SPECTRAL MEASURES IN LOCALLY CONVEX ALGEBRAS

BY

HELMUT H. SCHAEFER

The University of Michigan, U.S.A.(¹)

Introduction

The central subject of classical spectral theory has been the spectral representation of self-adjoint and normal operators in Hilbert space; after Hilbert had given a complete treatment of the bounded case, the spectral theory of unbounded self-adjoint and normal operators was developed by von Neumann and others. (An excellent account can be found in [23].) Abstractions of algebras of Hermitian and normal operators were considered by various authors, notably Stone [22] who characterized such algebras as algebras of continuous functions on a compact (Hausdorff) space. Investigations by Freudenthal [9] and Nakano [15] (especially papers 1 and 2), also leading to spectral theories, went in a different direction. Generalizations of unitary operators to reflexive Banach spaces were considered by Lorch [13]; Lorch also developed an operational calculus for those operators in reflexive Banach spaces that can be represented by a spectral measure [12]. (Taylor [24] developed such a calculus for closed operators on a Banach space whose spectrum does not cover the plane, but necessarily restricted to functions locally holomorphic on the spectrum. There are some recent results in this direction for operators on a locally convex space [26], [16].) The most extensive work on bounded and unbounded operators in a Banach space is due to Dunford, Schwartz, Bade and others (for a detailed bibliography, see [6] and [8]). Dunford considers operators that have a countably additive resolution of the identity, but may differ from a spectral operator (in the sense of Definition 4, Section 4 of this paper) by a quasi-nilpotent. A survey of this work is given in [6]. Other contributions were made by Bishop [3], and most recently results on spectral operators (in Dunford's sense) in locally convex spaces were announced by Tulcea [25].

Since the time when the spectral theory of bounded and unbounded normal operators in Hilbert space took definite shape, the theory of topological, in particular, of locally

(¹) Research sponsored in part by the U.S. Army Research Office.

convex vector spaces has made enormous progress; a presentation of this theory can be found in [4] and [11]. But in spite of this development, and in spite of the vast diversification of spectral theory some of the results of which have been mentioned, Hilbert space is still absolutely dominant when it comes to exhibiting reasonably large classes of linear mappings that can, with respect to their spectral behavior, be properly viewed as generalizations of finite matrices with linear elementary divisors. The theory presented in this paper is intended to show that spectral theory, in the sense presently discussed, is not intrinsically connected with orthogonality (and hence with the existence of an inner product), and can be developed for general locally convex spaces. On the other hand, it is not surprising that the concept of (partial) order plays a key role. To apply this tool one needs a study of the relations between order and topology in a topological vector space, such as was made in [14] and [18–20]. We establish in this paper, in terms of partial ordering, simple necessary and sufficient conditions for a closed (not necessarily continuous) operator on a locally convex space to be spectral (Definitions 3, 4, 5), that is, for it to be considered as a strict analogue of a self-adjoint (or normal) operator in Hilbert space. A few of the results contained in this paper, specialized to Banach algebras and spaces, have been announced in [21]. We proceed to give a brief survey of the five sections of the paper.

The central notion of this paper is that of spectral measure (Definition 2, Section 2). Spectral measures are certain vector-valued measures with values in a locally convex algebra A,(1) positive for a suitable ordering of A (Proposition 7). Thus we develop, in Section 1, some basic results on positive vector measures. The treatment is pursued only as far as its applications in subsequent sections require. A presentation of the general present-day theory of vector-valued measures can be found in [5], Chapter VI. It is apparent, though, that the notion of positivity, applied to vector measures, yields much richer results than the general theory.

Section 2 defines and discusses spectral measures on an arbitrary locally compact (Hausdorff) space X, with values in a locally convex algebra A. It is an important fact (Proposition 7) that every spectral measure is positive for a suitable ordering of A (in other words, that the range of a spectral measure is necessarily contained in a convex, weakly normal cone of A), since thus for the construction of spectral measures one may restrict attention to positive vector measures. The principal result is Theorem 2 which gives necessary and sufficient conditions for the existence of the Cartesian product of an arbitrary family of spectral measures.

Section 3 discusses the integration of spectral measures with respect to scalar-valued

(1) Formal definitions are given later. For locally convex algebras, see the beginning of Section 2.

(real or complex) bounded Baire functions on X; the integrals are called spectral elements of A (Definition 3). These are exactly the generalizations of operators similar to bounded normal operators when A is the algebra of operators on Hilbert space. However, since in a large part of the theory the assumption that A be the algebra of continuous endomorphisms of a topological vector space is entirely unessential, we have given the results in this section for arbitrary locally convex algebras. Each spectral element a can be represented by a unique spectral measure ν, defined on the Borel sets of the complex plane (respectively the real line) into A (Theorem 3); its support is the spectrum of a (Theorem 4). A simple necessary and sufficient condition is obtained for an $a \in A$ to be a real spectral element (Proposition 14), and for the elements of a subset $F \subset A$ to be presentable by a single spectral measure (Theorem 5). The algebra of all elements in A that are "functions" of a fixed spectral measure is, under a certain natural norm, isomorphic-isometric with the space of all complex-valued continuous functions on a compact Hausdorff space (Theorem 6, Corollary).

Section 4 discusses a number of special aspects when A is the algebra of continuous endomorphisms of a locally convex space E, in particular, the spectral behavior of spectral operators with compact spectrum (Definition 4), and shows that T is spectral in E if and only if its adjoint T' in E' is spectral (with respect to any topology on E' consistent with the dual system $\langle E, E' \rangle$) (Proposition 19). Here the adjoint T' and the map T^* (conjugate of a spectral T, cf. p. 157) can no longer be identified as is usually done in Hilbert space. Further, Theorem 7 exhibits quite general conditions under which certain subalgebras of the endomorphism algebra $\mathfrak{L}(E)$ can be represented simultaneously by a spectral measure (and hence are commutative). The existence of spectral operators, in particular, of compact spectral operators with infinite dimensional range, is closely related to the presence of absolute bases (in the sense of [10]), as Proposition 20 shows.

The final section studies the integration of spectral measures μ, with values in the algebra of weakly continuous endomorphisms of a locally convex space E, with respect to unbounded, complex-valued, Baire functions f on X. Every triple (X, f, μ) defines, in a natural way, a linear mapping T in E which is closed (Proposition 22) with dense domain D_T. (For Banach spaces and a more general class of mappings, an essentially equivalent definition of D_T is given by Bade [1].) The adjoint T' (Lemma 4) of a spectral operator $T \sim (X, f, \mu)$ is again spectral (Theorem 8), at least if E' is weakly semi-complete. As in the case of bounded spectrum, T can be represented by a spectral measure ν on the complex plane (Theorem 9) into $\mathfrak{L}(E)$; if, moreover, the resolvent set is non-empty then ν is unique and every $S \in \mathfrak{L}(E)$ commuting with T also commutes with ν. Theorem 10 gives necessary and sufficient conditions, in terms of order structures on $\mathfrak{L}(E)$, that a closed operator T

in E with dense domain and real spectrum be spectral. These conditions can be translated to the slightly more general case of spectral operators with complex spectrum.

The author is indebted to Professor L. J. Savage for several stimulating conversations on the subject of this paper.

Auxiliary Results on Ordered Vector Spaces

For the convenience of the reader, we collect in this section a number of basic definitions and theorems, without proofs, from the theory of ordered topological vector spaces. A vector space E over the real field $\mathbf{R}$ is ordered (=partially ordered) if for some pairs in $E \times E$, a reflexive and transitive relation "$\leqslant$" is defined such that $x \leqslant y$ (also denoted by $y \geqslant x$) implies $x+z \leqslant y+z$ for each $z \in E$ and $\lambda x \leqslant \lambda y$ for all $\lambda > 0$. The set $K = \{x: x \geqslant 0\}$, which is a convex cone in E containing its vertex 0, is called the positive cone of the ordered vector space E. It is well known and immediate that conversely, every such cone determines an ordering of E. An ordering of E induces, for every vector subspace $F \subset E$, an ordering of F and of E/F; similarly, if $\{E_\alpha : \alpha \in A\}$ is an arbitrary non-empty family of ordered vector spaces, an ordering is induced on their product $\prod_\alpha E_\alpha$ and direct vector sum $\bigoplus_\alpha E_\alpha$ [18, p. 119]. Further, let E_1 and E_2 be ordered vector spaces with positive cones K_1, K_2, and L a vector space of linear mappings on E_1 into E_2; L becomes an ordered vector space with respect to $K = \{T \in L: TK_1 \subset K_2\}$ as its positive cone. In particular, if $\langle E, F \rangle$ is a dual system [4, Chapter IV], E ordered with positive cone K, then F is ordered as a space $L(E, \mathbf{R})$ ($\mathbf{R}$ ordered as usual). The positive cone $H = \{y \in F: \langle x, y \rangle \geqslant 0 \text{ if } x \in K\}$ is called the dual (or conjugate) cone of K. An ordering of E is proper if "$\leqslant$" is anti-symmetric; the positive cone of a proper ordering of E is called a proper cone. A convex cone K (of vertex 0) in E is proper if and only if $K \cap -K = \{0\}$. The order structures of vector spaces considered in this paper will be proper unless the contrary is explicitly stated.

The preceding concepts carry over to vector spaces over the complex field $\mathbf{C}$ without difficulty. A vector space E over $\mathbf{C}$ is ordered if its underlying real space E_0 is ordered; all statements on E involving order concepts refer to E_0. As a consequence, if $\langle E, F \rangle$ is a complex dual system and E is ordered with positive cone K, the dual cone H of K will be identified with $\tilde{H} = \{y \in F: \operatorname{Re} \langle x, y \rangle \geqslant 0 \text{ for } x \in K\}$. We point out that the subset $\{z: \operatorname{Re} z \geqslant 0\}$ of $\mathbf{C}$ is not a proper cone in $\mathbf{C}$ (hence the corresponding ordering of $\mathbf{C}$ is not proper). For further details, see [19, Section 6]. Therefore, we shall henceforth assume a vector space to be defined over the complex field unless the contrary is stated, and by "cone" we shall understand a convex cone in E, containing its vertex 0. If a cone K and an order structure on E are mentioned in the same context, K will be the positive cone of the ordering in question.

Let E be an ordered vector space and a topological vector space. The positive cone K of E is normal [18, 20] if for some neighborhood base $\mathfrak{U}$ of 0, the relations $x \in U \in \mathfrak{U}$ and $0 \leqslant y \leqslant x$ imply $y \in U$. Equivalently, K is normal if the family of all full neighborhoods of 0 form a base, where $A \subset E$ is full [14] whenever $x, y \in A$ imply that the order interval $[x, y] = \{z: x \leqslant z \leqslant y\}$ is contained in A. Let ϕ denote a filter in E, then $[\phi] = \{[F]: F \in \phi\}$ is a filter base where $[F] = \bigcup \{[x,y]: x, y \in F\}$. Then K is normal in E if and only if $\lim \phi = 0$ implies $\lim [\phi] = 0$ for every filter ϕ on E [20, (1.a)]. If E is an ordered locally convex space,(1) it is easily seen that K is a normal cone in E if and only if there exists a family $\{p_\alpha: \alpha \in A\}$ of semi-norms, generating the topology of E, such that

$$p_\alpha(x+y) \geqslant p_\alpha(x) \quad (x, y \in K;\ \alpha \in A).$$

In a Hausdorff space, every normal cone is proper.

Let E be an ordered topological vector space, and let $\mathfrak{S}$ be a family of bounded subsets of E such that $E = \bigcup \{S: S \in \mathfrak{S}\}$. K is said to be an $\mathfrak{S}$-cone (strict $\mathfrak{S}$-cone) in E if for each $S \in \mathfrak{S}$, there exists an $S' \in \mathfrak{S}$ such that $S \subset \overline{S' \cap K - S' \cap K}$ $(S \subset (S' \cap K - S' \cap K))$. If E is locally convex, this property can be expressed by saying that the mapping

$$\mathfrak{S}' \to \{\overline{\Gamma(S \cap K)}: S \in \mathfrak{S}'\} \ (\mathfrak{S}' \to \{\Gamma(S \cap K): S \in \mathfrak{S}'\})$$

leaves the totality of fundamental systems $\{\mathfrak{S}'\}$ of $\mathfrak{S}$ invariant. (Here ΓA denotes the symmetric convex hull of $A \subset E$.) It is immediate that if K is a strict $\mathfrak{S}$-cone, then K is generating, i.e., $E = K - K$. Conversely, if $E = K - K$ is a (reflexive) Banach space, then K is a (strict) $\mathfrak{B}$-cone, $\mathfrak{B}$ the family of all bounded sets in E.

It is well known that if $\langle E, F \rangle$ is a dual system and $\mathfrak{S}$ is a family of weakly bounded subsets of F whose union is F, then the topology of uniform convergence on the sets of $\mathfrak{S}$ is a locally convex topology $\mathfrak{T}$ on E which is finer than the weak topology $\sigma(E, F)$. $\mathfrak{T}$ is called consistent with $\langle E, F \rangle$ if $E[\mathfrak{T}]' = F$, i.e., if the closed convex hull of each $S \in \mathfrak{S}$ is compact for $\sigma(F, E)$. The notions of normal cone and $\mathfrak{S}$-cone are dual as the following theorem shows whose proof may be found in [18, (1.5)]. We assume that E is ordered with positive cone K, H is the dual cone in F, and $\mathfrak{S}$ is a family of bounded subsets of F as above.

THEOREM A. *If H is an $\mathfrak{S}$-cone, then K is normal for the $\mathfrak{S}$-topology on E. Conversely, if K is normal for an $\mathfrak{S}$-topology consistent with $\langle E, F \rangle$, then H is a strict $\mathfrak{S}$-cone in F.*

(1) Notation and terminology will, with respect to locally convex vector spaces, in general follow Bourbaki [4]. However, the topologies considered in this paper will be Hausdorff topologies unless other wise stated.

The application of Theorem A to the system $\langle E, E' \rangle$ where E is locally convex, E' the (topological) dual of E, yields this corollary. We denote the dual cone of K now by K'.

COROLLARY. *If K is normal in E, then $E' = K' - K'$. Also, $E' = K' - K'$ if and only if K is weakly normal.*

Remark. If E is a normed space, then $E' = K' - K'$ if and only if K is normal for the norm topology of E [20, (1.c)].

Let E be an ordered locally convex space with positive cone K, and let M be a subset directed (filtering) for "$\leqslant$". The family of sections $M_x = \{y \in M : y \geqslant x\}$ $(x \in M)$ forms a filter base. The corresponding filter $\phi(M)$ is called the filter of sections of M.

THEOREM B. *Let K be normal in E, and let M be a non-empty directed subset of E. If $\phi(M)$ converges to $x_0 \in E$ weakly, then it converges to x_0 for the given topology on E.*

For the proof, see [19, (7.2)]. It can be shown that Theorem B is equivalent to the well-known theorem of Dini concerning the convergence of directed sets of continuous functions on a locally compact space.

Let E_1, E_2 be ordered locally convex spaces with respective positive cones K_1, K_2; assume that $\mathfrak{S}$ is a family of bounded subsets of E_1 whose union is E_1. Let $\mathfrak{L}(E_1, E_2)$ be the space of continuous linear mappings on E_1 into E_2, ordered with positive cone $\mathfrak{K} = \{T \in \mathfrak{L}(E_1 E_2): TK_1 \subset K_2\}$. We shall need the following result.

THEOREM C. *If K_1 is an $\mathfrak{S}$-cone in E_1, and if K_2 is normal in E_2, then $\mathfrak{K}$ is a normal cone for the $\mathfrak{S}$-topology on $\mathfrak{L}(E_1, E_2)$.*

The proof can be found in [19, (8.3)]. Finally, we list the following result concerning the continuity of positive linear forms on an ordered topological vector space E.

THEOREM D. *Let E be an ordered topological vector space with positive cone K. Each of the following assumptions implies that every positive linear form on E is continuous:*

(a) *K has non-empty interior*

(b) *E is metrizable and complete, K is closed and generating*

(c) *E is a bornological locally convex space, K is a semi-complete*(1) *strict $\mathfrak{B}$-cone*

While (*a*) is almost obvious, the proof of (*b*) can be found in [14, 5.5], and the proof of (*c*) in [18, (2.8)]. We note that each of the three assumptions in Theorem D implies that the dual cone K' of K is complete for the weak topology $\sigma(E', E)$.

(1) $A \subset E$ is semi-complete if every Cauchy sequence (for the uniformity on A induced by that of E) converges to a limit in A. For non-metrizable uniformities, semi-completeness is considerably weaker than completeness.

COROLLARY. *Let E, F be ordered locally convex spaces, such that the positive cone in F is weakly normal, and E satisfies any one of the conditions* (a), (b), (c) *of Theorem D. Then every positive linear mapping on E into F is continuous.*

Proof. Since every positive linear form on E is continuous, it follows that every positive linear map T is continuous for $\sigma(E, E')$ and $\sigma(F, F')$. If, under condition (a), x_0 is interior to K in E, it follows that the set $[-x_0, x_0]=\{z\in E: -x_0\leqslant z\leqslant x_0\}$ is mapped into $[-Tx_0, Tx_0]$ which is a bounded set in F since the positive cone in F is weakly normal. Hence T is continuous under (a). Under condition (b) or (c), we observe that the topology of E is necessarily the Mackey topology $\tau(E, E')$. Hence since T is weakly continuous, it is continuous for the given topologies on E and F.

1. Positive Vector Measures

Let X be a locally compact Hausdorff space, and denote by $\mathcal{C}(X)$ ($\mathcal{C}_{\mathbf{R}}(X)$) the vector space over $\mathbf{C}(\mathbf{R})$[1] of all complex-valued (real-valued) continuous functions on X with compact support. With respect to the subset of all real-valued non-negative functions as the positive cone, $\mathcal{C}(X)$ is an ordered vector space. When E is an ordered locally convex vector space with a weakly normal positive cone K, we shall show that every positive linear mapping φ on $\mathcal{C}(X)$ into E generates a positive vector measure on X (Definition 1, below).

For every compact subset $T\subset X$, let $\mathcal{C}(X; T)$ be the subspace of those elements in $\mathcal{C}(X)$ whose support is in T, endowed with the uniform topology. When $\mathcal{C}(X)$ is given the finest locally convex topology for which each of the injections $\mathcal{C}(X; T)\to\mathcal{C}(X)$ is continuous (equivalently, the topology of the inductive limit with respect to the directed family $\{\mathcal{C}(X); T)\}$), it follows from Theorem D, Corollary that every positive linear map on $\mathcal{C}(X)$ into E is continuous. In fact, each $\mathcal{C}_{\mathbf{R}}(X;T)$ satisfies condition (c) of Theorem D, and the underlying real space of $\mathcal{C}(X)$ is isomorphic with $\mathcal{C}_{\mathbf{R}}(X)\times\mathcal{C}_{\mathbf{R}}(X)$ where $\mathcal{C}_{\mathbf{R}}(X)$ is also endowed with the topology of the inductive limit.

Let $X=\mathbf{R}$ and assume that K is semi-complete and weakly normal in E. For every monotonic function $t\to x(t)$ on $\mathbf{R}$ into E, the Riemann–Stieltjes integral

$$\int f dx = \lim_n \sum_1^n f(\tau_\nu)[x(t_\nu)-x(t_{\nu-1})]$$

exists provided that for all n, $t_0\leqslant\tau_1\leqslant t_1\leqslant\ldots\leqslant\tau_n\leqslant t_n$ and $[t_0, t_n]$ contains the support of f, and that $\max_\nu|t_\nu-t_{\nu-1}|\to 0$ as $n\to\infty$. Thus x gives rise to a positive mapping

$$f\to\varphi(f)=\int f dx$$

(1) We denote the natural, real, complex numbers by $\mathbf{N}$, $\mathbf{R}$, $\mathbf{C}$ respectively.

on $\mathcal{C}(\mathbf{R})$ into E, and it can be shown that when K is weakly semi-complete, every positive linear mapping on $\mathcal{C}(\mathbf{R})$ into E is of this form.

