;\zeta)$ and using the fact that $\tilde{l}(z;\zeta)$ lies in A and that, therefore, (74) applies to it, we conclude:

$$\tilde{l}(\zeta;\eta)=\mathfrak{K}(\zeta,\bar{\eta})-\frac{1}{\pi}\iint_D\overline{\mathfrak{K}'(z;\bar{\zeta})}\,S'(z;\eta)\,d\tau_z. \tag{80}$$

Thus, the kernel $\tilde{l}(\zeta;\eta)$ and hence also $\mathfrak{L}(\zeta;\eta)$ can be obtained from the kernel $\mathfrak{K}(\zeta,\bar{\eta})$ by simple quadratures.

The complete analogy between our kernels $\mathfrak{K}$, $\mathfrak{L}$ and $\tilde{l}$ and the Bergman kernel K with its associate functions L and l [1] is quite evident. Many theorems and methods, valid for these well-known functions, can be translated without difficulty into the new situation. If $K_0(z;\bar{\zeta})$, $L_0(z;\zeta)$ and $l_0(z;\zeta)$ are the kernel functions for the space B of analytic functions $f'(z)$ with single-valued integral, we have

$$\frac{\partial^2\mathfrak{K}(z;\bar{\zeta})}{\partial z\,\partial\bar{\zeta}}=K_0(z;\bar{\zeta}),\qquad -\frac{\partial^2\mathfrak{L}(z;\zeta)}{\partial z\,\partial\zeta}=L_0(z;\zeta),\qquad -\frac{\partial^2\tilde{l}(z;\zeta)}{\partial z\,\partial\zeta}=l_0(z;\zeta), \tag{81}$$

One achievement of our study is therefore to have found a rational way to integrate the useful kernel functions of the domain D in both variables.

The transformation law for $\mathfrak{K}(z;\bar{\zeta})$ under conformal mapping has been given by (45). We can now derive from this law the corresponding formula (15) for $H(z;\zeta)$ and from formula (73):

$$\operatorname{Re}\{\mathfrak{L}^*(w;\omega)\}=\operatorname{Re}\left\{\mathfrak{L}(z;\zeta)+\frac{1}{2\pi N}\log f'(z)+\frac{1}{2\pi N}\log f'(\zeta)+\frac{1}{2\pi^2N^2}[w,z]\right\}. \tag{82}$$

It can be shown from the construction of $S(z;\zeta)$ and from (68′), (71) that $\mathfrak{L}(w;\omega)$ satisfies the transformation law:

$$2\pi N\,\mathfrak{L}^*(w;\omega)=2\pi N\,\mathfrak{L}(z;\zeta)+\log f'(z)+\log f'(\zeta)+\frac{1}{\pi N}[w,z]. \tag{82′}$$

7. A very clear understanding of the structure of the kernel $\mathfrak{K}$ and $\mathfrak{L}$ can be obtained by means of a special orthonormal system of eigen functions of the domain D. We consider the integral equation

$$w_\nu(z)=\frac{\lambda_\nu}{\pi}\iint_D\frac{\overline{w_\nu(\zeta)}}{(\zeta-z)^2}d\tau_\zeta \tag{83}$$

and ask for the eigen functions $w_\nu(z)$ of this problem, which belong to positive eigen values λ_ν. These eigen values are called the Fredholm eigen values of D [1, 23, 24, 25] and the $w_\nu(z)$ are the corresponding Fredholm eigen functions. It is known that the derivatives of all harmonic measures

$$v_j'(z) = \frac{1}{i}\frac{\partial \omega_j(z)}{\partial z} \tag{84}$$

are eigen functions of (83) belonging to the eigen value $\lambda = 1$. All other eigen functions $w_\nu(z)$ belong to eigen values $\lambda_\nu > 1$, and we shall only consider these "non-trivial" eigen functions. They are all orthogonal to the $v_j'(z)$, i.e.,

$$\iint_D w_\nu(z)\overline{v_j'(z)}\,d\tau = -\frac{1}{2}\int_C \omega_j w_\nu(z)\,dz = -\frac{1}{2}\int_{C_j} w_\nu\,dz = 0. \tag{85}$$

This shows that all non-trivial Fredholm eigen functions $w_\nu(z)$ belong to the class B of functions with single-valued integral. We introduce, therefore, the functions $W_\nu(z) \in A$ such that

$$W_\nu'(z) = w_\nu(z). \tag{86}$$

Observe that (86) and the normalization in A

$$\int_C \varkappa(s_z)\, W_\nu(z)\,ds_z = 0. \tag{87}$$

determine the $W_\nu(z)$ in unique manner.

The $\{w_\nu(z)\}$ being the set of eigen functions of (83) with $\lambda > 1$, we can assume without loss of generality:

$$\iint_D W_\nu'(z)\,\overline{W_\mu'(z)}\,d\tau = \delta_{\nu\mu} \tag{88}$$

and infer that the $\{W_\nu(z)\}$ form a complete orthonormal set in A. Hence, we have

$$\mathfrak{K}(z;\zeta) = \sum_{\nu=1}^{\infty} W_\nu(z)\,\overline{W_\nu(\zeta)}. \tag{89}$$

We can bring the integral equation (83) into the form:

$$W_\nu'(z) = \frac{\lambda_\nu}{2\pi i}\int_C \frac{\overline{W_\nu(\zeta)}}{(\zeta - z)^2}\,d\zeta. \tag{90}$$

In order to integrate (90) with respect to z, we make use of the equation

$$\int_C \varkappa(s_z)\frac{\partial S(z;\zeta)}{\partial \zeta}\,ds_z = 0 \tag{91}$$

which follows from (62). We find from (90) and (87):

$$W_\nu(z) = \frac{\lambda_\nu}{2\pi i}\int_C \overline{W_\nu(\zeta)}\frac{\partial S(z;\zeta)}{\partial \zeta}\,d\zeta. \tag{92}$$

Since all $W_\nu(z) \in A$, we have by (79) the identity

$$W_\nu(\zeta) = \iint_D \overline{\mathfrak{L}'(z;\zeta)}\, W_\nu(z)\, d\tau_z. \tag{93}$$

Integrating by parts, we obtain from this improper integral

$$\frac{1}{2i}\int_C \overline{\mathfrak{L}'(z;\zeta)}\, W_\nu(z)\, d\bar{z} = 0. \tag{93'}$$

We replace now $\mathfrak{L}(z;\zeta)$ by means of the representation (71) and find

$$\frac{1}{2i}\int_C \tilde{l}'(z;\zeta)\, \overline{W_\nu(z)}\, dz = \frac{1}{\lambda_\nu} W_\nu(\zeta), \tag{94}$$

that is

$$\iint_D \tilde{l}'(z;\zeta)\, \overline{W_\nu'(z)}\, d\tau_z = \frac{1}{\lambda_\nu} W_\nu(\zeta). \tag{94'}$$

Since $\tilde{l}(z;\zeta) \in A$, we find, therefore, that it has the Fourier development in the orthonormal system $\{W_\nu(z)\}$:

$$\tilde{l}(z;\zeta) = \sum_{\nu=1}^{\infty} \frac{W_\nu(z)\, W_\nu(\zeta)}{\lambda_\nu}. \tag{95}$$

Let $\hat{D}$ be the complement of D in the complex z-plane. Observe that the boundary values $W_\nu(\zeta)$ on C define a regular analytic function $\hat{W}_\nu^*(z)$ by means of (92) since $(\partial/\partial\zeta)\, S(z;\zeta)$ is defined everywhere. We have by Plemelj's formula the boundary relations

$$\hat{W}_\nu^*(z) = W_\nu(z) - \lambda_\nu \overline{W_\nu(z)} \quad (z \in C). \tag{96}$$

We see that

$$\int_C \varkappa(s_z)\, \hat{W}_\nu^*(z)\, ds_z = 0 \tag{96'}$$

holds. Using Cauchy's integral formula, we derive for $\hat{W}_\nu^*(z)$ the integral equation, valid in $\hat{D}$:

$$\hat{W}_\nu^*(z) = -\frac{\lambda_\nu}{2\pi i}\int_C [\hat{W}_\nu^*(\zeta)]^- \frac{\partial S(z;\zeta)}{\partial \zeta}\, d\zeta \quad (z \in \hat{D}), \tag{97}$$

where the integration over C is now to be understood as in the positive sence with respect to the complement $\hat{D}$.

Since it is also known [25] that

$$\iint_D |\hat{W}_\nu^{*\prime}(z)|^2\, d\tau = \lambda_\nu^2 - 1, \tag{98}$$

it is preferable to introduce the orthonormal eigen functions

$$\hat{W}_\nu(z) = \frac{i}{\sqrt{\lambda_\nu^2 - 1}} \hat{W}_\nu^*(z), \tag{99}$$

which form a complete set of eigen functions of the integral equation (92) with respect to $\hat{D}$. Every function analytic in each component of $\hat{D}$ and normalized by condition (96′) can be developed into a Fourier series of the $\hat{W}_\nu(z)$.

Let now $z \in \hat{D}$ and $\zeta \in D$. Then, for ζ fixed, the function $\partial S(z;\zeta)/\partial z$ will be regular analytic in $\hat{D}$ and may be developed into a Fourier series of the $\hat{W}_\nu(z)$. Indeed, its integral $S(z;\zeta)$ is single-valued in each component of $\hat{D}$. To calculate the Fourier coefficients of this function, we start with the equation

$$\sqrt{\lambda_\nu^2 - 1}\, \hat{W}_\nu(z) = \frac{\lambda_\nu}{2\pi} \int_C \overline{W_\nu(\zeta)} \frac{\partial S(z;\zeta)}{\partial \zeta} d\zeta \quad (z \in \hat{D}), \tag{100}$$

and its symmetric counterpart

$$\sqrt{\lambda_\nu^2 - 1}\, W_\nu(\zeta) = \frac{\lambda_\nu}{2\pi} \int_C [\hat{W}_\nu(z)]^- \frac{\partial S(z;\zeta)}{\partial z} dz \quad (\zeta \in D). \tag{100′}$$

The last equation can be transformed by integration by parts:

$$\iint_{\hat{D}} [\hat{W}_\nu'(z)]^- \cdot \frac{\partial S(z;\zeta)}{\partial z} d\tau_z = \frac{\pi}{i} \frac{\sqrt{\lambda_\nu^2 - 1}}{\lambda_\nu} W_\nu(\zeta). \tag{101}$$

Hence, we have the Fourier series:

$$\frac{\partial S(z;\zeta)}{\partial z} = \sum_{\nu=1}^{\infty} \frac{\pi}{i} \frac{\sqrt{\lambda_\nu^2 - 1}}{\lambda_\nu} \hat{W}_\nu'(z)\, W_\nu(\zeta), \tag{102}$$

valid for $z \in \hat{D}$, $z \in D$. In particular, we have the Parseval identity

$$\tilde{\Gamma}(\zeta;\bar{\eta}) = \frac{1}{\pi^2} \iint_{\hat{D}} \frac{\partial S(z;\zeta)}{\partial z} \cdot \left(\frac{\partial S(z;\eta)}{\partial z}\right)^- d\tau_z = \sum_{\nu=1}^{\infty} \left(1 - \frac{1}{\lambda_\nu^2}\right) W_\nu(\zeta)\, \overline{W_\nu(\eta)}. \tag{103}$$

This relation is very important since $\tilde{\Gamma}(\zeta;\bar{\eta})$ is a "geometric" quantity which depends only on the singularity function $S(z;\zeta)$ and can be obtained by quadratures. This function is now related to the eigen functions $W_\nu(z)$ and the eigen values λ_ν.

We derive from (89) and (95) the identity

$$\iint_D l(z;\zeta)\, \overline{l(z;\eta)}\, d\tau = \mathfrak{K}(\zeta;\bar{\eta}) - \tilde{\Gamma}(\zeta;\bar{\eta}). \tag{104}$$

This result is analogous to a corresponding formula for the Bergman kernel. It leads here as there to many estimates for the kernel and to a host of inequalities.

8. Let D be the exterior of the unit circle $|z|>1$. We have the complete set of orthonormal functions in the class A:

$$f_\nu(z)=\frac{1}{\sqrt{\pi\nu}}z^{-\nu} \quad (\nu=1,2,\dots). \tag{105}$$

Indeed, it is easily verified that

$$\iint_{|z|>1} f'_\nu(z)\overline{f'_\mu(z)}\,d\tau=\delta_{\nu\mu} \tag{105'}$$

and

$$\int_{|z|=1}\varkappa(s)f_\nu(z)\,ds=-\int_{|z|=1}f_\nu(z)\,ds=0. \tag{105''}$$

Hence, we find that in this case

$$\mathfrak{K}(z;\bar\zeta)=-\frac{1}{\pi}\log\left[1-\frac{1}{z\bar\zeta}\right]. \tag{106}$$

Because of (56), we have then

$$K_0(z;\bar\zeta)=\frac{1}{\pi}\frac{1}{(1-z\bar\zeta)^2}=\frac{1}{z^2\bar\zeta^2}\exp\{2\pi\,\mathfrak{K}(z;\bar\zeta)\}. \tag{107}$$

This leads to the partial differential equation for the kernel function

$$\frac{\partial^2}{\partial z\partial\bar\zeta}\log K_0(z;\bar\zeta)=2\pi K_0(z;\bar\zeta) \tag{108}$$

We wish to explain now the deeper reason for these relations which can be generalized to multiply-connected domains. We start with the identity (49) valid for $z(s)\in C_\varrho$ and differentiate it with respect to s. We find

$$\frac{\partial G_0(z;\zeta)}{\partial z}\dot z+\frac{\partial G_0(z;\zeta)}{\partial\bar z}\dot{\bar z}=0. \tag{109}$$

Since this equation holds identically in $\zeta\in D$, we may differentiate it in turn with respect to ζ. By virtue of (72) and (81), we obtain

$$L_0(\zeta;z)\dot z+K_0(\zeta;\bar z)\dot{\bar z}=0 \quad \text{for} \quad z\in C,\ \zeta\in D. \tag{110}$$

This boundary relation between the kernels K_0 and L_0 is the main reason for the many identities to which these functions give rise. We derive from (110) by logarithmic differentiation and by use of the symmetry properties of the kernels:

$$\left[\frac{d}{ds}\log K_0(z;\bar{\zeta})+\frac{\ddot{z}}{\dot{z}}\right]^- = \frac{d}{ds}\log L_0(z;\zeta)+\frac{\ddot{z}}{\dot{z}}. \tag{111}$$

Since $\ddot{z}/\dot{z}=i\varkappa(s)$ and because of the normalization (35) of the function class A, we have for every $f(z)\in A$:

$$\int_C f(z)\,d\,\overline{\log K_0(z;\bar{\zeta})}=\int_C f(z)\,d\log L_0(z;\zeta). \tag{112}$$

The right hand integral can be evaluated by means of the residue theorem.

It is known that the kernel $L_0(z;\zeta)$ of the class B of analytic functions with single-valued integral has a simple geometric meaning. Let $f_0(z;\zeta)$ and $f_{\frac{1}{2}\pi}(z;\zeta)$ be those univalent functions in D which map the domain onto the whole plane slit along rectilinear segments parallel to the real and imaginary axis, respectively, and normalized to have a simple pole for $z=\zeta$ with residue 1. Then

$$F(z;\zeta)=\frac{1}{2}[f_0(z;\zeta)+f_{\frac{1}{2}\pi}(z;\zeta)] \tag{113}$$

is likewise univalent in D and we have [20, 22]

$$L_0(z;\zeta)=F'(z;\zeta). \tag{113'}$$

Because of the univalence of $F(z;\zeta)$ its derivative has a double zero at infinity but is elsewhere different from zero in D. Since $L(z;\zeta)$ has a double pole for $z=\zeta$, we conclude from the residue theorem:

$$\frac{1}{2\pi i}\int_C f(z)\,d\log L_0(z;\zeta)=2[f(\infty)-f(\zeta)]. \tag{114}$$

We find from (110) by the argument principle and from the knowledge of all zeros and poles of $L_0(z;\zeta)$ that $K_0(z;\bar{\zeta})$ has $2N$ zeros in D. One, of course, is a double zero at infinity; the others define $2(N-1)$ analytic functions $m_k(\zeta)$ by the implicit equation:

$$K_0(\overline{m_k(\zeta)},\bar{\zeta})=0 \quad (k=1,2,\dots,2(N-1)). \tag{115}$$

We can write the identities (74) and (79) in the form

$$-\frac{1}{2i}\int_C f(z)\,d\,\overline{\mathfrak{K}(z;\bar{\zeta})}=f(\zeta) \tag{74'}$$

and

$$\int_C f(z)\,d\,\overline{\mathfrak{L}(z;\zeta)}=0. \tag{79'}$$

Hence, we conclude from (112) and (114);

$$\frac{1}{2i}\int_C f(z)\,d[2\,\mathfrak{K}(z;\zeta)-2\,\mathfrak{K}(z;\infty)]^- = \frac{1}{2\pi i}\int_C f(z)\,d\left[\log K_0(z;\bar{\zeta})+\pi\sum_{k=1}^{2N}\mathfrak{L}(z;\overline{m_k(\zeta)})\right]^- . \quad (116)$$

Consider now the expression

$$\log K_0(z;\bar{\zeta})+\pi\sum_{k=1}^{2N}\mathfrak{L}(z;\overline{m_k(\zeta)}). \quad (116')$$

It is regular analytic for $z\in D$ since the logarithmic poles in this combination destroy each other. It is not necessarily single-valued in D; but the periods of $\log K_0(z;\bar{\zeta})$ and of $\mathfrak{L}(z;\zeta)$ are integer multiples of $2\pi i/N$. Hence, the periods of the expression (116′) are pure imaginary and independent of ζ. We can, therefore, construct a combination

$$\log K_0(z;\bar{\zeta})+\pi\sum_{k=1}^{2N}\mathfrak{L}(z;\overline{m_k(\zeta)})+\sum_{j=1}^{N}a_j v_j(z)+b(\zeta) \quad (117)$$

which lies in the function space A. The $v_j(z)$ are the analytic completions of the harmonic measures $\omega_j(z)$ and make the combination single-valued in D. The term $b(\zeta)$ is an additive constant with respect to z which enforces the normalization (35) for the class A.

