# AN INEQUALITY IN THE GEOMETRY OF NUMBERS

BY

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## I

A theorem due to L. Fejes-Tóth [2] states that if $K_1, \ldots, K_n$ are $n$ non-overlapping convex domains each of which arises from a given convex domain $K$ by an area-preserving affine transformation and $H$ is a convex polygon having at most six sides which contains them then $A(H) \geqslant nh(K)$ where $A(H)$ denotes the area of $H$ and $h(K)$ is the area of the smallest polygon having at most six sides which can be circumscribed about $K$.

Restricting the domains $K_1, \ldots, K_n$ to be congruent and similarly situated, C. A. Rogers [4] obtains a similar result in which $H$ is any convex domain covering $K_1, \ldots, K_n$ and $h(K)$ is replaced by $d(K)$, the determinant of the closest lattice packing of $K$. Rogers's results depend on the following theorem:

THEOREM (Rogers). *Let $G$ be a plane, strictly convex, Jordan curve containing the origin, $O$, of a cartesian coordinate system in its interior. Denote by $G(P)$ the translate of $G$ which results from the translation which takes $O$ into $P$. Let $P_0, P_1, \ldots, P_n = P_0, P_{n+1}, \ldots, P_{n+m}$ be points which satisfy*

(1) *the polygon, $P_0P_1 \ldots P_n$ is a Jordan polygon, $\Pi$, bounding a domain, i.e. a closed, bounded, simply-connected set, $\Pi^*$;*

(2) *the domains bounded by $G(P_{r-1})$ and $G(P_r)$ have a common boundary point if $1 \leqslant r \leqslant n$;*

(3) *the points, $P_{n+1}, \ldots, P_{n+m}$ lie in $\Pi^*$;*

(4) *the domains bounded by $G(P_r)$ and $G(P_s)$ have no interior points in common if $1 \leqslant r < s \leqslant n + m$. Then*

$$\frac{A(\Pi^*)}{4\Delta(G)} \geqslant m + \frac{1}{2}n - 1$$

*where $A(\Pi^*)$ is the area of $\Pi^*$ and $\Delta(G)$ is the critical determinant of $G$.*

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Our main object in this paper is the proof of a more general theorem of which the above is a special case.

We consider the set of points $\{(\frac{1}{2}(x_1 - x_2), \frac{1}{2}(y_1 - y_2))\}$ where $(x_1, y_1)$ and $(x_2, y_2)$ are any two points in the domain bounded by $G$ and denote by $\Gamma$ the boundary of the set obtained in this way. $\Gamma$ is strictly convex, has $O$ as a centre of symmetry and, defining $\Gamma(P)$ in the same way as $G(P)$ above, if $G(P)$ and $G(Q)$ bound domains which touch so do $\Gamma(P)$ and $\Gamma(Q)$ and conversely (see [4], p. 313).

Let $\mu$ be the Minkowski distance function defined by $\Gamma$ (see, for example, Bonnesen-Fenchel [1], p. 21). Thus, for any two points, $A$ and $B$, in the plane, $\mu(A, B) = |AB|/|OR|$ where $|AB|$ denotes the Euclidean distance between $A$ and $B$ and $R$ is the point of $\Gamma$ such that the vector $\overrightarrow{OR}$ has the same direction as $\overrightarrow{AB}$.

The condition that the domains bounded by $\Gamma(P)$ and $\Gamma(Q)$ touch is equivalent to $\mu(P, Q) = 2$ while $\mu(P, Q) > 2$ is equivalent to their having no points in common. The same conditions with respect to $\frac{1}{2}\Gamma(P)$ and $\frac{1}{2}\Gamma(Q)$ are characterized by $\mu(P, Q) = 1$ and $\mu(P, Q) > 1$ respectively where $\frac{1}{2}\Gamma$ is the set consisting of the mid-points of the segments joining $O$ to the points of $\Gamma$. It will be simpler to deal with $\frac{1}{2}\Gamma$ and its translates. Indeed, if we replace $G$ by $\frac{1}{2}G$ in the above theorem $A(\Pi^*)$ and $\Delta(G)$ must each be multiplied by $\frac{1}{4}$ and the result is the same.