Let Δ_0 denote the σ-ring of subsets of X generated by all compact subsets of type G_δ, and Δ the σ-algebra generated by all closed subsets of X. If X and Y are locally compact Hausdorff spaces and f is a mapping on X into Y, then f is Δ_0-measurable (Δ-measurable) if f^{-1} maps the ring $\Delta_0(Y)$ into the ring $\Delta_0(X)$ (the algebra $\Delta(Y)$ into $\Delta(X)$). A Δ_0-measurable function will also be called Baire measurable, and Δ_0 the ring of Baire sets in X. By contrast, a Baire function on X is a member of the smallest subclass of Y^X which contains every continuous function and is closed under the formation of (simple) sequential limits. When $Y=\mathbf{C}$ and X is countable at infinity, every Baire function is both Baire measurable and Δ-measurable.

If φ is a positive linear map on $\mathcal{C}(X)$ into E and w is a continuous linear form on E, then

$$f \to \langle \varphi(f), w \rangle$$

is a complex measure μ_w on X in the sense of [5]. Denote by $\mathfrak{L}(\mu_w)$ the vector subspace of $\mathbf{C}^X$ of all Δ-measurable, μ_w-integrable [5] complex-valued functions on X, and set

$$\mathcal{H} = \bigcap \{\mathfrak{L}(\mu_w) \colon w \in E'\}.$$

Again, we consider $\mathcal{H}$ as ordered with positive cone $\{f \in \mathcal{H} \colon f$ is real-valued and $f(t) \geqslant 0$ for $t \in X\}$.

We denote by W the space of all linear forms

$$h \to \int h \, d\mu_w \quad (w \in E')$$

on $\mathcal{H}$, and consider $\mathcal{H}$ as equipped with the weak topology $\sigma(\mathcal{H}, W)$, which is in general not a Hausdorff topology. It is clear that $\mathcal{C}(X)$ is dense in $\mathcal{H}$ for this topology, and that φ is continuous on $\mathcal{C}(X)$ into E for $\sigma(\mathcal{C}(X), W)$ and $\sigma(E, E')$. Hence there exists a unique continuous extension μ, of φ, to $\mathcal{H}$ with values in the algebraic dual F of E'. It is natural to call $\mathcal{H}$ the space of all (Δ-measurable) μ-integrable functions, and to denote the value of μ at $h \in \mathcal{H}$ by $\mu(h)$, or $\int h \, d\mu$, or $\int h(t) \, d\mu(t)$. It is also clear that if $\delta \in \Delta$ is relatively compact, then $\chi_\delta \in \mathcal{H}$ (χ_δ the characteristic function of δ), and that the restriction of μ to these characteristic functions determines a set function, with values in H, on the ring of relatively compact Δ-sets which is countably additive with respect to the weak topology $\sigma(F, E')$. (Here H denotes the weak closure of K in F.) The sets $\delta \in \Delta$ for which $\chi_\delta \in \mathcal{H}$ will be called μ-integrable; we note that the μ-integrable sets form a subring of Δ. Without danger of confusion, we shall denote the set function $\delta \to \int \chi_\delta \, d\mu$ again by μ and call it a (positive) vector measure. $\delta \in \Delta$ is a null set (μ) if $\int \chi_\delta d\mu = 0$, and a statement (on points $t \in X$) will be said, as usual, to hold almost everywhere if it holds in $X \sim N$ where N is a null set (μ). Finally, μ is bounded if X is an integrable set, i.e., if $t \rightarrow 1$ is in $\mathcal{H}$.

Our primary objective in this section is to determine conditions under which μ is K-valued (where E is considered as a subspace of F), on the ring of bounded (i.e., relatively compact) Baire sets and under which μ is countably additive on this ring for the given topology on E. The answer will be that these assertions are true when K is weakly semi-complete, and normal for the given topology on E. If, in addition, μ is bounded then it is K-valued and countably additive on the σ-ring Δ_0 of all Baire sets in X. We begin the proof by establishing a relevant generalization of B, Levi's monotone convergence theorem. In Propositions 1–4 it will be assumed that K is normal in E and weakly semi-complete.

PROPOSITION 1. *Let $\{f_n\}$ be a monotone sequence in $\mathcal{H}$ such that $\{\mu(f_n)\}$ is a bounded sequence in E. There exists an $f \in \mu^{-1}(E)$ such that* $\lim f_n(t) = f(t)$ *a.e.* (μ) *in X, and*

$$\lim_n \mu(f_n) = \mu(f)$$

holds for the given topology on E.

Proof. Let $\delta_0 \subset X$ denote the set in which the sequence $\{f_n\}$ fails to converge properly. Since $\{\mu(f_n)\}$ is bounded, it follows that $\langle \mu(f_n), u \rangle = \int f_n d\mu_u$ forms a bounded monotone sequence of real numbers for every real, continuous positive linear form u on E. The functions f_n are Δ-measurable whence it follows that $\delta_0 \in \Delta$, and δ_0 is a null set for every μ_u by the classical monotone convergence theorem. Since, by the normality of K, every real continuous linear form is the difference of two non-negative ones (Theorem A, Corollary), it follows that δ_0 is a null set (μ). Defining $f \in \mathcal{H}$ by setting $f(t) = 0$ if $t \in \delta_0$ and $f(t) = \lim_n f_n(t)$ in $X \sim \delta_0$, one obtains $f_n \rightarrow f$ a.e. (μ). On the other hand, $\{\mu(f_n)\}$ is a weak Cauchy sequence in E, hence $\mu(f_n) \rightarrow z \in E$ for $\sigma(E, E')$ since K is weakly semi-complete. It is obvious that $z = \mu(f)$, and the convergence $\mu(f_n) \rightarrow \mu(f)$ for the given topology of E follows from Theorem B.

PROPOSITION 2. (Fatou's Lemma). *Let $\{f_n\} \subset \mathcal{H}$ be a sequence of non-negative functions such that the lower envelope of every non-empty finite subset of $\{f_n\}$ is in $\mu^{-1}(E)$, and such that*

$$\liminf_{n \rightarrow \infty} \langle \mu(f_n), u \rangle < +\infty$$

for every real, continuous positive linear form on E. There exists an $h \in \mathcal{H}$ such that $h(t) = \liminf_{n \rightarrow \infty} f_n(t)$ a.e. (μ), $\mu(h) \in E$ and

$$p[\mu(h)] \leqslant \liminf_{n \rightarrow \infty} p[\mu(f_n)]$$

for every continuous semi-norm p (on E) which is monotone on K.

Proof. For any pair (n, k) of positive integers with $k \geqslant n$, let $h_{n,k} = \inf\{f_\nu: n \leqslant \nu \leqslant k\}$. Since $h_{n,k} \in \mu^{-1}(E)$ by assumption, Proposition 1 implies that $h_n \in \mu^{-1}(E)$ where $h_n = \lim_k h_{n,k}$. The sequence $\{h_n\}$ is monotone, and the hypothesis implies that $\{\mu(h_n)\}$ is bounded. Thus, by Proposition 1, there exists $h \in \mu^{-1}(E)$ such that $h(t) = \lim_n h_n(t)$ (and hence $h(t) = \liminf_{n\to\infty} f_n(t)$) a.e. (μ). Now $0 \leqslant h_n \leqslant f_k$, and consequently $0 \leqslant \mu(h_n) \leqslant \mu(f_k)$, for all $k \geqslant n$; if p is monotone on K, then

$$p[\mu(h_n)] \leqslant \liminf_{k\to\infty} p[\mu(f_k)] \quad (n \in \mathbf{N}).$$

By Proposition 1, we have $\mu(h_n) \to \mu(h)$ for the given topology on E whence the assertion follows by the continuity of p.

An immediate consequence of the preceding propositions is the dominated convergence theorem of Lebesgue.

PROPOSITION 3. *Let $\{f_n\}$ be a sequence belonging to a sublattice of $\mathcal{H}$ contained in $\mu^{-1}(E)$, such that $|f_n| \leqslant g$ for some $g \in \mathcal{H}$ and all $n \in \mathbf{N}$. If f is a function (in $\mathcal{H}$) such that $f(t) = \lim_n f_n(t)$ a.e. (μ), then $f \in \mu^{-1}(E)$ and $\lim_n \mu(f_n) = \mu(f)$ in E.*

Proof. It follows from Proposition 2 that $|f_n - f| \in \mu^{-1}(E)$ and $f \in \mu^{-1}(E)$. Let $g_n = \sup\{|f_\nu - f| : \nu \geqslant n\}$, then $g_n \in \mu^{-1}(E)$ by Proposition 1. Since $\{g_n\}$ is non-increasing and $g_n(t) \to 0$ a.e. (μ), Proposition 1 implies that $\mu(g_n) \to 0$ in E. By the assumed normality of K and the positivity of μ, $0 \leqslant |f_n - f| \leqslant g_n$ implies that $\mu(|f_n - f|) \to 0$ in E, whence from $0 \leqslant |f_n - f| + (f_n - f) \leqslant 2|f_n - f|$ it follows that $\mu(f_n - f) \to 0$ in E.

For any subset $A \subset \mathcal{H}$, denote by $\tilde{A}$ the family of functions in $\mathcal{H}$ which are limits a.e. of some sequence in A which is dominated in $\mathcal{H}$. From Proposition 3, we obtain this corollary.

COROLLARY. *If A is a sublattice of $\mathcal{H}$ contained in $\mu^{-1}(E)$, then $\tilde{A}$ has the same property.*

PROPOSITION 4. *For arbitrary positive φ, the restriction of $\delta \to \mu(\delta)$ to the ring of bounded Baire sets is a K-valued set function which is countably additive for the given topology on E; if φ is continuous for the uniform topology on $\mathcal{C}(X)$, then the same result holds for the σ-ring Δ_0, and μ is bounded.*

Proof. Let $\mathfrak{A}$ denote the class of all vector sublattices of $\mathcal{H}$ contained in $\mu^{-1}(E)$; clearly, $\mathfrak{A}$ is inductive when ordered by inclusion. Denote by A a maximal element of $\mathfrak{A}$ containing $\mathcal{C}(X)$; from the corollary of Proposition 3, it follows that A is closed under the formation of simple limits of sequences dominated in $\mathcal{H}$. Since the characteristic function of every relatively compact Baire set is dominated by a member of $\mathcal{C}(X)$, it is clear that A contains every such function. The assertion that μ is countably additive is immediate from Proposition 1. Finally, the assumption that φ is continuous for the uniform topology on $\mathcal{C}(X)$ means that $\{\varphi(f_\alpha)\}$ is bounded in E for an arbitrary subset $\{f_\alpha\} \subset \mathcal{C}(X)$ of uniformly bounded functions, hence $t \to 1$ $(t \in X)$ is a member of $\mathcal{H}$. Thus, in this case, the characteristic function of every Baire set is in A which completes the proof.

We observe that even when φ is not continuous for the uniform topology on $\mathcal{C}(X)$, the measure $\delta \to \mu(\delta)$ can be extended to the σ-ring Δ_0 of all Baire sets. One has to adjoin an improper element ∞ to K and set $\mu(\delta) = \infty$ for every $\delta \in \Delta_0$ which is the union of an increasing sequence $\{\delta_n\}$ such that $\{\mu(\delta_n)\}$ is unbounded.

It is sometimes convenient to consider the measure $\delta \to \mu(\delta)$, rather than φ (i.e., the integrals with respect to μ of the functions in $\mathcal{C}(X)$), as given. Assume that μ is a set function, defined on the ring of relatively compact Baire sets into a locally convex space E, and countably additive for $\sigma(E, E')$. The integral $\int h d\mu$ may then be obtained as follows. If s is a simple Baire function of compact support, $s = \sum_{i=1}^n \alpha_i \chi_i$, define $\int s d\mu = \sum_{i=1}^n \alpha_i \mu(\delta_i)$. When $\mathcal{H}$ denotes, as above, the space of complex-valued functions on X that are Δ-measurable and integrable (in the sense of [5]) for each of the scalar measures $h \to \int h d\langle \mu, w\rangle$ $(w \in E')$—call this set W—then $s \to \int s d\mu$ has a unique continuous extension to $\mathcal{H}$ with respect to $\sigma(\mathcal{H}, W)$ and $\sigma(F, E')$. This extension, which is completely determined by its values for all simple Baire functions of compact support (equivalently, by its values for all $f \in \mathcal{C}(X)$), we shall call canonical. Let us agree on the following definition.

DEFINITION 1. *Let $\Gamma \subset \Delta$ denote a ring of subsets of X containing all bounded Baire sets. A mapping $\delta \to \mu(\delta)$ on Γ, with values in a locally convex space E and countably additive for $\sigma(E, E')$, is a vector measure if μ agrees on Γ with the canonical extension of its restriction to the bounded Baire sets.*

As a consequence of this definition, we shall not distinguish between two vector measures with values in E if they agree on the intersection of their respective domains.

As in the scalar case, by the support of a vector measure μ on X we understand the complement of the largest open set $U \subset X$ such that $\mu(f) = 0$ whenever U contains the support of $f \in \mathcal{C}(X)$.

If E is ordered with weakly normal positive cone K, a K-valued vector measure on X will be called positive. For positive vector measures, we obtain the following criterion of μ-integrability.

PROPOSITION 5. *Let μ be a positive vector measure and $\delta \in \Delta$. Denote by $\mathfrak{F}$ ($\mathfrak{G}$) the family of all compact (open) subsets of X contained in δ (containing δ), directed for $\subset$ ($\supset$). In order that δ be μ-integrable, it is necessary and sufficient that $\mathfrak{G}$ contains an* *integrable set. If this is the case, the limits* $\lim \mu(\mathfrak{F})$ *and* $\lim \mu(\mathfrak{G})$ *exist for* $\sigma(F, E')$ *and their common value is equal to* $\mu(\delta)$.

Proof. The proof is immediate from [5, Chapter IV, § 4, Theorem 4], and the corollary of Theorem A.

COROLLARY. *If* K *is normal in* E *and complete for* $\sigma(E, E')$, *then for every* μ*-integrable set the limits in Proposition 5 exist for the given topology on* E.

This is an immediate consequence of Proposition 5 and Theorem B. We summarize the main results of this section in the following theorem.

THEOREM 1. *Let* X *be a locally compact Hausdorff space, and let* E *denote an ordered locally convex vector space whose positive cone is normal and semi-complete for* $\sigma(E, E')$. *For each positive linear map* φ *on* $\mathcal{C}(X)$ *into* E, *there exists a unique positive vector measure* μ *on* X *such that*

$$\varphi(f) = \int f d\mu$$

for $f \in \mathcal{C}(X)$, *and conversely.* μ *is countably additive with respect to every consistent topology on* E *for which* K *is normal, and* μ *is bounded if and only if* φ *is continuous for the uniform topology on* $\mathcal{C}(X)$.

A vector lattice is countably order-complete if every countable, majorized subset has an upper bound; we note that for every vector measure on X, the space $\mathcal{H}$ is a countably order-complete vector lattice. Under the assumption of Theorem 1, the same is true of every maximal sublattice $A \subset \mathcal{H}$ such that $A \subset \mu^{-1}(E)$, as follows from Proposition 1. If we denote by $\mathcal{H}_N$ the space of all null functions (i.e., of all $h \in \mathcal{H}$ such that $\mu(|h|) = 0$), it is easily seen that $A/\mathcal{H}_N$ is a vector lattice which is at least countably order complete. It is then clear that the mapping $f \to \int f d\mu$ induces a homomorphism of $A/\mathcal{H}_N$ onto a subspace of E which is a countably order-complete vector lattice for an order finer than that induced by E; thus, if E is a vector lattice, this subspace is in general not a sublattice of E.

Let μ be a vector measure on X, defined on a domain $\Gamma(\mu)$ with values in a locally convex space E. Let f denote a Baire measurable function on X with values in a locally compact space Y. Denote by Γ_1 the family of sets $\{\varepsilon \subset Y: f^{-1}(\varepsilon) \in \Gamma(\mu)\}$, then if Γ_1 contains the bounded Baire sets,

$$\delta \to \nu(\delta) = \mu[f^{-1}(\delta)] \quad (\delta \in \Gamma_1)$$

defines a vector measure ν on Y into E; we shall denote it by $\nu = f(\mu)$ and write $\Gamma_1 = \Gamma(\nu)$. (It is clear that $\Gamma(\nu)$ is a ring, σ-ring or σ-algebra respectively, whenever $\Gamma(\mu)$ has this property.) ν is positive (or bounded) if μ is positive (or bounded). Concerning the supports S_μ and S_ν of μ and ν respectively, we have the following result; μ is assumed to be positive.

PROPOSITION 6. *If $\nu = f(\mu)$ where μ is bounded and f Δ-measurable, then*

$$S_\nu = \bigcap \{\overline{f(\delta)} : \delta \in \Delta \quad \text{and} \quad \tilde{\mu}(\delta) = \tilde{\mu}(X)\}.\text{(1)}$$

In particular, when f is continuous and S_μ compact, then $S_\nu = f(S_\mu)$.

Proof. When $\tilde{\mu}(\delta) = \mu(X)$ and $\delta_1 = \overline{f(\delta)}$, then $\tilde{\nu}(\delta_1) = \tilde{\mu}(X) = \tilde{\nu}(Y)$ and, since δ_1 is closed, $\delta_1 \supset S_\nu$. On the other hand, let $\varepsilon = f^{-1}(S_\nu)$. Then $\tilde{\nu}(S_\nu) = \tilde{\nu}(Y) = \tilde{\mu}(X) = \tilde{\mu}(\varepsilon)$ and $S_\nu = f(\varepsilon) = \overline{f(\varepsilon)}$ which completes the proof.

2. Spectral Measures

By a locally convex algebra we shall understand an algebra A (over C, unless otherwise stated) with unit e, such that the underlying vector space A_0 is a (Hausdorff) locally convex space, and multiplication is separately continuous; that is, for each fixed $b \in A$, $a \to ab$ and $a \to ba$ are continuous endomorphisms of A_0. We write $a \smile b$ when $ab = ba$.

A locally convex algebra A is ordered if A_0 is ordered with closed,(2) weakly normal positive cone K, and if multiplication in A is related to this ordering of A_0 by the condition:

(M) $e \in K$; $a, b \in K$ and $a \smile b$ imply $ab \in K$.

As in the case of a vector space, K is called the positive cone of A.

DEFINITION 2. *Let A denote a locally convex algebra, X a locally compact Hausdorff space. A spectral measure on X into A is a vector measure μ on X into A_0 such that:*

(i) *$\Gamma(\mu)$ is a σ-algebra, and $\mu(X) = e$.*

(ii) *$\mu(\delta \cap \varepsilon) = \mu(\delta)\mu(\varepsilon)$ for all $\delta, \varepsilon \in \Gamma(\mu)$.*

Condition (i) implies that every spectral measure is bounded (in the sense of Section 1); condition (ii) may be expressed by saying that a spectral measure is a weak σ-homomorphism of a Boolean σ-algebra of subsets of X, onto a Boolean σ-algebra of idempotents in A. The support $S(\mu)$ of a spectral measure will be called its spectrum. If X is the real (complex) number field under the usual topology, a spectral measure on X is called a real (complex) spectral measure.

PROPOSITION 7. *For every spectral measure μ on X into A, there exists an ordering of A with respect to which μ is positive.*

Proof. By the definition of positivity for vector measures (see page 135), we have to show that the range $\mu(\Gamma)$ of μ is contained in a weakly normal cone $K \subset A$ that satisfies

(1) $\tilde{\mu}$ denotes the canonical extension of μ to Δ.