We observe that

$$\int_C f(z)\,d\overline{v_j(z)} = -\int_C f(z)\,d v_j(z)=0 \quad (118)$$

since the real part of each $v_j(z)$ is constant on C. Hence, comparing terms on both sides of (116), we obtain the identity

$$2\pi[\mathfrak{K}(z;\bar{\zeta})-\mathfrak{K}(z;\infty)]=\log K_0(z;\bar{\zeta})+\pi\sum_{k=1}^{2N}\mathfrak{L}(z;\overline{m_k(\zeta)})+\sum_{j=1}^{N}a_j v_j(z)+b(\zeta). \quad (119)$$

We may express (119) in the more interesting form:

$$2\pi\,\mathfrak{K}(z;\bar{\zeta})=\log K_0(z;\bar{\zeta})+\pi\sum_{k=1}^{2N}\mathfrak{L}(z;\overline{m_k(\zeta)})+a(z)+b(\zeta). \quad (119')$$

This equation generalizes the relation (107), which we found in the case that D is the exterior of the unit circle. We may simplify (119′) by differentiating this equation with respect to z and $\bar{\zeta}$ and by using the identities (81):

$$2\pi K_0(z;\bar{\zeta})+\pi\sum_{k=1}^{2N}L_0(z;\overline{m_k(\zeta)})\,\overline{m_k'(\zeta)}=\frac{\partial^2}{\partial z\,\partial\bar{\zeta}}\log K_0(z;\bar{\zeta}). \quad (120)$$

This is the generalization of the identity (108) to the general case of multiply-connected domains.

The basic boundary relation (110) between the kernels $K_0(z;\bar{\zeta})$ and $L_0(z;\zeta)$ leads to the further consequence

$$\int_C f(z)\,d\,\overline{\log L_0(z;\zeta)} = \int_C f(z)\,d\,\log K_0(z;\bar{\zeta}) \tag{121}$$

valid for all $f(z)\in A$. Using the residue theorem and the definition (115) on the right hand, we find

$$\frac{1}{2\pi i}\int_C f(z)\,d\,\overline{\log L_0(z;\zeta)} = \sum_{k=1}^{2N} \overline{f(m_k(\zeta))} = -\frac{1}{2i}\int_C f(z)\,d\left[\sum_{k=1}^{2N} \mathfrak{K}(z; m_k(\zeta))\right]^{-}. \tag{122}$$

We introduce again a combination

$$\log L_0(z;\zeta) - 2\pi[\mathfrak{L}(z;\zeta) - \mathfrak{L}(z;\infty)] + \sum_{j=1}^{N} \tilde{a}_j v_j(z) + \tilde{b}(\zeta) \tag{123}$$

which is regular, lies in the function class A and can be used instead of $\log L_0(z;\zeta)$ in the left-hand term of (122). This is possible because of the identities (79′) and (118). We deduce now from the altered equation (122) that

$$\log L_0(z;\zeta) - 2\pi[\mathfrak{L}(z;\zeta) - \mathfrak{L}(z;\infty)] + \sum_{j=1}^{N} \tilde{a}_j v_j(z) + \tilde{b}(\zeta) = -\pi\sum_{k=1}^{2N} \mathfrak{K}(z; m_k(\zeta)). \tag{122′}$$

We differentiate this relation with respect to z and ζ and use the identities (81); we find

$$\frac{\partial^2}{\partial z\,\partial\zeta}\log L_0(z;\zeta) = -2\pi L_0(z;\zeta) - \pi\sum_{k=1}^{2N} K_0(z; m_k(\zeta))\, m_k'(\zeta). \tag{124}$$

This is the symmetric counterpart to identity (120). The right hand sum of (124) can be expressed as a single integral by means of the residue theorem. A simple calculation leads to the identity

$$\frac{\partial^2}{\partial z\,\partial\zeta}\log L_0(z;\zeta) = -2\pi L_0(z;\zeta) - \frac{1}{2i}\int_C K_0(z;\bar{t})\frac{\partial}{\partial\zeta}\log K_0(\zeta;\bar{t})\,d\bar{t}. \tag{124′}$$

Analogously, we may express (120) in integral form:

$$K_0(z;\bar{\zeta}) = \frac{1}{4\pi i}\int_C L_0(z;t)\frac{\partial}{\partial\bar{\zeta}}\log K_0(t;\bar{\zeta})\,dt. \tag{120′}$$

We hope that the elegant formulas obtained show how much the particular normalization (35) for the function class A is suited to the study of the logarithm of the kernel functions $K_0(z;\bar{\zeta})$ and $L_0(z;\zeta)$. Its use gives new insight into the differential equations satisfied by these important domain functions.

9. We return now to the connections $\Gamma(z;\zeta)$ which led originally to all our preceding considerations. From (2) and (73), we derive the representation of Γ in terms of the kernels

$$\Gamma(z;\zeta)=\pi N[\mathfrak{L}'(z;\zeta)+\mathfrak{K}'(z;\bar{\zeta})]. \tag{125}$$

While not as symmetric as the singular kernel $\mathfrak{L}(z;\zeta)$, the connection Γ has the great advantage of being single-valued in the domain. It is the most convenient building element in terms of which all important differentials of the domain can be constructed.

Let $q_n(z)$ be a regular analytic function of $z\in D$ which satisfies the boundary condition

$$q_n(z)\dot{z}^n>0 \quad \text{for} \quad z\in C. \tag{126}$$

We assume that under a conformal mapping of the domain the function $q_n(z)$ transforms according to (17), which guarantees that the boundary behavior (126) is unchanged under such a transformation. We shall call $q_n(z)$ a "positive differential" of the domain of order n.

Because of the argument principle, $q_n(z)$ has by (126) precisely $m=nN$ zeros in D, say $z_\nu(\nu=1,\dots,m)$. Taking the logarithmic derivative of (126) with respect to the arc length, we find

$$\frac{q_n'(z)}{q_n(z)}\dot{z}+n\frac{\ddot{z}}{\dot{z}}=\text{real}. \tag{127}$$

In view of the definition (I.1) of the curvature, this means

$$\operatorname{Im}\left\{-\frac{1}{n}\frac{q_n'(z)}{q_n(z)}\dot{z}\right\}=\varkappa(s). \tag{128}$$

The function

$$\Gamma_n(z)=-\frac{1}{n}\frac{q_n'(z)}{q_n(z)} \tag{129}$$

satisfies, therefore, the characteristic boundary condition (1) of a connection. It has poles at the zeros z_ν of $q_n(z)$ with the residue $-(m_\nu/n)$, where m_ν is the multiplicity of the zero point z_ν. Furthermore, we have

$$\operatorname{Re}\left\{\int_{C_\nu}\Gamma_n\,dz\right\}=-\frac{1}{n}\int_{C_\nu}d\log|q_n(z)|=0 \quad (\nu=1,\dots,N), \tag{130}$$

since $q_n(z)$ is single-valued in D. The properties enumerated of $\Gamma_n(z)$ are sufficient to characterize it in a unique way, as can be seen from our arguments in Section 1. Hence, we find

$$\Gamma_n(z)=\frac{1}{nN}\sum_{\nu=1}^{m}\Gamma(z;z_\nu), \tag{131}$$

where each zero of z_ν of $q_n(z)$ enters of summation as often as its multiplicity indicates. Thus, finally

$$\frac{q_n'(z)}{q_n(z)} = -\frac{1}{N}\sum_{\nu=1}^{m}\Gamma(z; z_\nu) \quad (m = nN). \tag{132}$$

This elegant representation formula allows us to construct all positive differentials of D of arbitrary order in terms of the connection $\Gamma(z;\zeta)$. Conversely, every combination of connections which has the form (128) with all poles z_ν inside of D will define a positive differential of the domain of order n.

In the same way, we may construct meromorphic positive differentials of D of all orders. The poles p_ν of such a differential will necessitate summands $-\Gamma(z;p_\nu)$ in the sum analogous to (132). The most important differential of this kind is the derivative of the Green's function:

$$i\,P'(z;\zeta) = \frac{\partial G(z;\zeta)}{\partial y} + i\frac{\partial G(z;\zeta)}{\partial x}. \tag{133}$$

It has a pole for $z=\zeta$ and $N+1$ zeros $z_\nu(\zeta)$ $(\nu = 1, \ldots, N+1)$ in D. Hence we find in analogy to (132)

$$\frac{P''(z;\zeta)}{P'(z;\zeta)} = \frac{1}{N}\left\{\Gamma(z;\zeta) - \sum_{\nu=1}^{N+1}\Gamma(z; z_\nu(\zeta))\right\}. \tag{134}$$

The usefulness of the connection $\Gamma(z;\zeta)$ in the general theory of domain functions of plane domains becomes quite evident from these simple formulas.

IV. Variational theory for connections

1. We introduced in Chapter III the domain functions $H(z;\zeta)$, $G_0(z;\zeta)$ and $\Gamma(z;\zeta)$ as solutions of various boundary value problems with respect to the variable z. We normalized them in such a way, that they were uniquely determined by boundary conditions and normalization, and that their dependence upon the location ζ of their singularity became harmonic and even, for the first two functions, symmetric. We also established how these functions transformed under a conformal mapping of the domain.

It is our aim in the present chapter to study the dependence of these important domain functions upon their domain of definition and to determine how they change under an arbitrary infinitesimal deformation of it. While the question of the character of H, G_0 and Γ as function of z and ζ belongs to the theory of harmonic functions, our new problem lies in the field of functional analysis and the calculus of variations.

Its treatment will lead us to new domain functions with useful transformations properties under conformal mapping; it also will lay the groundwork for the solution of various extremum problems which may be posed for the above domain functions.

For the infinitesimal deformation of the domain D, we shall use the same kinematics of variation as was used in Chapter II. We put again

$$z^* = z + \frac{\varrho^2 e^{i\alpha}}{z - z_0} \quad (z_0 \in D,\ \varrho > 0), \tag{1}$$

and consider the domain D^* which is determined by the boundary C^*, the image of the boundary C of D under the mapping (1). We shall denote by $H^*(z; \zeta)$, $G_0^*(z; \zeta)$ and $\Gamma^*(z; \zeta)$ the corresponding domain functions of D^* and wish to express them asymptotically in terms of the domain functions of D.

2. We start with the Neumann type function $H(z; \zeta)$. Let D_0 be the domain bounded by the boundary C of D and by the circumference $|z - z_0| = \varrho$, which we shall denote by c. We assume, of course, ϱ so small that c lies entirely in D. Next, we choose two arbitrary but fixed points ζ and η in D_0 and can assert in view of Green's identity:

$$H^*(\zeta^*; \eta^*) - H(\zeta; \eta) = \frac{1}{2\pi N} \int_{C+c} \left[H^*(z^*; \zeta^*) \frac{\partial H(z; \eta)}{\partial n} - H(z; \eta) \frac{\partial H(z^*; \zeta^*)}{\partial n} \right] ds. \tag{2}$$

Here, $z^*(z)$, $\zeta^*(\zeta)$ and $\eta^*(\eta)$ are related to z, ζ and η by means of formula (1). We use now the fact that the normal derivative of the H-function is given on C in terms of the local curvature as indicated by (III.3) Hence:

$$\begin{aligned} H^*(\zeta^*; \eta^*) - H(\zeta; \eta) = &-\frac{1}{2\pi N} \int_C \left[\varkappa(s_z)\, H^*(z^*; \zeta^*) - \varkappa^* H(z; \eta) \frac{ds^*}{ds} \right] ds \\ &+ \frac{1}{2\pi N} \int_c \left[H^*(z^*; \zeta^*) \frac{\partial H(z; \eta)}{\partial n} - H(z; \eta) \frac{\partial H^*(z^*; \zeta^*)}{\partial n} \right] ds. \end{aligned} \tag{3}$$

The second integral on the right side of (3) occurs frequently in the calculus of variations. It is of the following standard form: Let

$$f(z) = u(z) + i\, v(z), \quad g(z) = a(z) + i\, b(z) \tag{4}$$

be two functions of the complex variable z, analytic in a domain Δ which contains the circle c. u, v and a, b, respectively, denote the harmonic real and imaginary parts of these functions. We have, then, to evaluate the integral:

$$\frac{1}{2\pi} \int_c \left[u(z^*) \frac{\partial a(z)}{\partial n} - a(z) \frac{\partial u(z^*)}{\partial n} \right] ds = \mathrm{Re} \left\{ \frac{1}{2\pi i} \int_c f(z^*)\, d g(z) \right\}. \tag{5}$$

The right side integral is to be taken in the positive sense over the curve c. Since $f(z^*)$ and $g(z)$ are both analytic in some domain Δ^* which lies in Δ and which does not contain the interior of the c-circumference, we may take on the right side of (5) any fixed curve γ in Δ^* as the curve of integration, as long as it is homologous to c in Δ^*. We have on γ

$$f(z^*) = f(z) + \varrho^2 \frac{e^{i\alpha}}{z - z_0} f'(z) + \varrho^4 R(z) \tag{6}$$

where $R(z)$ is a remainder term analytic in Δ^*. We insert this development into (5) and derive from Cauchy's integral theorem:

$$\frac{1}{2\pi} \int_c \left[u(z^*) \frac{\partial a(z)}{\partial n} - a(z) \frac{\partial u(z^*)}{\partial n} \right] ds = \operatorname{Re} \{ e^{i\alpha} \varrho^2 f'(z_0) g'(z_0) \} + O(\varrho^4), \tag{7}$$

where the remainder term $O(\varrho^4)$ depends only on the behavior of $f(z^*)$ and $g(z)$ on the fixed (ϱ-independent) curve γ.

Applying this general identity to the second integral in (3), we obtain by means of (III.2).

$$\frac{1}{2\pi N} \int_c \left[H^* \frac{\partial H}{\partial n} - H \frac{\partial H^*}{\partial n} \right] ds = \operatorname{Re} \left\{ e^{i\alpha} \varrho^2 \frac{1}{N} \Gamma^*(z_0; \zeta^*) \Gamma(z_0; \eta) \right\} + O(\varrho^4). \tag{8}$$

We come now to the first integral on the right side of (3). We have by the transformation law (I.6) of the curvature:

$$\int_{C^*} \varkappa^* H(z; \eta) ds^* = \int_C H(z; \eta) \left[\varkappa(s_z) - \frac{\partial}{\partial n} \log \left| \frac{dz^*}{dz} \right| \right] ds. \tag{9}$$

Using the normalization (III.6) of H and the form (1) of $z^*(z)$, we arrive at

$$\int_{C^*} \varkappa^* H(z; \eta) ds^* = \operatorname{Re} \left\{ \int_C H(z; \eta) \frac{\partial}{\partial n} \left[\frac{\varrho^2 e^{i\alpha}}{(z - z_0)^2} \right] ds \right\} + O(\varrho^4). \tag{10}$$

We now make use of the identity

$$\frac{1}{2\pi} \int_C \left[H(z; \eta) \frac{\partial}{\partial n} \log \frac{1}{|z - z_0|} - \log \frac{1}{|z - z_0|} \frac{\partial}{\partial n} H(z; \eta) \right] ds = H(z_0; \eta) - N \log \frac{1}{|z_0 - \eta|}. \tag{11}$$

Differentiate this identity twice with respect to z_0 and use the value (III.3) for the normal derivative of H. We find:

$$\frac{1}{2\pi} \int_C H(z; \eta) \frac{\partial}{\partial n} \left[\frac{1}{(z - z_0)^2} \right] ds + \frac{1}{2\pi} \int_C \frac{\varkappa(s)\, ds}{(z - z_0)^2} = \Gamma'(z_0; \eta) - \frac{N}{(\eta - z_0)^2}. \tag{12}$$

Finally, we make use of the function $\nu(z)$ introduced in (III.59) and find from (10) and (12):

$$\frac{1}{2\pi}\int_{C^*}\varkappa^* H(z;\eta)\,ds^* = \mathrm{Re}\left\{\varrho^2 e^{i\alpha}\left[-N\nu''(z_0)+\Gamma'(z_0;\eta)-\frac{N}{(\eta-z_0)^2}\right]\right\}+O(\varrho^4). \tag{13}$$

By symmetry, we can conclude:

$$\frac{1}{2\pi}\int_{C}\varkappa H^*(z;\zeta^*)\,ds = \mathrm{Re}\left\{-\varrho^2 e^{i\alpha}\left[-N\nu''(z_0)+\Gamma^{*\prime}(z_0;\zeta)-\frac{N}{(\zeta-z_0)^2}\right]\right\}+O(\varrho^4). \tag{14}$$

Indeed, the transition from z^* to z is given by

$$z=z^*-\frac{\varrho^2 e^{i\alpha}}{z^*-z_0}+O(\varrho^4), \tag{1'}$$

which explains the opposite sign of $\varrho^2 e^{i\alpha}$ in (14). We should also have used $\nu^*(z)$ instead of $\nu(z)$ and ζ^* instead of ζ in the last term of (14); but because of the error term $O(\varrho^4)$ our formula remains valid in its present form.

We collect all terms necessary for (3) and find:

$$\begin{aligned}H^*(\zeta^*;\eta^*)-H(\zeta;\eta)=\mathrm{Re}\Big\{e^{i\alpha}\varrho^2\Big[&\frac{1}{N}\Gamma^*(z_0;\zeta^*)\Gamma(z_0;\eta)+\frac{1}{N}\Gamma'(z_0;\eta)\\&+\frac{1}{N}\Gamma^{*\prime}(z_0;\zeta)-2\nu''(z_0)-\frac{1}{(\eta-z_0)^2}-\frac{1}{(\zeta-z_0)^2}\Big]\Big\}+O(\varrho^4).\end{aligned} \tag{15}$$

We see, first of all, that the difference H^*-H can be estimated uniformly in D_0 to be of order ϱ^2. Hence, the same asymptotic behavior can be asserted for the derivative $\Gamma^*-\Gamma$ in each closed subdomain of D. We may, therefore, replace the term $\Gamma^*(z_0;\zeta^*)$ on the right-hand side of (15) by $\Gamma(z_0;\zeta)$ without affecting the validity of the asymptotic relation. Finally, in view of the transformation formula (III.15) for H under conformal mapping, we shall bring (15) into the more suggestive form

$$\begin{aligned}H^*(\zeta^*;\eta^*)-\left[H(\zeta;\eta)+\log\left|1-\frac{e^{i\alpha}\varrho^2}{(\zeta-z_0)^2}\right|+\log\left|1-\frac{e^{i\alpha}\varrho^2}{(\eta-z_0)^2}\right|\right]\\=\mathrm{Re}\left\{e^{i\alpha}\varrho^2\left[\frac{1}{N}(\Gamma(z_0;\zeta)\Gamma(z_0;\eta)+\Gamma'(z_0;\zeta)+\Gamma'(z_0;\eta))-2\nu''(z_0)\right]\right\}+O(\varrho^4).\end{aligned} \tag{16}$$

In order to understand the significance of the term $\nu''(z_0)$, let us consider the case that the domain D has exterior points. Let z_0 be such an exterior point; there, for ϱ small enough, the equation (1) will describe a conformal mapping of D into D^* which preserves the point at infinity. It is a matter of simple calculation to show that

$$[z^*, z] = -2\pi N \operatorname{Re}\{e^{i\alpha} \varrho^2 \nu''(z_0)\} + O(\varrho^4). \tag{17}$$

Thus, in this case we can identify the transformation formula (III.15) with the variational formula (16) if it is understood that all Γ-terms are to be deleted, if their argument lies outside of their domain D.

3. The variation of $H(\zeta; \eta)$ has introduced the combination of connections

$$R(z; \zeta, \eta) = \Gamma(z; \zeta)\,\Gamma(z; \eta) + \Gamma'(z; \zeta) + \Gamma'(z; \eta). \tag{18}$$

This term, which is a combination of products of connections and of their derivatives, is very reminiscent in structure to the Riemann tensor of curvature in differential geometry and it possesses some remarkable properties.

