APPLICATIONS OF ALMOST PERIODIC COMPACTIFICATIONS

BY

K. DELEEUW and I. GLICKSBERG

Stanford University, University of Notre Dame, U.S.A. (1)

1. Introduction

The theory of almost periodic functions on groups can be completely reduced to the study of continuous functions on compact topological groups by the introduction of the almost periodic compactification (see [1] or [14]). There are many possible constructions of the compactification; one of these is the following. If A is the space of almost periodic functions on a group G, and $\mathcal{B}(A)$ is the space of bounded linear operators on A, the compactification can be taken to be the closure in $\mathcal{B}(A)$, in the strong operator topology, of the group of right translates of A by elements of G.

This type of construction is of a very general nature and is peculiar neither to the strong operator topology nor to groups of operators. The purpose of this paper is to exhibit some extensions of this construction and applications of the resulting compactifications.

For example, in the above construction, if A is taken to be the space of weakly almost periodic functions (in the sense of [7]) on G, the closure in the weak operator topology of the right translates of G on A yields a compactification that is in general no longer a group but is a compact semigroup in which multiplication is separately continuous. This allows us to reduce, in a manner completely analogous to the almost periodic case, the theory of weakly almost periodic functions on groups to the study of continuous functions on such compact semigroups. As a consequence of the ideal structure for these semigroups (cf. Section 2), we indicate in Section 5 how the Eberlein theory of weakly almost periodic functions on locally compact abelian groups can be extended to a large class of groups and semigroups. In particular we show when and how a mean, and thus the possibility of Fourier analysis, arises for weakly almost periodic functions.

(1) Work supported in part by the United States Air Force Office of Scientific Research.

The first compactification that we study occurs in a more general situation. If S is a semigroup of operators on a Banach space B, and each element of B has weakly conditionally compact (1) orbit, the closure $\bar{S}$ of S in the weak operator topology will be a compact semigroup with separately continuous multiplication. Knowledge of the ideal structure of such semigroups allows us in Section 4 to extend the results of Jacobs in [12] and [13]; the simplest of these results is the following. If S is $\{T^n : n = 0, 1, \ldots\}$ then B is the direct sum of the closed linear subspace spanned by eigenvectors of T having eigenvalues of modulus 1 and a closed invariant linear subspace formed by all elements having 0 in the weak closure of their orbit.

In Section 6 the analogues of the results of Section 5 for almost periodic functions are indicated. Section 7 is concerned with applications to ergodic theory and Section 8 with other applications.

The results in this paper were announced in part in [6].

2. Structure of Compact Semigroups

This section is devoted to establishing the basic facts concerning topological semigroups that will be applied in what follows.

A *semigroup* is a set supplied with an associative binary composition that will be referred to as multiplication. If S is a semigroup that is at the same time a topological space, multiplication in S is said to be *separately continuous*, if for each σ in S the maps $\tau \to \sigma\tau$ and $\tau \to \tau\sigma$ of S into itself are continuous. The multiplication in S is said to be *jointly continuous* if the map $(\sigma, \tau) \to \sigma\tau$ of $S \times S$ into S is continuous.

An *identity* element of a semigroup S is an element e that satisfies $\sigma e = e\sigma = \sigma$ for all σ in S. A *topological semigroup* is a semigroup with identity which is a Hausdorff topological space in which the multiplication is separately continuous. There is considerable literature on topological semigroups in which the multiplication is assumed to be jointly continuous; see in particular [18]. It is necessary for us to consider semigroups satisfying the weaker hypothesis of separate continuity, since multiplication of operators on a Banach space is only separately continuous in the weak operator topology.

A *topological group* is a group supplied with a Hausdorff topology in which multiplication is jointly continuous and furthermore inversion is continuous, i.e., the map $\sigma \to \sigma^{-1}$ is continuous. A topological semigroup that is also a group need not be a

(1) We shall call a set conditionally compact if it has compact closure.

topological group since the multiplication may fail to be jointly continuous. Nevertheless we have the following theorem of Ellis (it is a special case of the main result of [9]) which is basic to our work in this paper. (We present a quite different proof of this theorem in the appendix based on a criterion for weak compactness in $C(X)$ due to Grothendieck.)

THEOREM 2.1. *A compact topological semigroup that is a group must be a topological group.*

If S is a semigroup with subsets D and E, we shall use the standard notation

$$DE = \{\sigma\tau : \sigma \in D,\ \tau \in E\}$$
$$\sigma E = \{\sigma\tau : \tau \in E\}$$
$$D\tau = \{\sigma\tau : \sigma \in D\}.$$

A non-empty subset D of S is called a *subsemigroup* of S if $DD \subset D$; D is called a *left ideal* if $SD \subset D$, a right ideal if $DS \subset D$, and a *two-sided ideal* if it is both. *A minimal left ideal* of S is a left ideal of S containing no other left ideal of S. Minimal right is defined similarly. We shall denote by $\mathfrak{L}(S)$ and $\mathfrak{R}(S)$ respectively the collections of all minimal left and all minimal right ideals of S. In general these may be empty collections, but if S is a compact topological semigroup minimal left and minimal right ideals exist.

LEMMA 2.2. *Let S be a compact topological semigroup. Then each left ideal of S contains at least one minimal left ideal of S and each minimal left ideal is closed. The same holds for right ideals.*

Proof. We shall prove only the assertion concerning left ideals. Let I be any left ideal of S and let Q be the collection of all closed left ideals of S contained in the given left ideal I. Q is a partially ordered set under the ordering of inclusion and is non-void since if $\sigma \in I$, $S\sigma$ is a closed left ideal contained in I. Let Q' be a subcollection of Q that is linearly ordered. Then $\bigcap_{J \in Q'} J$ is non-empty by the compactness of S and so is an ideal in Q that is contained in each J in Q'. Thus each linearly ordered subset of Q has a lower bound, and Zorn's lemma assures the existence of a minimal J_0 in Q. We shall show that J_0 is actually a minimal left ideal. Let J_1 be a left ideal contained in J_0 and let σ be an element in J_1. Then $S\sigma$ is a closed left ideal of S. Furthermore, $S\sigma \subset J_1 \subset J_0$ and since J_0 was minimal in Q, $S\sigma = J_0$ so J_1 must be J_0. Thus J_0 is a minimal left deal. It remains to show that any minimal left ideal J of S is closed. If σ is in J, then $S\sigma$ is a closed left ideal contained in J, which by the minimality of J must equal J. This completes the proof of Lemma 2.2.

If S is a semigroup, the intersection of all the two-sided ideals of S is called the *kernel* of S and denoted by $K(S)$. If $K(S)$ is non-empty, it is clearly the smallest two-sided ideal of S. The algebraic structure of $K(S)$ is known (see [3]) in the case that S has minimal right and minimal left ideals, and thus by Lemma 2.2 if S is a compact topological semigroup. Structure Theorem 2.3 and its corollaries which are established below are the basic results on topological semigroups for the applications made in the following sections. Almost all of Theorem 2.3 is contained in [3]; we include a proof for the sake of completeness.

An element e of a semigroup is called an *idempotent* if $ee=e$. If D is a subset of a semigroup, we shall denote by $\mathcal{E}(D)$ the set of idempotents in D.

THEOREM 2.3 (Structure theorem for the kernel.) *Let S be a compact topological semigroup. Then $K(S)$ is non-empty and*

(i) $\mathfrak{L}(S)=\{Se: e\in\mathcal{E}(K(S))\}$ *and* $\mathfrak{R}(S)=\{eS: e\in\mathcal{E}(K(S))\}$.

(ii) *If J_1 and J_2 are both in $\mathfrak{L}(S)$ or both in $\mathfrak{R}(S)$, and $J_1\cap J_2$ is non-empty then $J_1=J_2$.*

(iii) *If J is in $\mathfrak{L}(S)$, then $J\sigma=J$ for all σ in J. If J is in $\mathfrak{R}(S)$, then $\sigma J=J$ for all σ in J.*

(iv) $K(S)=\bigcup_{I\in\mathfrak{L}(S)} I=\bigcup_{I\in\mathfrak{R}(S)} I$.

(v) *If J_1 is in $\mathfrak{L}(S)$ and J_2 is in $\mathfrak{R}(S)$, then $J_1\cap J_2$ contains a unique idempotent. If that idempotent is e, $J_1\cap J_2=eSe$, and with e as identity $J_1\cap J_2$ is a compact topological group.*

Proof. (ii) and (iii). If J_1 and J_2 are in $\mathfrak{L}(S)$, $J_1\cap J_2$ is a left ideal, so $J_1=J_1\cap J_2=J_2$. If σ is in J_1, $J_1\sigma$ is a left ideal contained in J_1, so $J_1\sigma=J_1$. The same argument works for right ideals

(iv) If I is in $\mathfrak{L}(S)$ and τ is in S, then $I\tau$ is also in $\mathfrak{L}(S)$. For if J were a left ideal properly contained in $I\tau$, $I\cap\{\sigma:\sigma\tau\in J\}$ would be a left ideal properly contained in I. Thus $\bigcup_{\tau\in S} I\tau$ is a union of ideals in $\mathfrak{L}(S)$ and is a two-sided ideal. If I_1 is any two-sided ideal of S, and I is in $\mathfrak{L}(S)$, then $I=I_1I\subset I_1$, so I_1 contains $\bigcup_{\tau\in S} I\tau$ which must by definition be the kernel $K(S)$. Also any I_2 in $\mathfrak{L}(S)$ must be contained in $K(S)$, since $K(S)$ is a two-sided ideal, so by (ii), I_2 must be one of the $I\tau$. Thus $K(S)=\bigcup_{I\in\mathfrak{L}(S)} I$. The same argument applies to right ideals.

(i) and (v). By (ii) and (iv), we have the disjoint union

$$K(S) = \bigcup_{\substack{I \in \mathcal{L}(S) \\ J \in \mathcal{R}(S)}} I \cap J.$$

Choose $I \in \mathcal{L}(S)$, $J \in \mathcal{R}(S)$. Then $I \cap J$ contains JI so is non-empty, and if $\sigma \in I$, $\tau \in J$, then

$$(I \cap J)\sigma = I \cap J, \quad \tau(I \cap J) = I \cap J. \tag{2.1}$$

For it is clear that $(I \cap J)\sigma \subset I \cap J$, and if the inclusion were proper for some $J \in \mathcal{R}(S)$, then

$$I\sigma = \bigcup_{J \in \mathcal{R}(S)} (I \cap J)\sigma \subsetneqq \bigcup_{J \in \mathcal{R}(S)} (I \cap J) = I = I\sigma$$

is a contradiction. The second equality in (2.1) follows similarly. We shall now use (2.1) to show that $I \cap J$ is a group. It is clearly a subsemigroup so it suffices to show that it has a left identity and left inverses. If $\sigma \in I \cap J$, by (2.1) there is an element e in $I \cap J$ with $e\sigma = \sigma$, and also by (2.1), $\sigma(I \cap J) = I \cap J$, so e is a left identity. Since $(I \cap J)\sigma = I \cap J$ for each σ in $I \cap J$ by (2.1), left inverses exist. Thus $I \cap J$ is a group with e as identity element. Clearly $I = Se$ and $J = eS$ so (i) holds. Also

$$I \cap J = Se \cap eS \supset eSe = eI \supset e(I \cap J) = I \cap J$$

so $I \cap J = eSe$. That $I \cap J$ is a compact topological group now follows from Lemma 2.2 and Theorem 2.1.

More detailed information concerning the structure of $K(S)$ can be found in [19].

COROLLARY 2.4. *Let S be a compact topological semigroup. Then the following are equivalent:*

(i) *S has a unique minimal left (resp. right) ideal J.*

(ii) *$e_1 e_2 = e_1$, (resp. $e_1 e_2 = e_2$) for all e_1 and e_2 in $\mathcal{E}(K(S))$.*

If (i) *and* (ii) *hold, then*

$$J = K(S) = \bigcup_{e \in \mathcal{E}(K(S))} eS \quad (\text{or} = \bigcup_{e \in \mathcal{E}(K(S))} Se),$$

where the eS (resp. Se) are disjoint minimal right (resp. left) ideals of S that are compact topological groups.