(2) The closure of a weakly normal cone is weakly normal (Theorem A, Corollary).

condition (M) above. Denote by $\hat{A}$ the smallest subalgebra of A containing $\mu(\Gamma)$; it is clear that $\hat{A}$ is commutative. If K is the convex conical extension of $\mu(\Gamma)$ (i.e., the intersection of all (convex) cones in A containing $\mu(\Gamma)$), then $e \in K$ and obviously $ab \in K$ if $a, b \in K$. Thus there remains to show that K is weakly normal in A or, what amounts to the same, that K is normal in $\hat{A}$ for the weak topology, i.e., for the topology induced on $\hat{A}$ by $\sigma(A, A')$. By the corollary of Theorem A, it suffices to show that each real continuous linear form w on A may be represented as the difference of two real continuous linear forms that are non-negative on K. Let w be fixed;

$$\delta \to \langle \mu(\delta), w \rangle \quad (\delta \in \Gamma(\mu))$$

is a bounded, real-valued measure on X. By the Hahn–Jordan decomposition theorem, there exists two sets $\delta_i \in \Gamma(\mu)$ $(i=1, 2)$ such that $\delta_1 \cap \delta_2 = \phi$, $\delta_1 \cup \delta_2 = X$ and $\langle \mu(\delta), w \rangle \geqslant 0$ (or $\leqslant 0$) whenever $\delta \in \Gamma(\mu)$ is contained in δ_1 (or δ_2). Let $e_i = \mu(\delta_i)$ $(i=1, 2)$, then (since μ is a spectral measure) $e_1 e_2 = 0$ and $e_1 + e_2 = e$. Since $\hat{A}$ is commutative,

$$\hat{A} = A_1 \oplus A_2$$

where $A_i = \hat{A} e_i$, is the direct sum of the subalgebras A_1 and A_2. This sum is topological, because the projections $a \to a_i = a e_i (i=1,2)$ are continuous. (Moreover, $K = K_1 \oplus K_2$ where $K_i = K \cap A_i$.) Let us define the linear forms u_i on A by

$$a \to \langle a, u_1 \rangle = \langle a_1, w \rangle$$
$$a \to \langle a, u_2 \rangle = -\langle a_2, w \rangle.$$

Then the $u_i (i=1,2)$ are continuous and non-negative on K. Hence w has the decomposition $w = u_1 - u_2$ and the proposition is proved.

COROLLARY. *The range of every spectral measure is bounded.*

Proof. Let K be the positive cone of an ordering of A for which the spectral measure μ is positive. Thus $0 \leqslant \mu(\delta) \leqslant e$ for every $\delta \in \Gamma(\mu)$, hence $\mu(\Gamma) \subset [0, e]$. Since ($K$ being weakly normal) $[0, e]$ is bounded the corollary is proved.

Let X, Y be locally compact Hausdorff spaces. A function f on X into Y is Baire measurable (Section 1) when f^{-1} maps $\Delta_0(Y)$ into $\Delta_0(X)$; however, when Y is a vector space, it is convenient (and standard) to define as the Baire measurable functions those $f \in Y^X$ for which $f^{-1}(\delta) \cap N(f)$ is a Baire set whenever δ is a Baire set in Y, where $N(f) = \{t: f(t) \neq 0\}$. We follow this usage.

PROPOSITION 8. *If f is a Baire measurable function on X into Y and μ is a spectral measure on X, then $\nu = f(\mu)$ is a spectral measure on Y. If f is continuous and μ has compact spectrum, then ν also has compact spectrum and $S(\nu) = f[S(\mu)]$.*

Proof. Let $\Gamma(\mu)$ denote the domain of μ which is a σ-algebra of subsets of X; it is immediate that the family $\{\varepsilon\}$ of subsets of Y such that $f^{-1}(\varepsilon)\in\Gamma(\mu)$, is a σ-algebra of subsets of Y. It is now clear that $\nu=f(\mu)$ (cf. Section 1), when defined on $\Gamma(\nu)=\{\varepsilon\}$, is a spectral measure. The second assertion, which is a spectral mapping theorem, follows from Proposition 6.

Let $\{X_\iota\colon \iota\in I\}$ be an arbitrary (non-empty) collection of locally compact Hausdorff spaces, such that $X=\prod X_\iota$ is locally compact, and let Γ_ι be a σ-algebra of subsets of X_ι $(\iota\in I)$. A set $\delta=\prod\delta_\iota$ in $X=\prod X_\iota$ is elementary (with respect to $\{\Gamma_\iota\}$) if $\delta_\iota\in\Gamma_\iota(\iota\in I)$, and $\delta_\iota=X_\iota$ for all but finitely many $\iota\in I$. A spectral measure μ into A, with domain $\Gamma(\mu)$, on X is the (Cartesian) product of the family of spectral measures $\{\mu_\iota\colon \iota\in I\}$ on X_ι into A, with domains $\Gamma_\iota=\Gamma(\mu_\iota)$, if for each set $\delta\subset X$, elementary with respect to $\{\Gamma_\iota\}$, one has $\delta\in\Gamma(\mu)$ and

$$\mu(\delta)=\prod_{\iota\in I}\mu_\iota(\delta_\iota).$$

When μ is the product of $\{\mu_\iota\}$, we write $\mu=\bigotimes_{\iota\in I}\mu_\iota$.

PROPOSITION 9. *Let $X=\prod_{\iota\in I}X_\iota$ be a locally compact Hausdorff space, countable at infinity. Every spectral measure μ on X into A is the product of a family $\{\mu_\iota\}_{\iota\in I}$ of uniquely determined spectral measures on X_ι into A.*

Proof. Denote by f_ι the projection mapping of X onto $X_\iota(\iota\in I)$. Since f_ι is clearly a Baire measurable function, it follows from Proposition 8 that $\mu_\iota=f_\iota(\mu)$ is a spectral measure, with domain $\Gamma_\iota=\Gamma(\mu_\iota)$, on X_ι into A. Since μ is multiplicative on its domain Γ, it follows that for each set δ which is elementary with respect to $\{\Gamma_\iota\}$, one has $\mu(\delta)=\prod_{\iota\in I}\mu_\iota(\delta_\iota)$. The uniqueness (cf. Definition 1) is clear since for each Baire set $\delta_\iota\subset X_\iota$ (ι fixed), $\delta=\delta_\iota\times\prod_{\varkappa\neq\iota}X_\varkappa$ is an elementary set contained in $\Gamma(\mu)$, and hence $\mu_\iota(\delta_\iota)=\mu(\delta)$.

COROLLARY. *Each complex spectral measure is the product of two uniquely determined real spectral measures.*

When $\{X_\iota\}_{\iota\in I}$ is a collection of locally compact spaces and $\{\mu_\iota\}_{\iota\in I}$ a family of spectral measures with domains $\Gamma_\iota\subset\Delta(X_\iota)(\iota\in I)$ and ranges $\mu_\iota(\Gamma_\iota)\subset A$ where A is a locally convex algebra, we say that $\{\mu_\iota\}$ is commutative (or abelian) when $a_\iota\smile b_\varkappa$ for arbitrary elements $a_\iota\in\mu_\iota(\Gamma_\iota)$, $b_\varkappa\in\mu_\varkappa(\Gamma_\varkappa)\,(\iota,\varkappa\in I)$. Equivalently, $\{\mu_\iota\}$ is an abelian family if $\bigcup_{\iota\in I}\mu_\iota(\Gamma_\iota)$ is contained in a commutative subalgebra of A. By the product of the ranges $\mu_\iota(\Gamma_\iota)$, we shall understand the set of all products $a_{\iota_1}\dots a_{\iota_n}$ where $\{\iota_1,\dots,\iota_n\}$ is an arbitrary, finite (ordered) subset of I.

THEOREM 2. *Let $X=\prod_{\iota\in I} X_\iota$ denote a locally compact space, A a weakly semi-complete locally convex algebra, $\{\mu_\iota\}_{\iota\in I}$ a family of spectral measures with respective domains $\Gamma_\iota\subset\Delta(X_\iota)$ $(\iota\in I)$ and values in A. In order that there exist a (necessarily unique) spectral measure μ on X into A such that $\mu=\bigotimes_{\iota\in I}\mu_\iota$, it is necessary and sufficient that $\{\mu_\iota\}$ be abelian and that the product of the ranges $\mu_\iota(\Gamma_\iota)$ be contained in a weakly normal cone in A.*

Proof. The condition is necessary. If μ is a spectral measure on X into A there exists, by Proposition 7, a closed weakly normal cone $K\subset A$ such that $\mu(\Gamma)\subset K$ where Γ is the domain of μ. Hence, if $\mu=\bigotimes_{\iota\in I}\mu_\iota$, it follows that $\mu_\iota(\Gamma_\iota)\subset K(\Gamma_\iota$ the domain of μ_ι, $\iota\in I)$. Further, since μ is a spectral measure, from the definition of $\bigotimes_{\iota\in I}\mu_\iota$ it is clear that $\{\mu_\iota\}$ is an abelian family, and that the product of all ranges is in K.

Sufficiency. Let $\{\mu_\iota\}$ be an abelian family of spectral measures, with respective domains $\Gamma_\iota\subset\Delta(X_\iota)$, and with the product of their ranges contained in a weakly normal cone K which we assume as closed. Suppose first that X is compact. Let V denote the smallest vector subspace of $\mathcal{C}(X)$ containing all g such that

$$g(t)=g_{\iota_1}(t_{\iota_1})\dots g_{\iota_n}(t_{\iota_n})\quad(t\in X),\tag{*}$$

where $t=(t_\iota)$ and $\{\iota_1,\dots,\iota_n\}$ is an arbitrary, non-empty finite subset of I, and where $g_{\iota_k}\in\mathcal{C}(X_{\iota_k})(k=1,\dots,n)$; it is well known (and a consequence of the Stone–Weierstrass theorem) that V is uniformly dense in $\mathcal{C}(X)$. Denote by V_R the subspace of V containing all real-valued functions $g\in V$. Let ψ denote the unique linear mapping on V into A satisfying.

$$\psi(g)=\mu_{\iota_1}(g_{\iota_1})\dots\mu_{\iota_n}(g_{\iota_n})$$

for all g of type (*), $\mathcal{K}_0$ the conical extension of the set of functions g of type (*) with $g_{\iota_k}\geqslant 0\,(k=1,\dots,n)$, and S the subset $\{h\colon 0\leqslant h(t)\leqslant 1\}$ of V. Then $\mathcal{K}_0\cap S$ is dense in S for that normed topology on V_R whose unit ball is $\mathcal{K}_0\cap S-\mathcal{K}_0\cap S$ (cf. [5, Chapter III, § 5, Lemma 1]), ψ is bounded on $\mathcal{K}_0\cap S$ and hence $\psi(S)\subset K$; since for the uniform topology S has interior points in V_R, it follows that ψ is continuous on V_R, and hence on V, into A. Therefore, ψ has a continuous extension, φ, to $\mathcal{C}(X)$ which is positive for the ordering of A_0 whose positive cone is K. By Theorem 1, φ generates a unique vector measure μ on X into A_0 whose domain can be assumed as the σ-algebra generated by all subsets of X elementary with respect to $\{\Gamma_\iota\}$; if $\delta=\delta_\iota\times\prod_{k\neq\iota}X_k$, one has $\mu(\delta)=\mu_\iota(\delta)_\iota(\iota\in I)$. Thus since clearly $\mu(X)=e$, the proof for compact X will be complete when we show that μ is multiplicative on Γ. From the extension process (given in Section 1) and the separate continuity of multiplication in A for $\sigma(A,A')$ it follows first that $\mu(\delta)=\prod_{\iota\in I}\mu_\iota(\delta_\iota)$ for every elementary set δ (with respect to $\{\Gamma_\iota\}$). Thus if δ, ε are two elementary sets, $\mu(\delta\cap\varepsilon)=\mu(\delta)\mu(\varepsilon)$ since each

μ_ι is a spectral measure. Let Σ be a maximal family of members of Γ, containing all elementary sets, such that $\delta \in \Sigma$, $\varepsilon \in \Sigma$ imply $\mu(\delta \cap \varepsilon) = \mu(\delta)\mu(\varepsilon)$. It follows from Proposition 3 that Σ is closed under countable unions and intersections, hence that $\Sigma = \Gamma$. This completes the proof when X is compact.

For the general case, recall that if $X = \prod_{\iota \in I} X_\iota$ is locally compact, at most a finite number of the X are not compact. Thus $X = Y \times Z$ where Z is compact and $Y = X_1 \times \ldots \times X_n$, X_k locally compact ($k = 1, \ldots, n$). Since the density considerations above apply to any finite product of locally compact spaces, and since the uniqueness of μ is clear, the theorem is proved.

COROLLARY. *Let E be a Hilbert space, $\{\mu_\iota\}$ a family of spectral measures with compact spectra $X_\iota (\iota \in I)$, and values in the family of orthogonal projections on E. Then $\otimes_{\iota \in I} \mu_\iota$ exists if and only if $\{\mu_\iota\}$ is commutative.*

The corollary is immediate from Theorem 2 since the cone $\mathcal{K}$ of positive Hermitian endomorphisms of E is normal and weakly semi-complete for the topology of simple convergence, and since the product of any finite number of commuting elements of $\mathcal{K}$ is in $\mathcal{K}$.

If μ is a spectral measure on X with domain Γ and values in a (semi-complete) locally convex algebra A, it follows from Proposition 7, Corollary that $\int f d\mu \in A$ for every Baire measurable function which is bounded on the spectrum $S(\mu)$ of μ (for, by definition, Γ contains the σ-algebra generated by all Baire sets in X). By Proposition 7, there exists an ordering of A for which μ is positive; we assume A to be endowed with an order structure satisfying this condition, and we denote by $\mathcal{B}(\mu)$ the smallest subalgebra of $\mathbf{C}^{S(\mu)}$ containing all constant and all bounded Baire measurable functions.

PROPOSITION 10. *The mapping $f \to \int f d\mu$ is an order preserving homomorphism of the algebra $\mathcal{B}(\mu)$ into A, continuous for the uniform topology on $\mathcal{B}(\mu)$.*

Proof. Denote by $\mathcal{S}$ the family of simple Baire measurable (complex) functions on $S(\mu)$, augmented by the constant functions. $\mathcal{S}$ is dense in $\mathcal{B}(\mu)$ for the uniform topology, and from the multiplicativity of μ it follows that

$$\int fg d\mu = \int f d\mu \int g d\mu$$

for all $f, g \in \mathcal{S}$. Since μ is K-valued (where K is the positive cone in A) then $f \in \mathcal{S}$, $0 \leq f \leq 1$ implies

$$0 \leq \int f d\mu \leq \int d\mu = e,$$

hence $\mu(\{f: 0 \leq f \leq 1\})$ is contained in $[0, e]$ and therefore bounded in A. Thus $f \to \int f d\mu$

is continuous on $\mathcal{S}$ into A; its continuous extension to $\mathcal{B}(\mu)$ obviously coincides with the $\mu(f)$ defined in Section 1. This implies that $(f, g) \to \int fg\,d\mu$ is jointly continuous for the uniform topology on $\mathcal{B}(\mu)$. It is clear that $f \to \int f\,d\mu$ is order preserving (cf. the remarks subsequent to Theorem 1).

Remark. The preceding proof shows that in order to obtain the conclusion of Proposition 10, it is sufficient that μ is a spectral measure with values in a locally convex algebra A which is semi-complete for its given topology.

Let X denote a locally compact space, A a Banach algebra, μ a vector measure on X into the underlying B-space of A. We say a mapping f on X into A is a step function if $f = \sum_{i=1}^{n} a_i \chi_i$ where $a_i \in A$ and χ_i is the characteristic function of a Baire set $\delta_i \subset X$ $(i = 1, \ldots, n)$. If the mapping

$$f \to \sum_{i=1}^{n} a_i \, \mu(\delta_i)$$

is continuous for the uniform topology, it may be extended to a continuous map $f \to \mu(f) = \int f\,d\mu$ on $\mathcal{C}_A(X)$ into A.[1]

The preceding remarks apply to the present situation as follows. Let X be locally compact, μ a spectral measure on X into A. Under the uniform topology, $\mathcal{B}(\mu)$ is a Banach algebra. Clearly the subset of null functions $\mathcal{N} = \{f: \mu(|f|) = 0\}$ is a closed, two-sided ideal in $\mathcal{B}(\mu)$; hence $\mathcal{B}(\mu)/\mathcal{N}$ is a Banach algebra. Since $\mathcal{N}$ is the kernel of the homomorphism $\varphi: f \to \int f\,d\mu$ (by Prop. 10, $\int f\,d\mu = 0$ implies $\int |f|\,d\mu = 0$), $\mathcal{B}(\mu)/\mathcal{N}$ is algebraically isomorphic to $\hat{A} = \varphi(\mathcal{B}(\mu))$. Let us consider $\hat{A}$ under the norm carried over from $\mathcal{B}(\mu)/\mathcal{N}$; by the remarks above, $\int f\,d\mu$ is well defined, and a member of $\hat{A}$, for every $f \in \mathcal{C}_{\hat{A}}(X)$. Thus, in particular, when $X = X_1 \times X_2$ and $\mu = \mu_1 \otimes \mu_2$ (Proposition 9), the iterated integral

$$\int d\mu_1(t_1) \int f(t_1, t_2)\, d\mu_2(t_2)$$

is well defined. We shall need a Fubini theorem of the following type.

PROPOSITION 11. *Let $X = X_1 \times \ldots \times X_n$ be locally compact, μ a spectral measure on X into A with $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ (Prop. 9), and let π denote any permutation of $\{1, 2, \ldots, n\}$. For every complex-valued continuous function $t \to f(t) = f(t_1, \ldots, t_n)$ of compact support in X, one has*

$$\int f d\mu = \int d\mu_{\pi(1)} \ldots \int f d\mu_{\pi(n)}.$$

Proof. It will be sufficient to give the proof for $n = 2$; let $X = X_1 \times X_2$ and $\mu = \mu_1 \otimes \mu_2$. It is clear that the assertion holds for every function f in the tensor product $\mathcal{C}(X_1) \otimes \mathcal{C}(X_2)$.

---

[1] $\mathfrak{C}_A(X)$ denotes the space of continuous functions with compact support in X and values in A.

On the other hand, this space is uniformly dense in $\mathcal{C}(X)$. Let $f_n \in \mathcal{C}(X_1) \otimes \mathcal{C}(X_2)\,(n \in \mathbf{N})$ and $f_n \to f$ uniformly. Now, for the norm topology on $\hat{A}$ introduced in the preceding paragraph, $\int f_n(t_1, t_2)\,d\mu_2(t_2) \to \int f(t_1, t_2)\,d\mu_2(t)$ uniformly on X_1; hence

$$\int d\mu_1(t_1) \int f_n(t_1, t_2)\,d\mu_2(t_2) \to \int d\mu_1 \int f\,d\mu_2 = \int f\,d\mu$$

since $g \to \mu_1(g)$ is continuous on $\mathcal{C}_{\hat{A}}(X_1)$ into $\hat{A}$. This proves the assertion.

In the remainder of this paper, we shall use the term "*Baire function on X*" for every complex function which is in the algebra generated by functions either Baire measurable or constant on X; this usage disagrees with the usual meaning only when X is a locally compact space not countable at infinity.

3. Spectral Elements

We assume throughout this section that A is a semi-complete locally convex algebra in the sense explained at the beginning of Section 2. When μ is a spectral measure on a locally compact space X into A, let $B(\mu)\,(B_{\mathbf{R}}(\mu))$ denote the algebra of complex-valued (real-valued) Baire functions on X that are bounded on the spectrum of μ. We recall (Proposition 10) that $f \to \mu(f) = \int f\,d\mu$ is a homomorphism on $B(\mu)$ into A which induces a homomorphism of $B_{\mathbf{R}}(\mu)$ onto a real subalgebra of A.

Definition 3. *An element $a \in A$ is spectral (a real spectral element) if there exists a locally compact space X, a spectral measure μ on X into A, and $f \in B(\mu)\,(f \in B_{\mathbf{R}}(\mu))$ such that $a = \int f\,d\mu$.*

It is clear from this definition that every spectral element $a \in A$ can be represented as $a = a_1 + ia_2$ where a_1, a_2 are real spectral elements which commute; the latter is a consequence of Proposition 10. If $a = \int f\,d\mu$, let $\nu = f(\mu)$ (Proposition 6). By Proposition 8, ν is a spectral measure on the complex plane with compact spectrum (since f is bounded). Using the continuous linear forms on A, we conclude from a standard formula of measure theory that

$$\int f\,d\mu = \int z\,d\nu(z).$$

We call ν the complex spectral measure associated with the representation $a = \mu(f)$ of a.