We observe that by (III.1)

$$\Gamma(z; \zeta)\,\dot{z}(s) = r(s; \zeta) + i\,\varkappa(s) \quad (z = z(s) \in C), \tag{19}$$

where $r(s; \zeta)$ is a real-valued function of s which depends also on ζ. Differentiate this identity with respect to s and find by (I.1) and (19):

$$\Gamma'(z; \zeta)\,\dot{z}^2 = \dot{r}(s; \zeta) + i\,\dot{\varkappa}(s) - [r(s; \zeta) + i\,\varkappa(s)]\,i\,\varkappa(s). \tag{20}$$

Hence, $$\{\Gamma(z; \zeta)\,\Gamma(z; \eta) + \Gamma'(z; \zeta) + \Gamma'(z; \eta)\}\,\dot{z}^2$$
$$= r(s; \zeta)\,r(s; \eta) + \dot{r}(s; \zeta) + \dot{r}(z; \eta) + \varkappa(s)^2 + 2i\,\dot{\varkappa}(s). \quad (21$$

It is remarkable that the imaginary part of this expression does not depend on ζ or η, but only on the position of $z \in C$; that is:

$$\operatorname{Im}\{R(z; \zeta, \eta)\,\dot{z}^2\} = 2\,\frac{d\varkappa(s)}{ds} \quad (z = z(s) \in C). \tag{22}$$

Next, we wish to point out the very simple transformation law of $R(z; \zeta, \eta)$ under conformal mapping. Let us suppose that $\hat{z} = f(z)$ under a conformal mapping of D onto $\hat{D}$ with corresponding $\hat{R}(\hat{z}; \hat{\zeta}, \hat{\eta})$. From the transformation rule (III.16) for the connections Γ, we obtain after some rearrangement

$$\hat{R}(\hat{z}; \hat{\zeta}, \hat{\eta})\left(\frac{d\hat{z}}{dz}\right)^2 = R(z; \zeta, \eta) + 2\{\hat{z}; z\}, \tag{23}$$

where $$\{\hat{z}; z\} = \frac{\hat{z}'''}{\hat{z}'} - \frac{3}{2}\left(\frac{\hat{z}''}{\hat{z}'}\right)^2 \tag{24}$$

is the well-known Schwarzian differential parameter.

The domain function $R(z;\zeta,\eta)$ is regular analytic for $z\in D$, except for $z=\zeta$ and $z=\eta$ where it has a double pole. It is, therefore, of interest to point out a function of z which is regular analytic throughout D and which also has the transformation law (23). This function is

$$l_0(z;z)=-\left.\frac{\partial^2 l(z;\zeta)}{\partial z\,\partial\bar{\zeta}}\right|_{z=\zeta}. \tag{25}$$

Indeed, from the relations (III.71) and (III.81) follows

$$l_0(z;\zeta)=\frac{1}{\pi(z-\zeta)^2}-L_0(z;\zeta) \tag{26}$$

and the transformation law of $\mathfrak{L}(z;\zeta)$ under conformal mapping (III.82′) yields by differentiation:

$$\hat{L}_0(\hat{z};\hat{\zeta})\,\frac{d\hat{z}}{dz}\,\frac{d\hat{\zeta}}{d\zeta}=L_0(z;\zeta). \tag{27}$$

Hence,
$$\hat{l}_0(\hat{z};\hat{\zeta})\,\frac{d\hat{z}}{dz}\,\frac{d\hat{\zeta}}{d\zeta}=\frac{1}{\pi}\left[\frac{\frac{d\hat{z}}{dz}\,\frac{d\hat{\zeta}}{d\zeta}}{(\hat{z}-\hat{\zeta})^2}-\frac{1}{(z-\zeta)^2}\right]+l_0(z;\zeta) \tag{28}$$

Passing to the limit $\zeta\to z$, we find [1, 3]:

$$\hat{l}_0(\hat{z};\hat{z})\left(\frac{d\hat{z}}{dz}\right)^2=l_0(z;z)+\frac{1}{6\pi}\{\hat{z},z\}. \tag{29}$$

We recognize, therefore, from (23) and (29) that

$$R(z;\zeta,\eta)-12\pi\,l_0(z;z)=q(z;\zeta,\eta) \tag{30}$$

transforms according to

$$\hat{q}(\hat{z};\hat{\zeta},\hat{\eta})\left(\frac{d\hat{z}}{dz}\right)^2=q(z;\zeta,\eta), \tag{31}$$

i.e., as a quadratic differential.

It is known [3, 25] that for $z\in C$

$$\operatorname{Im}\{l_0(z;z)\,\dot{z}^2\}=\frac{1}{6\pi}\,\frac{d\varkappa(s)}{ds}. \tag{32}$$

Hence, the quadratic differential (30) satisfies the boundary condition

$$q(z;\zeta,\eta)\,\dot{z}^2=\text{real}\quad\text{for}\quad z\in C. \tag{33}$$

Thus, in order to construct of expressions $R(z; \zeta, \eta)$ it is sufficient to know all real quadratic differentials of D and the one regular function $l_0(z; z)$.

4. We shall now apply the formal relations of the preceding section in order to obtain a more general variational formula for the domain function $H(z: \zeta)$. We observe the identity

$$\frac{1}{2\pi i N}\int_C \frac{R(t; \zeta, \eta)}{t-z_0}\,dt = \frac{1}{N}R(z_0; \zeta, \eta) - \frac{\Gamma(\eta; \zeta)}{\eta - z_0} - \frac{\Gamma(\zeta; \eta)}{\zeta - z_0} - \frac{1}{(\zeta - z_0)^2} - \frac{1}{(\eta - z_0)^2} \tag{34}$$

which follows from the residue theorem and the known singularities of $R(t; \zeta, \eta)$ in D. We can, therefore, compress the variational formula (16) into:

$$H^*(\zeta; \eta) - H(\zeta; \eta) = \operatorname{Re}\left\{\frac{e^{i\alpha}\varrho^2}{2\pi i N}\int_C \frac{R(t; \zeta, \eta)}{t-z_0}\,dt\right\} - \operatorname{Re}\{2\,e^{i\alpha}\varrho^2\nu''(z_0)\} + O(\varrho^4) \tag{35}$$

We remark that each boundary point $t \in C$ is shifted by the variation (1) by the amount

$$\delta n = \operatorname{Re}\left\{\frac{e^{i\alpha}\varrho^2}{(t-z_0)\dot{t}}\dot{t}^{-1}\right\} \tag{36}$$

in the direction of the interior normal at t with respect to D. We have, by virtue of (22), for all $t \in C$:

$$R(t; \zeta, \eta)\,\dot{t}^2 = \operatorname{Re}\{R(t; \zeta, \eta)\,\dot{t}^2\} + 2i\,\dot{\varkappa}(s). \tag{37}$$

Hence, using (37) and the definition (III.59) of $\nu(z)$, we can express (35) as follows:

$$\delta H(\zeta; \eta) = \frac{1}{2\pi N}\int_C \operatorname{Re}\{R(t; \zeta, \eta)\,\dot{t}^2\}\,\delta n\,ds + \operatorname{Re}\left\{\frac{e^{i\alpha}\varrho^2}{2\pi N}\int_C 2\left[\frac{\dot{\varkappa}}{t-z_0} - \frac{i\varkappa}{(t-z_0)^2}\right]d\bar{t}\right\}. \tag{38}$$

We can apply integration by parts to the last integral in (38) and remove the term $\dot{\varkappa}$; indeed:

$$\int_C \dot{\varkappa}(s)\frac{\bar{\dot{t}}}{(t-z_0)}\,ds = \int_C \frac{\varkappa ds}{(t-z_0)^2} - \int_C \varkappa\,\frac{\bar{\ddot{t}}}{t-z_0}\,ds. \tag{38'}$$

Using (I.1) and (36), we arrive thus finally at:

$$\delta H(\zeta; \eta) = \frac{1}{2\pi N}\int_C \operatorname{Re}\{R(t; \zeta, \eta)\,\dot{t}^2\}\,\delta n\,ds - \frac{1}{\pi N}\int_C \varkappa^2\,\delta n\,ds. \tag{39}$$

We have expressed the variation of the domain function $H(\zeta; \eta)$ in a more geometrically understandable form. Since the expression is linear and homogeneous in

δn, we can assert the same variational law for the most general variation δn which can be built up from superposition of elementary variations (1). Formula (39) expresses the variation of $H(\zeta;\eta)$ in the Hadamard kinematics of deformation which is more intuitive and often preferable in applications.

5. The variation of $\Gamma(\zeta;\eta)$ under a deformation (1) of its domain D can be obtained from the corresponding variational formula (16) for $H(\zeta;\eta)$. Indeed, if we differentiate this formula with respect to ζ and make use of (III.2), we find:

$$\Gamma^*(\zeta^*;\eta^*)\frac{d\zeta^*}{d\zeta}=\Gamma(\zeta;\eta)+\frac{d}{d\zeta}\log\left(\frac{d\zeta^*}{d\zeta}\right)+\frac{\partial}{\partial\zeta}\operatorname{Re}\left\{2\,e^{i\alpha}\varrho^2\frac{1}{N}R(z_0;\zeta,\eta)\right\}+O(\varrho^4). \tag{40}$$

This result may be considerably simplified by the following formal considerations. From (III.72) and (III.73), we deduce the identity:

$$\frac{1}{N}\Gamma(z;\zeta)=\pi[\mathfrak{L}'(z;\zeta)+\mathfrak{K}'(z;\bar{\zeta})]=2i\frac{\partial}{\partial z}\operatorname{Im}\{P_0(\zeta;z)\}. \tag{41}$$

We may use this identity in order to transform $R(z;\zeta,\eta)$, defined by (18), as follows:

$$R(z;\zeta,\eta)=-4N^2\frac{\partial}{\partial z}\operatorname{Im}\{P_0(\zeta;z)\}\frac{\partial}{\partial z}\operatorname{Im}\{P_0(\eta;z)\}+2iN\frac{\partial^2}{\partial z^2}\operatorname{Im}\{P_0(\zeta;z)+P_0(\eta;z)\}. \tag{42}$$

We insert this representation of $R(z;\zeta,\eta)$ into the variational formula (40) for the connections and drop the part which is independent of ζ. We thus obtain:

$$\Gamma^*(\zeta^*;\eta^*)\frac{d\zeta^*}{d\zeta}=\Gamma(\zeta;\eta)+\frac{\partial}{\partial\zeta}\log\left(\frac{d\zeta^*}{d\zeta}\right)$$
$$+\frac{\partial}{\partial\zeta}\operatorname{Re}\left\{4i\,e^{i\alpha}\varrho^2\left[\frac{\partial^2}{\partial z^2}\operatorname{Im}\{P_0(\zeta;z)\}+\Gamma(z;\eta)\frac{\partial}{\partial z}\operatorname{Im}\{P_0(\zeta;z)\}\right]\right\}_{z=z_0}+O(\varrho^4). \tag{43}$$

We write out explicitly the real and imaginary parts as sums and differences of conjugate terms. Many of them are anti-analytic functions of ζ and are destroyed by the differentiation in ζ. Using the symmetry of $G_0(z;\zeta)$, we finally arrive at

$$\Gamma^*(\zeta^*;\eta^*)\frac{d\zeta^*}{d\zeta}=\Gamma(\zeta;\eta)+\frac{\partial}{\partial\zeta}\log\left(\frac{d\zeta^*}{d\zeta}\right)$$
$$+2i\frac{\partial}{\partial\zeta}\operatorname{Im}\{e^{i\alpha}\varrho^2[P_0''(z_0;\zeta)+\Gamma(z_0;\eta)P_0'(z_0;\zeta)]\}+O(\varrho^4). \tag{44}$$

The variational formula for the connections introduces thus a new domain function

$$\pi(z;\zeta,\eta)=P_0''(z;\zeta)+\Gamma(z;\eta)P_0'(z;\zeta). \tag{45}$$

Since $P_0'(z;\zeta)$ is an imaginary differential of the first order of D, i.e.,

$$P_0(z;\zeta)\,\dot z = \text{imaginary} \quad \text{for} \quad z \in C, \tag{46}$$

it follows from Section III.2 that $\pi(z;\zeta,\eta)$ is an imaginary quadratic differential of D; in fact, (45) is precisely the operation (III.18) which boosts the order of a differential by one unit. Thus, we can state that under a conformal mapping $\hat z(z)$

$$\hat\pi(\hat z;\hat\zeta,\hat\eta)\left(\frac{d\hat z}{dz}\right)^2 = \pi(z;\zeta,\eta), \tag{47}$$

and we have the boundary condition

$$\pi(z;\zeta,\eta)\,\dot z^2 = \text{imaginary} \quad \text{for} \quad z \in C. \tag{48}$$

Both results can also be verified directly.

It is interesting to observe how the variational formulas for the various domain functions lead in a natural way to expressions with a simple transformation law under conformal mapping and with a simple boundary behavior.

6. Having derived variational formulas for $H(z;\zeta)$ and $\Gamma(z;\zeta)$ under a deformation (1), we shall now give, for the sake of completeness, the corresponding variational formula for $G_0(z;\zeta)$. We proceed in precisely the same manner as we did in Section 2. We form the difference function

$$G_0^*(z^*;\zeta^*) - G_0(z;\zeta) = \Delta G_0(z;\zeta) \tag{49}$$

which is regular harmonic in the punched domain D_0. Because of (III.48), (III.49) and (III.50), we can assert

$$\Delta G_0(z;\zeta) = \Delta k_\varrho(\zeta) \quad \text{for} \quad z \in C_\varrho \tag{50}$$

$$\sum_{\varrho=1}^{N} \Delta k_\varrho(\zeta) = 0 \tag{50'}$$

$$\int_{C_\varrho} \frac{\partial}{\partial n_z}\Delta G_0(z;\zeta)\,ds_z = 0 \quad (\varrho = 1, 2, \ldots, N). \tag{50''}$$

It is, therefore, clear that for any two points ζ and η in D_0

$$\int_C \left\{G_0(z;\eta)\frac{\partial}{\partial n_z}\Delta G_0(z;\zeta) - \Delta G_0(z;\zeta)\frac{\partial}{\partial n_z}G_0(z;\eta)\right\} ds_z = 0. \tag{51}$$

Applying Green's identity to the integral (51) when extended over the boundary $C+c$ of D_0 and using (7) in order to evaluate the contribution of the integral over c, we find after obvious calculations:

$$G_0^*(\zeta^*;\eta^*)=G_0(\zeta;\eta)+\operatorname{Re}\{e^{i\alpha}\varrho^2 P_0'(z_0;\zeta)P_0'(z_0;\eta)\}+O(\varrho^4). \tag{52}$$

This is formally the exact variational law satisfied by the ordinary Green's function of the domain [21, 24].

7. Let us define the functionals of D

$$g(\zeta)=\lim_{z\to\zeta}\{G_0(z;\zeta)+\log|z-\zeta|\} \tag{53}$$

and

$$h(\zeta)=\lim_{z\to\zeta}\{H(z;\zeta)+N\log|z-\zeta|\}. \tag{54}$$

Under a conformal mapping $\hat{z}(z)$, the Green's function $G_0(z;\zeta)$ remains invariant. Hence, we obtain the transformation law for $g(\zeta)$:

$$\hat{g}(\hat{\zeta})=g(\zeta)+\log\left|\frac{d\hat{z}}{dz}\right|. \tag{55}$$

Similarly, we derive from (54) and the transformation behavior of $H(z;\zeta)$ (III.15) the equation

$$\hat{h}(\hat{\zeta})=h(\zeta)+(2+N)\log\left|\frac{d\hat{z}}{dz}\right|+\frac{1}{N\pi}[\hat{z},z]. \tag{56}$$

Observe that the combination

$$\mathfrak{H}(z;\zeta)=H(z;\zeta)-g(z)-g(\zeta)-N\,G_0(z;\zeta) \tag{57}$$

has the simple transformation law

$$\hat{\mathfrak{H}}(\hat{z};\hat{\zeta})=\mathfrak{H}(z;\zeta)+\frac{1}{\pi N}[\hat{z},z]. \tag{58}$$

$\mathfrak{H}$ is finite everywhere in D, but it is not a harmonic function of its variables. The main value of the functionals $g(\zeta)$, $h(\zeta)$ and $\mathfrak{H}(z;\zeta)$ lies in the applications to interesting extremum problems as we shall illustrate below.

We deduce from (16) and (52) the variational formulas

$$h^*(\zeta^*)-h(\zeta)=\operatorname{Re}\left\{e^{i\alpha}\varrho^2\left[\frac{1}{N}(\Gamma(z_0;\zeta)^2+2\,\Gamma'(z_0;\zeta))-\frac{N+2}{(\zeta-z_0)^2}-2\,\nu''(z_0)\right]\right\}+O(\varrho^4), \tag{59}$$

$$g^*(\zeta^*) - g(\zeta) = \mathrm{Re}\left\{e^{i\alpha}\varrho^2\left[P_0'(z_0;\zeta)^2 - \frac{1}{(\zeta - z_0)^2}\right]\right\} + O(\varrho^4), \tag{60}$$

and $\quad \mathfrak{H}^*(\zeta^*;\zeta^*) - \mathfrak{H}(\zeta,\zeta)$

$$= \mathrm{Re}\left\{e^{i\alpha}\varrho^2\left[\frac{1}{N}(\Gamma(z_0;\zeta)^2 + 2\,\Gamma'(z_0;\zeta)) - (N+2)\,P_0'(z_0;\zeta)^2 - 2\,\nu''(z_0)\right]\right\} + O(\varrho^4). \tag{61}$$

8. In order to illustrate the above results and to show how they can be applied, we consider now the very special case that the domain D, considered, is the exterior of the unit circle $|z| > 1$. In this case, it is immediate that

$$H(z;\zeta) = \log\frac{1}{|z-\zeta|} + \log\frac{1}{|1 - z\bar{\zeta}|} + 2\log|z\zeta| \tag{62}$$

and
$$G_0(z;\zeta) = \log\left|\frac{1 - z\bar{\zeta}}{z - \zeta}\right|. \tag{63}$$

Hence, in particular,
$$h(\zeta) = \log\frac{|\zeta|^4}{|\zeta|^2 - 1} \tag{62'}$$

and
$$g(\zeta) = \log(|\zeta|^2 - 1). \tag{63'}$$

By differentiating (62) and using (III.2), we find

$$\Gamma(z;\zeta) = -\frac{1}{z-\zeta} + \frac{\bar{\zeta}}{1 - z\bar{\zeta}} + \frac{2}{z}. \tag{64}$$

From (63), differentiation leads to

$$P_0'(z;\zeta) = -\frac{1}{z-\zeta} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} = \frac{|\zeta|^2 - 1}{(z-\zeta)(1 - z\bar{\zeta})}. \tag{65}$$

Further
$$\Gamma'(z;\zeta) = \frac{1}{(z-\zeta)^2} + \frac{\bar{\zeta}^2}{(1 - z\bar{\zeta})^2} - \frac{2}{z^2}. \tag{66}$$

Hence, it is verified by direct calculation that

$$\Gamma(z;\zeta)^2 + 2\,\Gamma'(z;\zeta) - 3\,P_0'(z;\zeta)^2 = -\frac{4(|\zeta|^2 + 1)}{z(z-\zeta)(1 - z\bar{\zeta})}. \tag{67}$$

Observe that the left side expression is suggested by the variational formula (61) for $\mathfrak{H}(\zeta;\zeta)$.