Proof. (i) implies (ii). If S has a unique minimal left ideal J, by Theorem 2.3,

$$J = K(S) = \bigcup_{e \in \mathcal{E}(K(S))} eS,$$

where each eS has the properties claimed. If e_1 and e_2 are in $\mathcal{E}(K(S))$, $e_1 e_2 \in e_1 S$.

e_1 is the identity of the group $e_1 S$ and thus commutes with $e_1 e_2$ so $(e_1 e_2)(e_1 e_2) = e_1(e_1 e_2) e_2 = e_1 e_2$, which must be e_1 since a group contains a unique idempotent.

(ii) implies (i). By Theorem 2.3 each minimal left ideal of S contains some e in $\mathcal{E}(K(S))$, and the minimal left ideals are disjoint. But if (ii) holds a left ideal containing one element of $\mathcal{E}(K(S))$ contains all of $\mathcal{E}(K(S))$. The proofs of the parenthetical insertions are completely analogous.

Corollary 2.5. *Let S be a compact topological semigroup that has a unique minimal left ideal J_1, and a unique minimal right ideal J_2 (if S is a commutative, this must always be the case). Then $J_1 = J_2 = K(S)$ which is a compact topological group.*

Proof. The only point needing proof is the parenthetical insertion. If S is commutative, and J_1 and J_2 are minimal ideals, $J_1 \cap J_2$ is non-empty since it contains $J_1 J_2$. Thus by (ii) of Theorem 2.3, $J_1 = J_2$, so S has a unique minimal ideal.

Corollary 2.6. *Let S be a topological semigroup, S' a subsemigroup that is compact. Then S' contains at least one idempotent. If S is a group S' is a subgroup.*

Proof. Let $S'' = S' \cup \{e\}$, where e is the identity element of S. S'' is a compact topological semigroup. S' is a left ideal of S'', which by Lemma 2.2 contains a minimal left ideal, which by Theorem 2.3 contains an idempotent. Suppose now that S is a group with identity e. By the first part of the corollary S' contains an idempotent which must be e. To show that S' is a group it suffices to show that $x S' = S'$ for all x in S'. By the first part of the corollary applied to $x S'$, there is an idempotent in $x S'$. Thus e is in $x S'$ and $S' = e S' \subset x S'$. Since $x \in S'$, $x S' \subset S'$ and $x S' = S'$ follows.

If S is a topological semigroup, we shall denote by $C(S)$ the Banach space of all complex valued bounded continuous functions on S, supplied the norm defined by

$$\|f\| = \sup_{\sigma \in S} |f(\sigma)|.$$

For each σ in S the translation maps R_σ and L_σ of $C(S)$ into itself are defined by

$$R_\sigma f(\tau) = f(\tau \sigma), \quad L_\sigma f(\tau) = f(\sigma \tau), \quad \text{all } \tau \text{ in } S.$$

Our next result will be of fundamental importance in the following sections. In part it is a consequence of the result of Grothendieck [10] that weak compactness and compactness in the topology of pointwise convergence agree on bounded subsets of a $C(X)$ for X compact.

THEOREM 2.7. *Let S be a topological semigroup, f a function in $C(S)$, and $O(f)=\{R_\sigma f : \sigma \in S\}$. Then*

(i) *If S is compact, $O(f)$ is weakly compact in $C(S)$.*

(ii) *If S is compact and its multiplication is jointly continuous, $O(f)$ is strongly compact in $C(S)$.*

(iii) *If $O(f)$ is strongly (resp. weakly) conditionally compact then $\sigma \to R_\sigma f$ is strongly (resp. weakly) continuous.*

Proof. (i) Separate continuity of multiplication insures the continuity of $\sigma \to R_\sigma f$ when $C(S)$ is taken in the topology of pointwise convergence. Thus $O(f)$ is compact in this topology as the continuous image of our compact S, and by the result of Grothendieck mentioned above, $O(f)$ is weakly compact.

(ii) We shall show first that $\sigma \to R_\sigma f$ is strongly continuous. The function F defined by $F(\tau,\sigma)=f(\tau\sigma)$ is continuous so that, for a fixed $\sigma_0 \in S$, for each $\tau \in S$ there is a neighborhood $V_\tau \times W_\tau$ of (τ, σ_0) in $S \times S$ on which F varies by less than ε. Covering S by finitely many V_τ, say $V_{\tau_1}, \dots, V_{\tau_n}$, we see that if σ is in $W=\bigcap_{i=1}^{n} W_{\tau_i}$, then

$$|f(\tau\sigma)-f(\tau\sigma_0)|<\varepsilon, \quad \text{all } \tau \text{ in } S.$$

This is precisely the continuity desired at σ_0. Thus $O(f)$ is strongly compact as the strongly continuous image of S.

(iii) (strong case). Let $\{\sigma_\gamma\}$ be a net in S converging to σ. Then $R_{\sigma_\gamma} f \to R_\sigma f$ pointwise, so the net $\{R_{\sigma_\gamma} f\}$ has at most the one strongly adherent point $R_\sigma f$. By the compactness of the closure of $O(f)$ this net must converge to $R_\sigma f$, and continuity follows. The same proof applies to the weak case.

Let D be a linear subspace of $C(S)$. Then a *mean* on D is an element m in the adjoint D^* of D which satisfies

$$\langle 1, m\rangle = 1, \qquad \langle f, m\rangle \geqslant 0 \text{ for } f \geqslant 0.$$

If D is invariant under right translation, a mean m on D is said to be *right invariant* if

$$\langle R_\sigma f, m\rangle = \langle f, m\rangle, \text{ all } f \text{ in } D,\ \sigma \text{ in } S.$$

Similarly one defines a left invariant mean via the L_σ; m is *invariant* if it (and D) are both right and left invariant.

If S is commutative or is a solvable group, $C(S)$ has an invariant mean (see [5] for this and further sufficient conditions).

For compact topological semigroups with jointly continuous multiplication the following result is in [17].

LEMMA 2.8. *Let S be a compact topological semigroup. Then the following are equivalent:*

(i) *S has a unique minimal left ideal.*

(ii) *$C(S)$ has a right invariant mean.*

The corresponding result holds for right ideals and left invariant means.

Proof. Assume that (i) is false. Let J_1 and J_2 be two distinct minimal left ideals of S. They are closed by Lemma 2.2 and disjoint by (ii) of Theorem 2.3. Let f in $C(S)$ satisfy

$$f(\sigma)=\begin{cases}0, & \sigma \in J_1\\ 1, & \sigma \in J_2.\end{cases}$$

Then for τ_1 in J_1 and τ_2 in J_2, $R_{\tau_1} f \equiv 0$ and $R_{\tau_2} f \equiv 1$, so a right invariant mean on $C(S)$ cannot exist. For the converse, assume that (i) is true. Then by Corollary 2.4, $K(S)$ is a union of compact topological groups that are right ideals. Normalized Haar measure on any one of these will be a right invariant mean for $C(S)$. This establishes the equivalence of (i) and (ii). The proof of the last assertion is completely analogous.

COROLLARY 2.9. *Let S be a compact topological semigroup. Then the following are equivalent:*

(i) *$K(S)$ is a compact topological group.*

(ii) *$C(S)$ has an invariant mean.*

(iii) *$C(S)$ has a right invariant mean and a left invariant mean.*

When these conditions hold, the invariant mean is unique and can be identified as the Haar integral over $K(S)$.

Proof. If $K(S)$ is a compact group, its normalized Haar measure provides an invariant mean since $K(S)$ is a two-sided ideal. Thus (i) implies (ii). Clearly (ii) implies (iii). If (iii) holds, by Lemma 2.8 S has a unique minimal left ideal and a unique minimal right ideal. Consequently $K(S)$ is a compact topological group by Corollary 2.5. Finally suppose m is an invariant mean. Then $\langle f, m\rangle = \int f\, d\mu$, where μ is a regular Borel measure on S. If μ is not supported by the group $K(S)$, there is a compact set $E \subset S$ disjoint from $K(S)$ with $\mu(E)>0$, and $\mu(K(S))<1$. Thus if the real valued function f in $C(S)$ has the constant value 1 on $K(S)$ and 0 on E, while $f \leqslant 1$ elsewhere, we have $\langle f, m \rangle = \int f \, d\mu < 1$. But for σ in $K(S)$, $R_\sigma f = 1$ and thus $1 = \langle 1, m \rangle = \langle R_\sigma f, m \rangle = \langle f, m \rangle < 1$, contradicting our assumption that μ is not supported by $K(S)$. Thus μ is supported by the group $K(S)$; trivially it is an invariant measure, and thus coincides with Haar measure.

LEMMA 2.10. *Let S and S' be topological semigroups with S' compact, and $\varphi: S \to S'$ be a continuous homomorphism, with $\varphi(S)$ dense in S'. Let $\tilde{\varphi}: C(S') \to C(S)$ be the dual map taking f into $f \circ \varphi$. Then $C(S')$ has a right (resp. left, two-sided) invariant mean if and only if $\tilde{\varphi}(C(S'))$ has a right (resp. left, two-sided) invariant mean.*

Proof. Since $\varphi(S)$ is dense, $\tilde{\varphi}$ is an isometry and $\tilde{\varphi}(C(S'))$ is a closed subspace of $C(S)$. Moreover, it is trivial that for f in $C(S')$,

$$R_\sigma(\tilde{\varphi} f) = \tilde{\varphi}(R_{\varphi(\sigma)} f),$$
$$L_\sigma(\tilde{\varphi} f) = \tilde{\varphi}(L_{\varphi(\sigma)} f) \quad \text{all } \sigma \in S.$$

Thus $\tilde{\varphi}(C(S'))$ is invariant. If m' is a right invariant mean on $C(S')$, and m is defined by

$$\langle \tilde{\varphi} f, m \rangle = \langle f, m' \rangle, \quad \text{all } f \text{ in } C(S'), \tag{2.2}$$

m is a right invariant mean on $\varphi(C(S'))$. For

$$\langle R_\sigma(\tilde{\varphi} f), m \rangle = \langle \tilde{\varphi}(R_{\varphi(\sigma)} f), m \rangle = \langle R_{\varphi(\sigma)} f, m' \rangle = \langle f, m' \rangle = \langle \tilde{\varphi} f, m \rangle, \quad \text{all } \sigma \in S.$$

On the other hand, if m is a right invariant mean on $\tilde{\varphi}(C(S'))$, we can define a mean m' on $C(S')$ by (2.2). m' satisfies

$$\langle R_{\varphi(\sigma)} f, m' \rangle = \langle \tilde{\varphi}(R_{\varphi(\sigma)} f), m \rangle = \langle R_\sigma \tilde{\varphi} f, m \rangle = \langle \tilde{\varphi} f, m \rangle = \langle f, m' \rangle, \quad \text{all } \sigma \in S.$$

Thus m' is invariant under the right translations produced by the dense subsemigroup $\varphi(S)$ of S'. That m' is right invariant on $C(S')$ now follows since $\tau \to R_\tau f$ is weakly continuous on S' by (i) and (iii) of Theorem 2.7. Similar proofs apply for left invariant means.

3. Compactifications of Semigroups of Operators

If B is a Banach space, we shall denote by $\mathcal{B}(B)$ the usual Banach algebra of bounded linear operators on B. The *weak operator topology* on $\mathcal{B}(B)$ is the weakest topology rendering all of the maps

$$T \to \langle T x, y \rangle, \quad x \in B, \ y \in B^*,$$

continuous, where $\langle \cdot, \cdot \rangle$ is the pairing between B and B^*. $\mathfrak{B}(B)$ is a topological semigroup under operator multiplication and the weak operator topology. For $U \to UV$ is clearly continuous and the continuity of $V \to UV$ follows from the identity $\langle UVx, y\rangle = \langle Vx, U^*y\rangle$. We shall speak of any subsemigroup of $\mathfrak{B}(B)$ containing the identity operator as a *semigroup of operators*.