DEFINITION 1. *Let $\Pi$ be a Jordan polygon and $E$ a finite set of points. We shall say that the pair $(\Pi, E)$ is "weakly admissible" if the following conditions are satisfied.*

(i) *The vertices of $\Pi$ are contained in $E$.*

(ii) *$E$ is contained in $\Pi^*$, the domain whose boundary is $\Pi$.*

(iii) *For any two points, $P$ and $Q$, in $E$, if the segment $PQ$ lies in $\Pi^*$ then $\mu(P, Q) \geqslant 1$.*

In particular, if $E$ consists merely of the vertices of $\Pi$ we shall say that $\Pi$ is a "weakly admissible polygon" when $(\Pi, E)$ is a weakly admissible pair.

Our main result is the following:

THEOREM 1. *Let $\mu$ be the Minkowski distance function defined by $\Gamma$ a convex, centrally symmetric, Jordan curve and let $(\Pi, E)$ be a weakly admissible pair then*

$$F(\Pi) = \frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} + 1 \geqslant n \qquad (\mathrm{I}_n)$$

*where $A(\Pi^*)$ is the area of $\Pi^*$, $M(\Pi)$ is the $\mu$-length of $\Pi$, $n$ is the number of points in $E$ and $\Delta$ is the critical determinant with respect to $\Gamma$.*

The existence of inequalities of the type $(\mathrm{I}_n)$ was suggested in a remark by H. Zassenhaus [6]. For the "star-shaped domain" $|xy| \leqslant |$ it has been shown by N. E. Smith [5] that

$$\frac{A(\Pi^*)}{\sqrt{5}} + \frac{N(\Pi)}{2} + 1 \geqslant n$$

where $N(\Pi)$ is the perimeter of $\Pi$ measured by the norm-distance $\nu((x_1, y_1), (x_2, y_2)) = |(x_2 - x_1)(y_2 - y_1)|^{\frac{1}{2}}$.

We shall first prove that Theorem 1 is true for weakly admissible polygons. The general case will then follow by induction on the number of points of $E$ contained in the interior of $\Pi$. The former case will occupy us for the greater part and it will be convenient to distinguish it as

THEOREM 2. *Let* $\Pi = P_1P_2 \ldots P_nP_{n+1}$ *where* $P_{n+1} = P_1$ *be a weakly admissible polygon, then*

$$F(\Pi) = \frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} + 1 \geqslant n.$$

The method we employ in the proof of Theorem 2 is by induction on $n$ and is based on the following observation. Let $P_iP_j$ be a diagonal of $\Pi$ (i.e. $P_i$ and $P_j$ are not consecutive and the open segment $P_iP_j - P_i - P_j$ is contained in the interior, $\Pi^* - \Pi$, of $\Pi$). Further let $\mu(P_i, P_j) = 1$. $P_iP_j$ divides $\Pi$ into two polygons, $\Pi_1$ and $\Pi_2$, which have, say, $n_1$ and $n_2$ vertices respectively. The assumption that Theorem 2 holds for polygons with fewer vertices than $\Pi$ yields $F(\Pi_1) \geqslant n_1$ and $F(\Pi_2) \geqslant n_2$ since $n_1$ and $n_2$ are each less than $n$. Adding these inequalities and noting that

$$A(\Pi_1^*) + A(\Pi_2^*) = A(\Pi^*),$$

$$M(\Pi_1) + M(\Pi_2) = M(\Pi) + 2\mu(P_i, P_j) = M(\Pi) + 2$$

and

$$n_1 + n_2 = n + 2$$

we obtain

$$\frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi) + 2}{2} + 2 \geqslant n + 2;$$

hence $F(\Pi) \geqslant n$.

Thus we would wish to show that if, in no weakly admissible polygon with $n$ vertices ($n \geqslant 4$), on which $F$ takes the value $F(\Pi)$, is there a diagonal of $\mu$-length equal to 1, then there exists a w. a. polygon, $\Pi'$, with $n$ vertices such that $F(\Pi') < F(\Pi)$ and $\Pi'$ does contain such a diagonal. For it would then suffice to prove the Theorem for $n = 3$ and when $n \geqslant 4$ for such polygons as contain a diagonal of unit $\mu$-length.

The above objective would be realised if we could show that, (i) in the absence of a diagonal of unit $\mu$-length in any polygon with $n$ vertices on which $F$ takes the value $F(\Pi)$ a local variation of the vertices of $\Pi$ exists under which $F(\Pi)$ is decreased while $\Pi$ remains weakly admissible; and (ii) $F$ attains a minimum on the set of w. a. polygons with $n$ vertices.