Let now
$$G_0(z) = \log|z|, \quad P_0(z) = \log z \tag{63''}$$

be the Green's function and its analytic completion with the source point at infinity. Clearly, $G_0(z)$ and $P_0(z)$ may be defined for every domain D and they are conformal invariants for all mappings which preserve the point at infinity. We can then combine (65) and (67) and write

$$\frac{|\zeta|^2+1}{z(z-\zeta)(1-z\bar{\zeta})} = \coth G_0(\zeta)\cdot P_0'(z)\, P_0'(z;\zeta). \tag{68}$$

Since by (III.72) $$L_0(z;\zeta) = -\frac{\partial^2 \mathfrak{L}(z;\zeta)}{\partial z\,\partial\zeta} = -\frac{2}{\pi}\frac{\partial^2 G_0(z;\zeta)}{\partial z\,\partial \zeta} \tag{69}$$

has in the case of the domain $|z|>1$ the value

$$L_0(z;\zeta) = \frac{1}{\pi(z-\zeta)^2}, \tag{69'}$$

we conclude from the relation (26) that in this case

$$l_0(z;\zeta) \equiv 0 \tag{70}$$

holds. Therefore, we may write (67) in the form:

$$\Gamma(z;\zeta)^2 + 2\,\Gamma'(z;\zeta) - 12\pi\, l_0(z;z) - 3\,P_0'(z;\zeta)^2 + 4\coth G_0(\zeta)\, P_0'(z)\, P_0'(z;\zeta) = 0. \tag{71}$$

This equation has been proved for the special domain $|z|>1$. But observe that $G_0(\zeta)$ is a conformal invariant for all mappings which preserve the point at infinity, that the first three terms in (71) form the quadratic differential $q(z;\zeta,\zeta)$ defined in (30) and that all other summands are likewise quadratic differentials of D. Since this covariant expression vanishes for one special domain, it must be identically zero. Thus, the identity (71) is proved for all simply-connected domains which contain the point at infinity.

From the variational equation (52), we find in the limit $\eta\to\infty$

$$G_0^*(\zeta^*) = G_0(\zeta) + \operatorname{Re}\{e^{i\alpha}\varrho^2 P_0'(z;\zeta)\, P_0'(z)\} + O(\varrho^4) \tag{72}$$

and, hence,

$$\log\sinh G_0^*(\zeta^*) = \log\sinh G_0(\zeta) + \operatorname{Re}\{e^{i\alpha}\varrho^2 \coth G_0(\zeta)\, P_0'(z_0)\, P_0'(z_0;\zeta)\} + O(\varrho^4). \tag{72'}$$

Consider then the functional

$$T(\zeta) = \mathfrak{H}(\zeta;\zeta) + 4\log\sinh G_0(\zeta). \tag{73}$$

In view of the variational formulas (61) and (72′) and because of the identity (71), we have for every simply-connected D which is varied according to (1):

$$T^*(\zeta^*) = T(\zeta) + \operatorname{Re}\{e^{i\alpha}\varrho^2[12\pi l_0(z_0; z_0) - 2\nu''(z_0)]\} + O(\varrho^4). \tag{74}$$

In Section III.7, we have defined the Fredholm eigen values $\lambda_\nu > 1$ of a domain D. One may consider the "Fredholm determinant"

$$\Delta = \prod_{\nu=1}^{\infty}\left(1 - \frac{1}{\lambda_\nu^2}\right). \tag{75}$$

It has been shown [25] that under a variation (1) the Fredholm determinant Δ varies according to

$$\log \Delta^* = \log \Delta - \operatorname{Re}\{e^{i\alpha}\varrho^2 2\pi l_0(z_0, z_0)\} + O(\varrho^4). \tag{76}$$

Combining (74) with (76), we find that under a variation (1)

$$\delta[\mathfrak{H}(\zeta;\zeta) + 4\log\sinh G_0(\zeta) - 6\log\Delta] = -2\operatorname{Re}\{e^{i\alpha}\varrho^2\nu''(z_0)\} + O(\varrho^4). \tag{77}$$

9. We have constructed a combination of various domain functions and functionals which have a very simple variational law (77). It is now surprising that we can construct another simple functional which has precisely the same variational expression.

Let $\tilde{D}$ be the complementary domain of the simply-connected domain D. $\tilde{D}$ is a finite domain and let $w = f(z)$ be univalent in $\tilde{D}$ and map this domain onto the unit circle $|w| < 1$. Let us define

$$[w, z]^{\sim} = \frac{1}{2}\iint_{\tilde{D}} [\nabla \log |w'|]^2 d\tau - \int_C \varkappa \log |w'|\, ds. \tag{78}$$

Observe the change in sign in the second term of (78). The reason is that we associate with the value s of the length parameter on C the curvature value $\varkappa(s)$ which belongs to the positive orientation relative to D; hence, the proper curvature relative to $\tilde{D}$ must be chosen as $-\varkappa(s)$.

The functional $[w, z]^{\sim}$ can be defined in $\tilde{D}$ for every conformal mapping $w = f(z)$ onto a finite domain in the w-plane. By the same formal reasoning as we used in Section I.9, it can be shown that in a chain of mappings $w(\zeta)$, $\zeta(z)$:

$$[w, z]^{\sim} = [w, \zeta]^{\sim} + [\zeta, z]^{\sim}. \tag{79}$$

The identity (79) can also be explained by means of the Neumann type function $\tilde{H}(z; \zeta)$ of the simply-connected finite domain $\tilde{D}$, which is characterized by the properties:

$$\tilde{H}(z;\zeta) = \log\frac{1}{z-\zeta} + \text{regular harmonic} \tag{80}$$

$$\frac{\partial \tilde{H}(z;\zeta)}{\partial \tilde{n}} = -\varkappa(s), \quad \tilde{n} = \text{interior normal relative to } \tilde{D} \tag{80'}$$

$$\int_C \varkappa(s_z)\,\tilde{H}(z;\zeta)\,ds_z = 0. \tag{80''}$$

As in Section III.2, we can derive the transformation law for this Neumann function:

$$\tilde{H}^*(w;\omega) = \tilde{H}(z;\zeta) - \log|f'(z)\,f'(\zeta)| + \frac{1}{\pi}[w,z]^{\sim}. \tag{81}$$

The additivity law (79) for the functional $[w,z]^{\sim}$ follows now directly from the group property of conformal mappings.

In the case that $\tilde{D}$ is the interior of the unit circle, we have

$$\tilde{H}(z;\zeta) = \log\frac{1}{|z-\zeta|} + \log\frac{1}{|1-z\bar{\zeta}|}. \tag{82}$$

There exists a three-parameter group of linear transformations

$$z^* = \sigma(z) = \frac{z-\tau}{1-\bar{\tau}z}\cdot e^{i\alpha} \tag{83}$$

which carry $\tilde{D}$ into itself. We verify that

$$\tilde{H}(z^*;\zeta^*) = \tilde{H}(z;\zeta) - \log|\sigma'(z)\,\sigma'(\zeta)|. \tag{84}$$

Hence, under every linear transformation of the unit circle onto itself we have

$$[w,z]^{\sim} = 0, \quad w = e^{i\alpha}\frac{z-\tau}{1-\bar{\tau}z}. \tag{85}$$

Let now $\tilde{D}$ be an arbitrary finite simply-connected domain and $z^* = \sigma(z)$ be a mapping of $\tilde{D}$ onto itself. Let $\eta = \varphi(z)$ be a fixed mapping of $\tilde{D}$ onto the unit circle; clearly $\eta^*(\eta)$ is a linear mapping of the unit circle into itself. Hence,

$$[z^*,z]^{\sim} = [z^*,\eta^*]^{\sim} + [\eta^*,\eta]^{\sim} + [\eta,z]^{\sim} = [z,\eta]^{\sim} + [\eta,z]^{\sim} = 0. \tag{86}$$

Hence, the symbol $[w,z]^{\sim}$ referring to the mapping of a domain D in the z-plane to a domain D^* in the w-plane depends only upon the domains D and D^* und is independent of the particular way in which this mapping is carried out. Hence, if we understand by $w = f(z)$ the mapping of $\tilde{D}$ onto the domain $|w| > 1$, the number $[w, z]^{\sim}$ is a functional of $\tilde{D}$ only.

Suppose now that we vary the infinite domain D by a deformation (1). This induces a corresponding deformation of the complement $\tilde{D}$. But since $z_0 \in D$ the correspondence $z^*(z)$ is a regular univalent mapping of $\tilde{D}$ into $\tilde{D}^*$. In particular, we have

$$[w, z^*]^{\sim} = [w, z]^{\sim} - [z^*, z]^{\sim}. \tag{87}$$

But by definition (78), we have

$$\begin{aligned}[z^*, z]^{\sim} &= \frac{1}{2}\iint_{\tilde{D}} \left(\nabla \log \left|1 - \frac{e^{i\alpha}\varrho^2}{(z-z_0)^2}\right|\right)^2 d\tau - \int_C \varkappa \log \left|1 - \frac{e^{i\alpha}\varrho^2}{(z-z_0)^2}\right| ds \\ &= \operatorname{Re}\left\{e^{i\alpha}\varrho^2 \int_C \frac{\varkappa(s)}{(z-z_0)^2} ds\right\} + O(\varrho^4). \end{aligned} \tag{88}$$

Thus, we find that the functional $[w, z]^{\sim}$ of $\tilde{D}$ satisfies the variational equation under the deformation (1):

$$\delta[w, z]^{\sim} = -\operatorname{Re}\{e^{i\alpha}\varrho^2 2\pi v''(z_0)\} + O(\varrho^4). \tag{89}$$

10. We have now constructed two functionals of the boundary curve C which possess the same variational formula. In order to discuss the meaning of this result, we wish to transform the functional (77) into a more suggestive expression. We calculate it first for the case that D is again the domain $|z| > 1$. Then, we have by (62′), (63′) and (67)

$$\mathfrak{H}(\zeta; \zeta) + 4 \log \sinh G_0(\zeta) = -4 \log 2. \tag{90}$$

Using the conformal invariance of $G_0(\zeta)$ and the transformation law (58) of the functional $\mathfrak{H}(\zeta; \zeta)$, we find that for an arbitrary simply-connected domain

$$\mathfrak{H}(\zeta; \zeta) + 4 \log \sinh G_0(\zeta) = -\frac{1}{\pi}[w, z] - 4 \log 2, \tag{91}$$

where $w(z)$ is the mapping function of D onto the exterior of the unit circle such that the point at infinity is preserved.

This leads as to the following interesting functional Φ which can be related to every simple closed curve C. We define

$$\Phi[C] = [w, z] + [w, z]^{\sim}, \tag{92}$$

where $[w, z]$ and $[w, z]^{\sim}$ are the expressions connected with the mapping of the exterior

D and the interior $\tilde{D}$ of C onto the exterior and the interior of the unit circle, respectively. Now, we can assert in view of (77) and (89): For every variation (1) of C

$$\delta\,\Phi[C] = 6\pi\,\delta \log \Delta \tag{93}$$

holds. In the case that C is the unit circle, we clearly have $\Phi[C]=0$ and $\Delta=1$. It appears, therefore, that we have proved the identity:

$$\prod_{\nu=1}^{\infty}\left(1-\frac{1}{\lambda_\nu^2}\right) = \exp\left\{\frac{1}{6\pi}([w,z]+[w,z]^{\tilde{}})\right\}. \tag{94}$$

From the knowledge of the mapping functions of the interior and the exterior of a curve onto circular domains, we can thus calculate its Fredholm determinant.

These considerations show the value of the variational technique, as well as the significance of the functionals $[w,z]$, to which we were led by our investigation. It is obvious that we have developed a formalism which is flexible enough to solve extremum problems with respect to the various functionals defined, and the solution of such problems will lead, as usual, to interesting existence and distortion theorems for the general theory of conformal mappings.

V. Connections on closed Riemann surfaces

1. In the theory of plane domains, we defined differentials, quadratic differentials, etc. by boundary behavior. For example, if D is a plane domain with analytic boundary C, then an analytic variable w defined on D is called a differential if

$$\operatorname{Im}\{w\dot{z}\}=0, \quad \left(\dot{z}=\frac{dz}{ds} \quad \text{on} \quad C\right) \tag{1}$$

on C. A plane domain is, of course, a Riemann surface which is provided with a global uniformizing parameter. Since all variables can be referred to the universal standard of comparison which this uniformizer affords, analysis can go much deeper. In shifting our interest to a more general Riemann surface, we are deprived of a universal coordinate system, and must fall back on the weaker device of considering transformation laws of sets of variables rather than boundary values.

For example, if $\mathfrak{R}$ is a Riemann surface, and $\mathfrak{U}=\{U_\alpha\}$ is a covering of $\mathfrak{R}$ such that to each $U_\alpha \in \mathfrak{U}$ there is associated a unique local uniformizing parameter $z_\alpha : U_\alpha \to \mathbf{C}$ (where $\mathbf{C}$ is the additive group of complex numbers, i.e., the complex numbers *not* including ∞), and z_α and z_β are analytically related on $U_\alpha \cap U_\beta$, then a differential (also called an Abelian differential) is defined by a set of variables $\{w_\alpha\}$, each w_α defined on U_α, such that we have

$$w_\alpha = w_\beta \frac{dz_\beta}{dz_\alpha} \tag{2}$$

in $U_\alpha \cap U_\beta$. More precisely, for each point $\mathfrak{p} \in U_\alpha \cap U_\beta$ we have

$$w_\alpha(z_\alpha(\mathfrak{p})) = w_\beta(z_\beta(\mathfrak{p})) \left(\frac{dz_\beta}{dz_\alpha}\right)_{\mathfrak{p}}. \tag{2'}$$

Any set of variables which satisfies these transformation rules is called an Abelian differential.

In Chapter III, we defined a domain function which we called a connection. The domain function was a variable $\Gamma(z)$ whose boundary behavior was characterized by the equation

$$\operatorname{Im}\{\Gamma(z)\dot{z}\} = \varkappa(s) \quad \text{for} \quad z = z(s) \in C. \tag{3}$$

Just as equation (2) represents the transformation laws arising from (1), we wish to find an equation representing the transformation laws arising from (3). We can do this as follows. Differentiating with respect to arc length in equation (1) we find

$$\operatorname{Im}\{w'\dot{z}^2 + w\ddot{z}\} = 0, \quad w' = \frac{dw}{dz}, \tag{4}$$

and dividing by $w\dot{z}$, which is real, we have

$$\operatorname{Im}\left\{\frac{w'}{w}\dot{z} + \frac{\ddot{z}}{\dot{z}}\right\} = 0. \tag{4'}$$

Since $\ddot{z}/\dot{z} = i\varkappa$, this shows that

$$\operatorname{Im}\left\{-\frac{w'}{w}\dot{z}\right\} = \varkappa, \tag{5}$$

or that $-(d/dz)\log w$ is a connection if w is a differential on D. This is the clue we need, for it suggests that we investigate the transformation laws of a set of variables $\{\Gamma_\alpha\}$ which are defined by

$$\Gamma_\alpha = -\frac{d}{dz_\alpha}\log w_\alpha, \tag{6}$$

where the variables $\{w_\alpha\}$ define an Abelian differential, i.e., satisfy (2). In fact, using equation (2) we find by a simple calculation that

$$\Gamma_\alpha = \Gamma_\beta\left(\frac{dz_\beta}{dz_\alpha}\right) - \frac{d}{dz_\alpha}\log\left(\frac{dz_\beta}{dz_\alpha}\right) \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{7}$$

We now make the definition that *any set of variables* $\{\Gamma_\alpha\}$ *which obeys the transformation rules* (7) *will be called a connection.* The existence of differentials and equation (6) assure us of the existence of connections.

2. An object which follows the transformation rules given by (7) deserves to be called a connection from the point of view of differential geometry since it allows us to define a covariant process of differentiation. In fact, if the collection $\{\varphi_\alpha\}$ defines an Abelian differential, then the collection $\{Q_\alpha\}$ where

$$Q_\alpha = \frac{d\varphi_\alpha}{dz_\alpha} + \Gamma_\alpha \varphi_\alpha, \quad \text{in} \quad U_\alpha \tag{8}$$

defines a *quadratic* differential, i.e.,

$$Q_\alpha = Q_\beta \left(\frac{dz_\beta}{dz_\alpha}\right)^2 \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{9}$$

This is verified by a simple computation using (2) and (7).

In fact, a collection of variables $\{\Psi_\alpha\}$ is said to define a differential of dimension ζ if they satisfy the transformation laws

$$\Psi_\alpha = \Psi_\beta \left(\frac{dz_\beta}{dz_\alpha}\right)^\zeta \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{2'}$$

(ζ can be any complex number.)

In this case the collection of variables $\{\eta_\alpha\}$ defined by

$$\eta_\alpha = \frac{d\Psi_\alpha}{dz_\alpha} + \zeta \Gamma_\alpha \Psi_\alpha, \quad \text{in} \quad U_\alpha, \tag{8'}$$

defines a differential of dimension $\zeta + 1$. These rules all correspond to rules for covariant differentiation in differential geometry and have led us to refer to the collection $\{\Gamma_\alpha\}$ as a connection.

The most general analytic connection is obtained as follows: let $\{\Gamma_\alpha\}$ be any analytic connection, then $\{\Gamma_\alpha + \varphi_\alpha\}$ is also a connection if $\{\varphi_\alpha\}$ defines an Abelian differential (whatever its singularities).

We arrive in this way at the most general connection, since any two connections $\{\Gamma_\alpha\}$ and $\{\Gamma^*_\alpha\}$ define a differential $\{\varphi_\alpha\}$ by

$$\Gamma^*_\alpha - \Gamma_\alpha = \varphi_\alpha. \tag{10}$$

That connections do indeed exist follows from the existence of differentials by the method of (6). But, of course, not every connection arises in this manner. (We shall look into this question presently.)