If S is a semigroup of operators on B, the *orbit* $O(x)$ of an element x of B is defined to be $\{Tx : T \in S\}$. S will be called *almost periodic* if each orbit has compact closure in the norm topology, and *weakly almost periodic* if each orbit has compact closure in the weak topology of B. For such semigroups S each orbit is bounded, so by the uniform boundedness theorem S is uniformly bounded, i.e., there is a constant M so that $\|T\| \leqslant M$ for all T in S.

If S is any semigroup of operators on B, we shall denote by $\bar{S}$ the closure of S in $\mathfrak{B}(B)$ in the weak operator topology. The following allows us to apply the results of Section 2 to the study of weakly almost periodic semigroups.

THEOREM 3.1. *Let S be a weakly almost periodic semigroup of operators. Then $\bar{S}$ is a compact topological semigroup under the weak operator topology.*

Proof. Since multiplication in $\mathfrak{B}(B)$ is separately continuous in the weak operator topology, the closure $\bar{S}$ of the semigroup S will be closed under multiplication and thus a topological semigroup. It remains to prove that it is compact. For each x in B, we shall denote by $O(x)^-$ the compact topological space formed by the closure of the orbit $O(x)$ in the weak topology. For each T in $\bar{S}$ and x in B, Tx is in $O(x)^-$. Let $\varrho : \bar{S} \to \prod_{x \in B} O(x)^-$ be induced by the maps $T \to Tx$. ϱ is 1–1 and is a homeomorphism because of the definitions of the weak topologies. Since by the Tychonoff theorem, $\prod_{x \in B} O(x)^-$ is compact, to show that $\bar{S}$ is compact it suffices to show that $\varrho(\bar{S})$ is closed, or equivalently, that each point in the closure of $\varrho(S)$ must be in $\varrho(\bar{S})$. Let $\{z_x\}_{x \in B}$ be a point in the closure of $\varrho(S)$. If $V : B \to B$ is defined by $Vx = z_x$, all $x \in B$, it is clear that V is in $\bar{S}$. Thus $\{z_x\}_{x \in B} = \varrho(V)$, and the proof is complete.

If S is actually almost periodic we can replace the weak topology and the weak operator topology in the above argument by the strong topology and the strong operator topology, respectively, to conclude that the strong operator closure of S is compact. Since it must remain compact in the weak operator topology it coincides with $\bar{S}$, on which both weak and strong operator topologies must coincide by compactness. As a consequence multiplication in $\bar{S}$ is jointly continuous. This follows since multiplication on bounded subsets of $\mathcal{B}(B)$ is jointly continuous in the strong operator topology by virtue of

$$\|UVx - U_0V_0x\| \leqslant \|UVx - UV_0x\| + \|UV_0x - U_0V_0x\| \leqslant \|U\|\,\|Vx - V_0x\| + \\ + \|(U - U_0)V_0x\|.$$

THEOREM 3.2. *Let S be an almost periodic semigroup of operators. Then the strong and weak operator topologies agree on $\bar{S}$, which is a compact topological semigroup in which multiplication is jointly continuous.*

Remark. In subsequent results our use of various crucial facts concerning the weak topology (as opposed, for example, to the weak* topology of B^*) will be quite apparent. It should, however, be pointed out that our construction of $\bar{S}$ cannot be imitated for "weak* almost periodic" semigroups on B^* (i.e., "uniformly bounded"). For although we have a weak* operator topology, the construction would fail (exactly) at the outset: $\mathcal{B}(B^*)$ is in general not a separately continuous semigroup.

4. Weakly Almost Periodic Semigroups of Operators

This section is devoted to an extension of the results of Jacobs in [12] and [13]. Throughout the section B is a fixed complex Banach space and S a fixed semigroup of operators on B which is weakly almost periodic in the sense of Section 3, i.e., each orbit $O(x) = \{Tx : T \in S\}$ is conditionally weakly compact. The weak operator closure $\bar{S}$ of S is, by Theorem 3.1, a compact topological semigroup in the weak operator topology, and the results of Section 2 applied to $\bar{S}$ will yield information concerning the action of S on B. In all of the following we shall consider S and $\bar{S}$ to be topologized with the weak operator topology.

We shall first define subsets B_r, B_0, and B_p of B introduced by Jacobs.

Recall that for each x in B, $O(x)^-$ is defined to be the weak closure of the orbit $\{Tx : T \in S\}$. Since $\bar{S}$ is compact in the weak operator topology, it is clear that $O(x)^- = \{Tx : T \in \bar{S}\}$.

DEFINITION OF B_r. *A point x of B is in B_r if for each y in $O(x)^-$, x is in $O(y)^-$ (or equivalently, $O(y)^- = O(x)^-$ for all y in $O(x)^-$).*

B_r is the set of reversible vectors in the sense of [12]. It is an $\bar{S}$-invariant subset of B but need not be a linear subspace.

LEMMA 4.1. *Let x be an element of B. Then the following are equivalent:*

(i) *x is in B_r.*

(ii) *For each U in $\bar{S}$ there is some V in $\bar{S}$ with $VUx=x$.*

(iii) *There is a projection E in the kernel $K(\bar{S})$ with $Ex=x$.*

Proof. The equivalence of (i) and (ii) is clear since $O(x)^- = \{Tx : T \in \bar{S}\}$. If (ii) holds,

$$\{V : V \in \bar{S},\ Vx=x\} \cap K(\bar{S}) \tag{4.1}$$

is non-empty since $K(\bar{S})$ is a left ideal. (4.1) is a compact subsemigroup of $\bar{S}$ and thus by Corollary 2.6 contains an idempotent. Thus (ii) implies (iii). Suppose now that E is a projection in $K(\bar{S})$ and $Ex=x$. $J=\{UE : U \in \bar{S}\}$ is a minimal left ideal of S by (i) of Theorem 2.3. If U is in $\bar{S}$, then UE is in J, so by (iii) of Theorem 2.3 there is a V in J with $VUE=E$. Then $VUx=VUEx=Ex=x$, so (iii) implies (ii).

Definition of B_0. *A point x of B is in B_0 if $O(x)^-$ contains* 0.

B_0 is the set of "Fluchtvectoren" in the sense of [12]. It is in general neither S-invariant nor a linear subspace.

Lemma 4.2. *Let x be an element of B. Then the following are equivalent.*

(i) *x is in B_0.*

(ii) *$Ux=0$ for some U in $\bar{S}$.*

(iii) *$Ex=0$ for some projection E in $K(\bar{S})$.*

Proof. The equivalence of (i) and (ii) is clear since $O(x)^- = \{Tx : T \in \bar{S}\}$. (iii) trivially implies (ii). That (ii) implies (iii) follows since $\{U : U \in \bar{S},\ Ux=0\}$ is a left ideal of $\bar{S}$ and thus by Lemma 2.2 and Theorem 2.3 contains an idempotent in $K(\bar{S})$.

Suppose now that E is any projection in $K(\bar{S})$. Then by the preceding two lemmas, the direct sum decomposition $B=EB+(I-E)B$ has the first factor a subset of B_r and the second a subset of B_0. Since these may be proper subsets the decomposition seems to be without interest in general. However, we shall see in Theorem 4.11 that if there is a *unique* projection E in $K(\bar{S})$ (this will occur, for example, when S is commutative), $EB=B_r$, $(I-E)B=B_0$ and the elements of B_r are almost periodic in the sense of [12]. In order to define this type of almost periodicity we need first a preliminary definition.

If R is any set of linear operators on B, and D is an R-invariant subspace of B, we shall denote by $R|D$ the set of linear operators on D obtained by restricting the operators in R to D; i.e., $U:D\to D$ is in $R|D$ if and only if there is a V in R with $Ux=Vx$ for all x in D. A finite dimensional S-invariant subspace D of B will be called a *unitary subspace* of B if $S|D$ is contained in a *bounded* group of operators on D (with, of course, the identity of the group being the identity operator on D).

D is a unitary subspace if and only if it is possible to choose an inner product on D so that all of the operators in $S|D$ are unitary. This is a consequence of the following well-known fact (see [20], p. 70).

LEMMA 4.3. *Let D be a finite dimensional complex linear space and G a bounded group of operators on D whose identity is the identity operator. Then it is possible to choose an inner product in D so that all of the operators in G are unitary.*

The following property of unitary subspaces will be needed later.

LEMMA 4.4. *Let D be a unitary subspace of B. Then $D \subset B_r$.*

Proof. If $S|D$ is contained in the bounded group G, $\tilde{S}|D$ will be contained in the closure $\bar{G}$ of G which is a compact topological group. $\tilde{S}|D$ is a subgroup of $\bar{G}$ by Corollary 2.6, and thus by Lemma 4.1, $D \subset B_r$.

DEFINITION OF B_p. *B_p is the closed linear subspace of B generated by the unitary subspaces.*

B_p is the set of almost periodic vectors in the sense of [12].

We shall see next that there are simpler equivalent definitions of B_p in the cases where S is either a group or commutative.

LEMMA 4.5. *If S is a group whose identity is the identity operator on B, B_p is the closed linear subspace of B generated by the finite dimensional S-invariant subspaces of B.*

Proof. If D is any finite dimensional S-invariant subspace of B, $S|D$ is a group which is bounded since S is uniformly bounded on B. Then D is a unitary subspace and the lemma follows.

LEMMA 4.6. *If S is commutative, B_p is the closed linear subspace of B spanned by the common eigenvectors of S that have eigenvalues of modulus* 1, *i.e., by those x in B that satisfy*

$$Tx = \lambda_T x, \quad |\lambda_T| = 1$$

for all T in S.

Proof. Each common eigenvector of the type described spans a one-dimensional S-invariant subspace of B that is unitary. Thus all such eigenvectors are in B_p. For the converse, let D be a unitary subspace of B. There is an inner product in D relative to which all of the operators in $S|D$ are unitary. It is well known (see [20]) that any commuting family of unitary operators can be simultaneously diagonalized. Thus D is spanned by common eigenvectors of the type described. This completes the proof of the lemma.

The following Lemma 4.7 is the key result that allows us to identify B_r with B_p in the circumstances of Theorem 4.10 and 4.11, and it is at this point that we introduce results depending on Theorem 2.1.

First, some comments concerning functions on compact topological groups and weak vector valued integration are necessary.

Let G be a compact topological group with identity element e and normalized Haar measure μ. We shall use below the well-known fact (see [20]) that G has an approximate identity $\{\varphi_\gamma\}$ consisting of trigonometric polynomials, i.e., a net φ_γ of functions in $C(G)$ having the following properties.

(i) $\lim\limits_{\gamma} \int \varphi_\gamma f \, d\mu = f(e)$, all f in $C(G)$.

(ii) Each φ_γ is in some finite dimensional left invariant subspace of $C(G)$.

Let X be a compact Hausdorff space and μ a regular Borel measure on X. If $f: X \to B$ is *weakly continuous*, $\int_X f(t)\, d\mu(t)$ is defined to be the unique element z of B that satisfies

$$\langle z, y\rangle = \int_X \langle f(t), y\rangle \, d\mu(t)$$

for all y in B^*. The existence of such an element is guaranted (see [2]) by the fact that the weakly closed convex hull of a weakly compact subset of B is weakly compact (see [7], Theorem 1.2). We shall use below standard properties of this vector-valued integral discussed in [2].

The use of weak integration in this context was suggested to us by H. Mirkil.