Our next objective is to show that this associated complex spectral measure is unique (i.e., independent of the given representation of a as a spectral integral); the proof which is essentially an adaptation to the present situation of a method due to Dunford [7], depends on several auxiliary results one of which is the assertion that if $a \in A$ commutes with a spectral element $c \in A$, then a commutes with every complex spectral measure associated with a representation of c. Let c, from now on through Theorem 3, denote a fixed spectral element; let ν be the complex spectral measure associated with an arbitrary (but fixed) representation of c. Hence $c=\int z d\nu(z)$; let us write

$$R(\lambda)=\int\frac{d\nu(\xi)}{\lambda-\xi} \quad (\lambda \notin S(\nu))$$

for λ not in the support of ν. Clearly $\lambda \to R(\lambda)$ is a locally holomorphic function on $\mathbf{C} \sim S(\nu)$ into A. We shall see later (Theorem 4) that $\mathbf{C} \sim S(\nu)$ is the exact domain of (local) analyticity of this function. For each $a \in A$, $\lambda \to R(\lambda)a$ is again locally holomorphic in the complement of $S(\nu)$, but if $a \neq e$, it may well be the case that $\lambda \to R(\lambda)a$ can be extended to a larger domain (e.g., to all of $\mathbf{C}$ if $a=0$). We shall say that f_a is an extension of $\lambda \to R(\lambda)a$ if $f_a(\lambda)=R(\lambda)a$ for all $\lambda \in \mathbf{C} \sim S(\nu)$ and if for all λ in its (open) domain D_{f_a}, f_a is locally holomorphic and satisfies the relation $(\lambda e - c) f_a(\lambda)=a$. Further, $\lambda \to R(\lambda)a$ is said to have the single-valued extension property when for every two extensions f_a and g_a, we have $f_a(\lambda)=g_a(\lambda)$ for all $\lambda \in D_f \cap D_g$. It is clear that when $\lambda \to R(\lambda)a$ has the single-valued extension property, there exists a unique maximal domain $D(a)$ to which Ra can be extended under preservation of the required properties.

LEMMA 1. *For every $a \in A$, $\lambda \to R(\lambda)a$ has the single-valued extension property.*

Proof. Let $b \in A$ satisfy the relation $(\lambda_0 e - c)b=0$. We shall show that $\nu(\delta)b=0$ for every Baire set in $\mathbf{C} \sim \{\lambda_0\}$, and that $\nu\{\lambda_0\}b=b$. Denote by ε a closed Baire set in $\mathbf{C}$ not containing λ_0. Then

$$j=\int_{\varepsilon}\frac{d\nu(\xi)}{\lambda_0-\xi}$$

exists and $j(\lambda_0 e - c)=\int_{\varepsilon} d\nu(\zeta)=\nu(\varepsilon)$ by Proposition 10 which implies that $\nu(\varepsilon)b=0$. It follows now from the countable additivity of ν that $\nu(\delta')b=0$ where $\delta'=\mathbf{C} \sim \{\lambda_0\}$. Hence, since $\nu(\mathbf{C})=e$, $\nu\{\lambda_0\}b=b$.

Assume now that f_a and g_a are extensions of $\lambda \to R(\lambda)a$, and consider an arbitrary point $\lambda_0 \in D_f \cap D_g$. Let $\{\lambda_n\}$ denote a sequence in $D_f \cap D_g$, such that $\lambda_n \to \lambda_0$ and $\lambda_n \neq \lambda_0$ for all $n \in \mathbf{N}$. Since, by definition of an extension, $(\lambda_n e - c)h(\lambda_n)=0$ where $h(\lambda)=f_a(\lambda)-g_a(\lambda)$ $(\lambda \in D_f \cap D_g)$, it follows from our previous observations that $\nu\{\lambda_0\}h(\lambda_n)=0$. This implies (by the continuity of h) that $\nu\{\lambda_0\}h(\lambda_0)=0$. On the other hand, we must have (since $(\lambda_0 e - c)h(\lambda_0)=0$) $\nu\{\lambda_0\}h(\lambda_0)=h(\lambda_0)$; hence $h(\lambda_0)=0$ as was to be shown.

If $\varrho(a)$ temporarily denotes the unique maximal domain to which $\lambda \to R(\lambda)a$ can be extended in the sense specified above, let us denote by $\sigma(a)$ the complement of $\varrho(a)$. It can easily be shown, using Liouville's theorem, that $\sigma(a)=\phi$ if and only if $a=0$.

LEMMA 2. *For every closed* $\delta \subset \mathbf{C}$, $\nu(\delta)A = \{a \in A: \sigma(a) \subset \delta\}$.

Proof. Let $a \in \nu(\delta)A$, i.e., let $a = \nu(\delta)b$ for some $b \in A$. Clearly $a = \nu(\delta)a$ since $\nu(\delta)$ is an idempotent. Set $R_\delta(\lambda)a = \nu(\delta)R(\lambda)a$; since ν commutes with $R(\lambda)$, we have $R(\lambda)\nu(\delta)a = R_\delta(\lambda)a$ for all $\lambda \notin S(\nu)$. Clearly $R_\delta(\cdot)a$ extends to $\varrho(a)$, and since for $\lambda \notin S(\nu)$ (Proposition 10)

$$R_\delta(\lambda) = \int_\delta \frac{d\nu(\xi)}{\lambda - \xi},$$

it follows that $\varrho(a)$ contains the complement of δ whence $\sigma(a) \subset \delta$. Assume, conversely, that $\sigma(a) \subset \delta$. Let ε be closed and $\varepsilon \cap \delta = \phi$. Denoting by ε' the complement of ε in $\mathbf{C}$, we obtain

$$R(\lambda)a = R_{\varepsilon'}(\lambda)a + R_\varepsilon(\lambda)a.$$

$\lambda \to R_\varepsilon(\lambda)a$ is holomorphic in a neighborhood of δ. On the other hand, $\lambda \to R(\lambda)a$ and *a fortiori* $\lambda \to R_{\varepsilon'}(\lambda)a$ have unique locally holomorphic extensions (Lemma 1) to a neighborhood of ε since $\sigma(a) \subset \delta$ by assumption. Because $\delta' = \mathbf{C} \sim \delta$ is the union of a countable number of closed sets ε_n, it follows that $R_{\delta'}(\cdot)a = R(\cdot)\mu(\delta')a$ has a unique extension to all of $\mathbf{C}$ which approaches 0 as $\lambda \to \infty$; thus, by the remark preceding this Lemma, $\mu(\delta')a = 0$ which implies $\mu(\delta)a = a$ and hence $a \in \mu(\delta)A$ which completes the proof.

PROPOSITION 12. *If c is a spectral element in A and $a \smile c$, then a commutes with the complex spectral measure associated with any representation of c.*

Proof. From $b \smile c$ it follows that $b \smile R(\lambda)$, $\lambda \notin S(\nu)$, where $c = \int z d\nu(z)$. Hence $R(\lambda)b\nu(\delta) = bR(\lambda)\nu(\delta) = bR_\delta(\lambda)$ and we conclude that $\sigma(b\nu(\delta)) \subset \delta$ whenever δ is closed. By Lemma 2, this implies that $\nu(\delta)b\nu(\delta) = b\nu(\delta)$. If δ_1 is a closed set with $\delta \cap \delta_1 = \phi$, then $\nu(\delta)b\nu(\delta_1) = \nu(\delta)\nu(\delta_1)b\nu(\delta_1) = 0$ since $\nu(\delta)\nu(\delta_1) = 0$. Since ν is countably additive, it follows that $\nu(\delta)b\nu(\delta') = 0$ where δ is closed and $\delta' = \mathbf{C} \sim \delta$. Now one has

$$\nu(\delta)b = \nu(\delta)b[\nu(\delta) + \nu(\delta')] = \nu(\delta)b\nu(\delta)$$

for every closed δ, hence $\nu(\delta)b = b\nu(\delta)$ for closed and, by the countable additivity of ν, for all Baire sets in $\mathbf{C}$.

THEOREM 3. *For each spectral element $c \in A$, there is one and only one complex spectral measure ν such that $c = \int z d\nu(z)$.*

Proof. If $c = \int f d\mu$ is a spectral element (Definition 3) where μ is a spectral measure on a locally compact space X into A, then $c = \int z d\nu(z)$ with $\nu = f(\mu)$ as we have noted earlier Let $c = \int z d\nu_1(z) = \int z d\nu_2(z)$ where ν_1, ν_2 are complex spectral measures. Proposition 12 then implies that ν_1 and ν_2 commute, since clearly c commutes with ν_1 and ν_2. Let δ be a closed set in C. Set $b=\nu_1(\delta)$. Applying Lemma 2 to b and $\nu=\nu_2$, we obtain $b=\nu_2(\delta)b$, hence $\nu_1(\delta)=\nu_2(\delta)\nu_1(\delta)$. Interchanging ν_1 and ν_2, we obtain $\nu_2(\delta)=\nu_1(\delta)\nu_2(\delta)$. Hence, since ν_1 and ν_2 commute, $\nu_1(\delta)=\nu_2(\delta)$ for every closed δ. Since ν_1 and ν_2 are countably additive by hypothesis, $\nu_1=\nu_2$.

Remark. It should be noted that for real spectral elements c, the unicity of the representation $c=\int t\,d\mu(t)$ where μ is a real spectral measure, can be obtained in a much simpler way. For Proposition 10 implies that $c^n=\int t^n d\mu(t)$, thus $\mu(P)$ is uniquely determined for all polynomials P of one real variable from which the unicity of μ follows immediately.

If $a=\int f\,d\mu$ (Definition 3) is a spectral element in a (semi-complete) locally convex algebra, then $\nu=f(\mu)$ which by Theorem 3 is independent of the particular representation of a, will be called the complex spectral measure associated with a. By Proposition 9, $\nu=\nu_1\otimes\nu_2$ where ν_1, ν_2 are uniquely determined real spectral measures. It is easy to see that a is a real spectral element if and only if ν_2 has spectrum $\{0\}$, and in this case we shall call ν_1 the real spectral measure associated with a.

It is now time to connect the notion of "spectrum" for a spectral measure with the algebraic meaning of the concept; in an algebra (with unit e) over the complex field C, the spectrum of an element $a\in A$ is usually understood to be the set of all $\lambda\in\mathbf{C}$ such that $\lambda e-a$ has no inverse. In contrast with the case of a Banach algebra, in a locally convex algebra A this purely algebraic definition is of little use, even when A is assumed as commutative, metrizable and complete with (jointly) continuous multiplication.

For example, let A be the product algebra of countably many copies of C, $A=\prod_1^\infty C_n$ ($C_n=\mathbf{C}$, $n\in\mathbf{N}$). If $a=(\lambda_1,\lambda_2,\ldots)$ where $\lambda_n>0$ and $\lim_n\lambda_n=0$, it is easily found that the set where $(\lambda e-a)^{-1}$ does not exist is $\{\lambda_n: n\in\mathbf{N}\}$, and hence is not closed. The "resolvent" $\lambda\to(\lambda e-a)^{-1}$ exists for all $\lambda\neq\lambda_n\,(n\in\mathbf{N})$, but it is holomorphic only when $\lambda\neq 0$. It can be shown by examples that if A is a locally convex algebra in the sense used here, the spectrum (as defined above) of an $a\in A$ may be empty, or consist of the entire plane, or consist of $\mathbf{C}\sim\{0\}$.

In view of the pathologies that were pointed out, we shall adopt this definition. If A is a locally convex algebra, the spectrum $\sigma(a)$ of a is the complement of the largest open subset of C in which $\lambda\to(\lambda e-a)^{-1}$ is locally holomorphic (i.e., in whose connected components $\lambda\to(\lambda e-a)^{-1}$ is holomorphic). We write $\varrho(a)=\mathbf{C}\sim\sigma(a)$ and call $\varrho(a)$ the resolvent set of a. The following result justifies the use of the term spectrum for the support of a spectral measure.

THEOREM 4. *For every spectral element $a\in A$, the spectrum $\sigma(a)$ is equal to the spectrum of its associated complex (respectively real) spectral measure.*

Proof. The assertion claims that if $a=\int z d\nu(z)$, then $\sigma(a)=S(\nu)$ where $S(\nu)$ denotes the spectrum of ν. We show first that $\sigma(a)\subset S(\nu)$. Let $\lambda_0\notin S(\nu)$; then, since A is semi-complete, the integral

$$\int\frac{d\nu(\xi)}{\lambda_0-\xi}$$

exists; by the homomorphism theorem (Proposition 10), it is clearly the resolvent $(\lambda_0 e-a)^{-1}$ of a at λ_0. Since $\lambda\to(\lambda e-a)^{-1}$ is holomorphic in a neighborhood of λ_0 (for $S(\nu)$ is closed), the first part of the proof is established. To prove the converse, $S(\nu)\subset\sigma(a)$, we proceed as follows. Let $\lambda_0\notin\sigma(a)$; there are closed circular disks K_1 and K_2, center at λ_0, with radii r_1, r_2 such that $0<r_1<r_2$ and $K_2\cap\sigma(a)=\phi$. Denote by ν_1 the restriction of ν to K_1 (more precisely, to the family of all Baire sets contained in K_1). It is clear that ν_1 is a vector measure with values in A, and so is ν_2 if ν_2 is the restriction of ν to $\mathbf{C}\sim K_1$; we have $\nu=\nu_1+\nu_2$ in an obvious sense. Further, if we denote by A_1 the smallest closed subalgebra of A that contains the range of ν_1, it is clear that $\nu(K_1)=e_1$ is the unit of A_1 and ν_1 is a spectral measure on K_1, with values in A_1. Let us set $a_1=\int z d\nu_1(z)$, then we have

$$R_1(\lambda)=(\lambda e_1-a_1)^{-1}\left(=\int\frac{d\nu_1(z)}{\lambda-z}\quad\text{if}\quad\lambda\notin S(\nu_1)\right)$$

for all $\lambda\notin\sigma(a_1)$. Further, setting $R(\lambda)=(\lambda e-a)^{-1}(\lambda\notin\sigma(a))$, and $R(\lambda)-R_1(\lambda)=R_2(\lambda)\,(=\int d\nu_2(z)/(\lambda-z)$ if $\lambda\notin S(\nu))$, we have $R(\lambda)=R_1(\lambda)+R_2(\lambda)$ for all λ not in either $\sigma(a)$, or $\sigma(a_1)$. If we denote by C_2 the boundary of K_2 (in $\mathbf{C}$), then R_1 is holomorphic in a neighborhood of C_2. On the other hand, R_2 is holomorphic in the interior of K_2 (since the support of ν_2 is contained in $\overline{\mathbf{C}\sim K_2}$), and R is holomorphic in K_2 by assumption. Thus R_1 has an analytic extension into the interior of K_2 which implies, by Cauchy's theorem, that

$$\int_{C_2}R_1(\lambda)\,d\lambda=0.$$

Now if $\tilde\lambda$ denotes Lebesgue measure on C_2, it is not difficult to verify that the product measure $\nu_1\otimes\tilde\lambda$ on $K_1\times C_2$ exists, and is a vector measure with values in A. One proves in a manner quite analogous to the proof of Proposition 11 that a Fubini theorem holds for $\nu_1\otimes\tilde\lambda$ (which will be needed only for the continuous function $(z,\lambda)\to(\lambda-z)^{-1}$ on $K_1\times C_2$ into $\mathbf{C}$). From the remark above, we conclude now that

$$\int_{C_2}R_1(\lambda)\,d\lambda=\int_{C_2}d\lambda\int_{K_1}\frac{d\nu_1(z)}{\lambda-z}=\int_{K_1}d\nu_1(z)\int_{C_2}\frac{d\lambda}{\lambda-z}=2\pi i\nu_1(K_1)=0,$$

hence that $\nu_1=0$. This implies that $\nu=\nu_2$ and, consequently, that the support $S(\nu)$ is contained in $\mathbf{C}\sim K_1$. Thus $\lambda_0\notin\sigma(a)$ implies that $\lambda_0\notin S(\nu)$ and the theorem is proved.

We note from the preceding proof that the restriction of a complex spectral measure ν to a closed set $\delta\subset\mathbf{C}$, is a spectral measure with respect to a suitably chosen subalgebra of A. (If $\nu(\delta)=0$, the process degenerates since the subalgebra $\{0\}$ has no proper unit.) Such a subalgebra may be constructed by taking the closure (in A) of the set of all elements $\int_\delta f d\nu$ where $f\in B(\nu)$.

Corollary. *If $\delta\subset\mathbf{C}$ is closed and ν a complex spectral measure, then the spectrum $\sigma(\int_\delta z d\nu(z))$ is equal to the support of the restriction of ν to δ, plus $\{0\}$ if $S(\nu)\not\subset\delta$.*

We shall now turn to a characterization of real spectral elements in an arbitrary, weakly semi-complete locally convex algebra A. We shall say that a is a positive spectral element if a is spectral and its spectrum is a subset of the non-negative reals. It is clear from Theorem 4 that every positive spectral element is real.

Proposition 13. *Let A be ordered such that the order interval $J=[0, e]$ is weakly semi-complete. Then every $a\in J$ is a positive spectral element of A such that $\sigma(a)\subset[0, 1]$, and whose associated real spectral measure is positive.*

Proof. Let $a\in J$, i.e., $0\leqslant a\leqslant e$. Denote by $P_{p,m}$ the polynomials

$$t\to P_{p,m}(t)=t^m(1-t)^{p-m}\quad(0\leqslant m\leqslant p,\ 0\leqslant t\leqslant 1).$$

To within a factor $\binom{p}{m}$, $P_{p,m}$ are the Bernstein polynomials on the real unit interval I. Set $\varphi(P_{p,m})=a^m(e-a)^{p-m}$ for arbitrary pairs (p,m) of integers with $p\geqslant m\geqslant 0$. If K denotes the positive cone in A, it follows from the definition of an ordered locally convex algebra (Sec. 2) that $\varphi(P_{p,m})\in K$ for all $p\geqslant m\geqslant 0$. Since every complex polynomial on $[0, 1]$ is a linear combination of Bernstein polynomials, we may consider φ as defined on the vector space V of polynomials on $[0, 1]$ into A. This mapping is continuous for the uniform topology on V into A. To see this, we use the well known fact that every non-negative (real) function on $[0, 1]$ can be uniformly approximated by linear combinations of Bernstein polynomials with non-negative coefficients. This implies that $\varphi(P)\in K$ for every non-negative polynomial P, hence $0\leqslant\varphi(P)\leqslant e$ for every $0\leqslant P\leqslant 1$; since this set has interior points in $V_{\mathbf{R}}$ and $[0, e]$ is bounded in A, φ is continuous on $V\cap C_{\mathbf{R}}(I)$ into A and hence on V into A. It follows that φ has a continuous extension to $C(I)$ (again denoted by φ) which is positive on $C(I)$ into A. Let N denote the subspace of $C(\mathbf{R})$ of all functions vanishing on $[0, 1]$; it is clear that $C(I)$ is algebraically isomorphic with $C(\mathbf{R})/N$ and that the natural mapping χ of $C(\mathbf{R})$ onto $C(\mathbf{R})/N$ is positive. Thus, identifying $C(I)$ with $C(\mathbf{R})/N$, the map

$$\psi=\varphi\circ\chi$$

defines a positive linear mapping on $C(\mathbf{R})$ into A. From Theorem 1 it follows now that there exists a unique positive vector measure μ extending ψ, with values in A when A is weakly semi-complete. Since, then, for every real Baire set δ one has $0 \leqslant \mu(\delta) \leqslant e$, we conclude that μ takes its values in $[0, e]$ if this interval is weakly semi-complete as we have assumed. It is also clear that $a = \int t d\mu(t)$; the proof will be complete if we can show that μ is a spectral measure. But it is clear, from the definition of φ on V, that $\varphi(P_1 P_2) = \varphi(P_1)\varphi(P_2)$ for any two polynomials on $[0, 1]$ and hence, by continuity, that φ is a homomorphism of the algebra $C(I)$ into A. Since N is an ideal in $C(\mathbf{R})$, it follows that χ is a homomorphism, and consequently so is ψ; thus, for any $f_1, f_2 \in C(\mathbf{R})$ we have $\psi(f_1 f_2) = \psi(f_1)\psi(f_2)$. If f is a bounded Baire function on $\mathbf{R}$, it is clear that $\mu(f) \in A$. Now every algebra $\mathcal{A}$ of bounded Baire functions on $\mathbf{R}$ which contains $C(\mathbf{R})$ and is maximal with respect to the property "If $f_1 \in \mathcal{A}$, $f_2 \in \mathcal{A}$ then $\mu(f_1 f_2) = \mu(f_1)\mu(f_2)$", is closed under the formation of simple limits of bounded sequences, as may be concluded from Proposition 3; hence, $\mathcal{A}$ contains all bounded Baire functions. Thus, μ is multiplicative and since $\int d\mu = \varphi(1) = e$ by definition of φ, a positive spectral measure with spectrum in I; the proof is complete.