3. It is clear that connections may have poles; thus let us define what we mean by $\varrho[\Gamma]$, the *residue* of a connection $\Gamma = \{\Gamma_\alpha\}$. Let $\mathfrak{p} \in U_\alpha \cap U_\beta$, and let $C \subset U_\alpha \cap U_\beta$ be a simple closed curve which bounds a *cell* $D \subset U_\alpha \cap U_\beta$ and let $\mathfrak{p} \in D$. We suppose that Γ is regular on C and in D except, possibly, at $\mathfrak{p}$. Then we have

$$\int_C \Gamma_\alpha dz_\alpha = \int_C \Gamma_\beta dz_\beta - \int_C d \log \left(\frac{dz_\beta}{dz_\alpha}\right) = \int_C \Gamma_\beta dz_\beta. \tag{11}$$

Thus we may define

$$\varrho[\Gamma; \mathfrak{p}] = \frac{1}{2\pi i}\int_C \Gamma_\alpha dz_\alpha \tag{12}$$

as the residue of Γ at $\mathfrak{p}$, and

$$\varrho[\Gamma] = \sum_{\mathfrak{p}\in\mathfrak{R}} \varrho[\Gamma; \mathfrak{p}], \tag{13}$$

as the residue of Γ. We can easily see that

$$\varrho[\Gamma] = \chi(\mathfrak{R}) = 2 - 2p, \tag{14}$$

where p is the genus of $\mathfrak{R}$ and $\chi(\mathfrak{R})$ is its Euler characteristic. For let Γ^* be a connection defined by

$$\Gamma^*_\alpha = -\frac{d}{dz_\alpha}\log \omega_\alpha, \tag{15}$$

where ω_α is an Abelian differential of the first kind. Then

$$\frac{1}{2\pi i}\int_C \Gamma^*_\alpha dz_\alpha = -(\text{the number of zeros of } \omega_\alpha \text{ inside } C), \tag{16}$$

so that
$$\varrho[\Gamma^*] = 2 - 2p. \tag{17}$$

But $\Gamma^* - \Gamma = \varphi$, an Abelian differential, and

$$\varrho[\Gamma + \varphi] = \varrho[\Gamma] + \varrho[\varphi] = \varrho[\Gamma], \tag{18}$$

so
$$\varrho[\Gamma^*] = \varrho[\Gamma]. \tag{19}$$

This shows that
$$\varrho[\Gamma] = 2 - 2p. \tag{14'}$$

We now see, therefore, that a connection must have at least one pole unless $\chi(\mathfrak{R})=0$, i.e., that $\mathfrak{R}$ is of genus 1.

As to the general structure of a connection, we can, in fact, demonstrate the following: *let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *be* n *points on the Riemann surface* $\mathfrak{R}$, *of genus* p, *let* $a_1, \ldots, a_n$ *be any* n *complex numbers such that* $a_1+\ldots+a_n=2-2p$, *and let* $k_1, \ldots, k_n$ *be* n *integers such that* $k_j \geqslant 1$, $j=1, \ldots, n$, *then there exists a connection* Γ *which has a pole of order* k_j *and residue* a_j *at* $\mathfrak{p}_j$, $j=1, \ldots, n$, *and is otherwise regular on* $\mathfrak{R}$. In order to see this let $\Gamma^{(1)}$ be any connection with simple poles at $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$. (We can always construct such a connection from a differential of the first kind.) We may next assume that $m=n$. For if $m<n$ we may increase the number of poles by adding differentials of the third kind (having one pole already in the set $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ and one pole free). If $n<m$ we may "kill" the poles at $m-n$ of the points $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ by adding differentials of the third kind. Let $\Gamma^{(2)}$ be a connection with simple poles $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, and suppose the residues are $b_1, \ldots, b_n$, respectively. Then there exists a differential of the third kind Ω which has simple poles at $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}_1, \ldots, \mathfrak{q}_n$ with residues $a_1, \ldots, a_n, -b_1, \ldots, -b_n$, respectively (since $a_1+\ldots+a_n=b_1+\ldots+b_n$). Therefore,

$$\Gamma^{(3)}=\Gamma^{(2)}+\Omega \tag{20}$$

is a connection having simple poles at $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ with residues $a_1, \ldots, a_n$, respectively. Next, suppose that $k_{j_1}, \ldots, k_{j_r}$ are the integers greater than 1. (Of course, there may be none! we may have $r=0$.) Let φ be a differential of the second kind with a pole of order k_{j_s} at $\mathfrak{p}_{j_s}$, $s=1, \ldots, r$. Then

$$\Gamma=\Gamma^{(3)}+\varphi \tag{21}$$

is a connection of the type asserted to exist.

4. In order to introduce some global considerations concerning connections, we find it convenient to use certain techniques of homology theory. In particular, the Čech cohomology groups will be of use to us [5, 11].

Corresponding to the covering $\mathfrak{U}$ of $\mathfrak{R}$, there is associated a complex $N(\mathfrak{U})$, the nerve of $\mathfrak{U}$. Since $\mathfrak{R}$ is a compact triangulable space, we can choose a $\mathfrak{U}$ such that

$$H^q(N(\mathfrak{U}), \mathbf{C}) \approx H^q(\mathfrak{R}, \mathbf{C}), \tag{22}$$

$q=0, 1, \ldots$. If to each pair $U_\alpha, U_\beta \in \mathfrak{U}$, such that $U_\alpha \cap U_\beta \neq 0$, we assign a complex number $c_{\alpha\beta}$ (so that $c_{\beta\alpha}=-c_{\alpha\beta}$), then we have defined a cochain on $N(\mathfrak{U})$.

Let $\{\omega_\alpha\}$ define an Abelian differential of the first kind on $\mathfrak{R}$. Let $\{f_\alpha\}$ be a collection of holomorphic functions defined on the $\{U_\alpha\}$ such that

$$\frac{df_\alpha}{dz_\alpha} = \omega_\alpha \tag{23}$$

then in $U_\alpha \cap U_\beta$ we have
$$f_\alpha - f_\beta = c_{\alpha\beta} = \text{const.} \tag{24}$$

Thus we have assigned a 1-cochain of $N(\mathfrak{U})$ to the collection $\{\omega_\alpha\}$. This cochain is actually a cocycle, for obviously we have

$$c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha} = 0. \tag{25}$$

(Note $c_{\gamma\alpha} = -c_{\alpha\gamma}$, etc.)

Next suppose $\{g_\alpha\}$ is another collection of holomorphic functions such that

$$\frac{dg_\alpha}{dz_\alpha} = \omega_\alpha. \tag{26}$$

Then $g_\alpha = f_\alpha + c_\alpha$, where c_α is a constant. Thus

$$g_\alpha - g_\beta = b_{\alpha\beta} = c_{\alpha\beta} + (c_\alpha - c_\beta).$$

The cochain $\{b_{\alpha\beta}\}$ is also a cocycle, which we see differs from $\{c_{\alpha\beta}\}$ be a coboundary, i.e., is cohomologous to $\{c_{\alpha\beta}\}$. Therefore, we have a unique cohomology class associated with $\{\omega_\alpha\}$.

The Abelian differentials of the first kind themselves form a group — a p-(complex) dimensional vector space over $\mathbf{C}$. Let us denote this group by $D^1(\mathfrak{R})$. The assignment of cohomology classes given above defines a homomorphism

$$h : D^1(\mathfrak{R}) \to H^1(\mathfrak{R}, \mathbf{C}). \tag{27}$$

The kernel of this homomorphism is zero, i.e., $h\{\omega_\alpha\} = 0$ means $\omega_\alpha \equiv 0$. For if $\{\omega_\alpha\} \to 0$ then we have holomorphic functions $\{f_\alpha\}$ such that

$$\frac{df_\alpha}{dz_\alpha} = \omega_\alpha \quad \text{in} \quad U_\alpha \tag{23'}$$

$$f_\alpha - f_\beta = 0 \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{24'}$$

This means $f_\alpha = f_\beta = f$ defined globally. Since $\mathfrak{R}$ is compact, $f = \text{const.}$ Therefore

$$\omega_\alpha = \frac{df}{dz_\alpha} \equiv 0. \tag{28}$$

Let the image of $D^1(\mathfrak{R})$ under h be denoted by $H^1_{\frac{1}{2}}(\mathfrak{R}, \mathbf{C})$. Then h is an isomorphism of $D^1(\mathfrak{R})$ onto $H^1_{\frac{1}{2}}(\mathfrak{R}, \mathbf{C})$.

We should note that if we let $\bar{D}^1(\mathfrak{R})$ denote the complex conjugates $\{\bar{\omega}_\alpha\}$ of the Abelian differentials, then we may also define (the definition extends directly and naturally as given above) h to map $\bar{D}^1(\mathfrak{R})$ *isomorphically into* $H^1(\mathfrak{R}, \mathbf{C})$. If we denote the image of $\bar{D}^1(\mathfrak{R})$ under h by $\bar{H}^1_{\frac{1}{2}}(\mathfrak{R}, \mathbf{C})$, it can, in fact, be shown that

$$H^1(\mathfrak{R}, \mathbf{C}) = H^1_{\frac{1}{2}}(\mathfrak{R}, \mathbf{C}) \oplus \bar{H}^1_{\frac{1}{2}}(\mathfrak{R}, \mathbf{C}), \tag{29}$$

so that the direct sum
$$D^1(\mathfrak{R}) \oplus \bar{D}^1(\mathfrak{R}) \tag{30}$$
is isomorphic to $H^1(\mathfrak{R}, \mathbf{C})$.

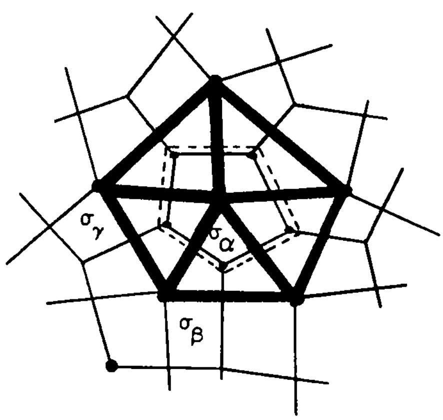
There are other methods of defining homomorphisms of $D^1(\mathfrak{R})$ into $H^1(\mathfrak{R}, \mathbf{C})$ which are interesting, and we shall now describe one of these.

The collections of functions $\{f_\alpha\}$ associated with $\{\omega_\alpha\}$ are functions defined on the neighborhoods of the covering $\mathfrak{U} = \{U_\alpha\}$.

Since $\mathfrak{R}$ is triangulable, we can associate a covering of $\mathfrak{R}$ with a given triangulation in a way which is conceptually very convenient. In particular, this covering $\mathfrak{U}$ will be finite and have the property that

$$H^q(N(\mathfrak{U}), G) \approx H^q(\mathfrak{R}, G) \tag{31}$$

for $q = 0, 1, \ldots$ and any coefficient group G. In order to do this, let $\mathfrak{K}$ be a simplicial complex covering $\mathfrak{R}$, i.e., a triangulation of $\mathfrak{R}$ and introduce a distance function on $\mathfrak{R}$ so that it becomes a metric space. (Any distance function, compatible with the topology of $\mathfrak{R}$, which makes $\mathfrak{R}$ a metric space, will be satisfactory.) Let $\mathfrak{K}^*$ be the dual cell complex of $\mathfrak{K}$, and let us label the



vertices (0-simplexes) of $\mathfrak{K}$ as $\{\sigma_\alpha\} = \sigma_\alpha, \sigma_\beta, \sigma_\gamma, \ldots$ etc. Let s_α denote the dual 2-cell of σ_α, and let U_α be an ε-neighborhood of s_α. We can choose ε so small that U_α and U_β do not intersect unless s_α and s_β have a common 1-cell on their boundaries, so that $N(\mathfrak{U})$ is naturally isomorphic to $\mathfrak{R}$. Such a covering can be termed *proper finite*, and we shall assume that $\mathfrak{U}$ is such a covering.

The function $\{f_\alpha\}$ can be extended to the star of U_α, i.e., f_α can be continued into each U_β such that $U_\alpha \cap U_\beta \neq 0$; for we have

$$f_\alpha(\mathfrak{p}) - f_\beta(\mathfrak{p}) = c_{\alpha\beta} \quad \text{for} \quad \mathfrak{p} \in U_\alpha \cap U_\beta, \tag{32}$$

thus we can define
$$f_\alpha(\mathfrak{p}) = f_\beta(\mathfrak{p}) + c_{\alpha\beta} \quad \text{for} \quad \mathfrak{p} \in U_\beta. \tag{33}$$

Having thus extended the function $\{f_\alpha\}$ let us define

$$\tau_{\alpha\beta} = f_\beta(\sigma_\beta) - f_\beta(\sigma_\alpha). \tag{34}$$

Now let us consider
$$\tau'_{\alpha\beta} = f_\alpha(\sigma_\beta) - f_\alpha(\sigma_\alpha), \tag{35}$$

then
$$\begin{aligned} \tau_{\alpha\beta} - \tau'_{\alpha\beta} &= f_\beta(\sigma_\beta) - f_\beta(\sigma_\alpha) - f_\alpha(\sigma_\beta) + f_\alpha(\sigma_\alpha) \\ &= f_\beta(\sigma_\beta) - f_\alpha(\sigma_\beta) - (f_\beta(\sigma_\alpha) - f_\alpha(\sigma_\alpha)) \\ &= c_{\alpha\beta} - c_{\alpha\beta} = 0. \end{aligned} \tag{36}$$

Thus, if $\sigma_{\alpha\beta}$ (the 1-simplex corresponding to (U_α, U_β) in $N(\mathfrak{U})$—we have $\partial\sigma_{\alpha\beta} = \sigma_\beta - \sigma_\alpha$) is a 1-simplex lying in the star of U_γ, then

$$f_\gamma(\mathfrak{p}) = f_\beta(\mathfrak{p}) - c_{\beta\gamma} \quad \text{for} \quad \mathfrak{p} \in U_\beta, \tag{37}$$

and
$$f_\gamma(\sigma_\beta) - f_\gamma(\sigma_\alpha) = f_\beta(\sigma_\beta) - c_{\beta\gamma} - f_\beta(\sigma_\alpha) + c_{\beta\gamma} = \tau_{\alpha\beta}, \tag{38}$$

thus we have
$$\tau_{\alpha\beta} = f_\gamma(\sigma_\beta) - f_\gamma(\sigma_\alpha), \tag{39}$$

where γ is such that $\sigma_{\alpha\beta}$ lies in the star of U_γ.

Now let τ be the cochain of $\mathfrak{R}$ (or equivalently of $N(\mathfrak{U})$) such that

$$\tau[\sigma_{\alpha\beta}] = \tau_{\alpha\beta}, \tag{40}$$

then we can easily see from the definition above that τ is a *cocycle*. For we have

$$\tau_{\alpha\beta} + \tau_{\beta\gamma} + \tau_{\gamma\alpha} = f_\beta(\sigma_\beta) - f_\beta(\sigma_\alpha) + f_\beta(\sigma_\gamma) - f_\beta(\sigma_\beta) + f_\beta(\sigma_\alpha) - f_\beta(\sigma_\gamma) = 0. \tag{41}$$

Again it can be seen that the cohomology class of τ depends only on the differential ω, and not on the particular choice of the functions f_α. For suppose that $\{g_\alpha\}$ is another set corresponding to $\{\omega_\alpha\}$, then

$$g_\alpha = f_\alpha + c_\alpha, \tag{42}$$

so we could have

$$\tau_{\alpha\beta} = g_\beta(\sigma_\beta) - g_\beta(\sigma_\alpha) = f_\beta(\sigma_\beta) + c_\beta - f_\beta(\sigma_\alpha) - c_\beta = \tau_{\alpha\beta}, \tag{43}$$

and we see that, in fact, the *cocycle*, τ itself does not depend on the system $\{f_\alpha\}$ but only on ω. Of course, τ does depend on the complex $\mathfrak{K}$; for if we shift the vertices of $\mathfrak{K}$, the numbers $\tau_{\alpha\beta}$ will change. Even so, τ still determines a unique cohomology class of $H^1(\mathfrak{R}, \mathbf{C})$, which we shall denote by τ—the ambiguity not being serious.

Finally, let us define

$$\xi_{\alpha\beta} = f_\beta(\sigma_\alpha) - f_\alpha(\sigma_\beta). \tag{44}$$

By adding

$$\tau_{\alpha\beta} = f_\beta(\sigma_\beta) - f_\beta(\sigma_\alpha), \tag{45}$$

and

$$\xi_{\alpha\beta} = f_\beta(\sigma_\alpha) - f_\alpha(\sigma_\beta), \tag{46}$$

we get

$$\tau_{\alpha\beta} + \xi_{\alpha\beta} = f_\beta(\sigma_\beta) - f_\alpha(\sigma_\beta) = -c_{\alpha\beta} \tag{47}$$

so we see that ξ, defined by $\xi[\sigma_{\alpha\beta}] = \xi_{\alpha\beta}$, is also a cocycle.

We shall show that the homomorphism

$$\tau : D^1(\mathfrak{R}) \to H^1(\mathfrak{R}, \mathbf{C}), \quad \text{where} \quad \tau(\omega) = \tau, \tag{48}$$

is an isomorphism into. Let $KI(c, z)$ denote the Kronecker index of a cocycle c, and a cycle z (of the dual dimension to c). Then we can prove that

$$\int_z \omega = KI(\tau(\omega), z), \tag{49}$$

where z is a 1-cycle on $\mathfrak{R}$. Since $KI(.,.)$ is a topological invariant, it is sufficient to work with cycles and cocycles of $\mathfrak{K}$. Let the values of z be defined by

$$z[\sigma_{\alpha\beta}] = k_{\alpha\beta}. \tag{50}$$

Then

$$\int_z \omega = \sum_{\sigma_{\alpha\beta}} k_{\alpha\beta} \int_{\sigma_{\alpha\beta}} \omega = \sum k_{\alpha\beta} \{f_\beta(\sigma_\beta) - f_\beta(\sigma_\alpha)\} = \sum k_{\alpha\beta}\, \tau_{\alpha\beta} = KI(\tau(\omega), z). \tag{51}$$

If we let $\omega_1, \ldots, \omega_p$ be a basis (over $\mathbf{C}$) in $D^1(\mathfrak{R})$, and $z_1, \ldots, z_{2p}$ be a one-dimensional (integral) homology basis of $\mathfrak{R}$, then

$$\pi_{jk} = KI(\tau(\omega_j), z_k) \tag{52}$$

are the elements of a period matrix of $\mathfrak{R}$.