LEMMA 4.7. *If $\bar{S}$ has a unique minimal right ideal and E is a projection in $K(\bar{S})$, $E(B) \subset B_p$.*

Proof. By Corollary 2.4, $K(\bar{S})$ is the unique minimal right ideal of $\bar{S}$ and

$$G = \{T E : T \in K(\bar{S})\} = \{T E : T \in \bar{S}\}$$

is a minimal left ideal and a compact topological group having identity E. Let φ_γ be an approximate identity on G consisting of trigonometric polynomials. For each γ, $T_\gamma : B \to B$ is defined by the vector-valued integrals

$$T_\gamma x = \int_G \varphi_\gamma(U)\, U x \, d\mu(U), \quad \text{all } x \text{ in } B.$$

For each x in B, $\lim_\gamma T_\gamma x = E x$ weakly, since

$$\langle T_\gamma x, y \rangle = \int_G \varphi_\gamma(U) \langle U x, y \rangle \, d\mu(U) \to \langle E x, y \rangle$$

for all y in B^*. B_p is by definition a strongly closed linear subspace of B, which is therefore weakly closed. Thus since for each x in B, $T_\gamma x \to E x$ weakly, to complete the proof of the lemma it suffices to show that each $T_\gamma x$ in B_p. Choose any element x in B. If F is any finite dimensional linear subspace of $C(G)$,

$$D_F = \left\{ \int_G f(U)\, U x \, d\mu(U) : f \in F \right\}$$

is a finite dimensional linear subspace of B; furthermore if F is left-invariant, D_F will be S-invariant. For if V is in S and $V_0 = VE$, $VU = V_0 U$ for all U in G, so for each f in F,

$$V \int_G f(U)\, U x \, d\mu(U) = \int_G f(U)\, V_0 U x \, d\mu(U) = \int_G f(V_0^{-1} W)\, W x \, d\mu(W)$$

which is in D_F. The same computation with $V = E$ shows E acts as the identity on D_F. Finally, if F is left-invariant, D_F is a unitary subspace of B. For E acts as the identity operator on D_F and $S | D_F$ is contained in the bounded group $G | D_F$, since for each V in S and z in D_F, $Vz = VEz$ and VE is in G. The proof is now complete, since x was an arbitrary element of B and each $T_\gamma x$ is in some D_F, a unitary subspace, and so in B_p.

A further definition and lemma are necessary before we can begin to establish the theorems of this section. $C_B(S)$ is defined to be the smallest uniformly closed subalgebra of $C(S)$ closed under complex conjugation and containing the constant functions and all f of the form

$$f(T) = \langle T x, y \rangle, \quad x \in B, \ y \in B^*. \tag{4.2}$$

LEMMA 4.8. *If $i : S \to \bar{S}$ is the injection map, the adjoint $i^* : C(\bar{S}) \to C(S)$ defined by $i^*(f) = f \circ i$ maps $C(\bar{S})$ onto $C_B(S)$.*

Proof. i^* is an isometry since S is dense in $\bar{S}$. Thus $i^*(C(\bar{S}))$ is a uniformly closed self-adjoint subalgebra containing 1. Since it also contains all f of the form (4.2), it must contain $C_B(S)$. For the converse, $(i^*)^{-1}(C_B(S))$ it a uniformly closed self-adjoint subalgebra of $C(\bar{S})$ that contains 1 and separates points. So it must be all of $C(\bar{S})$ by the Stone–Weierstrass Theorem. We can now establish our theorems.

THEOREM 4.9. *Let B be a Banach space and S a weakly almost periodic semigroup of operators on B. Then the following are equivalent:*

(i) *$C_B(S)$ has a right invariant mean.*
(ii) *$\bar{S}$ has a unique minimal left ideal.*
(iii) *$E_1E_2 = E_1$ for all projections E_1 and E_2 in $K(\bar{S})$.*
(iv) *B_0 is a closed S-invariant linear subspace of B.*

Proof. By Lemmas 2.10 and 4.8, (i) holds if and only if $C(\bar{S})$ has a right invariant mean, which by Lemma 2.8 and Corollary 2.4 is equivalent to (ii) and (iii).

Assume now that (iii) holds. Then all projections in $K(\bar{S})$ have the same kernel, so by Lemma 4.2 B_0 is a closed linear subspace of B. To establish (iv) it remains to show that B_0 is S-invariant. Choose x in B_0 and U in S. By Lemma 2.2 and Theorem 2.3 there is a V in $\bar{S}$ which is such that VU is a projection in $K(\bar{S})$. But by (iii) and Lemma 4.2, x is in the kernel of each such projection so $VUx = 0$. Thus Ux is in B_0 so (iv) must hold.

Assume now that (iv) holds. Since B_0 is a closed linear subspace it is weakly closed. Thus B_0 is $\bar{S}$-invariant. Let E_1 and E_2 be projections in $K(\bar{S})$. For any x in B, $E_2(I - E_2)x = 0$ so $(I - E_2)x$ is in B_0. By the $\bar{S}$-invariance of B_0, $E_1(I - E_2)x$ must also in B_0. But by Lemma 4.1, $E_1(I - E_2)x$ is in B_r and thus must be 0 since by their definitions $B_0 \cap B_r = \{0\}$. Thus $E_1(I - E_2)x = 0$ for all x in B so $E_1 = E_1E_2$ and (iii) holds. This completes the proof of Theorem 4.9.

THEOREM 4.10. *Let B be a Banach space and S a weakly almost periodic semigroup of operators on B. Then the following are equivalent:*

(i) *$C_B(S)$ has a left invariant mean.*
(ii) *$\bar{S}$ has a unique minimal right ideal.*
(iii) *$E_1E_2 = E_2$ for all projections E_1 and E_2 in $K(\bar{S})$.*
(iv) *$B_r = B_p$.*

Proof. The equivalence of (i), (ii) and (iii) follows as in the proof of Theorem 4.9.

Assume now that (i), (ii) and (iii) hold. By (iii), all of the projections in $K(\bar{S})$ have the same range, so by Lemma 4.1, B_r is a closed linear subspace of B. Thus by Lemma 4.4, $B_p \subset B_r$. But by (ii) and Lemmas 4.1 and 4.7, $B_r \subset B_p$ so $B_r = B_p$ and (iv) is established.

Assume now that (iv) holds. By the definition of unitary subspace, if E is a projection in $\bar{S}$ and D is a unitary subspace of B, $Ex = x$ for all x in D. As a consequence the elements of B_p are fixed under all projections in $\bar{S}$. Now let E_1 and E_2 be projections in $K(\bar{S})$ and x be an element of B. By Lemma 4.1, $E_2 x$ is in B_r and thus in B_p by (iv). Since E_1 leaves B_p pointwise fixed, $E_1 E_2 x = E_2 x$, so $E_1 E_2 = E_2$ and (iii) is established. This completes the proof of Theorem 4.10.

The following main theorem of this section is now a simple consequence of the preceding two results. That (iv) holds if S is commutative and B reflexive is the main result of [12].

THEOREM 4.11. *Let B be a Banach space and S a weakly almost periodic semigroup of operators on B. Then the following are equivalent:*

(i) *$C_B(S)$ has a two-sided invariant mean.*

(ii) *$K(\bar{S})$ is a compact topological group.*

(iii) *$K(\bar{S})$ contains a unique projection.*

(iv) *B_0 is a closed S-invariant subspace of B, $B_p = B_r$ and B is the direct sum of B_p and B_0.*

Proof. The equivalence of (ii) and (iii) follows from Theorem 2.3. The equivalence of (i) and (ii) follows from Lemmas 4.8 and 2.10 and Corollary 2.9. That (iv) implies (iii) follows from the corresponding parts of Theorems 4.9 and 4.10. So it remains to show that (iii) implies (iv). Because of Theorems 4.9 and 4.10, all that needs to be established is that B is the direct sum of B_p and B_0. Since $B_p = B_r$ and $B_0 \cap B_r = \{0\}$, it suffices to show that each x in B has some representation of the form $x = x_r + x_0$ with x_r in B_r and x_0 in B_0. If E is the projection in $K(\bar{S})$, $x = Ex + (I - E)x$ is such a representation by Lemmas 4.1 and 4.2. This completes the proof of Theorem 4.11.

If S is commutative, $C(S)$ has a two-sided invariant mean (see [5]) so (i) through (iv) of Theorem 4.11 hold. We discuss next conditions on B and S of a quite different nature which guarantee that the assertions of Theorems 4.9, 4.10 and 4.11 hold.

A Banach space is called *strictly convex* if $\|x\| = \|y\| = 1$ and $x \neq y$ imply $\|x + y\| < 1$.

COROLLARY 4.12. *Assume that B is strictly convex and that $\|T\| \leqslant 1$ for all T in S. Then* (i), (ii), (iii) *and* (iv) *of Theorem* 4.10 *hold.*

Proof. If x is in B_r and U is in $\bar{S}$, by Lemma 4.1 there is a V in $\bar{S}$ with $VUx = x$. Thus $\|Ux\|$ must equal $\|x\|$ for all x in B_r. If E is a projection in $\bar{S}$ and

x is in B_r, E leaves x fixed. For if this were not the case we would have

$$\|x\| = \|Ex\| = \left\| E\left(\frac{Ex+x}{2}\right)\right\| \leqslant \left\|\frac{Ex+x}{2}\right\| < \|x\|.$$

Thus if E_1 and E_2 are projections in $K(\bar{S})$, $E_1 E_2 x = E_2 x$ for all x in B, since by Lemma 4.1 $E_2 x$ is in B_r. It follows that $E_1 E_2 = E_2$ and (iii) of Theorem 4.10 holds.

COROLLARY 4.13. *Assume that B^* is strictly convex and that $\|T\| \leqslant 1$ for all T in S. Then* (i), (ii). (iii) *and* (iv) *of Theorem* 4.9 *hold.*

Proof. The argument is essentially that of Corollary 4.12 applied to the adjoints of the operators in $\bar{S}$. Let E_1 and E_2 be any two projections in $K(\bar{S})$. By Theorem 2.3 there is a V in $\bar{S}$ with $E_1 E_2 V = E_1$. Thus if y is in B^*, $V^* E_2^* E_1^* y = E_1^* y$ so

$$\|E_1^* y\| = \|V^* E_2^* E_1^* y\| \leqslant \|E_2^* E_1^* y\| \leqslant \|E_1^* y\|$$

and thus $\|E_2^* E_1^* y\| = \|E_1^* y\|$. If for some y in B^*, $E_2^* E_1^* y \neq E_1^* y$, we would have the contradiction

$$\|E_1^* y\| = \|E_2^* E_1^* y\| = \tfrac{1}{2}\|E_2^*(E_2^* E_1^* y + E_1^* y)\| \leqslant \tfrac{1}{2}\|E_2^* E_1^* y + E_1^* y\| < \|E_1^* y\|.$$

Thus $E_2^* E_1^* = E_1^*$ so $E_1 E_2 = E_1$ and (iii) of Theorem 4.9 holds.

Putting together these two results we obtain

COROLLARY 4.14. *Assume that B and B^* are strictly convex and that $\|T\| \leqslant 1$ for all T in S. Then* (i), (ii), (iii) *and* (iv) *of Theorem* 4.11 *hold.*

For the case of B and B^* reflexive and strictly convex, and with one of them uniformly convex, this is the main result of [13].

5. Weakly Almost Periodic Functions

Let S be a topological semigroup. A function f in $C(S)$ is said to be *almost periodic* if $\{R_\sigma f : \sigma \in S\}$ is conditionally compact in the strong topology of $C(S)$; f is said to be *weakly almost periodic* if $\{R_\sigma f : \sigma \in S\}$ is conditionally compact in the weak topology of $C(S)$. The corresponding definitions involving left translates are equivalent; for almost periodic functions this is proved as on page 167 of [14], and for weakly almost periodic functions the equivalence is Proposition 7 of [10]. We shall denote the set of almost periodic functions on S by $A(S)$ and the set of weakly almost periodic functions on S by $W(S)$. By Theorem 4.2 of [7], $A(S)$ and $W(S)$ are invariant closed linear subspaces of $C(S)$ and are thus Banach spaces. In the present section we shall confine our attention mainly to $W(S)$. The corresponding results for $A(S)$ are discussed in Section 6.

In some cases $A(S)$ or $W(S)$ is all of $C(S)$. Indeed by Theorem 2.7 we have

THEOREM 5.1. *If S is a compact topological semigroup, $W(S)=C(S)$. If furthermore the multiplication of S is jointly continuous, $A(S)=C(S)$.*

The following indicates how part of $A(S)$ or $W(S)$ can be obtained when S is not compact.