The second requirement suggests that we adjoin to the class of w. a. polygons certain limiting ones. Thus, if $\{P_{r1} \ldots P_{rn} P_{r1} \mid r = 1, 2, \ldots\}$ is a sequence of w. a. polygons and $\lim_{r \to \infty} P_{ri} = P_i (i = 1, 2, \ldots, n)$ we adjoin the polygon $P_1 \ldots P_n P_1$ and prove it to be weakly admissible.

Such a limiting polygon may contain singular vertices, such a vertex being one which is contained in a side other than those of which it is an end point. In Figures $1a$–$1d$ we illustrate examples of these.

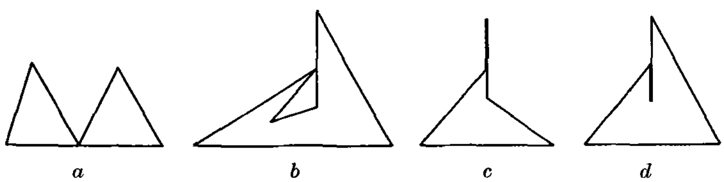



Fig. 1.

Certain singular vertices (e.g. Figs. $1a$, $1c$) lead to a decomposition of the polygon and to an application of the inductional assumption in a similar way to that described when a diagonal has $\mu$-length 1. Others (e.g. Figs. $1b$, $1d$), however, present a new difficulty. The variation of these or of the ends of the sides in which they lie is restricted in so far as a neighbouring polygon may, in a sense, be self-overlapping (Fig. 2). To overcome this difficulty we are led to enlarge the class of weakly admissible polygons still further to include polygons of this type. It will be seen, however, that we are not obliged to allow the angles, as we shall define them, in these polygons to exceed $2\pi$.

In the next section we shall give a precise definition of a class of polygons which meets the requirements we have just outlined. Furthermore we shall show that these polygons possess such properties as are necessary for the proof of Theorem 2 by the method described.

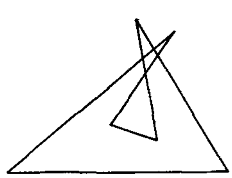


Fig. 2.

## II

In what follows we shall regard the angle at a vertex of a triangle, say $B$ in $ABC$, as the intersection of the half-plane, $p$, which is bounded by the line through $A$ and $B$ and which contains $C$ and the half-plane, $q$, bounded by the line through $C$ and $B$ and containing $A$. If $A'B'C'$ is the image of $ABC$ under a barycentric, orientation-preserving mapping, the angle at $B'$ is the intersection of half-planes which correspond to $p$ and $q$, i.e. the half-plane which is on the same side of $A'B'$ as $p$ is of $AB$ and that which is on the same side of $B'C'$ as $q$ is of $BC$. If, in particular, $A'B'C'$ is an improper triangle, i.e. one in which two angles are zero and the third, say $\angle B'$, is $\pi$, the angle at $B'$ is well defined by the mapping as one of the half-planes bounded by the line through $A'$, $B'$ and $C'$.

By a vertex triangulation of a domain, $K^*$, bounded by a convex Jordan polygon, $K$, with $n$ vertices we shall mean a set of $n-2$ non-overlapping triangles which cover $K^*$ and whose vertices are the vertices of $K$.

DEFINITION 2. *We shall say that a closed polygon,* $\Pi$, *belongs to the class* $\mathfrak{A}$, *if it is the image of a convex, Jordan polygon,* $K$, *with vertices* $P_1, \ldots, P_n, P_{n+1} = P_1$, *under a single-valued mapping,* $\Theta$, *which is defined on* $K^*$ *and has the properties:*

1) $\Theta$ *maps the triangles,* $T_1, \ldots, T_{n-2}$, *of a vertex triangulation of* $K^*$ *barycentrically onto triangles which may be improper, i.e. with angles* $0$, $0$ *and* $\pi$;

2) $\Theta$ *is orientation preserving;*

3) *if* $T_i$ *and* $T_j$ $(i \neq j)$ *have a common vertex, say* $P_r$, *then the angles at* $\Theta P_r$ *in* $\Theta T_i$ *and* $\Theta T_j$ *have no common interior points.*

In addition we define

(i) $\Theta P_i (i = 1, \ldots, n)$ *to be the vertices of* $\Pi$;