We can, however, convert the assertion above into a more interesting statement. Let us define a 0-cochain k by

$$k[\sigma_\alpha] = f_\alpha(\sigma_\alpha) = k_\alpha. \tag{53}$$

The coboundary of this cochain is given by

$$\delta k[\sigma_{\alpha\beta}] = k[\sigma_\beta - \sigma_\alpha] = f_\beta(\sigma_\beta) - f_\alpha(\sigma_\alpha). \tag{54}$$

Therefore, we may conclude that

$$f_\beta(\sigma_\beta) - f_\alpha(\sigma_\alpha) = f_\beta(\sigma_\beta) - f_\beta(\sigma_\alpha) - c_{\alpha\beta} = \tau_{\alpha\beta} - c_{\alpha\beta}, \tag{55}$$

or

$$\tau_{\alpha\beta} = c_{\alpha\beta} + k_\beta - k_\alpha, \tag{56}$$

i.e., τ is cohomologous to c. But this means that we can now write

$$\int_z \omega = KI(h(\omega), z). \tag{57}$$

Now choose a basis $(\nu) = (\omega_1, \dots, \omega_p)$ in $D^1(\mathfrak{R})$. Then we can define an homomorphism

$$h_*^{(\nu)} : H_1(\mathfrak{R}, \mathbf{Z}) \to D^1(\mathfrak{R}) \tag{58}$$

by

$$h_*^{(\nu)}(z) = \sum_{j=1}^{p} KI(h(\omega_j), z)\, \omega_j. \tag{59}$$

It is clear that $h_*^{(\nu)}$ is an isomorphism into, for if $h_*^{(\nu)}(z) = 0$, then $KI(h(\omega), z) = 0$ for all ω, i.e., $z = 0$. We next form the factor group

$$J(\mathfrak{R}) = \frac{D^1(\mathfrak{R})}{h_*^{(\nu)} H_1(\mathfrak{R}, \mathbf{Z})}. \tag{60}$$

This group is clearly a p complex dimensional torus viz. a p-dimensional Abelian variety. $J(\mathfrak{R})$ is the *Jacobi variety* of $\mathfrak{R}$.

5. In order to facilitate our further discussion of connections, we introduce some terminology. A connection will be called *normal* if each of its singularities has an integer residue. We define D_Γ, *the divisor of a connection* Γ, to be the zero-cycle of $\mathfrak{R}$ which has the value $\varrho[\Gamma; \mathfrak{p}]$ at $\mathfrak{p}$ (viz., the residue of Γ at $\mathfrak{p}$). Thus, we see that to say Γ is normal means that D_Γ is an integral divisor.

A connection will be called *simple* if its only singularities are simple poles. Since any connection can be expressed as the sum of a simple connection and an Abelian differential of the second kind, we see that the divisor of a connection is determined entirely by the "simple part" of the connection.

A connection will be called *canonical* if it can be expressed as

$$\Gamma_\alpha = -\frac{d}{dz_\alpha} \log \omega_\alpha \tag{60'}$$

where ω_α is an Abelian differential. And a connection will be called *elementary* if it has only one singular point. An elementary connection is clearly normal since $\varrho[\Gamma]=2-2p$.

We are now going to assign a cohomology class to each normal connection Γ; this class will be an element of $H^1(\Re; \mathbf{C}^*)$. We note that the coefficient group used is $\mathbf{C}^*$, the multiplicative group of non-vanishing complex numbers. When this group is used, cochains will be written in a multiplicative notation rather than the usual additive notation used when the coefficient group is $\mathbf{C}$. For example, if $\{A_{\alpha\beta}\}$ is a cochain, the condition for this cochain to be a cocycle is that $A_{\alpha\beta}\, A_{\beta\gamma}\, A_{\gamma\alpha}=1$.

Let Γ be represented by $\{\Gamma_\alpha\}$, and let

$$\mathfrak{L}_\alpha = -\int \Gamma_\alpha\, dz_\alpha \tag{61}$$

be the indefinite integral of $-\Gamma_\alpha$ in U_α. Also we let

$$\Psi_\alpha = e^{\mathfrak{L}_\alpha}. \tag{62}$$

Although $\mathfrak{L}_\alpha$ may have logarithmic singularities and thus not be single-valued in U_α, we see that, since Γ is normal, Ψ_α is single-valued in U_α. Also we have

$$\mathfrak{L}_\alpha - \mathfrak{L}_\beta = b_{\alpha\beta} + \log\left(\frac{dz_\beta}{dz_\alpha}\right), \tag{63}$$

where $b_{\alpha\beta}$ is a complex number. Even though $b_{\alpha\beta}$ is not determined uniquely, it is determined up to an additive term $2\pi i\, n_{\alpha\beta}$, so that the non-zero complex number

$$A_{\alpha\beta} = e^{b_{\alpha\beta}} \tag{64}$$

is uniquely determined. We now have

$$\frac{\Psi_\alpha}{\Psi_\beta} = A_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right). \tag{65}$$

It is easily seen that $\{A_{\alpha\beta}\}$ is a cocycle on $N(\mathfrak{U})$ with coefficients in $\mathbf{C}^*$. Of course, this cocycle is not uniquely determined by the connection Γ since the functions $\mathfrak{L}_\alpha$ are determined only up to an additive constant. Thus if $\mathfrak{L}^*_\alpha$ is another determination of $-\int \Gamma_\alpha\, dz_\alpha$, then

$$\mathfrak{L}^*_\alpha = \mathfrak{L}_\alpha + b_\alpha. \tag{66}$$

Let $A_\alpha = e^{b_\alpha}$, then if $\{A^*_{\alpha\beta}\}$ is the cocycle corresponding to the functions $\mathfrak{L}^*_\alpha$, we see that $A^*_{\alpha\beta} = A_{\alpha\beta}(A_\alpha/A_\beta)$, so $\{A^*_{\alpha\beta}\}$ is cohomologous to $\{A_{\alpha\beta}\}$. This shows that Γ determines a unique cohomology class of $H^1(N(\mathfrak{U}), \mathbf{C}^*)$. It follows from standard techniques of Čech cohomology theory that this cohomology class in turn determines a unique element of $H^1(\mathfrak{R}, \mathbf{C}^*)$. We shall let (Γ) denote this element of $H^1(\mathfrak{R}, \mathbf{C}^*)$, and $(\Gamma) \in H^1(\mathfrak{R}, \mathbf{C}^*)$ is called the cohomology class of the connection Γ.

Naturally, one will ask the question: given connections Γ and Γ^*, when is $(\Gamma) = (\Gamma^*)$? We can proceed to answer this as follows. Let $\mathfrak{L}_\alpha = -\int \Gamma_\alpha\, dz_\alpha$ and $\mathfrak{L}^*_\alpha = -\int \Gamma^*_\alpha\, dz_\alpha$ be the indefinite integrals of the connections over the U_α, and let $\Psi_\alpha = e^{\mathfrak{L}_\alpha}$, $\tilde{\Psi}_\alpha = e^{\mathfrak{L}^*_\alpha}$ in U_α, then (Γ) and (Γ^*) are determined by

$$\frac{\Psi_\alpha}{\Psi_\beta} = A_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right) \quad \text{and} \quad \frac{\tilde{\Psi}_\alpha}{\tilde{\Psi}_\beta} = A^*_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right). \tag{68}$$

Since $(\Gamma) = (\Gamma^*)$ we have
$$A_{\alpha\beta} = \frac{A_\alpha}{A_\beta} A^*_{\alpha\beta}. \tag{69}$$

Let $\Psi^*_\alpha = A_\alpha \tilde{\Psi}_\alpha$, then
$$\Psi^*_\alpha = A_\alpha e^{\mathfrak{L}^*_\alpha} = e^{\mathfrak{L}^*_\alpha + b_\alpha} \tag{70}$$

Since $\mathfrak{L}^*_\alpha$ is determined only up to an additive constant, we see that the Ψ^*_α will serve to determine (Γ^*) just as well as the $\tilde{\Psi}_\alpha$. But now we have

$$\frac{\Psi_\alpha}{\Psi_\beta} = \frac{\Psi^*_\alpha}{\Psi^*_\beta}, \tag{71}$$

so if we let
$$f_\alpha = \frac{\Psi_\alpha}{\Psi^*_\alpha}, \tag{72}$$

then in $U_\alpha \cap U_\beta$ we have
$$f_\alpha(\mathfrak{p}) = f_\beta(\mathfrak{p}). \tag{73}$$

Define a function $f(\mathfrak{p})$ on all of $\mathfrak{R}$ by

$$f(\mathfrak{p}) = f_\alpha(\mathfrak{p}) \quad \text{for} \quad \mathfrak{p} \in U_\alpha. \tag{74}$$

Thus f is single valued (since Γ is *normal*) and meromorphic on $\mathfrak{R}$.

We can now write
$$\log\,(f\Psi^*_\alpha) = \log \Psi_\alpha, \tag{75}$$

thus
$$-\frac{d}{dz_\alpha}\log \Psi^*_\alpha - \frac{1}{f}\frac{df}{dz_\alpha} = -\frac{d}{dz_\alpha}\log \Psi_\alpha. \tag{76}$$

This shows that
$$\Gamma^*_\alpha = \Gamma_\alpha + \frac{1}{f}\frac{df}{dz_\alpha}, \tag{77}$$

or symbolically
$$\Gamma^* = \Gamma + \frac{df}{f}, \tag{77'}$$

where f is a meromorphic function on $\mathfrak{R}$. Clearly, if this last relation holds between Γ and Γ^* then we have $(\Gamma) = (\Gamma^*)$.

Suppose now that Γ is a simple connection with its singularity at $\zeta \in \mathfrak{R}$, and Γ^* is another simple connection with its singularity at ζ also. It then follows that $(\Gamma) \neq (\Gamma^*)$. For their difference would be an Abelian differential of the first kind, which cannot be written as $d \log f$, where f is meromorphic.

We may also ask: when is a connection canonical? The answer is surprisingly simple. *A connection* Γ *is canonical if and only if it is normal and* $(\Gamma)=1$. The "only if" part is immediate from the definition of (Γ). Now suppose Γ is normal and $(\Gamma)=1$. Let

$$\Psi_\alpha = e^{-\int \Gamma_\alpha \, dz_\alpha}, \tag{78}$$

then

$$\frac{\Psi_\alpha}{\Psi_\beta} = A_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right). \tag{79}$$

The Ψ_α are single-valued since Γ is normal, and $A_{\alpha\beta} = A_\alpha / A_\beta$, since $(\Gamma)=1$. Thus Ψ_α / A_α is an Abelian differential and

$$\Gamma_\alpha = -\frac{d}{dz_\alpha} \log \frac{\Psi_\alpha}{A_\alpha} = -\frac{d}{dz_\alpha} \log \Psi_\alpha, \tag{80}$$

which proves our statement.

6. In this next section we shall make some references to the theory of complex line bundles. This theory is not essential to subsequent development of this paper and can be omitted if the reader so desires. The theory of complex line bundles (as we shall use it) has been developed by K. Kodaira and D. C. Spencer in [13]. Another development is contained in [10].

A line bundle is determined over $\mathfrak{R}$ if we are given a collection $\{\theta_{\alpha\beta}\}$ of non-vanishing holomorphic functions $\theta_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbf{C}^*$. We say that two such line bundles $\{\theta^*_{\alpha\beta}\}$ and $\{\theta_{\alpha\beta}\}$ are equivalent if there exist non-vanishing holomorphic functions $\lambda_\alpha : U_\alpha \to \mathbf{C}^*$ such that

$$\theta^*_{\alpha\beta} = \theta_{\alpha\beta} \frac{\lambda_\alpha}{\lambda_\beta}. \tag{81}$$

Each line bundle $\{\theta_{\alpha\beta}\}$ determines (again by standard techniques in Čech theory) a unique element of $H^1(\mathfrak{R}, \Omega^*)$, where Ω^* is the sheaf (faisceau) of germs of non-vanishing holomorphic functions on $\mathfrak{R}$.

It is known that

$$H^1(\mathfrak{R}, \Omega^*) \approx \mathbf{T}_p \times \mathbf{Z}, \tag{82}$$

where $\mathbf{T}_p$ is a $2p$-dimensional torus group, and $\mathbf{Z}$ is the additive group of integers.

In fact, $H^1(\mathfrak{R}, \Omega^*)$ even carries a natural complex structure and in this structure

$$\mathbf{T}_p \approx J(\mathfrak{R}), \tag{83}$$

the Jacobi variety of $\mathfrak{R}$ [28]. (In higher dimensions, the variety is called the Picard variety.)

Since constants are clearly holomorphic, a cocycle $\{A_{\alpha\beta}\}$ on $N(\mathfrak{U})$ with coefficients in $\mathbf{C}^*$ determines a bundle. Thus each connection Γ determines a bundle, $b(\Gamma) \in H^1(\mathfrak{R}, \Omega^*)$. Naturally, we wish to know when $b(\Gamma) = b(\Gamma^*)$ for two connections Γ and Γ^*. The condition is quite similar to the previous one concerning the equality of cohomology classes.

Given two connections Γ *and* Γ^*, *we have* $b(\Gamma) = b(\Gamma^*)$ *if and only if*

$$\Gamma^* = \Gamma + d \log f + \omega, \tag{84}$$

where f is a meromorphic function on $\mathfrak{R}$, and ω is an Abelian differential of the first kind.

In order to see this, we first note that if $\{\lambda_\alpha\}$ is a system of non-vanishing holomorphic functions such that

$$\lambda_\alpha = \lambda_\beta B_{\alpha\beta} \quad \text{in} \quad U_\alpha \cap U_\beta,$$

where $B_{\alpha\beta}$ is a cocycle of $N(\mathfrak{U})$ with coefficients in $\mathbf{C}^*$, then $d/dz_\alpha \log \lambda_\alpha$ defines an Abelian differential of the first kind. For we have

$$\frac{d}{dz_\alpha} \log \lambda_\alpha = \frac{d}{dz_\alpha} \log \lambda_\beta + \frac{d}{dz_\alpha} \log B_{\alpha\beta} = \left(\frac{dz_\beta}{dz_\alpha}\right) \frac{d}{dz_\beta} \log \lambda_\beta.$$

In order to prove (84), if (Γ) is represented by $\{A_{\alpha\beta}\}$ and $\{\Gamma^*\}$ by $\{A^*_{\alpha\beta}\}$, we have

$$\frac{A_{\alpha\beta}}{A^*_{\alpha\beta}} = \frac{\lambda_\alpha}{\lambda_\beta} \quad \text{in} \quad U_\alpha \cap U_\beta, \tag{85}$$

where λ_α is holomorphic and non-vanishing in U_α, and λ_β is holomorphic and non-vanishing in U_β, since $b(\Gamma) = b(\Gamma^*)$. As in (68), we have

$$\frac{\Psi_\alpha}{\Psi_\beta} = A_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right), \quad \frac{\Psi^*_\alpha}{\Psi^*_\beta} = A^*_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right), \tag{68'}$$

so that

$$\frac{\Psi_\alpha}{\Psi^*_\alpha} \cdot \frac{\Psi^*_\beta}{\Psi_\beta} = \frac{A_{\alpha\beta}}{A^*_{\alpha\beta}} = \frac{\lambda_\alpha}{\lambda_\beta}. \tag{86}$$

This means that
$$f_\alpha = \frac{\Psi_\alpha}{\lambda_\alpha \Psi_\alpha^*} = \frac{\Psi_\beta}{\lambda_\beta \Psi_\beta^*} = f_\beta \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{87}$$

So the $\{f_\alpha\}$ determine a meromorphic function f on $\mathfrak{R}$ where $f = f_\alpha$ in U_α. We can now write
$$\Psi_\alpha = \Psi_\alpha^* f / \lambda_\alpha \quad \text{in} \quad U_\alpha, \tag{88}$$
or
$$-\log \Psi_\alpha^* = -\log \Psi_\alpha + \log f + \log \lambda_\alpha, \tag{89}$$

and upon differentiation we have
$$-\frac{d}{dz_\alpha}\log \Psi_\alpha^* = -\frac{d}{dz_\alpha}\log \Psi_\alpha + \frac{d}{dz_\alpha}\log f + \frac{d}{dz_\alpha}\log \lambda_\alpha, \tag{90}$$

which is the explicit expression of equation (84).

As an application of this, we have immediately, if Γ_ζ and Γ_ζ^* are elementary connections (with singularity at $\zeta \in \mathfrak{R}$) then
$$b(\Gamma_\zeta) = b(\Gamma_\zeta^*), \tag{91}$$

since $\Gamma_\zeta^* - \Gamma_\zeta$ is an Abelian differential of the first kind. Thus, if we define the mapping
$$b : \zeta \to b(\Gamma_\zeta), \tag{92}$$

this maps the Riemann surface $\mathfrak{R}$ into the Jacobi variety of $\mathfrak{R}$.

A natural question is: which elements of $H^1(\mathfrak{R}, \mathbf{C}^*)$ are cohomology classes of connections? The answer is simple—all! For, as observed, each cocycle $\{A_{\alpha\beta}\}$ determines a bundle and each bundle has a (meromorphic) cross section [14]. Let $\{\Psi_\alpha\}$ determine a cross section of the bundle defined by $\{A_{\alpha\beta}(dz_\beta/dz_\alpha)\}$, then
$$\frac{\Psi_\alpha}{\Psi_\beta} = A_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right), \tag{65}$$

so $-(d/dz_\alpha)\log \Psi_\alpha = \Gamma_\alpha$ defines a connection. (In fact, a *normal* connection.)

As remarked earlier, each constant is holomorphic, so we have the inclusion mapping
$$\iota : \mathbf{C}^* \to \Omega^*. \tag{93}$$

This induces
$$\iota^* : H^1(\mathfrak{R}, \mathbf{C}^*) \to H^1(\mathfrak{R}, \Omega^*). \tag{94}$$

We have also defined an homomorphism h of $D^1(\mathfrak{R})$ into $H^1(\mathfrak{R}, \mathbf{C})$ by (27) and through exponentiation $e : \mathbf{C} \to \mathbf{C}^*$, we have the homomorphism
$$e^* : D^1(\mathfrak{R}) \to H^1(\mathfrak{R}, \mathbf{C}^*).$$

By the result expressed in equation (84), we see that the kernel of the homomorphism ι^* is $e^*(D^1(\mathfrak{R}))$, i.e., we have

$$\frac{H^1(\mathfrak{R}, \mathbf{C}^*)}{e^*(D^1(\mathfrak{R}))} \approx J(\mathfrak{R}). \tag{95}$$

7. Having developed a method of differentiation which is covariant with respect to local uniformizing parameters, it is natural to inquire about anti-differentiation, or integration. Locally, this is quite simple since we have only one independent variable. In fact, leaving aside the questions of multiple valuedness, if ζ is any complex number, and $\{\Psi_\alpha\}$ defines a differential of dimension $\zeta + 1$, we can find a differential $\{\varphi_\alpha\}$ of dimension ζ such that

$$\frac{d\varphi_\alpha}{dz_\alpha} + \zeta\,\Gamma_\alpha\,\varphi_\alpha = \Psi_\alpha \tag{96}$$

as follows. Using the integrating factor $\omega_\alpha^{-\zeta}$ where

$$\omega_\alpha = e^{-\int \Gamma_\alpha\, dz_\alpha}, \tag{97}$$

we have

$$\varphi_\alpha = \omega_\alpha^\zeta \int \frac{\Psi_\alpha}{\omega_\alpha^\zeta}\, dz_\alpha. \tag{98}$$

A simple formal calculation shows that $\{\varphi_\alpha\}$ defines a differential of dimension ζ.

In order to indicate how one can proceed in a global manner, we shall limit ourselves to the integration of quadratic differentials, i.e., the case where $\zeta = 1$, since this will illustrate all of the essentials.