LEMMA 5.2. *Let S and S' be topological semigroups, $\varrho: S \to S'$ a continuous homomorphism and $\tilde{\varrho}: C(S') \to C(S)$ the induced mapping defined by*

$$\tilde{\varrho} f = f \circ \varrho, \quad \text{all } f \text{ in } C(S').$$

Then $\tilde{\varrho}(W(S')) \subset W(S)$ and $\tilde{\varrho}(A(S')) \subset A(S)$. If S' is compact, $\tilde{\varrho}(C(S')) \subset W(S)$, and if in addition the multiplication in S' is jointly continuous, $\tilde{\varrho}(C(S')) \subset A(S)$.

Proof. Since ϱ is a homomorphism, if f is in $C(S')$,

$$R_\sigma(\tilde{\varrho} f) = \tilde{\varrho}(R_{\varrho(\sigma)} f), \quad \text{all } \sigma \text{ in } S. \tag{5.1}$$

Thus

$$\{R_\sigma(\tilde{\varrho} f) : \sigma \in S\} \tag{5.2}$$

is contained in the image under $\tilde{\varrho}$ of

$$\{R_\tau f : \tau \in S'\}. \tag{5.3}$$

$\tilde{\varrho}$ is continuous, and thus weakly continuous, so (5.2) will be conditionally compact (resp. weakly conditionally compact) if (5.3) is conditionally compact (resp. weakly conditionally compact). Thus $\tilde{\varrho}(A(S')) \subset A(S)$ and $\tilde{\varrho}(W(S')) \subset W(S)$. The assertions of the lemma referring to the case where S' is compact now follows from Theorem 5.1.

As a simple application we have the following. If S is a locally compact group, and S' is its one-point compactification, with the multiplication of S extended by $\infty = \sigma\infty = \infty\sigma$, S' is a compact topological semigroup. Thus Lemma 5.2 yields Eberlein's result that $C_0(G) \subset W(G)$.

Actually, as we show below in Theorem 5.3, all the functions in $W(S)$ are induced by a continuous homomorphism $\varrho: S \to S'$, with S' a compact topological semigroup.

To obtain this result we proceed as follows. The restrictions (1) of the translation operators R_σ to the Banach space $W(S)$ clearly form a weakly almost periodic semigroup of operators in the sense of Section 3. The weak operator closure of this semigroup is by Theorem 3.1 a compact topological semigroup in the weak operator topology. It will be denoted by S^w and called the *weakly almost periodic compactification* of S. This is justified by

THEOREM 5.3. *The homomorphism $R: S \to S^w$ defined by $R(\sigma) = R_\sigma$ is continuous. The induced map $\tilde{R}: C(S^w) \to C(S)$ is an algebra isomorphism of $C(S^w)$ onto $W(S)$.*

Proof. Observe first that by the Hahn-Banach theorem the weak topology of $W(S)$ is identical with the topology induced on it by the weak topology of $C(S)$. By (iii) of Theorem 2.7, for each f in $W(S)$, the map $\sigma \to R_\sigma f$ of S into $W(S)$ is continuous into that topology. Thus R is continuous as S^w has the weak operator topology. Since R is a homomorphism, by Lemma 5.2 the induced map $\tilde{R}: C(S^w) \to C(S)$ defined by $\tilde{R}h = h \circ R$ takes $C(S^w)$ into $W(S)$. To show that $\tilde{R}$ is onto $W(S)$, let f be any function in $W(S)$. Let m_e be the unit point mass at the identity element e of S. If the function h on S^w is defined by

$$h(T) = \langle Tf, m_e \rangle = Tf(e), \quad \text{all } T \text{ in } S^w,$$

h is continuous and

$$h \circ R(\sigma) = h(R_\sigma) = R_\sigma f(e) = f(\sigma), \quad \text{all } \sigma \text{ in } S,$$

so $\tilde{R}h = f$. Thus $\tilde{R}$ is onto. Since $R(S)$ is dense in S^w, $\tilde{R}$ is 1–1 and an isometry. Moreover, $\tilde{R}$ evidently preserves the ordinary multiplication of functions, so $W(S)$ is an algebra, and $\tilde{R}$ an algebra isomorphism, completing our proof.

Because of Theorems 5.1, 5.3 and Lemma 5.2, the multiplication in S^w cannot be jointly continuous if $W(S) \neq A(S)$. Thus for example if S is a locally compact but non-compact group the multiplication in S^w is not jointly continuous since $C_0(S) \subset W(S)$, while clearly $C_0(S)$ is not contained in $A(S)$; furthermore S^w certainly cannot be a group in this example in view of Theorem 2.1.

Before discussing the main results of this section, which are the consequences for $W(S)$ of the existence of the compactification S^w, we show in Theorem 5.5 that our compactification has the expected property of homomorphism extension. This yields as Corollary 5.6 a characterization of S^w.

(1) In the following we shall also use the symbol R_σ to denote the restriction of the translation mapping to some subspace of $C(S)$.

LEMMA 5.4. *If S is a compact topological semigroup, then $\sigma \to R_\sigma$ is a topological isomorphism of S with S^w.*

Proof. By Theorem 5.3 the map is a continuous homomorphism. By Theorem 5.1, $W(S) = C(S)$ so the map is clearly 1–1. Thus since S is compact, the map is a homeomorphism, mapping S onto a dense compact subset of S^w, i.e., onto S^w.

THEOREM 5.5. *Let S and S' be topological semigroups, and $\varrho : S \to S'$ be a continuous homomorphism. Then there is a continuous homomorphism $\varrho^w : S^w \to S'^w$ for which $\varrho^w(R_\sigma) = R_{\varrho(\sigma)}$, all σ in S.*

Proof. Lemma 5.2 shows the induced map $\tilde{\varrho}$ takes $W(S')$ into $W(S)$. Consider first the case in which $\varrho(S)$ is dense in S'. Then $\tilde{\varrho}$ is an isometry, so $\tilde{\varrho}(W(S'))$ is closed, and the adjoint $\tilde{\varrho}^*$ of $\tilde{\varrho}$ is a map of $W(S')^*$ *onto* $W(S)^*$. Let $\mathcal{B}(W(S))$ and $\mathcal{B}(W(S'))$ be the algebras of bounded linear operators on $W(S)$ and $W(S')$ respectively, taken in the weak operator topologies. Denote by $\mathcal{B}_\varrho$ the subset of $\mathcal{B}(W(S))$ consisting of all T which leave invariant the closed (and therefore weakly closed) subspace $\tilde{\varrho}(W(S'))$ of $W(S)$. $\mathcal{B}_\varrho$ is clearly a closed subalgebra of $\mathcal{B}(W(S))$ that contains S^w. Since $\tilde{\varrho}$ is an isometry, each T in $\mathcal{B}_\varrho$ induces a corresponding map ψ_T in $\mathcal{B}(W(S'))$ that is characterized by

$$\tilde{\varrho}\,\psi_T = T\,\tilde{\varrho}.$$

It is simple to check that $T \to \psi_T$ is an algebra homomorphism of $\mathcal{B}_\varrho$ into $\mathcal{B}(W(S'))$. It is continuous for the weak operator topologies since $\tilde{\varrho}^*$ is onto; for if f is in $W(S')$ and m' in $W(S')^*$, when m in $W(S)^*$ is chosen so that $\varrho^* m = m'$ we have

$$\langle \psi_T f, m' \rangle = \langle \psi_T f, \tilde{\varrho}^* m \rangle = \langle \tilde{\varrho}\,\psi_T f, m \rangle = \langle T\,\tilde{\varrho} f, m \rangle.$$

Now by (5.1)
$$R_\sigma(\tilde{\varrho} f) = \tilde{\varrho}(R_{\varrho(\sigma)} f),$$

all f in $W(S')$, so $\psi_{R_\sigma} = R_{\varrho(\sigma)}$. Thus if $\varrho^w : S^w \to \mathcal{B}(W(S))$ is defined to be the restriction of the map $T \to \psi_T$ to S^w, $\varrho^w(R_\sigma) = R_{\varrho(\sigma)}$. Since ϱ^w is continuous, it maps S^w, which is the closure of $\{R_\sigma : \sigma \in S\}$ into

$$\text{closure } \{R_{\varrho(\sigma)} : \sigma \in S\} \subset \text{closure } \{R_\tau : \tau \in S'\} = S'^w.$$

This completes the proof for the special case where $\varrho(S)$ is dense in S'.

To obtain the general case is now quite simple. Let $\varrho_1 : S \to S'^w$ be the continuous homomorphism defined by

$$\varrho_1(\sigma) = R_{\varrho(\sigma)}, \quad \text{all } \sigma \text{ in } S.$$

Let S_1 be the compact topological semigroup that is the closure of the range of ϱ_1. We can now apply the special case that has been established to the map $\varrho_1 : S \to S_1$. It yields a continuous homomorphism $\varrho_1^w : S^w \to S_1^w$ satisfying $\varrho_1^w(R_\sigma) = R_{\varrho_1(\sigma)}$ for each σ in S. Let $\psi : S_1 \to S_1^w$ be defined by $\psi(\eta) = R_\eta$ for all η in S_1. Since S_1 is compact, by Lemma 5.4, ψ is a topological isomorphism. Thus the composite map $\varrho^w : S^w \to S'^w$ defined by $\varrho^w = \psi^{-1} \circ \varrho_1^w$ is a continuous homomorphism. For each σ in S it satisfies

$$\varrho^w(R_\sigma) = \psi^{-1} \circ \varrho_1^w(R_\sigma) = \psi^{-1}(R_{\varrho_1(\sigma)}) = \varrho_1(\sigma) = R_{\varrho(\sigma)},$$

and thus is the desired mapping. The proof is complete.

It is not at all apparent that one can easily find a proof of Theorem 5.5 avoiding the special case; however, the special case itself has considerable value and because of it we shall make the following definition. If $\varrho : S \to S'$ is a continuous homomorphism with $\varrho(S)$ dense in S', we shall say S is *densely represented* in S' by ϱ.

COROLLARY 5.6. *Let S be densely represented by ϱ in the compact topological semigroup S', and suppose the induced map $\tilde{\varrho}$ defined by $\tilde{\varrho} f = f \circ \varrho$ takes $C(S')$ onto $W(S)$. Then there is a topological isomorphism φ of S^w onto S' for which $\varphi(R_\sigma) = \varrho(\sigma)$ for all σ in S. (That is, we can identify S^w as the unique compact semigroup in which S can be densely represented so that all elements of $W(S)$ extend continuously.)*

Proof. By Theorem 5.5 there is a continuous homomorphism $\varrho^w : S^w \to S'^w$ that satisfies $\varrho^w(R_\sigma) = R_{\varrho(\sigma)}$, all σ in S. Let $\psi : S'^w \to S'$ be the topological isomorphism, whose existence is guaranteed by Lemma 5.4, that satisfies $\psi(R_\tau) = \tau$, all τ in S'. $\varphi : S^w \to S'$ is defined to be the composite mapping $\psi \circ \varrho^w$. It is a continuous homomorphism and satisfies $\varphi(R_\sigma) = \varrho(\sigma)$, all σ in S. Since $\varrho(S)$ is dense in S', $\varrho^w(S^w)$ is a dense compact subset of S'^w and thus all of S'^w. ψ is also onto so φ must be onto. Since S^w is compact, it remains to show that φ is 1–1. Let $\tilde{\varphi} : C(S') \to C(S^w)$ be the map induced by φ, i.e., $\tilde{\varphi} f = f \circ \varphi$, all f in $C(S')$. Then

$$\tilde{\varphi} f(R_\sigma) = f(\varphi(R_\sigma)) = f(\varrho(\sigma)) = \tilde{\varrho} f(\sigma), \quad \text{all } \sigma \text{ in } S,$$

so $\tilde{\varphi}$ is the composite of $\tilde{\varrho}$ with the natural isomorphism of $W(S)$ and $C(S^w)$. Since $\tilde{\varrho}$ has been assumed to be onto, $\tilde{\varphi}$ is onto and thus φ must be 1–1.