COROLLARY. *Let A be weakly semi-complete; in order that $a \in A$ be a positive spectral element, it is necessary and sufficient that, for a suitable $\gamma > 0$, the convex conical extension of the set*

$$\{a^m(\gamma e - a)^n : m, n \geqslant 0\}$$

be a weakly normal cone in A.

Proof. The condition is necessary. For if a is positive, then $a = \int t d\nu(t)$ where ν is a real spectral measure with non-negative compact support. Choose $\gamma > 0$ so that $S(\nu) \subset [0, \gamma]$. By Proposition 7, there exists an ordering of A such that ν is positive; it is then clear that for this ordering,

$$0 \leqslant a = \int_{S(\nu)} t d\nu(t) \leqslant \gamma \int d\nu = \gamma e;$$

hence, by the definition of an ordered locally convex algebra, we obtain $(a^m(\gamma e - a)^n) \in K$ for all integers $m, n \geqslant 0$ which implies the assertion. Conversely, let the condition be satisfied; if K is the closed conical hull of the set in question, then K is the positive cone for an ordering of A and $0 \leqslant a \leqslant \gamma e$; thus, if $a_1 = \gamma^{-1} a$, $0 \leqslant a_1 \leqslant e$ and a_1 is a positive spectral element by Proposition 13. Clearly then so is a, whence it follows that the condition is sufficient.

PROPOSITION 14. *Let A be weakly semi-complete. $a \in A$ is a real spectral element if and only if there exists an ordering of A with respect to which a is in the real linear hull of* $[0, e]$.

Proof. We note first that, given an ordering of A, $a \in A$ is in the linear hull of $[0, e] = \{b: 0 \leqslant b \leqslant e\}$ if and only if there exist real constants c_1, c_2 such that $c_1 < c_2$ and $c_1 e \leqslant a \leqslant c_2 e$.

(The simple proof is left to the reader.) Assume now that the condition is satisfied. Then we have $c_1 e \leqslant a \leqslant c_2 e$ with $c_1 < c_2$; if $a_1 = a - c_1 e$, then $0 \leqslant a_1 \leqslant (c_2 - c_1)e$. Proposition 13 now implies that there exists a unique real spectral measure ν (Theorem 3) such that $a_1 = \int t d\nu(t)$; since $\int d\nu = e$, we obtain

$$a = \int (t - c_1) d\nu(t)$$

which shows that a is a real spectral element. Conversely, if a is a real spectral element, let ν denote the associated real spectral measure with spectrum $S(\nu)$. Since $S(\nu)$ is compact, there exist real numbers c_1, c_2 with $c_1 < c_2$ and $S(\nu) \subset [c_1, c_2]$. By Proposition 7, there exists an ordering of A with respect to which ν is positive. Since $f \to \int f d\nu$ is order preserving (recall that the positive cone K in A is closed), it is immediate that for this order

$$c_1 e = c_1 \int d\nu \leqslant \int t d\nu(t) \leqslant c_2 \int d\nu = c_2 e.$$

The theorem is proved.

Examples

1. Let $I \neq \phi$ be an arbitrary index set, and let A be the algebraical and topological product $\prod_{\iota \in I} C_\iota (C_\iota = \mathbf{C}, \iota \in I)$. With $K = \{a : a_\iota \geqslant 0, \iota \in I\}$ as its positive cone, A is a weakly complete locally convex algebra. Each element a for which $\{a_\iota\}$ is bounded, is spectral; $\sigma(a) = \overline{\{a_\iota\}}$. In the sense of Definition 5, every $a \in A$ is spectral; this is in accordance with the fact that A is the algebra of all complex-valued functions on I.

2. Let A be a real Banach operator algebra with unit e; let us assume that A is weakly semicomplete for the topology of simple convergence, and that the operator norm is monotone on K where K denotes the simple closure of the set S whose elements are finite sums of squares; we show that with K as its positive cone, A is an ordered (real) Banach algebra. It is only necessary to show (since the assumptions above imply that K is normal) that if $a, b \in K$ and $a \smile b$, then $ab \in K$. Since for all a with $\|e - a\| \leqslant 1$, $a^{\frac{1}{2}}$ exists (as the principal value of the corresponding binomial series), e is interior to K. Thus to show that $ab \in K$ when $a \smile b$ and $a, b \in K$, we can assume that $0 \leqslant a \leqslant e$, $0 \leqslant b \leqslant e$. Now $0 \leqslant e - a \leqslant e$, hence $\|e - a\| \leqslant \|e\| = 1$ so that $a^{\frac{1}{2}}$ exists, and likewise $b^{\frac{1}{2}}$ exists. Clearly $a^{\frac{1}{2}} \smile b^{\frac{1}{2}}$; then $ab = (a^{\frac{1}{2}} b^{\frac{1}{2}})^2 \in K$.(1) Hence every element of an algebra satisfying the assumption made above, is a real spectral element; for since e is an interior point of K, A is identical with the real linear hull of $[0, e]$.

Further examples will be considered in Section 4.

(1) The argument shows that in fact every $a \in K$ is a square. An example of the present case is furnished by each simply closed Banach algebra of Hermitian operators on a Hilbert space.

Information concerning the possible choice of the constants c_1, c_2 is contained in the following proposition. We recall that a partial order O_1 on a set Q is finer than another O_2, if $x \leqslant y(O_1)$ implies $x \leqslant y(O_2)$ for all $x, y \in Q$. When Q is a vector space, "O_1 finer than O_2" is equivalent with $K_1 \subset K_2$ for the respective positive cones; in particular, the coarsest ordering O finer than all orderings in a family $\{O_\alpha\}$ is determined by $K = \cap_\alpha K_\alpha$ as its positive cone.

PROPOSITION 15. *Let $a \in A$ be a real spectral element, ν its associated spectral measure. Then $S(\nu) \subset [c_1, c_2]$ if and only if $c_1 e \leqslant a \leqslant c_2 e$ for the finest order on A for which ν is positive.*

Proof. If ν is positive for an ordering of A, it follows as in the second part of the proof of Proposition 14 that $S(\nu) \subset [c_1, c_2]$ implies $c_1 e \leqslant a \leqslant c_2 e$ for the order in question; hence the condition is necessary. Now let $c_1 e \leqslant a \leqslant c_2 e$ for the finest ordering of A for which ν is positive; if this relation can be satisfied with $c_1 = c_2$, then $a = c_1 e$ and ν, which by Theorem 3 is unique, is a one point measure; thus clearly $S(\nu) = \{c_1\}$. Let $c_1 e \leqslant a \leqslant c_2 e$ where $c_1 < c_2$; a simple argument reduces this case to $c_1 = 0$, $c_2 = 1$. Then ν may be constructed as in the proof of Proposition 13; with the notation adopted there, we have to show that $\psi(f) = 0$ whenever $f \in \mathfrak{C}(\mathbf{R})$ is such that its support is contained in the complement (in $\mathbf{R}$) of $[0, 1]$. But if f is supported by $\mathbf{R} \sim [0, 1]$ then $f \in N$ and hence $\chi(f) = 0$, and consequently $\psi(f) = 0$; thus $S(\nu) \subset [0, 1]$ and the proof is complete.

If A is a weakly semi-complete locally convex algebra, $a = \int f d\mu$ a spectral element of A, then for every bounded complex Baire function g, the integral $\int g \circ f d\mu$ defines another spectral element b; since

$$\int g \circ f d\mu = \int g(z) d\nu(z)$$

where ν is the complex spectral measure associated with a, b is defined unambiguously (i.e., independent of any particular representation of a as a spectral integral), and denoted by $b = g(a)$. (It is clear that it suffices to assume that g is a bounded Baire function on $\sigma(a)$ into $\mathbf{C}$.) Briefly but somewhat imprecisely, we shall say that b "is a function of a" and that $a = \int f d\mu$ is a "function of μ"; obviously $b = g(a)$ is a function of μ. The correspondence $g \to g(a)$, which by Proposition 10 is a homomorphism of the algebra of complex Baire functions bounded on $\sigma(a)$, into A, is usually referred to as an operational calculus.

PROPOSITION 16. (Spectral Mapping Theorem.) *If a is a spectral element and g is a continuous complex valued function on $\sigma(a)$, then $\sigma[g(a)] = g[\sigma(a)]$.*

The proof is immediate from Proposition 8 and Theorem 4. By Proposition 6, a corresponding result holds for arbitrary Baire functions bounded on $\sigma(a)$.

We note in particular that the consideration of functions of a spectral element permits us to introduce an operation of conjugation in the class of all spectral elements in A. For if $a=\int z d\nu(z)$ where ν is the associated complex spectral measure of a, one may define $a^*=\int \bar{z} d\nu(z)\,(z=\xi+i\eta,\ \bar{z}=\xi-i\eta$ where $\xi,\eta\in\mathbf{R})$; clearly $a\to a^*$ is an involution. Similarly, ν is the product $\nu_1\otimes\nu_2$ of two uniquely determined real spectral measures (Proposition 9, Corollary), and by the Fubini theorem (Proposition 11) we obtain

$$a=\int z d\nu(z)=\int (\xi+i\eta)d(\nu_1\otimes\nu_2)(\xi,\eta)=\int \xi d\nu_1(\xi)+i\int \eta d\nu_2(\eta);$$

here the first (second) integral may be referred to as the real (imaginary) part of a. If $a=\int f d\mu$ where f_1+if_2 and $f_i\in B_{\mathbf{R}}(\mu)\,(i=1,2)$ then clearly $\nu_1=f_1(\mu)$ and $\nu_2=f_2(\mu)$ in the sense of Proposition 8. For $a=\int f_1 d\mu+i\int f_2 d\mu$ implies that

$$a=\int t_1 d\lambda_1(t_1)+i\int t_2 d\lambda_2(t_2)$$

where $\lambda_i=f_i(\mu)$ are real spectral measures (Proposition 8). By Proposition 7 and Theorem 2, $\lambda_1\otimes\lambda_2$ is a complex spectral measure whence, by Proposition 11,

$$a=\int (t_1+it_2)d(\lambda_1\otimes\lambda_2)(t_1,t_2).$$

By Theorem 3, $\lambda_1\otimes\lambda_2=\nu=\nu_1\otimes\nu_2$ thus Proposition 9 implies that $\lambda_i=\nu_i\,(i=1,2)$. Hence, if $a=\int f d\mu$, then $a_1=\int f_1 d\mu$ and $a_2=\int f_2 d\mu$ are *the* real and imaginary parts of a. One denotes by $|a|$ the (real) spectral element $\int |f| d\mu$ and, if $a=\int f d\mu$ is a real spectral element, one can define $a^+=\int f^+ d\mu$ and $a^-=\int f^- d\mu$ where f is real valued and $f^+=\sup\{f,0\}$, $f^-=\sup\{-f,0\}$. Obviously, a^+ and a^- are positive spectral elements, and $a=a^+-a^-$, $|a|=a^++a^-$.

If $F\subset A$ and all elements of F are functions of a single spectral measure μ, we say for briefness that F is presentable by μ. In an ordered algebra we call $J=[0,e]$ the unit interval. J spans A if each $a\in A$ is a linear combination of elements of J. The following theorem is an extension of Proposition 14.

THEOREM 5. *Let F be a non-empty subset of the weakly semi-complete locally convex algebra A. In order that F be a family of real spectral elements presentable by a single spectral measure μ, it is necessary and sufficient that F be abelian, and that F be contained in an ordered real subalgebra spanned by its unit interval.*

Proof. Necessity. Let $F=\{a_\iota\colon \iota\in I\}$. If $a_\iota=\int f_\iota d\mu\,(\iota\in I)$, then it follows, since a_ι are real spectral elements, that each f_ι is real-valued a.e. (μ). Hence it is no essential restriction to assume f_ι as real $(\iota\in I)$. But then, by Proposition 10, F is contained in the image under $f\to\int f d\mu$ of $B_{\mathbf{R}}(\mu)$, which is a real, commutative subalgebra of A with the required properties.

Sufficiency. Let $\tilde{A}$ be a real subalgebra of A which is ordered, and spanned by $J = [0, e]$. (We assume without loss of generality that $e \in \tilde{A}$.) Clearly $\tilde{A}$, by means of the canonical imbedding $\tilde{A} \to A$, induces an ordering of A. If $a_\iota \in F$ is fixed, it follows from Proposition 14 that a_ι is a (real) spectral element, hence that $a_\iota = \int t d\nu_\iota(t)$ for a (unique) real spectral measure ν_ι that takes its values in the positive cone K of A. From Proposition 12 it follows, since F is abelian, that $\{\nu_\iota : \iota \in I\}$ is an abelian family. Let X_ι denote the compact subspace of R which is the spectrum $\sigma(a_\iota)(\iota \in I)$. If $X = \prod_{\iota \in I} X_\iota$ is the topological product of these spaces, then by Theorem 2 there exists a unique spectral measure ν on X into A such that $\nu = \otimes_{\iota \in I} \nu_\iota$. If f_ι denotes the projection mapping of X onto $X_\iota(\iota \in I)$, then clearly $a_\iota = \int t d\nu_\iota(t) = \int f_\iota d\nu$. Hence F is presentable by ν and the theorem is proved.

Remark. Let F denote any non-empty family of spectral elements in A, and F_1 (or F_2) the set of real (or imaginary) parts of the elements of F. In order that F be presentable by a spectral measure μ, it is necessary and sufficient that $F_1 \cup F_2$ be presentable (or, equivalently, that F_1 and F_2 be presentable and the product of two representing spectral measures exists).

COROLLARY 1. *Every subalgebra of A which is (algebraically) isomorphic with a $*$-algebra B of bounded complex functions containing 1,*(1) *such that the non-negative functions in B are mapped onto a weakly normal cone in A, is presentable by a spectral measure (and hence consists entirely of spectral elements).*

In the discussion following Proposition 16, we have seen that every spectral element $a \in A$ may be decomposed (independently of any given representation of a) into a sum $a_1 + ia_2$ where a_1, a_2 are real spectral elements. Conversely, given two real spectral elements a_1 and a_2, when is it true that $a_1 + ia_2$ is spectral? A sufficient condition is supplied by the following corollary.

COROLLARY 2. *Let A be an ordered, weakly semi-complete locally convex algebra. If $\{a_1, \ldots, a_n\}$ is an abelian family contained in the real span of $[0, e]$, then $\varphi(a_1, \ldots, a_n)$ is a spectral element of A for every complex polynomial φ in n indeterminates.*

Proof. By Theorem 5, there exists a spectral measure μ and functions $f_\nu \in B_{\mathbf{R}}(\mu)$ such that $a_\nu = \int f_\nu d\mu (\nu = 1, \ldots, n)$. It is then clear from Proposition 10 that

$$\varphi(a_1, \ldots, a_n) = \int \varphi(f_1, \ldots, f_n) d\mu$$

which proves the assertion.

(1) By a $*$-algebra we mean an algebra B of complex valued functions on a set S, such that $f \in B$ implies $\bar{f} \in B$.

The method used in the proof of Theorem 5 enables us to obtain a converse of Corollary 1, namely to characterize certain subalgebras of a weakly semi-complete locally convex algebra as algebras of all continuous functions on a compact Hausdorff space. The result obtained is related to Theorems 14, 17, 18 of Dunford [7], but our construction differs from Dunford's which uses the structure spaces (i.e., the spaces of maximal ideals) of the algebras involved. Let V be an Archimedean ordered real vector space with order unit e_0 (cf. [18], Section 4), then V is the (real) linear hull of its "unit interval" $[0, e_0]$. The order topology on V (l.c.), which is the finest locally convex topology on V for which the positive cone is normal, is normable in the presence of an order unit; the gauge function of $[-e_0, e_0]$ is a generating norm. If A_0 is a real commutative algebra with unit e, such that the underlying vector space is Archimedean ordered, the positive cone K is invariant under multiplication and e is an order unit, then it is easily verified that A_0 is an ordered normed algebra for its order norm (the gauge function of $[-e, e]$). The complexification $A = A_0 + iA_0$, for the order whose positive cone is the positive cone in A_0, is again an ordered normed algebra, the order norm $a \to \|a\|_0$ on A_0 being extended to A by

$$\|a + ib\|_0 = \sup_{0 \leqslant \theta \leqslant 2\pi} \|a \cos \theta + b \sin \theta\|_0.$$

THEOREM 6. *Let A be a weakly semi-complete locally convex algebra, $\hat{A}$ a closed commutative ordered subalgebra spanned by its unit interval. Then $\hat{A}$, under its order norm, is isomorphic-isometric with the algebra of all continuous (complex) functions on a compact Hausdorff space.*

Proof. Denote by K the positive cone in $\hat{A}$; since K is invariant under multiplication, $A_0 = K - K$ is a real subalgebra of $\hat{A}$. By Proposition 14, every $a \in A_0$ is a real spectral element; denote its associated real spectral measure by ν_a. Each ν_a takes its values in K, and by Proposition 12 $\{\nu_a : a \in A_0\}$ is an abelian family. Denote by X the compact space $\prod\{\sigma(a) : a \in A_0\}$, by μ the product $\otimes\{\nu_a : a \in A_0\}$ (Theorem 2), and by S the support of μ in X. $a = \int f_a d\mu$ for all $a \in A_0$ when f_a denotes the projection of S onto $\sigma(a)$. On the other hand, $\int f d\mu \in A_0$ for every continuous real function on S since K is closed in $\hat{A}$ for the induced topology. Thus $f \to \int f d\mu$ is a linear mapping on $C_{\mathbf{R}}(S)$ onto A_0; we show that this mapping is norm-preserving where $C_{\mathbf{R}}(S)$ is equipped with the usual sup-norm and A_0 with its order norm. But the order norm is determined by $\|a\|_0 = \{\inf \lambda : -\lambda e \leqslant a \leqslant \lambda e\}$; hence, by Proposition 15, $\|a\|_0 = \sup\{|\lambda| : \lambda \in \sigma(a)\}$. Now if $a = \int f d\mu$ with $f \in C_{\mathbf{R}}(S)$, $\sigma(a) = f(S)$ by Proposition 8 and Theorem 4. It follows that $\|f\| = \|a\|_0$ hence $C_{\mathbf{R}}(S)$ and A_0 are isometric under $f \to \int f d\mu$. Since for complex-valued $f = f_1 + if_2 \in C(S)$, $\|f\| = \sup\{\|f_1 \cos \theta + f_2 \sin \theta\| : 0 \leqslant \theta \leqslant 2\pi\}$, the mapping $f \to \int f d\mu$ is an isometry of $C(S)$ onto $\hat{A}$ which completes the proof of the theorem.

Remark. In the preceding proof, the condition that $\hat{A}$ be closed in A can be replaced by the weaker assumption that the positive cone K of $\hat{A}$ be complete for the order topology on $\hat{A}$.

If $F(\mu)$ is the subalgebra of A consisting of all elements in A that are presentable by a fixed spectral measure μ, then the order norm on $F(\mu)$, for the order whose positive cone is $K = \{\mu(f): f \geqslant 0, f \in B_{\mathbf{R}}(\mu)\}$, is simply the spectral radius of $a \in F(\mu)$: $\|a\|_0 = r(a)$. With the aid of Proposition 6, it is not difficult to show that K is complete for the order topology on $F(\mu)$. Thus, in view of the remark above, the following result holds (A is assumed as semi-complete).

COROLLARY. *Every subalgebra $F(\mu)$ of A (μ a fixed spectral measure), equipped with the norm $a \to r(a)$, is isomorphic-isometric to $\mathcal{C}(S)$ for a compact Hausdorff space S.*

4. Spectral Operators

Let E denote a locally convex vector space (over C) with elements $x, y, \ldots$; the set of all continuous endomorphisms of E is, with the algebraic operations defined in the natural way, an algebra $\mathfrak{L}(E)$; we shall denote elements of $\mathfrak{L}(E)$ by $S, T, \ldots$, and the unit of $\mathfrak{L}(E)$ by I. If $\mathfrak{S}$ is a family of bounded subsets of E (cf. Section 0) whose union is E, then $\mathfrak{L}(E)$ (more precisely, the underlying vector space of $\mathfrak{L}(E)$) becomes a locally convex space for the topology of uniform convergence on the sets of $\mathfrak{S}$ (the $\mathfrak{S}$-topology); it is clearly no restriction to assume that $\mathfrak{S}$ is saturated, i.e., hereditary, and invariant under the formation of closed convex circled hulls of finite unions of its elements. It is quickly verified that, if $\mathfrak{S}$ is a saturated family which is left invariant by each $T \in \mathfrak{L}(E)$, then $\mathfrak{L}(E)$ is a locally convex algebra for the $\mathfrak{S}$-topology, in the sense explained in Section 2. $\mathfrak{L}(E)$ is a locally convex algebra for the $\mathfrak{S}$-topologies considered in the sequel.