Let $\{Q_\alpha\}$ define a quadratic differential of any dimension, and let $\{\Gamma_\alpha\}$ be a *canonical* connection. Let $\{\omega_\alpha\}$ define an Abelian differential such that $\Gamma_\alpha = -(d/dz_\alpha)\log \omega_\alpha$. We define

$$F_\alpha = \int \frac{Q_\alpha}{\omega_\alpha}\, dz_\alpha, \tag{99}$$

an indefinite integral over U_α.

The transformation laws for $\{Q_\alpha\}$ and $\{\omega_\alpha\}$ show that

$$F_\alpha - F_\beta = c_{\alpha\beta} + 2\pi i\, n_{\alpha\beta}, \tag{100}$$

where $c_{\alpha\beta} = -c_{\beta\alpha}$ is a complex number, and $n_{\alpha\beta} = -n_{\beta\alpha}$ is an integer. The F_α may not all be single valued (they may have logarithmic singularities in some U_α, which accounts for the terms $2\pi i\, n_{\alpha\beta}$; a different $n_{\alpha\beta}$ will be required for each branch of a multiple valued function). The non-vanishing complex number

$$M_{\alpha\beta} = e^{c_{\alpha\beta} + 2\pi i\, n_{\alpha\beta}} \tag{101}$$

is, however, uniquely determined. Since

$$\left.\begin{aligned} &M_{\beta\alpha} = M_{\alpha\beta}^{-1} \\ &M_{\alpha\beta}\, M_{\beta\gamma}\, M_{\gamma\alpha} = 1 \quad \text{for} \quad U_\alpha \cap U_\beta \cap U_\gamma \neq 0, \end{aligned}\right\} \tag{102}$$

and

we see that $M_{\alpha\beta}$ determines an element of $H^1(\mathfrak{R}, \mathbf{C}^*)$.

We now let

$$\varphi_\alpha = \omega_\alpha F_\alpha, \tag{103}$$

so that we have

$$\frac{\varphi_\alpha}{\varphi_\beta} = \frac{\omega_\alpha}{\omega_\beta} \frac{F_\alpha}{F_\beta} = \frac{dz_\beta}{dz_\alpha} \frac{F_\alpha}{F_\beta}. \tag{104}$$

If $F_\alpha / F_\beta = 1$ for all $U_\alpha \cap U_\beta$ then $\{\varphi_\alpha\}$ defines an Abelian differential. In any event, the $\{\varphi_\alpha\}$ define an analytic variable which is the product of the Abelian differential $\{\omega_\alpha\}$ with an Abelian integral.

The assignment of the cohomology class of $(M_{\alpha\beta})$ to the quadratic differential $\{Q_\alpha\}$ can be denoted by $h_\Gamma(Q_\alpha)$. (The subscript Γ indicates the dependence of this assignment on Γ.) If $D^2(\mathfrak{R})$ denotes the linear space of all quadratic differentials (not necessarily regular) on $\mathfrak{R}$, we can state that

$$h_\Gamma : D^2(\mathfrak{R}) \to H^1(\mathfrak{R}, \mathbf{C}^*) \tag{105}$$

is an homomorphism.

The homomorphism

$$e : \mathbf{C} \to \mathbf{C}^* \tag{106}$$

defined by $e : z \to e^{2\pi i z}$ induces an homomorphism

$$e^* : H^1(\mathfrak{R}, \mathbf{C}) \to H^1(\mathfrak{R}, \mathbf{C}^*). \tag{107}$$

Thus $e^{*-1} h_\Gamma(Q_\alpha)$ denotes a subset of $H^1(\mathfrak{R}, \mathbf{C})$. (The elements of this subset are the ones defined by $(c_{\alpha\beta} + 2\pi i\, n_{\alpha\beta})$, for all choices of $n_{\alpha\beta}$.) Let $z \in H_1(\mathfrak{R}, \mathbf{Z})$, and form

$$KI(e^{*-1} h_\Gamma(Q_\alpha), z). \tag{108}$$

This is a set of complex numbers, which we may refer to as the *periods* of $\{Q_\alpha\}$ around z. If all the periods of $\{Q_\alpha\}$ lie in the kernel of e, then

$$\varphi_\alpha = \omega_\alpha F \tag{103'}$$

where F is the logarithm of a meromorphic function on $\mathfrak{R}$. For in this case, we would have

$$F_\alpha - F_\beta = 2\pi i\, n_{\alpha\beta} \quad \text{in} \quad U_\alpha \cap U_\beta \tag{100'}$$

so that
$$G_\alpha = e^{F_\alpha} \tag{109}$$
would define a meromorphic function G on $\mathfrak{R}$ since
$$G_\alpha = G_\beta \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{110}$$
Thus we can define $F = \log G$, and we have $F_\alpha = F$ in U_α, which is what we asserted.

If we let $D^2_\Gamma(\mathfrak{R})$ denote the space of quadratic differentials of the first kind which have zeros (of at least as high an order) at the zeros of $\{\omega_\alpha\}$ we can define
$$h_\Gamma(Q_\alpha) \in H^1(\mathfrak{R}, \mathbf{C}), \tag{111}$$
so that
$$h_\Gamma : D^2_\Gamma(\mathfrak{R}) \to H^1(\mathfrak{R}, \mathbf{C}). \tag{112}$$
For in this case, each F_α will be regular and single-valued, so we need not use $\mathbf{C}^*$ as coefficients. In this case
$$\varphi_\alpha = \omega_\alpha F \tag{103''}$$
where F is an Abelian integral of the first kind. Again, we can define the period of $\{Q_\alpha\}$ around z to be
$$KI(h_\Gamma(Q_\alpha), z). \tag{108'}$$
Since $D^2_\Gamma(\mathfrak{R})$ is a vector space of (complex) dimension p (the genus of $\mathfrak{R}$), much of the development given for $D^1(\mathfrak{R})$ will carry over to $D^2_\Gamma(\mathfrak{R})$.

Finally, we can modify our approach so that it will apply to the linear space $D^2_{[\Gamma]}(\mathfrak{R})$, the quadratic differentials of $\mathfrak{R}$ which are regular except (possibly) on the divisor of Γ. (Actually, only $[\Gamma]$, the carrier of the divisor, is of importance.)

In this case each
$$F_\alpha = \int \frac{Q_\alpha}{\omega_\alpha} dz_\alpha \tag{99'}$$
is single valued if $U_\alpha \cap [\Gamma] = 0$. We define only such F_α, then as before we let $F_\alpha - F_\beta = c_{\alpha\beta}$, and this defines
$$h_\Gamma : D^2_{[\Gamma]}(\mathfrak{R}) \to H^1(\mathfrak{R} \bmod [\Gamma], \mathbf{C}). \tag{113}$$
By considering elements z of $H_1(\mathfrak{R} \bmod [\Gamma], \mathbf{Z})$ and $KI(.,.)$ defined for these groups, we have periods of elements of $D^2_{[\Gamma]}(\mathfrak{R})$ defined on $\mathfrak{R} - [\Gamma]$.

Up to this point, we have restricted ourselves to canonical connections. Certain complications arise if the connection Γ is not canonical. One wishes to deal with elementary connections as much as possible, however, and except in a few trivial cases, an elementary connection is never canonical. Thus we shall indicate how to deal with this complication.

Let us remark first, however, on our preference for dealing with elementary connections. If $\Gamma_\alpha(z_\alpha;\zeta)$ is an elementary connection having its simple pole at $\zeta\in\mathfrak{R}$, we can treat ζ as a parameter, and ask for the behavior of $\Gamma(z;\zeta)$ with respect to ζ. This behavior is not uniquely determined, but we can choose a $\Gamma(z;\zeta)$ with calculable behavior as follows. Let $\Gamma_0(z;\zeta_0)$ be any elementary connection, and let $\eta(z;\zeta,\zeta_0)$ be a differential of the third kind with residue $2-2p$ at ζ and $2p-2$ at ζ_0. Choose some homology basis (which is a fundamental system) $A_1,\ldots,A_p$; $B_1,\ldots,B_p$ on $\mathfrak{R}$. We can now choose an $\eta(z;\zeta,\zeta_0)$ whose periods around the A-loops are all zero. This $\eta(z;\zeta,\zeta_0)$ is *unique*. We now form

$$\Gamma(z;\zeta)=\Gamma_0(z;\zeta_0)+\eta(z;\zeta,\zeta_0). \tag{114}$$

This elementary connection $\Gamma(z;\zeta)$ is defined for every $\zeta\in\mathfrak{R}$, and its dependence on ζ can be calculated from that of $\eta(z;\zeta,\zeta_0)$.

Now let $\{\Gamma_\alpha\}$ denote any normal connection, then we have

$$\Psi_\alpha=e^{\Omega_\alpha}=e^{-\int\Gamma_\alpha dz_\alpha}. \tag{62'}$$

We proceed as before for the integration of quadratic differentials.

$$\varphi_\alpha=\Psi_\alpha F_\alpha, \tag{115}$$

where

$$F_\alpha=\int\frac{Q_\alpha}{\Psi_\alpha}dz_\alpha. \tag{116}$$

But now we have

$$\frac{\varphi_\alpha}{\varphi_\beta}=\frac{\Psi_\alpha}{\Psi_\beta}\frac{F_\alpha}{F_\beta}=A_{\alpha\beta}\left(\frac{dz_\beta}{dz_\alpha}\right)\frac{F_\alpha}{F_\beta}. \tag{117}$$

Therefore,

$$A_{\alpha\beta}F_\alpha=F_\beta \quad\text{in}\quad U_\alpha\cap U_\beta \tag{118}$$

in order for $\{\varphi_\alpha\}$ to be an Abelian differential.

The transformation laws for the $\{F_\alpha\}$ are

$$A_{\alpha\beta}(F_\alpha+c_{\alpha\beta})=F_\beta \quad\text{in } U_\alpha\cap U_\beta. \tag{119}$$

The $\{c_{\alpha\beta}\}$ define a new type of cochain (we note that, in general, $c_{\beta\alpha}\neq -c_{\alpha\beta}$, but $c_{\alpha\beta}=-A_{\beta\alpha}c_{\beta\alpha}$ always). The condition that a cochain $\{c_{\alpha\beta}\}$ defines a cocycle is given by

$$c_{\alpha\beta}+A_{\beta\alpha}c_{\beta\gamma}+A_{\gamma\alpha}c_{\gamma\alpha}=0. \tag{120}$$

Two cocycles $c_{\alpha\beta}$ and $c'_{\alpha\beta}$ are to be considered equivalent if there exists a zero dimensional cochain $\{c_\alpha\}$ such that

$$c_{\alpha\beta}-c'_{\alpha\beta}=c_\alpha-A_{\beta\alpha}c_\beta. \tag{121}$$

The equivalence classes of cocycles form a group which we denote by $H^1_\Gamma(\mathfrak{R}, \mathbf{C})$. This group depends only on (Γ) and $\mathfrak{R}$, and not on any special representation of (Γ).

Thus we can define

$$h_\Gamma : D^2_\Gamma(\mathfrak{R}) \to H^1_\Gamma(\mathfrak{R}, \mathbf{C}), \tag{122}$$

and

$$h_\Gamma : D^2_{[\Gamma]}(\mathfrak{R}) \to H^1_\Gamma(\mathfrak{R} \bmod [\Gamma], \mathbf{C}) \tag{123}$$

as before.

The extension of all these notions to treat differentials of any integral dimension is immediate. Non-integral dimensions introduce certain technicalities concerned with the multivaluedness arising from using non-integral exponents.

VI. Variation of connections on Riemann surfaces

1. We wish to derive in this chapter a variational formula for connections on a Riemann surface $\mathfrak{R}$ which is the generalization of formula (IV.44) valid for planar domains. In order to do so, we will have to define a method of infinitesimal variation for Riemann surfaces which is analogous to the variational kinematics described by (IV.1). Let $\mathfrak{p}_0$ be a point on the surface; we choose a uniformizer z in a neighborhood of $\mathfrak{p}_0$ such that $z = z_0$ corresponds to $\mathfrak{p}_0$. We consider the conformal mapping

$$z^* = z + \frac{e^{i\alpha} \varrho^2}{z - z_0} \tag{1}$$

which maps the circumference $|z - z_0| = \varrho$ into the linear slit $S_\varrho \equiv \langle z_0 - 2\varrho e^{i\alpha}, z_0 + 2\varrho e^{i\alpha} \rangle$. We define $\mathfrak{R}^*$ as that Riemann surface which consists of all points of $\mathfrak{R}$ except those which correspond to points z inside of the circle $|z - z_0| < \varrho$; this surface is closed by identifying the points $z = z_0 + \varrho e^{i\varphi}$ and $z = z_0 + \varrho e^{i(\alpha - \varphi)}$ which have the same image z^* on S_ϱ. Thus, the hole of $\mathfrak{R}$ created by the removal of the circle $|z - z_0| < \varrho$ is closed and z^* may be used as a uniformizer on $\mathfrak{R}^*$ for the remaining points of the piece of $\mathfrak{R}$ where previously z served as uniformizer. The variation of $\mathfrak{R}$ thus obtained leads to a Riemann surface $\mathfrak{R}^*$ which is arbitrarily near to $\mathfrak{R}$ in the sense that the differentials of $\mathfrak{R}^*$ differ from the differentials of $\mathfrak{R}$ arbitrarily little except in the immediate neighborhood of the point $\mathfrak{p}_0$. The variational formulas for the various canonical differentials of $\mathfrak{R}$ have been given by Schiffer and Spencer [26].

Let now $\Gamma(\mathfrak{p}; \mathfrak{q})$ be an elementary connection of $\mathfrak{R}$ and let $\Gamma^*(\mathfrak{p}; \mathfrak{q})$ by the corresponding connection of $\mathfrak{R}^*$, the Riemann surface obtained from $\mathfrak{R}$ by a variation of the above type with $\mathfrak{p}_0$ different from $\mathfrak{p}$ and $\mathfrak{q}$. We denote by $\mathfrak{R}_0$ that part of $\mathfrak{R}$ which is obtained by removal of all points which correspond to the circle $|z - z_0| < \varrho$

in the z-uniformizer neighborhood. We may consider $\mathfrak{R}_0$ also as a part of $\mathfrak{R}^*$ if we refer the points in the z^*-uniformizer neighborhood to those of $\mathfrak{R}_0$ by the relation (1).

We define now the expression

$$\Gamma^*(\mathfrak{p};\, \mathfrak{q}) - \Gamma(\mathfrak{p};\, \mathfrak{q}) = \Delta(\mathfrak{p};\, \mathfrak{q}) \tag{2}$$

for $\mathfrak{p}, \mathfrak{q} \neq \mathfrak{p}_0$. If ϱ is small enough, both points will lie in $\mathfrak{R}_0$. In this Riemann domain, $\Delta(\mathfrak{p};\, \mathfrak{q})$ is regular analytic since the poles of Γ at $\mathfrak{q}$ have cancelled by subtraction and the expression transforms like a differential

$$\Delta_\alpha(z_\alpha;\, \mathfrak{q})\frac{dz_\alpha}{dz_\beta} = \Delta_\beta(z_\beta;\, \mathfrak{q}), \tag{3}$$

since both connections Γ^* and Γ have the transformation law

$$\Gamma_\alpha \frac{dz_\alpha}{dz_\beta} = \Gamma_\beta - \frac{d}{dz_\beta} \log \frac{dz_\beta}{dz_\alpha}. \tag{3'}$$

Let $A_\nu, B_\nu\, (\nu = 1, \ldots, p)$ be a set of canonical conjugate cross-cuts of $\mathfrak{R}$ which lie entirely in $\mathfrak{R}_0$. They may then also serve as cross-cut system for $\mathfrak{R}^*$. We denote by $t(\mathfrak{p};\, \mathfrak{q})$ an integral of the second kind with a simple pole of residue 1 at $\mathfrak{q}$ and normalized with respect to the cross-cut system such that

$$\oint_{A_\nu} dt(\mathfrak{p};\, \mathfrak{q}) = 0, \quad \oint_{B_\nu} dt(\mathfrak{p};\, \mathfrak{q}) = w_\nu'(\mathfrak{q}) \tag{4}$$

where the $w_\nu'(\mathfrak{q})$ are differentials of the first kind.

The system A_ν, B_ν cuts $\mathfrak{R}$ into a simply-connected domain. It transforms, therefore, $\mathfrak{R}_0$ into a doubly-connected domain with the boundary continua $b = \sum(A_\nu + B_\nu)$ and the curve c which corresponds to the circumference $|z - z_0| = \varrho$. Since $t(\mathfrak{p};\, \mathfrak{q})$ is single-valued in the cut-up domain thus obtained, we can apply the residue theorem and find:

$$\frac{1}{2\pi i}\oint_{b+c} \Delta(\mathfrak{p};\, \mathfrak{q})\, t(\mathfrak{p};\, \mathfrak{r})\, d\mathfrak{p} = \Delta(\mathfrak{r};\, \mathfrak{q}). \tag{5}$$

Observe that Δ is a differential on $\mathfrak{R}_0$ such that the left side line integral is independent of the choice of the uniformizer. Each cut of the cross-cut system is run through twice in opposite directions. On each edge the proper determination of the integrand is to be taken. By normalization (4), $t(\mathfrak{p};\, \mathfrak{r})$ has the same value on both edges of each B_ν-cut; but it has the saltus $w_\nu'(\mathfrak{r})$ across each cut A_ν. Hence (5) yields

$$\frac{1}{2\pi i} \sum_\nu w_\nu'(\mathfrak{r}) \oint_{A_\nu} \Delta(\mathfrak{p};\, \mathfrak{q})\, d\mathfrak{p} + \frac{1}{2\pi i}\oint_c \Delta(\mathfrak{p};\, \mathfrak{q})\, t(\mathfrak{p};\, \mathfrak{r})\, d\mathfrak{p} = \Delta(\mathfrak{r};\, \mathfrak{q}). \tag{6}$$

Until now we defined Γ and Γ^* only by the characteristic connection properties, i.e., by their singularities at $\mathfrak{q}$ and their transformation law (3'). We might add to each an arbitrary differential of the first kind without affecting these properties. In order to obtain a variational formula for the connections, we have to couple them more closely and this can be done by the requirement:

$$\oint_{A_\nu} [\Gamma^*(\mathfrak{p};\ \mathfrak{q}) - \Gamma(\mathfrak{p};\ \mathfrak{q})]\, d\mathfrak{p} = \oint_{A_\nu} \Delta(\mathfrak{p};\ \mathfrak{q})\, d\mathfrak{p} = 0. \tag{7}$$

While the integral of each connection over the A_ν is parameter dependent, the integral of the variation is uniformizer independent and serves to single out the proper $\Gamma^*(\mathfrak{p}: \mathfrak{q})$ to the given connection $\Gamma(\mathfrak{p};\ \mathfrak{q})$.

Now, formula (6) simplifies to

$$\Delta(\mathfrak{r};\ \mathfrak{q}) = \frac{1}{2\pi i} \oint_c \Delta(\mathfrak{p};\ \mathfrak{q})\, t(\mathfrak{p};\ \mathfrak{r})\, d\mathfrak{p}. \tag{8}$$

The significance of this formula lies in the fact that $\Delta(\mathfrak{r};\ \mathfrak{q})$ can be calculated for all points in $\mathfrak{R}_0$ by evaluating the right-hand integral over the circumference c which lies in the z-uniformizer neighborhood of $\mathfrak{R}$.