We can now proceed to the main results of this section. The first, Theorem 5.7 below, is essentially Theorem 4.11 for the case where B is $W(S)$ and the semigroup of operators is $\{R_\sigma : \sigma \in S\}$. Before stating this result we recall the notation of Section 4 in this context. $W(S)_0$ is the set of all f in $W(S)$ having 0 in the weak closure of $\{R_\sigma f : \sigma \in S\}$. $W(S)_r$ is the set of all f in $W(S)$ which are such that f is in the weak closure of $\{R_\sigma h : \sigma \in S\}$ wherever h is in the weak closure of $\{R_\sigma f : \sigma \in S\}$. One further definition is necessary before describing $W(S)_p$. If H is a Hilbert space and $\sigma \to U_\sigma$ is a unitary representation [1] of S on H, a function f on S of the form

$$f(\sigma) = (U_\sigma x, y), \quad \text{all } \sigma \in S,$$

for some x and y in H, is called a *coefficient* of the representation. If f is a coefficient of a finite dimensional unitary representation of S, f is in $A(S)$ and thus $W(S)$. Furthermore, it will be contained in a subspace of $W(S)$ that is unitary in the sense of Section 4 and thus in $W(S)_p$. Conversely any unitary subspace of $W(S)$ consists entirely of coefficients of finite dimensional unitary representation of S. Thus $W(S)_p$ is precisely the closed linear subspace of $W(S)$ spanned by the coefficients of finite dimensional unitary representations of S.

THEOREM 5.7. *Let S be a topological semigroup. Then the following are equivalent:*

(i) *$W(S)$ has an invariant mean.*

(ii) *$K(S^w)$, the kernel of S^w, is a compact topological group.*

(iii) *$W(S)_0$ is a closed translation invariant linear subspace of $W(S)$, $W(S)_p = W(S)_r$, and $W(S)$ is the direct sum of $W(S)_0$ and $W(S)_p$.*

Proof. By Theorem 5.3 and Lemma 2.10, (i) holds if and only if $C(S^w)$ has an invariant mean. And by Corollary 2.9 this is equivalent to (ii). By Theorem 4.11, (ii) is equivalent to (iii) modified to assert the invariance of $W(S)_0$ under only the rihgt translations $\{R_\sigma : \sigma \in S\}$. So to complete the proof it remains to show that $W(S)_0$ is automatically invariant under left translations. This follows since right translates commute with left translates; if f is in $W(S)_0$ and τ is in S, 0 is in the weak closure of $\{R_\sigma f : \sigma \in S\}$, so 0 is in the weak closure of $\{R_\sigma L_\tau f : \sigma \in S\} = L_\tau(\{R_\sigma f : \sigma \in S\})$, and $L_\tau f$ must be in $W(S)_0$.

If $W(S)$ has an invariant mean, Corollary 2.9 can be used to identify the mean.

THEOREM 5.8. *Assume that $W(S)$ has an invariant mean m. Then if $\tilde{R} : C(S^w) \to W(S)$ is the isomorphism of Theorem 5.3*

$$\langle \tilde{R} h, m \rangle = \int_{K(S^w)} h \, d\mu, \quad \text{all } h \text{ in } C(S^w),$$

where μ is the normalized Haar measure on the compact group $K(S^w)$. In particular m is the unique invariant mean on $W(S)$.

---

[1] A unitary representation is a strongly continuous homomorphism into the unitary group.

Proof. If m is an invariant mean on $W(S)$, the construction of Lemma 2.10 applied to the case $S' = S^w$ and $\varphi = R$ yields an invariant mean m' on $C(S^w)$ that satisfies

$$\langle \tilde{R} h, m \rangle = \langle h, m' \rangle, \quad \text{all } h \text{ in } C(S^w).$$

But by Corollary 2.9, m' is normalized Haar measure on $K(S^w)$.

If $W(S)$ has an invariant mean m, a Fourier analysis of $W(S)$ relative to m can be established. For S a locally compact Abelian group this was carried out in [7]. It is evident from Theorem 5.8 that this Fourier analysis of $W(S)$ is identical with the Fourier analysis of the restriction of $C(S^w)$ to the kernel $K(S^w)$ relative to Haar measure on $K(S^w)$. We omit the details.

When $W(S)$ has an invariant mean, $W(S)_0$ can be identified in terms of this mean.

COROLLARY 5.9. *Assume that $W(S)$ has an invariant mean m. Then $W(S)_0$ is*

$$\{f : f \in W(S), \quad \langle |f|^2, m \rangle = 0\}. \tag{5.4}$$

Proof. Let $\tilde{R} : C(S^w) \to W(S)$ be the isomorphism of Theorem 5.3. By Theorem 5.8, (5.4) is the same as

$$\{\tilde{R} h : h \in C(S^w), \quad h \equiv 0 \text{ on } K(S^w)\}. \tag{5.5}$$

Let e be the identity of S and E the identity of $K(S^w)$. Recall that by the proof of Theorem 5.3, if h is in $C(S^w)$, $h(T) = T\tilde{R}h(e)$ for all T in S^w. Let h be $\equiv 0$ on $K(S^w)$. Then since $K(S^w)$ is a left ideal, for all σ in S, $E\tilde{R}h(\sigma) = R_\sigma E\tilde{R}h(e) = h(R_\sigma E) = 0$. Thus by Lemma 4.2, $\tilde{R}h$ is in $W(S)_0$. This shows that (5.5) is contained in $W(S)_0$. For the reverse, assume that $\tilde{R}h$ is in $W(S)_0$. Then by Lemma 4.2, $E\tilde{R}h = 0$. For any T in $K(S^w)$, $TE = T$ and thus $h(T) = T\tilde{R}h(e) = 0$, so $h \equiv 0$ on $K(S^w)$. Thus $W(S)_0$ is contained in (5.5) and the proof is completed.

If our topological semigroup S is algebraically a group, $W(S)_p$ is identical with the space of almost periodic functions on S.

LEMMA 5.10. *If S is a group, $W(S)_p = A(S)$.*

Proof. We have observed that $W(S)_p$ is spanned by the coefficients of finite dimensional unitary representations of S. Since such coefficients are in $A(S)$, and $A(S)$ is a closed linear subspace of $C(S)$, $W(S)_p \subset A(S)$. For the reverse, suppose that f is in $A(S)$. Then f is in $A(S^d)$, where S^d is the group S supplied with the discrete topology. The theory of almost periodic functions on groups (see [15]) assures us that the uniformly closed invariant subspace of $A(S^d)$ generated by f is spanned in the norm topology of $A(S^d)$ by the coefficients of finite dimensional unitary representations of S^d that it contains. Each of these coefficients will be continuous on S since f is, and so will be in $W(S)_p$. Thus f can be uniformly approximated by functions in $W(S)_p$, and since $W(S)_p$ is closed, f is in $W(S)_p$. This completes the proof.

As a consequence of Theorem 5.7, Corollary 5.9 and Lemma 5.10, we have the following (which for locally compact Abelian groups is due to Eberlein [8]).

THEOREM 5.11. *Let G be a topological semigroup which is algebraically a group. If $W(G)$ has an invariant mean m, then $W(G)$ is the direct sum of $A(G)$ and the subspace $W(G)_0=\{f: f\in W(G), \langle |f|^2, m\rangle=0\}$.*

In Theorem 5.14 below we indicate a method for constructing topological groups G which have invariant means for $W(G)$. First two lemmas are necessary.

LEMMA 5.12. *Let G be a topological group, H a closed normal subgroup, and let f be a function in $W(G)$. Then the restriction of f to H is in $W(H)$. If f is constant on cosets of H, and the function $\tilde{f}$ is defined on G/H by $f(\sigma)=\tilde{f}(\sigma H)$, all σ in G, then $\tilde{f}$ is in $W(G/H)$.*

Proof. The first assertion follows from Lemma 5.2 applied to the injection map $H\to G$. Let $C^H(G)$ consist of all functions in $C(G)$ that are constant on cosets of H. The natural map $G\to G/H$ induces an onto isomorphism $\varphi: C(G/H)\to C^H(G)$, defined by

$$(\varphi(\tilde{f}))(\sigma)=\tilde{f}(\sigma H), \quad \text{all } \tilde{f}\in C(G/H),\ \sigma\in G.$$

Let f be a function in $C^H(G)\cap W(G)$ and suppose that $f=\varphi(\tilde{f})$. By the Hahn–Banach theorem the weak topology of $C^H(G)$ is identical with the topology induced on it by the weak topology of $C(G)$. Since f is in $W(G)$, the subset $\{R_\sigma f: \sigma\in G\}$ of $C^H(G)$ is weakly conditionally compact in that topology. Thus its image $\{R_\tau \tilde{f}: \tau\in G/H\}$ under φ^{-1} is weakly conditionally compact in $C(G/H)$, so $\tilde{f}$ is in $W(G/H)$ and the proof is complete.

LEMMA 5.13. *Let S be a topological semigroup, f in $W(S)$ and m in $W(S)^*$. Then if the function h is defined by*

$$h(\sigma)=\langle R_\sigma f, m\rangle, \quad \textit{all } \sigma \textit{ in } S,$$

h is in $W(S)$.

Proof. For each f in $W(S)$, let $\varphi(f)$ be the function on S defined by

$$(\varphi(f))(\sigma)=\langle R_\sigma f, m\rangle, \quad \text{all } \sigma \text{ in } S.$$

By Theorem 2.7, $\varphi(f)$ is in $C(S)$. It is simple to check that $\varphi: W(S) \to C(S)$ is a bounded linear operator and that $R_\sigma \varphi(f) = \varphi(R_\sigma f)$ for all σ in S. Thus $\{R_\sigma \varphi(f) : \sigma \in S\}$ is the weakly continuous image of the conditionally weakly compact set $\{R_\sigma f : \sigma \in S\}$ and so is itself conditionally weakly compact. This shows that φ maps $W(S)$ into itself and the proof is complete.

THEOREM 5.14. *Let G be a topological group, and let $\{G_\alpha\}_{\alpha<\gamma}$ be a well-ordered increasing family of (not necessarily closed) subgroups satisfying*

(i) $G_\alpha^* = \bigcup_{\beta<\alpha} G_\beta$ *is a relatively closed normal subgroup of G_α, $\alpha < \gamma$.*

(ii) $W(G_\alpha / G_\alpha^*)$ *has an invariant mean.*

(iii) $G_\gamma^* = G$, $G_1 = \{e\}$, *e the identity of G.*

Then $W(G)$ has an invariant mean.

Proof. We shall show first that $W(G)$ has a right invariant mean. Let M be the set of all means on $W(G)$, i.e., the set of all m in $W(G)^*$ that satisfy $\langle 1, m \rangle = 1$ and $\langle f, m \rangle \geqslant 0$ if $f \geqslant 0$. For each $\alpha < \gamma$ let $M_\alpha = \{m : m \in M, \langle R_\sigma f, m \rangle = \langle f, m \rangle$, all $f \in W(G)$, $\sigma \in G_\alpha\}$. Each M_α is weak* compact (although possibly void) and $M_1 = M$. For each $\alpha \leqslant \gamma$ let $M^\alpha = \bigcap_{\beta<\alpha} M_\beta$. M^γ is the set of right invariant means for $W(G)$. $M^\alpha \neq \phi$ if and only if $M_\beta \neq \phi$ for each $\beta < \alpha$, by the weak* compactness of the M_β. Thus to show that $M^\gamma \neq \phi$ it suffices to show that $M_\alpha \neq \phi$ for each $\alpha < \gamma$ and since $M_1 \neq \phi$, it suffices to establish the induction step: $M_\beta \neq \phi$ for all $\beta < \alpha$ (or equivalently $M^\alpha \neq \phi$) implies $M_\alpha \neq \phi$.