Thus the results of the two preceding sections apply, in particular, to endomorphism algebras of locally convex spaces. The results in this section are explicitly based on the assumption that the elements under consideration are endomorphisms. They are true for general algebras to the extent that they do not depend upon the fact that $\mathfrak{L}(E)$ is the algebra of *all* continuous endomorphisms of E, since every locally convex algebra is an endomorphism algebra on its own underlying locally convex space. But it should be noted that spectral properties of an element depend, in general, upon the algebra in which it is imbedded. Also, if E is an ordered locally convex space then under certain conditions (see below) $\mathfrak{L}(E)$, equipped with an $\mathfrak{S}$-topology, is an ordered locally convex algebra for the order induced on $\mathfrak{L}(E)$; if A is an ordered locally convex algebra, then the order induced on A by its underlying ordered vector space is, in general, distinct from the given order on A.

DEFINITION 4. *A continuous endomorphism T of E is a spectral operator (on E) if T is a spectral element of $\mathfrak{L}(E)$ under the topology of simple convergence.*

By Definition 3, T is a spectral operator on E if there exists a spectral measure μ, on some locally compact space X, with values in A such that $T=\int f d\mu$ where f is a bounded (complex-valued) Baire measurable function. If $\Gamma(\mu)$ denotes the domain of μ, it follows from Definition 1 that μ is countably additive on $\Gamma(\mu)$ with respect to the weak topology on $\mathfrak{L}(E)$ associated with the topology of simple convergence; since the dual of $\mathfrak{L}(E)$ for this latter topology is isomorphic with $E \otimes E'$ ([4], Chapter IV, § 2, Proposition 11), it follows that the additivity requirement on μ is expressed by the relation

$$\langle \mu(\delta)\, x, x' \rangle = \sum_{n=1}^{\infty} \langle \mu(\delta_n)\, x, x' \rangle$$

where $\{\delta_n\}$ is a disjoint sequence in $\Gamma(\mu)$, $\delta = \bigcup_1^\infty \delta_n$ and x (or x') are arbitrary elements in E (or E'). We shall also say that T is a spectral operator for an $\mathfrak{S}$-topology if μ is countably additive for this topology on $\mathfrak{L}(E)$. A real spectral operator is, in accordance with Definition 3 and Theorem 4, a spectral operator with real spectrum. We note that every spectral operator has compact spectrum $\sigma(T)$, hence a non-empty resolvent set $\varrho(T)$. Thus if E is a Banach space, a spectral operator on E is a bounded spectral operator of scalar type in the sense of Dunford [6]. Unless the contrary is expressly stated, we assume in the sequel that the algebra $\mathfrak{L}(E)$ is equipped with the topology of simple convergence and semi-complete for this topology; if an ordering is considered on $\mathfrak{L}(E)$, we denote by J the unit interval $[0, I]$, and by $L_{\mathbf{R}}(J)\,(L_{\mathbf{C}}(J))$ the real (complex) linear extension of J.

PROPOSITION 18. *If T is a spectral operator on E, then $T=T_1+iT_2$ where T_1 and T_2 commute, and are in $L_{\mathbf{R}}(J)$ for a suitable ordering of $\mathfrak{L}(E)$. When $\mathfrak{L}(E)$ is weakly semi-complete, the condition is also sufficient for T to be spectral.*

Proof. If $T=\int f d\mu$, then μ is positive for some ordering of $\mathfrak{L}(E)$ (Proposition 7) and $T=\int f_1 d\mu + i \int f_2 d\mu$ where f_1 (f_2) is the real (imaginary) part of f. Clearly $T_i = \int f_i d\mu$ ($i=1,2$) commute and are contained in $L_{\mathbf{R}}(J)$ for every order on $\mathfrak{L}(E)$ for which μ is positive. Conversely, when $\mathfrak{L}(E)$ is weakly semi-complete and $T=T_1+iT_2$ where T_1, T_2 have the required properties, if follows from Theorem 5 that T_1, T_2 are real spectral operators presentable by a spectral measure μ which implies that T is spectral.

Examples

1. Let E denote a complex Hilbert space of arbitrary dimension. The set $\mathcal{K}=\{T: \langle Tx,x\rangle \geqslant 0 \text{ for } x\in E\}$ is a convex cone of vertex 0 in $\mathcal{L}(E)$. It is well known and easy to verify that this cone, which consists of all positive Hermitian elements of $\mathcal{L}(E)$, is normal for the topology of simple convergence, and that $T_1 T_2 \geqslant 0$ if T_1, T_2 are commuting positive elements. Further, $L_{\mathbf{R}}(J)$ is identical with the real vector subspace of $\mathcal{L}(E)$ containing all Hermitian operators. Also, if T is a normal operator, then $T=T_1+iT_2$ where T_1, T_2 commute and are Hermitian; every such operator is normal. Hence every normal operator is spectral, for $\mathcal{L}(E)$ is weakly semi-complete. Thus Proposition 18 contains the spectral theorem for normal operators in Hilbert space. Among all spectral operators in Hilbert space, the normal operators are distinguished by the fact that they are presentable by a spectral measure that takes its values in $\mathcal{K}$. If T is an arbitrary spectral operator on Hilbert space, then by a result due to Mackey (see Wermer [27]) there exists an automorphism S of E such that $S^{-1}TS$ is normal, hence the associated complex spectral measure of T takes its values in $S\mathcal{K}S^{-1}$ which is a cone determining an ordering of $\mathcal{L}(E)$, and which is isomorphic with $\mathcal{K}$. Hence all spectral measures with values in $\mathcal{L}(E)$ take their values in convex cones isomorphic with $\mathcal{K}$. We note that each spectral measure is countably additive for the strong operator topology.

2. Let $E=\lambda$, where λ is a perfect space (vollkommener Raum) in the sense of Köthe [11], equipped with its normal topology. $\mathcal{L}(\lambda)$ is an ordered locally convex algebra for the order structure induced by the natural order of λ. It follows (since λ is weakly semi-complete) that J is weakly semi-complete in $\mathcal{L}(\lambda)$. Hence every element in $L_{\mathbf{R}}(J)$ is a spectral operator; clearly the elements of $L_{\mathbf{R}}(J)$ are the operators representable by diagonal matrices with bounded real entries. Obviously, all elements in $L_{\mathbf{C}}(J)$ are spectral; more generally, every continuous endomorphism of λ which is similar to an operator with bounded diagonal matrix, is a spectral element of $\mathcal{L}(\lambda)$. We remark that this example includes all spaces $l_p (1 \leqslant p < \infty)$, and similar remarks are valid for l_∞.

If $T=\int f d\mu$ is a spectral operator on E and g is a complex function continuous on $\sigma(T)$ then, as we have seen in Section 3, $g(T)=\int g\circ f d\mu$ is unambiguously defined. In particular, each spectral operator T has a conjugate $T^*=\int \bar{f} d\mu$ where $\bar{f}$ is the complex conjugate of f. We have noted in the first of the preceding examples that normal (Hermitian) operators are a special class of complex (real) spectral operators on Hilbert space, namely those whose associated complex (real) spectral measures take their values in the closed conical extension $\mathcal{K}$ of the set of all orthogonal projections. Since the unitary operators in

Hilbert space are those normal operators whose spectrum is contained in the unit circle $\{\xi: |\xi| = 1\}$, it appears natural to call an automorphism U of a locally convex space E pseudo-unitary if it is spectral and if $\sigma(U) \subset \{\xi: |\xi| = 1\}$. We obtain the following characterization of pseudo-unitary operators:

A spectral operator $U \in \mathcal{L}(E)$ is pseudo-unitary if and only if $UU^* = I$.

For let U be pseudo-unitary; then $U = \int z d\nu(z)$ where, by Theorem 4, the support $S(\nu)$ of ν is contained in the unit circle. By Proposition 10, $UU^* = \int |\xi|^2 d\nu = \int d\nu = I$. Conversely, if $UU^* = I$ and $f(\xi) = |\xi|^2$, we have $f(U) = \int |\xi|^2 d\nu = I$ since $f(U) = UU^*$; hence by the spectral mapping theorem (Proposition 16), $f[\sigma(U)] = \sigma(I) = \{1\}$; thus $\sigma(U) \subset \{\xi: |\xi| = 1\}$ as was to be shown.

One of the important properties of normal operators in Hilbert space is that they have no residual spectrum; that is, if T is normal and $\lambda \in \sigma(T)$, then either λ is in the point spectrum of T, or else the range $(\lambda - T)E$ is dense in E. In other words, if λ is any complex number, then either λ is in the point spectrum of both T and its adjoint T', or $(\lambda - T)$ and $(\lambda - T')$ are both one-to-one. Here the adjoint of T is the endomorphism T' of the topological dual E' of E which satisfies

$$\langle Tx, x' \rangle = \langle x, T'x' \rangle$$

for all $x \in E$, $x' \in E'$. While in Hilbert space, T' and T^* may be identified (more precisely, $T^* \to T'$ is an anti-isomorphism of $\mathcal{L}(E)$ onto $\mathcal{L}(E')$), we have to distinguish between T^* and T' when T is a spectral operator on a general locally convex space. We have $\sigma(T) = \sigma(T')$ and $\overline{\sigma(T)} = \sigma(T^*)$ as it should be. In the next proposition, the dual E' of E is equipped with any locally convex topology consistent with the dual system $\langle E, E' \rangle$.

PROPOSITION 19. *$T \in \mathcal{L}(E)$ is spectral if and only if T' is a spectral operator on E'. When T is spectral, then $\sigma(T) = \sigma(T')$ and every pole of the resolvent of T is simple. If ν denotes the complex spectral measure associated with T, then $\nu\{\lambda\}E = (\lambda - T)^{-1}\{0\}$ for every λ, and $\nu\{\lambda\} \neq 0$ if and only if $(\lambda - T)E$ is not dense in E.*

Proof. Let T be a spectral operator on E, ν its associated complex spectral measure. Clearly $\delta \to \nu'(\delta)$, δ an arbitrary Baire set in $\mathbf{C}$, is a complex spectral measure with values in $\mathcal{L}(E')$ such that $S(\nu) = S(\nu')$ for the respective spectra of ν and ν'. Since

$$\int \xi d\langle \nu(\xi)x, x' \rangle = \int \xi d\langle x, \nu'(\xi)x' \rangle$$

for all $x \in E$ and $x' \in E'$, it is clear that the existence of $\int \xi d\nu(\xi) \in \mathcal{L}(E)$ implies the existence of $\int \xi d\nu'(\xi)$ in $\mathcal{L}(E')$. From Theorem 4 we obtain $\sigma(T) = \sigma(T')$. It is clear, since the dual of E' is E by assumption, that T and T' may be interchanged. If T is spectral and λ_0 a pole of the resolvent, then for $\lambda \in \varrho(T)$ one has

$$(\lambda - T)^{-1} = \int \frac{d\nu(\xi)}{\lambda - \xi} = \frac{\nu\{\lambda_0\}}{\lambda - \lambda_0} + \int_{\sigma'} \frac{d\nu(\xi)}{\lambda - \xi},$$

where $\sigma' = \sigma(T) \sim \{\lambda_0\}$. Hence λ_0 is simple, and $\nu\{\lambda_0\} E = (\lambda_0 - T)^{-1}\{0\}$. We show that this latter relation holds for any complex λ. Let λ be fixed and $N = (\lambda - T)^{-1}\{0\}$. If $\nu\{\lambda\} x = x$ for some $x \in E$, then

$$(\lambda - T)x = \int (\lambda - \xi) d\nu(\xi) \cdot \nu\{\lambda\} x = 0,$$

hence $x \in N$ and $\nu\{\lambda\} E \subset N$. To prove the reverse inclusion, we note that $\nu(\delta)N \subset N$ for any δ since ν commutes with T; hence $\delta \to \hat{\nu}(\delta)$, where $\hat{\nu}(\delta)$ is the restriction of $\nu(\delta)$ to N, is a spectral measure with values in $\mathcal{L}(N)$. Now $0 = \int (\lambda - t) d\hat{\nu}(t) = \int t d\hat{\nu}(\lambda - t)$, which shows (Theorem 4) that the spectral measure $\delta \to \hat{\nu}(\lambda - \delta)$ has its support equal to $\{0\}$, whence $\nu\{\lambda\} N = N$. But then $\nu\{\lambda\} E \supset N$ as we wanted to show.

To prove the final assertion, we note that $(\lambda - T)E$ is dense in E if and only if $(\lambda - T')$ is one-to-one in E' which, by what we have shown, is equivalent to $\nu'\{\lambda\} = 0$. But $\nu'\{\lambda\} = 0$ is equivalent with $\nu\{\lambda\} = 0$ and the proposition is proved.

Let V denote a vector lattice. If V^+ denotes the vector space of all linear forms on V that are differences of non-negative linear forms (respectively differences of linear forms with non-negative real parts), a theorem due to F. Riesz asserts that V^+ is an order complete vector lattice. More generally, if V is ordered with generating positive cone K such that $0 \leqslant w \leqslant u + v$ with $u, v \in K$ implies $w = x + y$ where $0 \leqslant x \leqslant u$, $0 \leqslant y \leqslant v$, the same assertion is valid ([20], Section 13). We shall say of such a cone that K has the decomposition property; it is well known and easy to verify that in every vector lattice, the positive cone has the decomposition property. If E is a locally convex space, K a convex cone with compact base in E, then K is the closed convex hull of its extremal rays ([11], p. 342). A cone K satisfying this assertion will be said to have the Krein–Milman property. Recently Choquet has shown(1) that every cone which is the projective limit of a sequence of cones with compact base, has the Krein–Milman property. We recall that a ray $\{\lambda x : \lambda \geqslant 0\}$ where $0 \neq x \in K$, is extremal if $x = y + z$ with $y, z \in K$ implies that $y = \varrho x$, $z = (1 - \varrho)x$ $(0 \leqslant \varrho \leqslant 1)$. Before proving the main result of this section, we establish this lemma.

LEMMA 3. *Let E denote a (properly) ordered locally convex space whose positive cone is closed and generating, and has either the decomposition or the Krein–Milman property. If P_i are continuous projections on E such that $0 \leqslant P_i \leqslant I$ $(i = 1, 2)$ for the induced order on $\mathcal{L}(E)$, then P_1 and P_2 commute.*

(1) *C. R. Acad. Sci. Paris* 250 (1960), 2495–2497.

Proof. Assume first that $K \subset E$ has the first of the properties stated. Since K is closed and proper, E^+ is total over E [18, (1.7)]. When Q_1, Q_2 denote the adjoints of P_1, P_2 respectively for the dual system $\langle E, E^+ \rangle$ (note that P_1, P_2 are $\sigma(E, E^+)$-continuous), it is clear that $0 \leq Q_i \leq I$ $(i = 1, 2)$ for the ordering induced on $\mathfrak{L}(E^+)$ by the lattice ordering of E^+. If $F = Q_1 E^+$, $G = (I - Q_1) E^+$, then $E^+ = F + G$ is a band decomposition of E^+. (A band H is an order-complete sublattice of E^+, such that $x \in H$ and $|y| \leq |x|$ imply $y \in H$, and which contains the upper bound of every subset $M \subset H$ that is majorized in E^+.) Since $0 \leq Q_2 \leq I$ and since F, G are bands, it follows that $Q_2 F \subset F$ and $Q_2 G \subset G$. From this it follows easily that $Q_1 Q_2 = Q_2 Q_1$, and hence that $P_1 P_2 = P_2 P_1$. Assume now that $K \subset E$ has the Krein–Milman property. If $x \neq 0$ generates an extremal ray of K and P_1 is a projection with $0 \leq P_1 \leq I$, then (since $0 \leq P_1 x \leq x$) we must have either $P_1 x = x$ or else $P_1 x = 0$. As the same argument applies to P_2, we obtain $P_1 P_2 x = P_2 P_1 x$ for every x on an extremal ray. By assumption, the convex hull of these rays is dense in K; hence, by continuity, P_1 and P_2 commute on K. Since K is generating in E, the proof is complete.

Remark. There are cones which have the Krein-Milman but not the decomposition property. For example, if K is a closed and proper right circular cone in $\mathbf{R}^3$, K is clearly the closed convex hull of its extremal rays but does not have the decomposition property [20, (13.3)]. Since, on the other hand, every generating closed proper cone $K \subset \mathbf{R}^n$ which has the decomposition property is the positive cone for a lattice ordering of $\mathbf{R}^n$, it is clear that K has the Krein–Milman property. This property is also shared by the positive cones of the lattices $l_p (1 \leq p < \infty)$, but in general the positive cone of a locally convex vector lattice does not have the Krein–Milman property. Examples are furnished by the spaces $L_p(\mu) (p \geq 1$, μ Lebesgue measure on $\mathbf{R})$, and $\mathcal{C}(X)$ where X is compact but contains no isolated points.

We recall that a locally convex vector lattice is a locally convex space and a vector lattice, such that the positive cone is normal and the lattice operations are continuous.

THEOREM 7. *Let E denote a locally convex vector lattice, or an ordered locally convex space whose positive cone K is weakly normal, generating and has the Krein–Milman property. Assume in addition that K is weakly semi-complete, and that every positive linear form on E is continuous.*(1) *If $\mathfrak{L}(E)$ denotes the algebra of weakly continuous endomorphisms of E, J the unit interval in $\mathfrak{L}(E)$ for the induced order, then the vector subspace $L_{\mathcal{C}}(J)$ of $\mathfrak{L}(E)$ is a subalgebra that is presentable by a positive spectral measure.*

Proof. We consider the algebra $\mathfrak{L}(E)$ under the topology of simple convergence where E is equipped with the weak topology. Since K is weakly normal and generating, it follows

(1) Cf. Section 0, Theorem D.

from Theorem C that the positive cone $\mathcal{K}$ is normal in $\mathfrak{L}(E)$. Since every positive linear form on E is continuous, every positive endomorphism of E is weakly continuous which implies that $\mathcal{K}$ is (weakly) semi-complete. Since, by Lemma 3, any two projections in J commute, the family of all projections in J forms a Boolean σ-algebra $\mathcal{B}$. Let $T \in L_{\mathbf{C}}(J)$, then $T = T_1 - T_2 + i(T_3 - T_4)$ where a suitable positive multiple of each $T_i (i = 1, ..., 4)$ is in J. By Proposition 13, each T_i is a real spectral operator whose associated real spectral measure takes its values in $\mathcal{B}$ (note that K is closed in E, hence $\mathcal{K}$ and J are closed in $\mathfrak{L}(E)$ so that J is semi-complete). Since $\mathcal{B}$ is abelian, it follows that the family $F = L_{\mathbf{R}}(J)$ satisfies the assumptions of Theorem 5; the hypothesis that $\mathfrak{L}(E)$ be weakly semi-complete is dispensable since $\mathcal{K}$ has that property. Consequently F is presentable by a spectral measure μ taking its values in $\mathcal{B}$; this implies (cf. Theorem 5, Remark) that $L_{\mathbf{C}}(J)$ is a subalgebra of $\mathfrak{L}(E)$ presentable by μ.

COROLLARY. *Under the assumptions of the theorem $L_{\mathbf{C}}(J)$, when equipped with the norm $T \to r(T)$ (r the spectral radius of T), is isomorphic-isometric with an algebra $\mathcal{C}(S)$ for compact S.*

This is immediate from Theorem 6, Corollary.

Remarks and Examples

1. It may, of course, occur that $L_{\mathbf{C}}(J) = \{\alpha I: \alpha \in \mathbf{C}\}$. Thus if K is the cone $\{x: x_3 \geq 0, x_1^2 + x_2^2 \leq x_3^2\}$ in $\mathbf{R}^3$, it turns out that 0 and I are the only projections in J, hence $L_{\mathbf{C}}(J)$ is trivial. An infinite-dimensional example of this character is furnished by the space E' of functions of bounded variation on [0, 1] vanishing at 0; if E' is equipped with its Mackey topology $\tau(E', E)$ (as the dual of $E = \mathcal{C}(0, 1)$), it is weakly semi-complete and lattice-ordered but 0 and I are the only continuous projections in J. (It should be observed that for τ, however, E' is not a locally convex lattice, and not every positive linear form is continuous.) That there are no other projections in J follows from the fact that there are no such projections on $\mathcal{C}(0, 1)$. On the other hand, if E is an order-complete locally convex lattice and $x \geq 0$, there exists a projection $P \in J$ such that $Px = x$ and $Py = 0$ whenever $\inf(x, |y|) = 0$. A wide class of examples where $L_{\mathbf{C}}(J)$ is always of infinite dimension are the perfect spaces of Köthe [11] (see Ex. 2 after Proposition 18).