(ii) $\angle \Theta P_i$, *the angle in* $\Pi$ *at* $\Theta P_i$, *to be the sum of the angles at* $\Theta P_i$ *in those triangles,* $\Theta T_j$, *for which* $T_j$ *has* $P_i$ *as a vertex*;

(iii) $\Theta P_i \Theta P_{i+1}$ $(i = 1, \ldots, n)$ *to be the sides of* $\Pi$, *and the* $\mu$-*perimeter of* $\Pi$ *to be*

$$M(\Pi) = \sum_{i=1}^{n} \mu(\Theta P_i, \Theta P_{i+1}).$$

(iv) $A(\Pi^*)$, *the area associated with* $\Pi$, *as the sum of the areas of the triangles* $\Theta T_i$ $(i = 1, \ldots, n-2)$.

We note in connection with (ii) that, as a consequence of the conditions of Definition 1, $0 \leqslant \angle \Theta P_i \leqslant 2\pi (i = 1, \ldots, n)$. Regarding (iii) we note that $M(\Pi)$ depends only on

$\Theta P_1, \ldots, \Theta P_n$. With $(x, y)$ as coordinates in a cartesian system we can express $A(\Pi^*)$ according to (iv) as

$$\sum_{i=1}^{n-2} \frac{1}{2} \int_{\Theta T_i} x\,dy - y\,dx,$$

the sense of each path of integration being such that the integrals are each non-negative. Accordingly, in the sum, common sides of the triangles $\Theta T_1, \ldots, \Theta T_{n-2}$ are traversed back and forth precisely once; thus

$$A(\Pi^*) = \tfrac{1}{2} \int_{\Pi} x\,dy - y\,dx$$

and $A(\Pi^*)$ depends only on $\Theta P_1, \ldots, \Theta P_n$.

Definition 3. *We define the straight segment $\Theta P_i \Theta P_j (|i-j| \neq 1)$ to be a diagonal of $\Pi$ if there exists a path, $\lambda$, in $K^*$ whose end points are $P_i$ and $P_j$ and whose inner points are inner points of $K^*$ such that $\Theta$ is a sense preserving mapping of $\lambda$ onto $\Theta P_i \Theta P_j$.*

We may assume that $\lambda$ is a simple polygonal path whose vertices are contained in the common sides of those triangles in the vertex triangulation of $K^*$ associated with $\Theta$ which cover $\lambda$. For if $A$ and $B$ are two points of $\lambda$ contained in the sides of such a triangle, say $T_k$, the straight segment, $AB$, is mapped by $\Theta$ onto the straight segment $\Theta A\, \Theta B$ and we can replace the section of $\lambda$ between $A$ and $B$ by $AB$. Indeed, this section is already $AB$ if $\Theta T_k$ is proper.

Definition 4. *A polygon in $\mathfrak{A}$ is defined to be weakly admissible if its sides and diagonals are each of $\mu$-length not less than 1. We shall denote the subclass of polygons in $\mathfrak{A}$ with this property by $\mathfrak{A}(\mu)$.*

Theorem 3. *Let $K = P_1 \ldots P_n P_{n+1}, P_{n+1} = P_1$ be a convex Jordan polygon and $\Theta$ a mapping of $K^*$ such that $\Theta K$ belongs to $\mathfrak{A}(\mu)$.*

*Let $\Theta P_i \Theta P_j$ be a diagonal of $\Theta K$ and $K_1$ and $K_2$ denote the polygons into which $P_i P_j$ divides $K$.*

*There exist mappings, $\Theta_1$ of $K_1^*$ and $\Theta_2$ of $K_2^*$, such that $\Theta_1$ agrees with $\Theta$ on $K_1 - P_i P_j$ and $\Theta_1 K_1$ is in $\mathfrak{A}(\mu)$; similarly for $\Theta_2$.*

*Proof.* Let $n = 4$ and suppose that $\Theta$ is barycentric on the triangles $P_1P_2P_3$ and $P_1P_3P_4$. If the diagonal $\Theta P_i \Theta P_j$ is $\Theta P_1 \Theta P_3$ the restrictions of $\Theta$ to these triangles are the mappings $\Theta_1$ and $\Theta_2$ of the theorem. If $\Theta P_2 \Theta P_4$ is a diagonal, then $\mu(\Theta P_2, \Theta P_4) \geqslant 1$ and we may take $\Theta_1$ and $\Theta_2$ to be the barycentric mappings of $P_1P_2P_4$ onto $\Theta P_1 \Theta P_2 \Theta P_4$ and $P_2P_3P_4$ onto $\Theta P_2 \Theta P_3 \Theta P_4$ respectively.