2. We remember that the circumference c is to be followed in the positive sense with respect to $\mathfrak{R}_0$ when integrating in (8). Since $\Gamma(\mathfrak{p};\ \mathfrak{q})$ and $t(\mathfrak{p};\ \mathfrak{r})$ are both regular analytic functions of z for $|z - z_0| \leqslant \varrho$, we may use Cauchy's integral theorem and reduce (8) to

$$\Delta(\mathfrak{r};\ \mathfrak{q}) = \frac{1}{2\pi i} \oint_c \Gamma^*(\mathfrak{p};\ \mathfrak{q})\, t(\mathfrak{p};\ \mathfrak{r})\, d\mathfrak{p}. \tag{8'}$$

We observe that $\Gamma^*(\mathfrak{p};\ \mathfrak{q})$ is a regular analytic function of the uniformizer z^* and admits a Taylor series development in this variable. But the integral (8') can be evaluated more conveniently in the variable z. Hence, we use the transformation formula (3') in order to express Γ^* as follows:

$$\Gamma^*_z(\mathfrak{p};\ \mathfrak{q}) = \Gamma^*_{z^*}(\mathfrak{p};\ \mathfrak{q}) \left[1 - \frac{e^{i\alpha}\varrho^2}{(z - z_0)^2}\right] - \frac{2e^{i\alpha}\varrho^2}{(z - z_0)^3} + O(\varrho^3). \tag{9}$$

Let
$$\Gamma^*_{z^*}(\mathfrak{p};\ \mathfrak{q}) = F^*(z^*) \tag{10}$$

by the analytic representation of Γ^* in terms of the uniformizer z^*. Using the relation (1) between z and z^*, we find

$$\Gamma^*_z(\mathfrak{p};\ \mathfrak{q}) = F^*(z) + \frac{e^{i\alpha}\varrho^2}{z - z_0} F^{*\prime}(z) - \frac{e^{i\alpha}\varrho^2}{(z - z_0)^2} F^*(z) - \frac{2e^{i\alpha}\varrho^2}{(z - z_0)^3} + O(\varrho^3). \tag{11}$$

Thus, (8′) becomes

$$\Gamma^*(\mathfrak{r};\mathfrak{q})-\Gamma(\mathfrak{r};\mathfrak{q})=\frac{1}{2\pi i}\oint_c t(\mathfrak{p};\mathfrak{r})\left\{F^*(z)+e^{i\alpha}\varrho^2\left[\frac{F^{*\prime}(z)}{z-z_0}-\frac{F^*(z)}{(z-z_0)^2}-\frac{2}{(z-z_0)^3}\right]+O(\varrho^3)\right\}dz. \quad (12)$$

All terms written out are well-defined inside the circumference c, and the integral may be evaluated by means of the residue theorem. The term $O(\varrho^3)$ is a regular analytic function of $\mathfrak{r}$ and $\mathfrak{q}$ outside of c which has the explicit factor ϱ^3. It may be evaluated by integration over a fixed concentric circle with radius a as long as a is so small that this circle lies in the neighborhood of z_0 considered. Thus, a depends on $\mathfrak{r}$, $\mathfrak{q}$ and $\mathfrak{p}_0$ but not on ϱ. Hence, the $O(\varrho^3)$-term behaves uniformly as a remainder term for $\varrho\to 0$ as long as $\mathfrak{r}$, $\mathfrak{q}$ lie in a closed subdomain of $\mathfrak{R}_0$. By a simple calculation, we can evaluate the right hand integral in (12) and find

$$\Gamma^*(\mathfrak{r};\mathfrak{q})-\Gamma(\mathfrak{r};\mathfrak{q})=e^{i\alpha}\varrho^2[F^*(z_0)\,t'(\mathfrak{p}_0;\mathfrak{r})+t''(\mathfrak{p}_0;\mathfrak{r})]+O(\varrho^3). \quad (13)$$

Let $|z-z_0|=a$ be a fixed circle in $\mathfrak{R}_0$ in the uniformizer neighborhood of z. Clearly, by definition (10)

$$F^*(z_0)=\frac{1}{2\pi i}\oint_{|z^*-z_0|=a}\frac{F^*(z^*)}{z^*-z_0}dz^*=\frac{1}{2\pi i}\oint_{|z^*-z_0|=a}\frac{\Gamma^*(\mathfrak{p};\mathfrak{q})}{z^*-z_0}dz^*. \quad (14)$$

Holding a fixed, we deduce from (13) that $\Gamma^*-\Gamma=O(\varrho^2)$ as $\varrho\to 0$. Hence,

$$F^*(z_0)=\frac{1}{2\pi i}\oint_{|z-z_0|=a}\frac{\Gamma(\mathfrak{p};\mathfrak{q})}{z-z_0}dz+O(\varrho^2)=\Gamma(\mathfrak{p}_0;\mathfrak{q})+O(\varrho^2). \quad (14')$$

This leads to the final variational result

$$\Gamma^*(\mathfrak{r};\mathfrak{q})-\Gamma(\mathfrak{r};\mathfrak{q})=e^{i\alpha}\varrho^2[\Gamma(\mathfrak{p}_0;\mathfrak{q})\,t'(\mathfrak{p}_0;\mathfrak{r})+t''(\mathfrak{p}_0;\mathfrak{r})]+O(\varrho^3). \quad (15)$$

This elegant variational formula expresses the change of Γ in terms of the connection itself and differentials of the second kind and corresponds to formula (IV.44). In order to show the analogy of this formula to (IV.16), we shall now express (15) in terms of connections only.

It is well known that

$$t(\mathfrak{p};\mathfrak{s})-t(\mathfrak{q};\mathfrak{s})=w'(\mathfrak{s};\mathfrak{p},\mathfrak{q}) \quad (16)$$

is a differential of the third kind in $\mathfrak{s}$ with poles at $\mathfrak{p}$ and $\mathfrak{q}$ and with residues ± 1, respectively [9, 21, 26, 30]. Hence, we can write the difference of two connections

$$\Gamma(\mathfrak{s};\mathfrak{p})-\Gamma(\mathfrak{s};\mathfrak{q})=(2p-2)\,[t(\mathfrak{p};\mathfrak{s})-t(\mathfrak{q};\mathfrak{s})] \quad (17)$$

provided that

$$\oint_{A_\nu}[\Gamma(\mathfrak{s};\mathfrak{p})-\Gamma(\mathfrak{s};\mathfrak{q})]\,ds=0. \quad (18)$$

We have also the symmetry law

$$t'(\mathfrak{s};\,\mathfrak{r})=t'(\mathfrak{r};\,\mathfrak{s}),\tag{19}$$

where the dash denotes differentiation with respect to the first argument. Hence,

$$\begin{aligned}t'(\mathfrak{p}_0;\,\mathfrak{r})=t'(\mathfrak{r};\,\mathfrak{p}_0)&=\frac{\partial}{\partial\mathfrak{r}}[t(\mathfrak{r};\,\mathfrak{p}_0)-t(\mathfrak{p};\,\mathfrak{p}_0)]\\&=\frac{1}{2p-2}\frac{\partial}{\partial\mathfrak{r}}[\Gamma(\mathfrak{p}_0;\,\mathfrak{r})-\Gamma(\mathfrak{p}_0;\,\mathfrak{p})]\\&=\frac{1}{2p-2}\frac{\partial}{\partial\mathfrak{r}}\Gamma(\mathfrak{p}_0;\,\mathfrak{r}).\end{aligned}\tag{20}$$

Similarly, $$t''(\mathfrak{p}_0;\,\mathfrak{r})=\frac{1}{2p-2}\frac{\partial}{\partial\mathfrak{p}_0}\left[\frac{\partial}{\partial\mathfrak{r}}\Gamma(\mathfrak{p}_0;\,\mathfrak{r})\right]=\frac{1}{2p-2}\frac{\partial}{\partial\mathfrak{r}}\Gamma'(\mathfrak{p}_0;\,\mathfrak{r}).\tag{20'}$$

Thus, finally, we may bring (15) into the form

$$\Gamma^*(\mathfrak{r};\,\mathfrak{q})-\Gamma(\mathfrak{r};\,\mathfrak{q})=\frac{1}{2p-2}\frac{\partial}{\partial\mathfrak{r}}\{e^{i\alpha}\varrho^2[\Gamma(\mathfrak{p}_0;\,\mathfrak{r})\,\Gamma(\mathfrak{p}_0;\,\mathfrak{q})+\Gamma'(\mathfrak{p}_0;\,\mathfrak{r})+\Gamma'(\mathfrak{p}_0;\,\mathfrak{q})]\}+O(\varrho^3).\tag{21}$$

3. This symmetric variational formula suggests the existence of a symmetric expression $H(\mathfrak{r};\,\mathfrak{q})$ which is regular analytic in each uniformizer neighborhood except for a logarithmic pole for $\mathfrak{r}=\mathfrak{q}$ and such that

$$\Gamma(\mathfrak{r};\,\mathfrak{q})=\frac{\partial}{\partial\mathfrak{r}}H(\mathfrak{r};\,\mathfrak{q}).\tag{22}$$

This function $H(\mathfrak{r};\,\mathfrak{q})$ would then possess a particularly simple variational formula.

It is, indeed, possible to construct such a function $H(\mathfrak{r};\,\mathfrak{q})$ from the canonical integrals of the Riemann surface $\mathfrak{R}$. Let $w(\mathfrak{p};\,\mathfrak{r},\,\mathfrak{s})$ be an integral of the third kind, which has its logarithmic poles at $\mathfrak{r}$ and $\mathfrak{s}$ with residues $+1$ and -1, respectively, and which is normalized in such a way that it has zero periods with respect to the crosscuts A_ν of the canonical cut-system considered. It is well known [9, 26] that the expression

$$W(\mathfrak{p},\,\mathfrak{q};\,\mathfrak{r},\,\mathfrak{s})=w(\mathfrak{p};\,\mathfrak{r},\,\mathfrak{s})-w(\mathfrak{q};\,\mathfrak{r},\,\mathfrak{s})\tag{23}$$

is symmetric in the pair of argument points $\mathfrak{p}$, $\mathfrak{q}$ and the pair of parameter points $\mathfrak{r}$, $\mathfrak{s}$.

We choose an arbitrary but fixed differential of the first kind $w'(\mathfrak{p})$ and denote its zeros by $\mathfrak{p}_\nu$ $(\nu=1,\,2,\,\ldots,\,2p-2)$. We construct then the sum

$$\Lambda(\mathfrak{p},\,\mathfrak{q})=\frac{1}{(2p-2)(2p-3)}\sum_{\nu\neq\mu}W(\mathfrak{p},\,\mathfrak{p}_\nu;\,\mathfrak{q},\,\mathfrak{p}_\mu),\quad \nu,\,\mu=1,\,\ldots,\,2p-2.\tag{24}$$

Evidently, $\Lambda(\mathfrak{p}, \mathfrak{q})$ is a differential symmetric in $\mathfrak{p}$ and $\mathfrak{q}$ which has a logarithmic pole for $\mathfrak{p}=\mathfrak{q}$ with residue 1. For $\mathfrak{p}=\mathfrak{p}_\nu$, we have a logarithmic pole with residue $-(2p-2)^{-1}$. Hence, the expression

$$H(\mathfrak{p}; \mathfrak{q}) = [\Lambda(\mathfrak{p}, \mathfrak{q}) \cdot (2p-2) + \log w'(\mathfrak{p}) + \log w'(\mathfrak{q})] \tag{25}$$

has precisely all the properties demanded of the function $H(\mathfrak{p}, \mathfrak{q})$ introduced in (22). Indeed, $H(\mathfrak{p}, \mathfrak{q})$ is symmetric in $\mathfrak{p}$ and $\mathfrak{q}$, is locally analytic in both variables, except for $\mathfrak{p}=\mathfrak{q}$ where it has a logarithmic pole with residue $(2-2p)$. From its very construction it follows that its derivative with respect to $\mathfrak{p}$ is a connection which has a simple pole at $\mathfrak{p}=\mathfrak{q}$ with the residue $(2-2p)$. Hence, the relation (22) is verified.

It is seen that $H(\mathfrak{r}; \mathfrak{q})$ is determined only up to an additive constant by (22) and its symmetry. Hence, if we wish to derive from (21) a variational formula for $H(\mathfrak{r}, \mathfrak{q})$ by integration, we will likewise run into the problem of a proper constant of integration. We may dispose of this constant in such a way that the following elegant variational formula holds:

$$H^*(\mathfrak{r}; \mathfrak{q}) - H(\mathfrak{r}; \mathfrak{q}) = \frac{1}{2p-2} e^{i\alpha} \varrho^2 [H'(\mathfrak{p}_0; \mathfrak{r}) H'(\mathfrak{p}_0; \mathfrak{q}) + H''(\mathfrak{p}_0; \mathfrak{r}) + H''(\mathfrak{p}_0; \mathfrak{q})] + O(\varrho^3). \tag{26}$$

This formula corresponds to (IV.16) in the case of planar domains.

An important consequence of the identity (22) and the symmetry of $H(\mathfrak{r}; \mathfrak{q})$ is the fact that the connection $\Gamma(\mathfrak{r}; \mathfrak{q})$ depends analytically upon its parameter $\mathfrak{q}$. It is evident from (22) and the construction (25) of $H(\mathfrak{r}; \mathfrak{q})$ that given an arbitrary connection Γ with the only pole $\mathfrak{q}$, we can imbed it into a family $\Gamma(\mathfrak{r}; \mathfrak{q})$ of connections depending analytically upon $\mathfrak{q}$ by the normalization

$$\oint_{A_\nu} [\Gamma(\mathfrak{r}; \mathfrak{q}) - \Gamma(\mathfrak{r}; \mathfrak{q}_1)]\, d\mathfrak{r} = 0 \tag{27}$$

valid for fixed $\mathfrak{q}_1$ and arbitrary $\mathfrak{q}$. This result is, of course, implied in the representation (V.114). Finally, attention should be drawn to the expression

$$E(\mathfrak{p}; \mathfrak{q}) = \exp\{-H(\mathfrak{p}; \mathfrak{q})\}. \tag{28}$$

In view of (25), $E(\mathfrak{p}; \mathfrak{q})$ transforms like a differential in $\mathfrak{p}$ and a differential in $\mathfrak{q}$; it has no periods with respect to the crosscuts A_ν and has multiplicative periods with respect to the crosscuts B_ν. It has no poles and vanishes only at $\mathfrak{p}=\mathfrak{q}$ with the order $2p-2$. By these properties, the multiplicative double differential $E(\mathfrak{p}; \mathfrak{q})$ is determined up to a constant factor. Because of its symmetry in parameter and argument, it appears particularly convenient as the element with which to construct the various differentials and functions on $\mathfrak{R}$.

References

[1]. S. BERGMAN & M. SCHIFFER, Kernel functions and conformal mapping. *Compositio Math.*, 8 (1951), 205–249.

[2]. ——, *Kernel functions in mathematical physics*. New York, 1952.

[3]. R. COURANT, *Dirichlet's principle, conformal mapping and minimal surfaces*. New York, 1950.

[4]. R. COURANT & B. MANEL & M. SHIFFMAN, A general theorem on conformal mapping of multiply-connected domains. *Proc. Nat. Acad. Sci. U.S.A.*, 26 (1940), 503–507.

[5]. S. EILENBERG & N. STEENROD, *Foundations of algebraic topology*. Princeton, 1952.

[6]. A. HAMMERSTEIN, Nichtlineare Integralgleichungen nebst Anwendungen, *Acta Math.*, 54 (1930), 117–176.

[7]. G. H. HARDY & J. E. LITTLEWOOD & G. PÓLYA, *Inequalities*. Cambridge, 1934.

[8]. N. S. HAWLEY, Connections on closed Riemann surfaces, *Tech. Rep.* No. 72, Nonr-225(11), *Appl. Math. and Stat. Lab.*, Stanford University, 1958.

[9]. K. HENSEL & G. LANDSBERG, *Theorie der algebraischen Funktionen einer Variabeln*. Leipzig, 1902.

[10]. F. HIRZEBRUCH, *Neue topologische Methoden in der algebraischen Geometrie*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 9 (1956).

[11]. W. HUREWICZ & H. WALLMAN, *Dimension theory*. Princeton, 1941.

[12]. A. HURWITZ & R. COURANT, *Funktionentheorie*. Berlin, 1929.

[13]. K. KODAIRA & D. C. SPENCER: Groups of complex line bundles over compact Kähler varieties. *Proc. Nat. Acad. Sci. U.S.A.*, 39 (1953), 868–872.

[14]. ——, Divisor class groups on algebraic varieties. *Proc. Nat. Acad. Sci. U.S.A.*, 39 (1953), 872–877.

[15]. P. KOEBE, Über die konforme Abbildung mehrfach-zusammenhängender Bereiche. *Jahresber. D. Math. Ver.*, 19 (1910), 339–348.

[16]. S. LEFSCHETZ, Selected topics in algebraic geometry. *Bull. Nat. Res. Council*, 63 (1925), Chapter XVII.

[17]. L. LICHTENSTEIN, *Neuere Entwicklung der Potentialtheorie, Konforme Abbildung*. Encycl. math. Wissensch. II, C. 3 (1918).

[18]. G. PÓLYA & G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis*. Berlin, 1925.

[19]. F. RIESZ & B. SZ. NAGY, *Functional analysis*. New York, 1955.

[20]. M. SCHIFFER: The span of multiply-connected domains. *Duke Math. J.*, 10 (1943), 209–216.

[21]. ——, Hadamard's formula and variation of domain functions. *Amer. J. Math.*, 68 (1946), 417–448.

[22]. ——, The kernel function of an orthonormal system. *Duke Math. J.*, 13 (1946), 529–540.

[23]. ——, The Fredholm eigen values of plane domains. *Pacific J. Math.*, 7 (1957), 1187–1225.

[24]. ——, Application of variational methods in the theory of conformal mappings. *Proc. Symp. Appl. Math.*, 8 (1958), 93–113.

[25]. ——, Fredholm eigen values of multiply-connected domains. *Pacific J. Math.*, 9 (1959), 211–269.

[26]. M. SCHIFFER & D. C. SPENCER, *Functionals of finite Riemann surfaces*. Princeton, 1954.

[27]. E. SCHOTTKY, Über die konforme Abbildung mehrfach-zusammenhängender ebener Flächen. *J. Reine Angew. Math.*, 83 (1877), 300–351.

[28]. F. SEVERI, *Vorlesungen über algebraische Geometrie*. Leipzig, 1911.

[29]. H. WEYL, *Space, time, matter*. London, 1922.

[30]. H. WEYL, *Die Idee der Riemannschen Fläche*. Stuttgart, 1955.

Received August 17, 1961