2. If, in addition to the conditions of Theorem 7, E is a Banach space, then the positve cone in $\mathfrak{L}(E)$ is normal for the topology of simple convergence (the "strong operator topology") by Theorem C. Hence it follows from Theorem 1 that all elements of $L_{\mathbf{C}}(J)$ are spectral for this topology. More generally, every $T \in L_{\mathbf{C}}(J)$ is spectral for every $\mathfrak{S}$-topology on $\mathfrak{L}(E)$ for which $\mathcal{K}$ is normal and weakly semi-complete.

3. Another class of examples for Theorem 7 is furnished by those weakly semi-complete (separable) locally convex spaces that have an absolute basis; $\{x_n\}$ is an absolute basis of

E when each $x \in E$ has a unique expansion $x = \sum_1^\infty c_n x_n$ which converges unconditionally to x for the given topology on E. ($\{x_n\}$ is a weak absolute basis if the series converges unconditionally for $\sigma(E, E')$; for tonnelé spaces, the two notions coincide by the Banach–Steinhaus theorem.) It is easily seen that the cone $K = \{x\colon c_n \geqslant 0,\ n \in \mathrm{N}\}$ determines a lattice ordering on E, is weakly normal, generating and has the Krein–Milman property. Clearly each $x \to c_n(x)$ is a continuous linear form on E, hence $c_n(x) = \langle x, x_n' \rangle$ where $x_n' \in E'(n \in \mathrm{N})$. If $T \in \mathfrak{L}(E)$, then T has a matrix representation $(t_{m,n})$ where $t_{m,n} = \langle Tx_m, x_n' \rangle$ $(m, n \in \mathrm{N})$. If we assume in addition that every positive linear form on E (for the order whose positive cone is K) is continuous (which, by Theorem D, is automatically the case when E is a Fréchet space), then it follows from Theorem 7 that every $T \in \mathfrak{L}(E)$ representable by a diagonal matrix with bounded entries is spectral; more generally, every operator T such that $S^{-1}TS$ has such a matrix where S is an automorphism of E, is spectral. (Clearly, then, T has a bounded diagonal matrix with respect to the basis $\{Sx_n\}$.) For compact operators, we obtain a partial converse. Recall that T is compact if $T(U)$ is relatively compact for some 0-neighborhood U in E; an eigenvalue of T is simple if it has multiplicity one.

PROPOSITION 20. *Let T be a compact operator in E with dense range and simple eigenvalues. T is a spectral operator if and only if E has a weak absolute basis with respect to which T has a diagonal matrix.*

Proof. When T is compact, it is well known that $\sigma(T)$ consists of a finite or denumerable number $\{\lambda_n\colon n = 1, 2, \ldots\}$ of eigenvalues such that $\lim_n \lambda_n = 0$ if the sequence is infinite. Also $0 \in \sigma(T)$ except perhaps when E is finite-dimensional. If T satisfies the assumptions made and is spectral, then $\nu\{0\} = 0$ and $\nu\{\lambda_n\} = P_n$ where P_n are mutually orthogonal projections of rank 1; here ν denotes the associated complex spectral measure of T. Since ν is countably additive and $\int_{\sigma(T)} d\nu = I$, it follows that $\{x_n\}$ is a weak absolute basis of E where x_n spans the range of $P_n (n \in \mathrm{N})$. Clearly the matrix of T with respect to $\{x_n\}$ is $(\lambda_n \delta_{mn})$. Conversely, if $\{x_n\}$ is a weak absolute basis of E with respect to which T has the matrix $(\lambda_n \delta_{mn})$, then for every subset $M \subset \mathrm{N}$ we define

$$\mu(M) = \sum_{n \in M} P_n,$$

denoting by P_n the projection of E for which $P_n x_n = x_n$, $P_n x_m = 0$ $(m \neq n)$. Since $\{x_n\}$ is an absolute basis, μ is a spectral measure on the discrete space N into $\mathfrak{L}(E)$ (Definition 2) such that $T = \int t\, d\mu(t)$ which proves the assertion.

COROLLARY. *On the space $\mathcal{C}(0, 1)$, there exists no compact spectral operator with dense range and simple eigenvalues.*

This is an immediate consequence of Proposition 20 and a result of Karlin [10] to the effect that the space $C(0, 1)$ possesses no absolute basis. We note that the method of proof of Proposition 20 extends at once to the case where all but a finite number of eigenvalues are simple.

5. Spectral Operators with Unbounded Spectrum

Let E be a locally convex vector space (over $\mathbf{C}$), $\mathcal{L}(E)$ the algebra of weakly continuous endomorphisms of E, equipped with the topology of simple convergence. Unless otherwise stated, we assume in this section that E is weakly semi-complete, and that $\mathcal{L}(E)$ is weakly semi-complete. By a linear mapping in E, we understand a linear mapping T defined on a vector subspace $D_T \subset E$, with values in E. Such a mapping is closed if the graph $\{(x, Tx)$: $x \in D_T\}$ is closed in $E \times E$; it is said to commute with an $S \in \mathcal{L}(E)$ if $ST \subseteq TS$, i.e. if TS (defined on $S^{-1}(D_T)$) is an extension of ST (defined on D_T). The resolvent set $\varrho(T)$ is defined as the largest open set in $\mathbf{C}$ such that $(\lambda - T)^{-1}$ exists, is continuous with dense domain, and is such that its continuous extension $R(\lambda)$ is holomorphic in a neighborhood of λ. The spectrum $\sigma(T)$ then is the complement of $\varrho(T)$ in $\mathbf{C}$. We shall extend, in this section, the concept of spectral operator to a class of mappings with non-compact spectrum. The members of this class are, consequently, in general not elements of $\mathcal{L}(E)$.

Let (X, f, μ) be given, where X is a locally compact Hausdorff space, μ a spectral measure on X into $\mathcal{L}(E)$, and f an arbitrary complex valued Baire function on X. The triple (X, f, μ) defines a linear mapping in E as follows. If $x \in E$ is fixed, $\delta \to \mu(\delta)x = \mu_x(\delta)$ is a vector measure on X (Definition 1); denote by D_f the subset of E such that for $x \in D_f$, f is μ_x-integrable (Section 1) and $\mu_x(f) = \int f d\mu_x \in E$. Clearly D_f is non-empty (since $0 \in D_f$) and a vector subspace of E; thus $x \to \mu_x(f)$ defines a linear mapping T in E with domain $D_T = D_f$. We shall write, to denote this association, $T \sim (X, f, \mu)$ or $T = \int f d\mu$, or $T = \int f(t) d\mu(t)$.

Definition 5. *A linear mapping T in E is spectral (or a spectral operator) if there exists (X, f, μ) such that $T \sim (X, f. \mu)$.*

We note that every spectral operator in the sense of Definition 4 is spectral in the present sense; if we want to refer specifically to a member T of the subclass singled out by Definition 4, we shall say that T is a spectral operator on E with bounded spectrum. When $\{P_n\}$ is a sequence of mutually orthogonal projections in $\mathcal{L}(E)$ such that $I = \sum_1^\infty P_n$ weakly, we shall say that $\{P_n\}$ decomposes E continuously and write $E = \sum_1^\infty E_n$, where $E_n = P_n E (n \in \mathbf{N})$.

PROPOSITION 21. *If T is a spectral mapping in E, then D_T is dense, and there exists a continuous decomposition $E=\sum_1^\infty E_n$ such that the restriction T_n of T to E_n is a spectral operator on E_n with bounded spectrum ($n\in\mathrm{N}$), and such that $Tx=\sum_1^\infty T_nP_nx$, the series being weakly convergent if and only if $x\in D_T$.*

Proof. Let $T\sim(X,f,\mu)$, $\delta_n=f^{-1}[n-1\leqslant|\xi|<n], P_n=\mu(\delta_n)(n\in\mathrm{N})$. Since μ is a spectral measure, $\{P_n\}$ decomposes E continuously. If $x\in E_n$, then $\mu(\delta_n)x=x$ and $\mu_x(\delta)=\mu(\delta)\mu(\delta_n)x=\mu(\delta\cap\delta_n)x$ for every Baire set δ in X; hence f is integrable with respect to μ_x which implies that $x\in D_T$; thus $E_n\subset D_T$ for every n. Hence $\bigoplus_1^\infty E_n$ (the algebraic direct sum) is contained in D_T whence it follows that D_T is dense in E. If μ_n denotes the spectral measure whose values are the restrictions to E_n of the values of μ, then clearly $T_n=\int f_n d\mu_n$ where $f_n=f\chi_n$ (χ_n the characteristic function of δ_n). Hence each T_n is a spectral operator with bounded spectrum on E_n ($n\in\mathrm{N}$). Finally, when $x\in D_T$, it follows that $\lim_p\sum_1^p\int f_n d\mu_x=\int f d\mu_x$ for $\sigma(E,E')$, thus $Tx=\sum_1^\infty T_nP_nx$. Conversely, since $\sum_1^\infty f_n(t)=f(t)$ everywhere in X, the convergence of $\sum_1^\infty\int f_n d\mu_x$ for $\sigma(E,E')$ implies that f is μ_x-integrable with $\int f d\mu_x\in E$.(1) Therefore $x\in D_T$.

PROPOSITION 22. *Every spectral operator in E is closed.*

Proof. Let $T\sim(X,f,\mu)$, let $\{P_n\}$ denote a decomposition of E satisfying the assertions of Proposition 21, and let $S_n=TP_n$ ($n\in\mathrm{N}$). We show first that each P_n commutes with T, that is, $P_nT\subseteq TP_n$. If $x\in D_T$, then

$$P_nTx=P_n\int f d\mu_x=\sum_{m=1}^{\infty}P_n\int f_m\,d\mu_x=\sum_{m=1}^{\infty}\int f_m\,d\mu_{(P_nx)}=TP_nx,$$

where we have used the notation of the preceding proof. To show that T is closed, let $x\to\tilde{x}$ where x is restricted to D_T; i.e., let x converge to $\tilde{x}\in E$ along the trace on D_T of the neighborhood filter of $\tilde{x}$. Further assume that $Tx\to\tilde{y}$. Since P_n is continuous, it follows that $x_n=P_nx\to P_n\tilde{x}=\tilde{x}_n$, and $y_n=P_nTx\to P_n\tilde{y}=\tilde{y}_n$. Since P_n and T commute, we obtain $P_nTx=S_nx$ and, S_n being continuous, $S_n\tilde{x}=\tilde{y}_n$. It follows that

$$\tilde{y}=\sum_1^\infty\tilde{y}_n=\sum_1^\infty TP_n\tilde{x}$$

which, by Proposition 21, shows that $\tilde{x}\in D_T$ and $T\tilde{x}=\tilde{y}$. Hence T is closed.

Our next objective is to show that every spectral mapping T has an adjoint T' which is spectral in E'. We have to discuss the notion of adjoint of mappings not necessarily continuous on E since it appears that for general locally convex spaces, this concept has

(1) Cf. the proof of Lemma 5 below.

not been considered in the literature. Let T be a linear map, with values in a locally convex space E, and defined on a vector subspace D_T of E. One considers the subset E'_T of E' for whose elements x',

$$x \to \langle Tx, x' \rangle$$

is continuous on D_T. If D_T is dense in E (as we shall henceforth assume), then x' determines uniquely a $y' \in E'$ such that $\langle Tx, x' \rangle = \langle x, y' \rangle$ for $x \in D_T$. Since E'_T is a vector subspace of E' and since $x' \to y'$ is obviously linear, $x' \to y'$ determines a linear mapping T' with domain $D_{T'} = E'_T$ and values in E'. T' is called the adjoint of T. We collect a number of facts needed later in the following lemma, whose proof is a straightforward generalization of a method designed by von Neumann (see [23]). E' is considered under any locally convex topology consistent with $\langle E, E' \rangle$.

LEMMA 4. *Let T be linear in E with dense domain D_T. The adjoint T' is closed; in order that $D_{T'}$ be dense in E', it is necessary and sufficient that T has a closed extension. In this case, T'' is the smallest closed extension of T; hence $T = T''$ when T is closed. If $T_1 \subseteq T_2$, then $T'_2 \subseteq T'_1$.*

Proof. When $E \times E$ denotes the (algebraical and topological) product of E with itself, then the dual of $E \times E$ can be identified with $E' \times E'$, the canonical bilinear form on $(E \times E) \times (E' \times E')$ being

$$(z, z') \to \langle z, z' \rangle = \langle x, x' \rangle + \langle y, y' \rangle$$

where $z = (x, y) \in E \times E$, $z' = (x', y') \in E' \times E'$. The mapping ϕ: $(x', y') \to (y', -x')$ is an automorphism of $E' \times E'$ such that $\phi^2 = -I$. From this and the relation

$$\langle Tx, x' \rangle - \langle x, T'x' \rangle = 0 \quad (x \in D_T, x' \in D_{T'})$$

one concludes that the graph $G_{T'}$ which satisfies the relation $\phi[G_{T'}] = G_T^0$, is closed. Hence T' is a closed map. It is further clear that $T_1 \subseteq T_2$ implies $T'_2 \subseteq T'_1$.

When $x \in (D_{T'})^0$, then $(x, 0) \in (G_{T'})^0$ and $(0, -x) \in [\phi(G_{T'})^0] = G_T^{00}$. If T is closed, G_T is closed hence $G_T^{00} = G_T$ by the bipolar theorem. It follows that $(0, -x) \in G_T$ which implies $x = 0$, hence that $D_{T'}$ is dense. Thus if T is not closed but has a closed extension S, then $S' \subseteq T'$ and it follows that $D_{T'}$ is dense in E'. Conversely, if $D_{T'}$ is dense for some T, then T'' is defined by $y \to -x$ for every pair $(x, y) \in G_{T'}^0$, and the graph of T'' is the closure of G_T in $E \times E$. Clearly T'' is an extension of T, and the smallest closed extension possible. Thus $T = T''$ when T is closed.

Lemma 5. *Let $T \sim (X,f,\mu)$. If $x' \in D_{T'}$, then f is $\mu'_{x'}$-integrable, where μ' denotes the adjoint spectral measure of μ, and $\int f d\mu'_{x'} \in E'$ when E' is $\sigma(E',E)$-semi-complete.*

Proof. Without loss of generality, we assume that f is real-valued and $f \geqslant 0$. Let $x'_0 \in D_{T'}$ be fixed. By definition of $D_{T'}$, the linear form $x \to \int f d\langle \mu x, x'_0 \rangle$ is continuous on D_T. Denote by φ the real part of the continuous extension of this linear form to E.

We show first that the set $\{\mu'(\delta)x'_0 : \delta$ a set in the domain $\Gamma(\mu)\}$ is contained in a weakly normal cone in E'. For this, it is sufficient (Theorem A, Corollary) to show that each $x \in E$ is the difference, $x = x_1 - x_2$, of two elements such that $\mathrm{Re}\langle x_i, \mu'(\delta)x'_0 \rangle \geqslant 0$ ($i = 1,2$; $\delta \in \Gamma(\mu)$). By the Hahn–Jordan decomposition theorem, there exists $\varepsilon_1 \in \Gamma(\mu)$ such that $\mathrm{Re}\langle \mu(\delta)x, x'_0 \rangle \geqslant 0$ for all $\delta \subset \varepsilon_1$, and $\leqslant 0$ for all $\delta \subset \varepsilon_2 = X \sim \varepsilon_1$, since $\delta \to \mathrm{Re}\langle \mu(\delta)x, x'_0 \rangle$ is a (totally finite) real-valued measure on the σ-algebra $\Gamma(\mu)$. Since μ is a spectral measure, $x_1 = \mu(\varepsilon_1)x$ and $x_2 = -\mu(\varepsilon_2)x$ furnish a decomposition $x = x_1 - x_2$ of the required type. (Note that ε_1 and ε_2 depend on x.)

We show next that φ is positive on K, where K is the dual cone in E of the convex conical extension of $\{\mu'(\delta)x'_0 : \delta \in \Gamma(\mu)\}$. As we have shown, $E = K - K$. It is also true that $D_T = D_T \cap K - D_T \cap K$ since if f is μ_x-integrable, f is μ_{x_1}-integrable where $x_1 = \mu(\varepsilon_1)x$. Now φ is positive on $D_T \cap K$, hence on K since φ is continuous and, by the bipolar theorem, K is the weak closure of $K \cap D_T$ in E.

Finally, let $\{f_m\}$ be an increasing sequence of bounded Baire functions, such that $f_m \geqslant 0$ ($m \in \mathrm{N}$) and $\lim_m f_m = f$. If we set $\varphi_m(x) = \mathrm{Re}\int f_m d\langle \mu x, x'_0 \rangle$, then clearly $0 \leqslant \varphi_m(x) \leqslant \varphi(x)$ on K. This implies that $\{\varphi_m\}$ is a bounded monotone sequence whence it follows (Proposition 1) that f is $\mu'_{x'_0}$-integrable. Since $\int f_m d\mu'_{x'_0} \in E'$, the weak semi-completeness of E' implies that $\int f d\mu'_{x'_0} \in E'$.

Theorem 8. *If T is a spectral operator in E and E' is weakly semi-complete, then T' is a spectral operator in E'.*

Proof. Let $T \sim (X,f,\mu)$. Since the domain D_T is dense in E (Proposition 21), it follows from Proposition 22 and Lemma 4 that the adjoint T' of T in E' exists, and is a closed linear mapping with dense domain $D_{T'}$. When $x' \in D_{T'}$, it follows from Lemma 5 that f is $\mu'_{x'}$-integrable, hence in view of

$$\langle Tx, x' \rangle = \langle \int f d\mu_x, x' \rangle = \langle x, \int f d\mu'_{x'} \rangle = \langle x, T'x' \rangle$$

($x \in D_T$, $x' \in D_{T'}$), it follows that

$$T'x' = \int f d\mu'_{x'} \quad (x' \in D_{T'}).$$

Conversely, if f is $\mu'_{x'}$-integrable with $\int f d\mu'_{x'} \in E'$ for some $x' \in E'$, then clearly $x \to \langle x, \int f d\mu'_{x'} \rangle = \int f d\langle \mu x, x' \rangle$ is continuous on D_T, hence $x' \in D_{T'}$. It follows now that $T' \sim (X, f, \mu')$ which by Definition 5 shows that T' is a spectral operator.

When $T \sim (X, f, \mu)$, then $f(\mu)$ is a complex spectral measure (Proposition 8). If by $\tilde{1}$ we denote the identical mapping on the complex plane $\mathbf{C}$ and let $\nu = f(\mu)$, it is not difficult to verify that $T \sim (\mathbf{C}, \tilde{1}, \nu)$. We are not able to show that ν is uniquely determined by T unless the resolvent set $\varrho(T)$ is non-empty; in this case, ν will be called the complex spectral measure associated with T.