So let us assume that M^α is non-empty. Choose a mean m in M^α. Then

$$\langle R_\sigma f, m \rangle = \langle f, m \rangle, \quad \text{all } f \in W(G),\ \sigma \in G_\alpha^*. \tag{5.6}$$

Let $\varphi: W(G) \to W(G)$ be the map (whose existence is guaranteed by Lemma 5.13) defined by

$$(\varphi(f))(\sigma) = \langle R_\sigma f, m \rangle, \quad \text{all } \sigma \in G. \tag{5.7}$$

By (5.6), each $\varphi(f)$ is constant on right cosets of G_α^* in G. This together with Lemma 5.12 guarantees the existence of a map $\psi: W(G) \to W(G_\alpha / G_\alpha^*)$ that satisfies

$$(\psi(f))(G_\alpha^* \sigma) = (\varphi(f))(\sigma), \quad \text{all } f \in W(G),\ \sigma \in G_\alpha. \tag{5.8}$$

Now let m_1 be an invariant mean for $W(G_\alpha / G_\alpha^*)$. The mapping $f \to \langle \psi(f), m_1 \rangle$ is a continuous linear functional on $W(G)$ that takes 1 into 1 and preserves positivity, and so it is a mean on $W(G)$. If it is denoted by m', it is simple to check, using

(5.7), (5.8) and the invariance of m_1 that

$$\langle R_\sigma f, m'\rangle = \langle f, m'\rangle, \quad \text{all } f \in W(G), \sigma \in G_\alpha.$$

Thus m' is in M_α and the proof of the induction step is completed, so $W(G)$ has a right invariant mean. A similar argument shows that $W(G)$ has a left invariant mean. Thus by Theorem 5.3, Lemma 2.10 and Corollary 2.9, $W(G)$ has an invariant mean.

If G is a compact or commutative topological group, $C(G)$ and thus $W(G)$ has an invariant mean. This fact, together with Theorem 5.14 indicates that the class of topological groups G having an invariant mean for $W(G)$ is quite large. But we know of no examples of such groups besides those constructed from compact and commutative groups using Theorem 5.14. In particular we do not know whether the free group G_2 with two generators has an invariant mean for $W(G_2)$ ($C(G_2)$ has no invariant mean, see [4], p. 290). However, for a group G, $W(G)$ has an invariant mean if it merely has a one-sided invariant mean, as one can see from the following considerations.

For f in $W(G)$ let f^* be defined by $f^*(\sigma) = f(\sigma^{-1})$. In view of the equivalence of right and left weak almost periodicity, f^* is in $W(G)$ and $f \to f^*$ appears as a (real) involution on $W(G)$; let $m \to m^*$ be the dual involution on $W(G)^*$. The identity

$$\langle R_{\sigma^{-1}} f, m\rangle = \langle R_{\sigma^{-1}} f^{**}, m\rangle = \langle (L_\sigma f^*)^*, m\rangle = \langle L_\sigma f^*, m^*\rangle$$

shows that one has a homeomorphism mapping the weak operator closure S_w of $\{L_\sigma : \sigma \in G\}$ onto S^w which extends $L_\sigma \to R_{\sigma^{-1}}$. Moreover, as is easily seen S_w is exactly an anti-isomorph of S^w, with an extension of $R_\sigma \to L_\sigma$ providing the anti-isomorphism. Hence $R_\sigma \to L_\sigma \to R_{\sigma^{-1}}$ has a continuous extension φ mapping S^w anti-isomorphically onto itself. Since φ then interchanges minimal left and minimal right ideals, uniqueness of either insures that $K(S^w)$ is a group.

Finally let us point out that it seems likely that the analogue of Theorem 5.14 with $C(G)$ in place of $W(G)$ is not true in general. However, a proof similar to that given above shows that the analogue holds if G is discrete.

6. Almost Periodic Functions

Let S be a topological semigroup. The restrictions of the right translation operators R_σ to the Banach space $A(S)$ of almost periodic functions on S clearly form an almost periodic semigroup of operators in the sense of Section 3. The strong operator closure of this semigroup is by Theorem 3.2 a compact topological semigroup, having jointly continuous multiplication, in the strong (or equivalently weak) operator topology. It will be denoted by S^a and called the *almost periodic compactification* of S. The justification for this is to be found in the fact that *all the results* 5.3 *through* 5.9 *carry over to the present situation; indeed in their statements one need only replace* $W(S)$ *by* $A(S)$, S^w *by* S^a, *and "compact semigroup" by "compact semigroup with jointly continuous multiplication" to obtain valid results.* Since the proofs are simple modifications (and in some cases simplifications) we shall not give them here. However, for reference purposes we state formally three results, the analogues of Theorems 5.3 and 5.5 and Corollary 5.6.

THEOREM 6.1. *The homomorphism* $R: S \to S^a$ *defined by* $R(\sigma) = R_\sigma$ *is continuous. The induced map* $\tilde{R}: C(S^a) \to C(S)$ *is an algebra isomorphism of* $C(S^a)$ *onto* $A(S)$.

THEOREM 6.2. *Let* S *and* S' *be topological semigroups, and* $\varrho: S \to S'$ *a continuous homomorphism. Then there is a continuous homomorphism* $\varrho^a: S^a \to S'^a$ *for which* $\varrho^a(R_\sigma) = R_{\varrho(\sigma)}$.

THEOREM 6.3. *Let* S *be densely represented by* ϱ *in the compact topological semigroup* S' *having jointly continuous, multiplication, and suppose the induced map* $\tilde{\varrho}$ *defined by* $\tilde{\varrho} f = f \circ \varrho$ *takes* $C(S')$ *onto* $A(S)$. *Then there is a topological isomorphism* φ *of* S^a *onto* S' *for which* $\varphi(R_\varrho) = \varrho(\sigma)$ *for all* σ *in* S.

In other words we can identify S^a as the unique compact semigroup having jointly continuous multiplication in which S can be densely represented so that all elements of $A(S)$ extend continuously.

We see now that the distinction between S^w and S^a as "compactifications" rests primarily on the distinction between separate and joint continuity of multiplication. We can view S^w as the maximal "compactification" of S reflecting its algebraic structure and S^a as the maximal "jointly continuous compactification" of this sort.

Some of the results of Section 5 do, of course, say nothing new for almost periodic functions (5.11 and 5.14 for example). But this is not the case for the decomposition theorem, Theorem 5.7. For example if S is the topological semigroup formed by the non-negative reals under addition, the analogue of (iii) of Theorem 5.7 states in this case that $A(S)$ is the direct sum of the space of restrictions to S of almost periodic functions on the full line (which is clearly our $A(S)_p$) and $C_0(S)$ (the space of all continuous functions tending to zero at infinity, which clearly forms the set of f in $C(S)$ having 0 in the strong closure of their orbits). Thus in particular every f in $A(S)$ can be approximated in the uniform norm by linear combinations of exponentials. This raises the question of when one might expect, on semigroups, an analogue of the classical approximation theorem for almost periodic functions on groups. One natural formulation (see [6]) fails in general (some important cases in which it holds will be covered in a subsequent paper), due to the lack of an analogue of the Peter–Weyl theorem. In the group case the latter combines with the Stone–Weierstrass theorem (and the fact that G^a is a group when G is) to yield approximation; in the semigroup case the Peter–Weyl theorem must be replaced by a detailed investigation of the structure of S^a.

7. Convex Semigroups of Operators and Ergodic Theory (1)

In this section applications of the results of Section 4 to ergodic theory are given. Throughout B will be a fixed Banach space and S a semigroup of operators on B that is weakly almost periodic and *convex* in the sense that

$$\{\lambda U + \mu V : \lambda \geqslant 0,\ \mu \geqslant 0,\ \lambda + \mu = 1\}$$

is contained in S whenever U and V are in S. The results that we obtain are related to, and were suggested by, those of [11].

Recall that for each x in B, $O(x)^-$ is defined to be the weak closure of the orbit $O(x) = \{Tx : T \in S\}$ and that $O(x)^- = \{Tx : T \in \bar{S}\}$. Since S is weakly almost periodic, each $O(x)^-$ is a weakly compact subset of B. Furthermore, each $O(x)$ is convex so the weak closure $O(x)^-$ is convex and is identical with its strong closure.

In ergodic theory the following conditions on S and B are of interest:

I. If 0 is in $O(x)^-$, then 0 is in $O(Tx)^-$ for each T in S; equivalently, B_0 is S-invariant. (2)

II. Each $O(x)^-$ contains at least one fixed point of S.

III. Each $O(x)^-$ contains exactly one fixed point of S.

The main result below is that I is satisfied if and only if the conditions of Theorem 4.9 hold, and that the same relation subsists between II and Theorem 4.10 and between III and Theorem 4.11. We also establish that B_p is the set of fixed points of S and that the kernel $K(\bar{S})$ consists entirely of projections. First we need a lemma.

---

(1) After this paper was prepared for publication the authors learned of some recent results of H. Cohen and H. S. Collins which contain several of the results of this section (see Affine Semigroups, *Trans. Amer. Math. Soc.*, 93 (1959), 97–113).

(2) The proof of Theorem 7.4 shows that I is also equivalent to the $\bar{S}$-invariance of B_0. This is the condition that S act "ergodically" on B in the sense of [11].

LEMMA 7.1. *Let D be a Banach space and G a subset of the algebra $\mathcal{B}(D)$ of bounded linear operators on D that is a group under operator multiplication. Assume furthermore that G is convex and is compact in the weak operator topology. Then G consists of a single projection.*

Proof. For each V in G the map $U \to VU$ of $\mathcal{B}(D)$ into itself is linear and is a 1–1 map of G onto itself. Thus is takes extreme points of G onto extreme points. Since by the Krein-Milman Theorem G has at least one extreme point, all points of G must be extreme, so G consists of only one operator which clearly must be a projection.

THEOREM 7.2. *If S is a convex weakly almost periodic semigroup, $K(\bar{S})$ consists entirely of projections.*

Proof. By Theorem 3.1, the weak operator closure $\bar{S}$ is a compact topological semigroup in the weak operator topology. Since S is convex, $\bar{S}$ will be convex. By Theorem 2.3, $K(\bar{S}) = \bigcup E\bar{S}E$, where the union is over all projections E in $K(\bar{S})$, and each $E\bar{S}E$ compact and a group. Since $\bar{S}$ is convex, each $E\bar{S}E$ is convex, so by Lemma 7.1 consists of E alone.

LEMMA 7.3. *If S is a convex weakly almost periodic semigroup, B_p is the set of fixed points of S.*

Proof. It is clear that each fixed point of S is in B_p. For the converse let D be a unitary subspace of B. As in the proof of Lemma 4.4, $\bar{S}|D$ is a group in $\mathcal{B}(D)$ containing the identity operator. $\bar{S}|D$ is compact and convex since $\bar{S}$ is compact and convex, so by Lemma 7.1, $\bar{S}|D$ contains only the identity operator. Thus S e aves each unitary subspace of B pointwise fixed and as a consequence B_p consists entirely of fixed points.

THEOREM 7.4. *Let S be a convex weakly almost periodic semigroup. Then*

I *holds if and only if* (i) *through* (iv) *of Theorem* 4.9 *hold;*

II *holds if and only if* (i) *through* (iv) *of Theorem* 4.10 *hold;*

III *holds if and only if* (i) *through* (iv) *of Theorem* 4.11 *hold.*

Proof. I implies 4.9. We shall assume that B_0 is S-invariant and prove that it is a closed linear subspace. Note that, since S and $O(x)$ are convex, if x is in B_0, 0 is in the *strong* closure of $O(x)$. Choose K so that $\|T\| \leqslant K$ for all T in S. Now let x and y be in B_0 and choose $\varepsilon > 0$. There is a U in S with $\|Ux\| < \varepsilon/2K$ and since B_0 is S-invariant, there is a V in S with $\|VUy\| < \frac{1}{2}\varepsilon$. Then

$$\| VU(x+y) \| \leqslant K \| Ux \| + \| VUy \| < \varepsilon,$$

so $x+y$ is in B_0 and B_0 is a linear subspace. To show that B_0 is closed, let x be a point in the closure of B_0 and choose $\varepsilon > 0$. There is a y in B_0 with $\| x-y \| < \varepsilon/2K$ and a U in S with $\| Uy \| < \frac{1}{2}\varepsilon$. Thus

$$\| Ux \| \leqslant \| U(x-y) \| + \| Uy \| < \varepsilon,$$

and x must be in B_0. That 4.9 implies I is trivial.