THEOREM 9. *If $T \sim (\mathbf{C}, \tilde{1}, \nu)$ is a representation of T by means of a complex spectral measure ν, then the support $S(\nu)$ is equal to $\sigma(T)$. If T is spectral and $\varrho(T) \neq \phi$, then ν is unique and commutes with every $Q \in \mathfrak{L}(E)$ that commutes with T.*

Proof. If T is spectral but $\sigma(T)$ is not the entire plane, an inspection of the proof of Theorem 4 shows that the conclusion $\sigma(T) = S(\nu)$ remains valid. If $S(\nu)$ is the entire plane then the second part of that proof still applies while the inclusion $\sigma(T) \subset S(\nu)$ is trivial. We assume now that $\varrho(T)$ is non-empty. Let $\lambda_0 \in \varrho(T)$; since T is closed, $(\lambda_0 - T)^{-1} = R(\lambda_0)$ is closed, hence defined and continuous on E with range D_T. With the aid of Proposition 21, it is easily verified that

$$R(\lambda_0) = \int \frac{d\nu(\xi)}{\lambda_0 - \xi}.$$

Hence $R(\lambda_0) \in \mathfrak{L}(E)$ is a spectral operator (Definition 4) with bounded spectrum; when f denotes the bounded continuous function $\xi \to (\lambda_0 - \xi)^{-1}$ on $S(\nu)$, then $R(\lambda_0) = \int \xi \, d\varrho(\xi)$ where $\varrho = f(\nu)$ is the associated complex spectral measure of $R(\lambda_0)$. If $T \sim (\mathbf{C}, \tilde{1}, \hat{\nu})$ is a second representation of T, then also $R(\lambda_0) = \int \xi \, d\hat{\varrho}(\xi)$ where $\hat{\varrho} = f(\hat{\nu})$. By Theorem 3, $\varrho = \hat{\varrho}$; if δ is a bounded Baire set, then so is $\varepsilon = f(\delta)$ and it follows that $\nu(\delta) = \varrho(\varepsilon) = \hat{\varrho}(\varepsilon) = \hat{\nu}(\delta)$ whence it follows that $\nu = \hat{\nu}$. Similarly, if $Q \in \mathfrak{L}(E)$ commutes with T, then $Q \smile R(\lambda_0)$ and hence, by Proposition 12, Q commutes with ϱ and, therefore, also with ν. The proof is complete.

COROLLARY. *If T is the limit of a sequence $\{T_n\}$ of spectral operators with bounded spectrum in the sense of Proposition 21, then $\sigma(T) = \overline{\bigcup_n \sigma(T_n)}$.*

The corollary is clear from the preceding theorem and the corollary of Theorem 4.

Remark. If $\sigma(T)$ covers the complex plane, the unicity of ν in $T = \int \xi \, d\nu(\xi)$ can be proved, for example, when it is known that for every projection $P \in \mathfrak{L}(E)$, $PT \subseteq TP$ implies that P commutes with $\int \bar{\xi} \, d\nu(\xi)$. From this it follows that in Hilbert space, every representation $T \sim (\mathbf{C}, \tilde{1}, \nu)$ is unique.

If μ is a spectral measure on a locally compact space X, with values in $\mathfrak{L}(E)$, then the class of all spectral operators T in E such that $T\sim(X,f,\mu)$ for some complex Baire function on X is, in general, not an algebra under the usual definition of addition and multiplication. However, by a consideration similar to the proof of Proposition 21, one concludes that if $T_1=\int f_1 d\mu$, $T_2=\int f_2 d\mu$, then $\alpha T_1=\int \alpha f_1 d\mu$ and $T_1+T_2\subseteq\int(f_1+f_2)d\mu$, $T_1T_2\subseteq\int f_1f_2 d\mu$. Thus, in place of Proposition 10, we obtain the following result.

PROPOSITION 23. *Let* $T_i=\int \xi d\nu_i(\xi)\,(i=1,\dots,n)$ *be spectral operators in* E *such that* $\nu=\bigotimes_{i=1}^n \nu_i$ *exists* (Theorem 2). *If* φ *is any complex polynomial in* n *indeterminates, then*

$$T=\varphi(T_1,T_2,\dots,T_n)$$

has a closed extension which is a spectral operator in E.

The definition of bounded functions of spectral operators with bounded spectrum (Section 3) can be extended to define $g(T)$ where T is any spectral operator in E and g an arbitrary complex-valued Baire function on C. When $T\sim(X,f,\mu)$, the natural definition of $g(T)$ is $g(T)=\int g\circ f d\mu=\int g(\xi)d\nu(\xi)$ where $\nu=f(\mu)$ (Proposition 8). However, unless ν is uniquely determined by T, we cannot be certain that this definition is unambiguous. (It is, to be sure, unambiguous for entire analytic functions even when $\varrho(T)=\phi$.) Thus, except when g is an entire function, we consider $g(T)$ as defined when $\varrho(T)$ is non-empty; then $g(T)=\int g(\xi)d\nu(\xi)$ where ν is the associated complex spectral measure of T (Theorem 9). Let G_{C} denote the algebra of complex Baire functions that are bounded on bounded subsets of $\sigma(T)$; if ν denotes the associated complex spectral measure of T, and $\{\delta_n\}$ a sequence of disjoint bounded Baire sets in C whose union is C, then $P_n=\nu(\delta_n)\,(n\in\mathrm{N})$ defines a continuous decomposition of E such that for each n, the restriction of $g(T)$ to $E_n=P_nE$ is a spectral operator with bounded spectrum for every $g\in G_{\mathrm{C}}$. From this it follows by Proposition 21 that $g(T)h(T)=(gh)(T)$ for every pair $g,h\in G_{\mathrm{C}}$, and thus that $g\to g(T)$ preserves multiplication on G_{C}.

If T is a spectral operator in E, $T\sim(\mathrm{C},\tilde{1},\nu)$, then $T=T_1+iT_2$ where $T_1=\int \mathrm{Re}(\xi)d\nu(\xi)$ and $T_2=\int \mathrm{Im}(\xi)d\nu(\xi)$. Clearly T_1,T_2 are spectral operators with real spectrum. When $\varrho(T)\neq\phi$, this decomposition is unique in the following sense: If $T=S_1+iS_2$ where S_1,S_2 are real spectral operators such that the product $\nu_1\otimes\nu_2$ of their associated real spectral measures exists, then $S_1=T_1$ and $S_2=T_2$. The proof follows from Theorem 9 and a consideration entirely analogous to that preceding Theorem 5.

It follows from Definition 5 that, roughly speaking, there exist as many spectral operators in E as there are spectral measures with values in $\mathfrak{L}(E)$; thus the conditions of

Theorem 7, in conjunction with the subsequent examples, give a means to construct spectral operators with unbounded spectrum. We shall, however, in analogy to Section 3 characterize spectral operators directly with the aid of order structures on $\mathfrak{L}(E)$; to avoid clumsy formulations, we restrict ourselves to real spectral operators (that is, to spectral operators with real spectrum).

THEOREM 10. *Let T be a closed operator in E with dense domain D_T. In order that T be a real spectral operator, it is necessary and sufficient that $(I+T^2)^{-1}$ exists, and that there is an ordering of $\mathfrak{L}(E)$ for which $0 \leqslant (I+T^2)^{-1} \leqslant I$ and $-I \leqslant T(I+T^2)^{-1} \leqslant I$.*

Proof. The condition is necessary. Let $T = \int t\,d\nu(t)$ where ν is the associated real spectral measure of T. T is closed and so is $T^2 = \int t^2 d\nu(t)$ by Proposition 22 and the remarks following Proposition 23 (these will repeatedly be used in this proof). It follows immediately that $(I+T^2)^{-1}$ exists and that

$$(I+T^2)^{-1} = \int \frac{d\nu(t)}{1+t^2}.$$

Further, since $(I+T^2)^{-1} \in \mathfrak{L}(E)$, we have $T(I+T^2)^{-1} = \int (t/[1+t^2])d\nu(t)$. By Proposition 7, there exists an ordering of $\mathfrak{L}(E)$ for which ν is positive. It is clear that for any such ordering T satisfies the inequalities listed in the theorem.

The condition is sufficient. Set $R = (I+T^2)^{-1}$, $S = T(I+T^2)^{-1}$. We show first that $RT \subseteq TR$ (i.e., that R and T commute). If $x \in D_T, y = Rx$, then $x = y + T^2y$ which shows that $y \in D_T$. Since R^{-1} exists, $RTx = TRx$ for $x \in D_T$ is equivalent with $Tx = R^{-1}TRx$; if $Rx = y$, then from $x = y + T^2y$ it follows that $y \in D_{T^3}$ when $x \in D_T$. Now $Tx = Ty + T^3y = R^{-1}Ty = R^{-1}TRx$ which proves the assertion. From this it follows that R and S commute. By Proposition 14, $R = \int t_1 d\nu_1(t_1)$ and $S = \int t_2 d\nu_2(t_2)$ and the associated spectral measures ν_1 and ν_2 commute by Proposition 12. Since ν_1, ν_2 take their values in the positive cone of $\mathfrak{L}(E)$ for a certain ordering of $\mathfrak{L}(E)$ by assumption, Theorem 2 shows that the complex spectral measure $\nu = \nu_1 \otimes \nu_2$ exists. Also, since R^{-1} exists, $\nu_1\{0\} = 0$, and, integrating over $\mathbf{C} \sim \{0\}$, $R^{-1} = \int \lambda_1^{-1} d\nu_1(\lambda_1) = \int \lambda_1^{-1} d\nu(\lambda_1, \lambda_2)$ in the sense of Definition 5. As $S = \int \lambda_2 d\nu(\lambda_1, \lambda_2)$, it follows from Proposition 23 that

$$T = SR^{-1} \subseteq \int f d\nu,$$

where we have set $f(\lambda_1, \lambda_2) = \lambda_1^{-1}\lambda_2$ for $\lambda_1 \neq 0, f(0, \lambda_2) = 0$. Since S is in $\mathfrak{L}(E)$, it follows that in the last inclusion equality must hold. This shows that $T = \int f d\nu$ where f is real-valued; by Definition 5 the proof is complete.

Remark. Instead of using the method of reduction adopted above, we could have proved an analogous theorem utilizing the Cayley transform $\lambda \to (\lambda - i)/(\lambda + i)$ to characterize real spectral operators. (This method is the original one, due to von Neumann (see [17]).) The result would have been that a closed linear map T is a real spectral operator if and only if its Cayley transform $(T - iI)(T + iI)^{-1}$ is pseudo-unitary (Section 4, p. 158).

Examples

1. The theorem clearly includes the case of arbitrary self-adjoint operators in Hilbert space, after which it is shaped. Since every normal operator in Hilbert space has a canonical decomposition $T = T_1 + iT_2$ where T_1, T_2 are its self-adjoint real components, which commute, the spectral representation of normal operators in Hilbert space is contained in the combination of Theorem 10 and Proposition 23. Every spectral operator in Hilbert space is similar to a normal operator (cf. Example 1, following Proposition 18). Regarding the unicity of the representation $T \sim (\mathbf{C}, \tilde{1}, \nu)$, cf. the remark after Theorem 9.

2. If λ is a perfect space (Example 2, following Proposition 18), then every diagonal matrix with arbitrary complex (real) entries defines a spectral (real spectral) operator; the property of being spectral is, of course, invariant under similarity. We should like to point out, however, a contrast of the case of operators on a Banach space with the general situation. Every operator T on a Banach space with $\sigma(T)$ unbounded is necessarily discontinuous (hence, when closed, not defined everywhere). If we denote by ω and Ω, respectively, the space of all complex functions on a countable respectively uncountable set, equipped with the product topology, then every diagonal matrix defines a continuous endomorphism T of ω (or Ω). If A is any non-empty subset of $\mathbf{C}$, then clearly we may choose T such that the point spectrum of T is dense in A (for ω) respectively contains A (for Ω).

3. Let E denote a weakly semi-complete space with absolute basis $\{x_n\}$; denote by $[x_n, x'_m]$ the corresponding biorthogonal system in $E \times E'$. If $\{\lambda_n\}$ is an arbitrary sequence of complex (real) numbers, then

$$x \to Tx = \sum_{1}^{\infty} \lambda_n \langle x, x'_n \rangle x_n,$$

defined on the set D_T for whose elements the series converges, is a spectral (real spectral) operator on E.

A spectral operator T in E with real spectrum will, in accordance with the case where $\sigma(T)$ is compact, be called positive if $\sigma(T) \subset \mathbf{R}^+$; it is clear that if T is a real spectral operator with $\sigma(T)$ bounded below, then $\alpha I + T$ is positive for suitable $\alpha \geqslant 0$. (If $\sigma(T)$ is bounded above, then $\alpha I - T$ is positive for some $\alpha \geqslant 0$.) Thus semi-bounded self-adjoint operators are special cases of spectral operators with semi-bounded real spectrum; they are distinguished by the fact that their associated spectral measures have values that are orthogonal projections in Hilbert space. Positive spectral operators are characterized by a condition which is simpler than that of Theorem 10.

PROPOSITION 24. *A closed operator T in E with dense domain D_T is a positive spectral operator if and only if $(I+T)^{-1}$ exists as a member of $\mathfrak{L}(E)$, and $0 \leqslant (I+T)^{-1} \leqslant I$ for a suitable order on $\mathfrak{L}(E)$.*

Proof. If $T = \int t d\nu(t)$ where $\sigma(T) \subset \mathbf{R}^+$, then $(I+T)^{-1} \subseteq \int d\nu(t)/(1+t)$ since the support $S(\nu) = \sigma(T)$ by Theorem 9. Since $\int (1+t)^{-1} d\nu(t) \in \mathfrak{L}(E)$ and T, hence $(I+T)^{-1}$, is closed, equality must hold. By Proposition 7, there exists an order on $\mathfrak{L}(E)$ for which ν is positive; hence (since $S(\nu) \subset \mathbf{R}^+) 0 \leqslant (I+T)^{-1} \leqslant I$ for this ordering. Thus the condition is necessary.

To prove its sufficiency, we note that $(I+T)^{-1} = \int t d\nu(t)$ by Proposition 13, and $S(\nu) \subset [0,1]$. Let $f(t) = t^{-1}(1-t)$ for $t > 0$, $f(t) = 0$ for $t \leqslant 0$. Then clearly $T \subseteq \int f d\nu$ and again equality must hold since T is closed. By Definition 5, T is spectral and, since $f(\mathbf{R}) \subset \mathbf{R}^+$, it follows that $\sigma(T) \subset \mathbf{R}^+$ (Theorem 9, Proposition 6) which completes the proof.

COROLLARY. *Let E be an ordered locally convex space satisfying the assumptions of Theorem 7. Let T be a closed operator on E with dense domain D such that $D = D \cap K - D \cap K$. If $D \cap K$ is mapped into K by T and $(1, 1)$ onto K by $I+T$, then T is a positive spectral operator in E.*

Proof. It follows from the assumptions that $R = (I+T)^{-1}$ exists. Moreover, $RK \subset K$ hence R is weakly continuous in E. Thus if E is equipped with the weak topology, $R \in \mathfrak{L}(E)$ and, since $(I+T)x = x + Tx \geqslant x$ for all $x \in D \cap K$, $0 \leqslant R \leqslant I$ for the induced order on $\mathfrak{L}(E)$. It follows now from Proposition 24 that T is a positive spectral operator in E.

We note that if E' is weakly semi-complete (E being assumed as weakly semi-complete throughout this section), then by Theorem 8 the assertions of Proposition 19 remain in force for all spectral operators in E. For operators with non-empty resolvent set we obtain, in addition, the following result.

PROPOSITION 25. *Let T be a closed operator in E with dense domain and $\varrho(T) \neq \phi$. T is spectral if and only if for each $\lambda \in \varrho(T)$, $(\lambda - T)^{-1}$ is spectral with compact spectrum. Moreover, if T is spectral, then T has no closed proper extension which is a spectral operator.*

Proof. If $T \sim (\mathbf{C}, \tilde{\mathbf{I}}, \nu)$ then clearly $(\lambda - T)^{-1} = \int (\lambda - \xi)^{-1} d\nu(\xi)$ for each $\lambda \in \varrho(T)$ so that $(\lambda - T)^{-1}$ is spectral (Definition 4). Conversely, if $(\lambda_0 - T)^{-1} = \int \xi d\mu(\xi)$ for some $\lambda_0 \in \varrho(T)$, one obviously has

$$\lambda_0 - T \subseteq \int_{\xi \neq \lambda_0} \frac{d\mu(\xi)}{\lambda_0 - \xi}$$

(note that $\mu\{\lambda_0\} = 0$ since $\lambda_0 - T$ is one-to-one). Here equality must hold since T is closed. Thus T is a spectral operator (Definition 5),

$$T = \int f d\mu,$$

where $f(\xi) = 0$ for $\xi = \lambda_0$, $f(\xi) = \lambda_0 + (\xi - \lambda_0)^{-1}$ for $\xi \neq \lambda_0$. Let $U \supseteq T$ where U is a spectral operator. Let $\lambda_0 \in \varrho(T)$. If $\lambda_0 \in \varrho(U)$, then it is clear that $U = T$ since U is closed. In fact if $U \neq T$, the only remaining possibility is that every $\lambda \in \varrho(T)$ is in the point spectrum of U. Hence, since U is assumed to be spectral, it follows (cf. Proposition 19) that λ_0 is in the point spectrum of U'. Since $U' \subseteq T'$ by Lemma 4, λ_0 is in the point spectrum of T' which clearly contradicts $\lambda_0 \in \varrho(T)$. Hence $U = T$.

It is also clear that if T is spectral with non-empty resolvent set, no proper contraction of T can be a spectral operator.

References

[1]. BADE, W. G., Unbounded spectral operators. *Pacific J. Math.*, 4 (1954), 373–392.

[2]. —— Weak and strong limits of spectral operators. *Ibidem*, 393–413.

[3]. BISHOP, E., Spectral theory for operators on a Banach space. *Trans. Amer. Math. Soc.*, 86 (1957), 414–445.

[4]. BOURBAKI, N., *Espaces vectoriels topologiques*. Chaps. I–II, III–IV. *Actualités scientifiques et industrielles*, 1189–1229. Paris 1953, 1955.

[5]. —— *Integration*. Chaps. I–IV, V, VI. *Actualités scientifiques et industrielles*, 1175, 1244, 1281. Paris 1952, 1956, 1959.

[6]. DUNFORD, N., A survey of the theory of spectral operators. *Bull. Amer. Math. Soc.*, 64 (1958), 217–274.

[7]. —— Spectral operators. *Pacific J. Math.*, 4 (1954), 321–354.

[8]. DUNFORD, N. and SCHWARTZ, J., *Linear Operators*. Part I, *General Theory*. New York, 1958. Part II, *Spectral Theory* (to appear).

[9]. FREUDENTHAL, H., Teilweise geordnete Moduln. *Proc. Akad. Wetensch.* 39 (1936), 641–651.

[10]. KARLIN, S., Bases in Banach spaces. *Duke Math.* J., 15 (1948), 971–985.

[11]. KÖTHE, G., *Topologische lineare Räume*. Berlin–Göttingen–Heidelberg, 1960.

[12]. LORCH, E. R., On a calculus of operators in reflexive vector spaces. *Trans. Amer. Math. Soc.*, 45 (1939), 217–234.

[13]. —— The integral representation of weakly almost periodic transformations in reflexive vector spaces. *Trans. Amer. Math. Soc.*, 49 (1941), 18–40.

[14]. NAMIOKA, I., Partially ordered linear topological spaces. *Memoirs Amer. Math. Soc.*, no. 24 (1957).

[15]. NAKANO, H., *Semi-Ordered Linear Spaces*. (Tokyo 1955).

[16]. NEUBAUER, G., *Zur Spektraltheorie in lokalkonvexen Algebren*. Diss., Heidelberg, 1960.

[17]. NEUMANN, J. VON, Allgemeine Eigenwerttheorie hermitischer Funktionaloperatoren. *Math. Ann.*, 102 (1929), 49–131.

[18]. SCHAEFER, H., Halbgeordnete lokalkonvexe Vektorräume. *Math. Ann.*, 135 (1958), 115–141.
[19]. —— Halbgeordnete lokalkonvexe Vektorräume. II. *Math. Ann.*, 138 (1959), 259–286.
[20]. —— Halbgeordnete lokalkonvexe Vektorräume. III. *Math. Ann.*, 141 (1960), 113–142.
[21]. —— A New class of spectral operators. *Bull. Amer. Math. Soc.*, 67 (1961), 154–155.
[22]. STONE, M. H., A general theory of spectra. I, II. *Proc. Nat. Acad. Sci. U.S.A.*, 26 (1940), 280–283; 27 (1941), 83–87.
[23]. SZ.-NAGY, B., *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes.* Berlin, 1942.
[24]. TAYLOR, A. E., Spectral theory of closed distributive operators. *Acta Math.*, 84 (1950), 189–224.
[25]. TULCEA, I., Spectral operators on locally convex spaces. *Bull. Amer. Math. Soc.*, 67 (1961), 125–128.
[26]. WAELBROECK, L., Le calcul symbolique dans les algebres commutatives. *J. Math. Pures Appl.*, 33 (1954), 147–186.
[27]. WERMER, J., Commuting spectral measures on Hilbert space. *Pacific J. Math.*, 4 (1954), 355–361.

Receieed May 25, 1961