II implies 4.10. By Lemma 7.3, B_p consists entirely of fixed points, so certainly $B_p \subset B_r$. For the reverse inclusion, choose x a point in B_r. By II, $\{Tx : T \in \bar{S}\}$ contains a fixed point Ux. By Lemma 4.1, there is a V in $\bar{S}$ with $x = VUx = Ux$. Thus x is a fixed point and consequently in B_p.

4.10 implies II. Let x be in B, and E be a projection in $K(\bar{S})$. Then Ex is in $O(x)^-$, and by Lemma 4.1, (iv) of Theorem 4.10 and Lemma 7.3, Ex is a fixed point.

III implies 4.11. Since III is stronger than II, by "II implies 4.10" above, $B_r = B_p$, which by Lemma 7.3 is the set of fixed points of S. If (iii) of Theorem 4.11 did not hold, there would be two distinct projections E_1 and E_2 in $K(\bar{S})$. Then for any x in B with $E_1 x \neq E_2 x$, $O(x)^-$ would contain the two elements $E_1 x$ and $E_2 x$ which are in B_r by Lemma 4.1 and thus fixed points.

4.11 implies III. By "4.10 implies II" above, each $O(x)^-$ contains at least one fixed point. Let u and v be fixed points in $O(x)^-$. Choose U and V in $\bar{S}$ so $Ux = u$ and $Vx = v$. Then $U(x-u) = 0$ so $x-u$ is in B_0. B_0 is a closed linear subspace and thus is weakly closed, so since it is S-invariant it must be $\bar{S}$-invariant. As a consequence, $V(x-u) = v-u$ is in B_0. But $v-u$ is a fixed point so $v = u$. This completes the proof of Theorem 7.4.

8. Miscellaneous Applications

In this section we present two applications of the results of Sections 2 and 4. The first shows that weak and strong compactness are equivalent for groups of operators.

THEOREM 8.1. *Let B be a Banach space, $\mathfrak{B}(B)$ the algebra of bounded linear operators on B, and S a subset of $\mathfrak{B}(B)$. Suppose that S is a group under operator multiplication and is compact in the weak operator topology. Let E be the identity element of S. Then B_0 is the kernel of E, B_p is the range of E, and S is compact in the strong operator topology.*

Proof. Since S is a group and is compact in the weak operator topology, $S=\bar{S}=K(\bar{S})$ and the conditions of Theorem 4.11 hold. By Lemma 4.2, B_0 is the kernel of E, and by Lemma 4.1, the range of E is $B_r=B_p$. Thus it remains only to show that S is compact in the strong operator topology. By Theorem 3.2 it suffices to show that

$$\{x : x \in B,\ O(x)^- \text{ strongly compact}\} \tag{8.1}$$

is all of B. By Theorem 4.2 of [7], (8.1) is a closed linear subspace of B. Since each element of B_0 is annihilated by S, (8.1) contains B_0. And since each unitary subspace of B is contained in (8.1), (8.1) contains B_p. Thus (8.1) is all of B, since by (iv) of Theorem 4.11, B is the direct sum of B_p and B_0.

The following is the analogue in our context of the well-known fact that a closed subsemigroup of a compact group is a subgroup.

COROLLARY 8.2. *Let S be a subset of $\mathfrak{B}(B)$ that consists of invertible operators. Suppose that S is compact in the weak operator topology and closed under operator multiplication. Then S is a group that is compact in the strong operator topology, and $B=B_p$.*

Proof. By Corollary 2.6, S contains a projection, which must be the identity operator since S consists entirely of invertibles. Thus S, with the weak operator topology, is a compact topological semigroup and Theorem 2.3 can be applied. Let E be a projection $K(S)$. Since E is invertible it must be the identity operator, so $S=K(S)$. Since $K(S)$ contains a unique projection, by Theorem 2.3 it is a group. The remainder of the proof follows from Theorem 8.1.

Appendix

This section is devoted to a proof of Theorem 2.1: Every compact topological semigroup that is algebraically a group is a topological group, i.e., has jointly continuous multiplication and continuous inversion. First we need two lemmas.

LEMMA A.1. *Let G be a compact topological semigroup that is algebraically a group. Then $C(G)$ has an invariant mean m that satisfies $\langle f, m\rangle > 0$ if $f \geq 0$ and $f \not\equiv 0$.*

Proof. The proof of the existence of m is identical with that given for compact topological groups in [16], pp. 91–99, except for the construction of the sequence $g_1, g_2, \dots, g_n, \dots$. One has to replace the three paragraphs starting near the bottom of p. 93 with "Let us denote by Δ ..." and ending on p. 94 with "..., that $s=0$", with the following argument.

Δ is defined to be the subset of $C(G)$ consisting of all $n^{-1}\sum_{i=1}^{n} R_{\sigma i} f$. It is clear that its closure $\bar{\Delta}$ in the norm topology is convex. Since a strongly closed convex set is weakly closed, $\bar{\Delta}$ is weakly closed and thus the weakly closed convex hull of $\{R_\sigma f : \sigma \in G\}$. By Theorem 2.7, $\{R_\sigma f : \sigma \in G\}$ is weakly conditionally compact so by Theorem 1.2 of [7], $\bar{\Delta}$ is weakly compact and thus compact in the topology of pointwise convergence. For each h in $C(G)$, define $S(h)$ to be

$$\sup_{\sigma,\tau \in G} |h(\sigma) - h(\tau)|.$$

Let $s = \inf_{h \in \Delta} S(h) = \inf_{h \in \bar{\Delta}} S(h)$. In order to complete the argument we must find a function g in $C(G)$ with $S(g) = s$ and a sequence $\{g_n\}$ in Δ with $g_n \to g$ uniformly. But for any $s' > s$, $\{h : h \in \bar{\Delta},\ S(h) \leqslant s'\}$ is non-empty and closed in the topology of pointwise convergence (in which $\bar{\Delta}$ is compact), so some g in $\bar{\Delta}$ yields $S(g) = s$. And since $\bar{\Delta}$ is the strong closure of Δ, there is a sequence $\{g_n\}$ in Δ converging uniformly to g.

LEMMA A. 2. *Let S be a compact topological semigroup and let f be in $C(S)$. Then the subset*

$$\{|R_\sigma f - R_\tau f| : \sigma,\ \tau \in S\} \tag{A. 1}$$

of $C(S)$ is compact in the weak topology. Furthermore, the weak topology and the topology of pointwise convergence agree on (A. 1).

Proof. The map $(\sigma, \tau) \to |R_\sigma f - R_\tau f|$ of $S \times S$ into $C(S)$ supplied with the topology of pointwise convergence is continuous. Thus (A. 1) is compact in the topology of pointwise convergence. Since it is bounded, by Theorem 5 of [10] it is compact in the weak topology, and since the two topologies are comparable, they must agree on (A. 1).

We can now proceed to the proof of Theorem 2.1. Let e be the identity element of G. Separate continuity shows the topology of G is defined by the base of neighborhoods of e, and thus it suffices to show the map $(\tau, \sigma) \to \tau^{-1}\sigma$ of $G \times G$ into G is continuous at (e, e). Suppose that this is not the case. Then there exists a neighborhood W of e which has the property that for each neighborhood U of e there is some σ_U and τ_U in U with $\eta_U = \tau_U^{-1}\sigma_U$ not in W. Let N be the directed set of neighborhoods of e with the usual ordering, i.e., $U > V$ if $U \subset V$. Let η be a cluster point of the net $\{\eta_U\}$. Such an η must exist by the compactness of G. By the choice of the σ_U and τ_U, η is outside of W and thus $\eta \neq e$. Choose f in $C(G)$ so that $f(e) \neq f(\eta)$. Then $f \neq R_\eta f$ so by Lemma A. 1,

$$\langle |f - R_\eta f|, m \rangle > 0. \tag{A. 2}$$

Since $\lim_N \sigma_U = e$ and $\lim_N \tau_U \eta = \eta$,

$$\lim_N |R_{\sigma_U} f - R_{\tau_U \eta} f| = |f - R_\eta f| \tag{A. 3}$$

in the topology pointwise convergence. Thus by Lemma A. 2, (A. 3) also holds in the weak topology of $C(G)$. In particular

$$\lim_N \langle |R_{\sigma_U} f - R_{\tau_U \eta} f|, m \rangle = \langle |f - R_\eta f|, m \rangle > 0, \tag{A. 4}$$

by (A. 2). For each U in N, by invariance of m, and since $\eta_U = \tau_U^{-1} \sigma_U$,

$$\langle |R_{\sigma_U} f - R_{\tau_U \eta} f|, m \rangle = \langle R_{\tau_U^{-1}} |R_{\sigma_U} f - R_{\tau_U \eta} f|, m \rangle = \langle |R_{\eta_U} f - R_\eta f|, m \rangle.$$

Thus by (A. 4)

$$\lim_N \langle |R_{\eta_U} f - R_\eta f|, m \rangle > 0. \tag{A. 5}$$

We shall obtain a contradiction to (A. 5) by showing that the net $\{\langle |R_{\eta_U} f - R_\eta f|, m \rangle\}$ of real numbers has 0 as a cluster point. Since η is a cluster point of the net $\{\eta_U\}$, the function $0 = |R_\eta f - R_\eta f|$ is a cluster point of the net $|R_{\eta_U} f - R_\eta f|$ in the topology of pointwise convergence. Thus by Lemma A. 2 it is also a cluster point in the weak topology of $C(G)$; as a consequence 0 is a cluster point of the net $\langle |R_{\eta_U} f - R_\eta f|, m \rangle$. This is our contradiction and completes the proof.

References

[1]. H. Anzai & S. Kakutani, Bohr compactifications of locally compact abelian groups I and II. *Proc. Imperial Academy, Tokyo*, 19 (1943), 476–480; 533–539.

[2]. N. Bourbaki, *Intégration*. Éléments de Mathématique, XIII, Book VI, Paris, 1952.

[3]. A. H. Clifford, Semigroups containing minimal ideals. *Amer. J. Math.*, 70 (1948), 521–526.

[4]. M. M. Day, Means for the bounded functions and ergodicity of the bounded representations of semigroups. *Trans. Amer. Math. Soc.*, 69 (1950), 276–291,

[5]. ——, Amenable semigroups. *Illinois J. of Math.*, 1 (1958), 509–543.

[6]. K. deLeeuw & I. Glicksberg, Almost periodic compactifications. *Bull. Amer. Math. Soc.*, 65 (1959), 134–139.

[7]. W. F. Eberlein, Abstract ergodic theorems and weakly almost periodic functions. *Trans. Amer. Math. Soc.*, 67 (1949), 217–240.

[8]. ——, The point spectrum of weakly almost periodic functions. *Michigan Math. J.*, 3 (1955–56), 137–139.

[9]. R. ELLIS, Locally compact transformation groups. *Duke Math. J.*, 24 (1957), 119–126.
[10]. A. GROTHENDIECK, Critères de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.*, 74 (1952), 168–186.
[11]. R. HEYNEMAN, *Duality in general ergodic theory*. Thesis, University of California, 1958.
[12]. K. JACOBS, Ergodentheorie und fastperiodische Funktionen auf Halbgruppen. *Math. Z.*, 64 (1956), 298–338.
[13]. ——, Fastperiodizitätseigenschaften allgemeiner Halbgruppen in Banachräumen. *Math. Z.*, 67 (1957), 83–92.
[14]. L. LOOMIS, *Abstract Harmonic Analysis*. New York, 1953.
[15]. W. MAAK, *Fastperiodische Funktionen*. Berlin, 1950.
[16]. L. PONTRJAGIN, *Topological Groups*. Princeton, 1939.
[17]. W. G. ROSEN, On invariant means over compact semigroups. *Proc. Amer. Math. Soc.*, 7 (1956), 1076–1082.
[18]. A. D. WALLACE, The structure of topological semigroups. *Bull. Amer. Math. Soc.*, 61 (1955), 95–112.
[19]. ——, The Rees–Suschkewitsch structure theorem for compact simple semigroups. *Proc. Nat. Acad. Sci. U.S.A.*, 42 (1956), 430–432.
[20]. A. WEIL, *L'intégration dans les groupes topologiques et ses applications*. Paris, 1940.

Received January 4, 1960