Let us assume that the theorem is true for polygons in $\mathfrak{A}(\mu)$ with $m$ vertices, $4 < m < n$.

There is no loss of generality in taking $i < j$ and letting $K_1 = P_1 \ldots P_i P_j P_{j+1} \ldots P_n P_1$ and $K_2 = P_i P_{i+1} \ldots P_{j-1} P_j P_i$.

We shall be concerned with those of the diagonals of $K$ which are sides of the triangles in the triangulation of $K^*$ associated with $\Theta$ and shall refer to these as $\Theta$-diagonals of $K$.

Let $\lambda$ be a polygonal path in $K^*$ such that $\Theta\lambda = \Theta P_i \Theta P_j$. To prove that $\Theta_1$ exists we examine $\lambda$ in relation to those $\Theta$-diagonals of $K$ which have an end point amongst the vertices of $K_2$. We consider separately the case in which there exists a $\Theta$-diagonal both of whose end points are vertices of $K_2$ and that in which there is no such $\Theta$-diagonal.

(a) Let $P_r P_s$ be a $\Theta$-diagonal, $i \leqslant r < s \leqslant j$; i.e. $P_r$ and $P_s$ are both vertices of $K_2$. Further, let $P_r P_s P_t$ be the triangle in the vertex triangulation of $K^*$ which has $P_r P_s$ as a side and lies in $K'^*$ where $K' = P_s \ldots P_n P_1 \ldots P_r P_s$. We examine the possibility that $\lambda$ contains points in $P_r P_s$; this is certainly the case if it contains points in $K^* - K'^*$. There is a point, say $A$, in $P_r P_s P_t$ which is nearest along $\lambda$ to $P_i$ and a point, say $B$, in $P_r P_s P_t$ which is nearest to $P_j$. Since $\Theta$ maps $AB$ onto a straight segment we can replace the section of $\lambda$ between $A$ and $B$ by $AB$. The resulting path is one which coincides with $P_r P_s$ if $i = r$ and $j = s$, for then $A \equiv P_i$ and $B \equiv P_j$, or its interior points lie in the interior of $K'^*$. If $\lambda$ contains no points of $P_r P_s$ its interior points lie in the interior of $K'^*$. Thus we can assume under all circumstances that either $\lambda$ coincides with $P_r P_s$ or its interior points lie in the interior of $K'^*$.

Let $\Theta'$ be the restriction of $\Theta$ to $K'^*$. Clearly $\Theta' K'$ belongs to $a$. $\Theta' K'$ is in fact in $\mathfrak{A}(\mu)$. For a diagonal of $\Theta' K'$ is a diagonal of $\Theta K$ and therefore of $\mu$-length not less than 1 while the sides of $\Theta' K'$, being amongst those of $\Theta K$ and in addition $\Theta P_r \Theta P_s$ which is a diagonal of $\Theta K$, have $\mu$-length not less than 1. Furthermore $\Theta' P_i \Theta' P_j$ is a diagonal of $\Theta' K'$ or a side of $\Theta' K'$. The latter is certainly the case if $K'$ has only three vertices for then $r = 1$ and $s = n - 1$ and we must have $i = r$ and $j = s$. If $\Theta' P_i \Theta' P_j$ is a side of $\Theta' K'$ then $\Theta'$ is the mapping $\Theta_1$ which we seek. If $\Theta' P_i \Theta' P_j$ is a diagonal of $\Theta' K'$, $K'$ has more than three vertices and we apply the inductional assumption to $\Theta' K'$ to obtain the existence of $\Theta_1$.

(b) Suppose that there is no $\Theta$-diagonal of $K$ both of whose end points are vertices of $K_2$.

We shall show that all of the points, $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$, lie in one of the open half-planes bounded by $L$, the straight line containing $\Theta P_i$ and $\Theta P_j$ or in $L$ itself.

If $j - i = 2$ the above assertion is trivial. Let $j - i > 2$ and consider $P_r P_{r+1}$ where $i < r < j - 1$. The triangle in the vertex triangulation of $K^*$ with side $P_r P_{r+1}$ is such that its third vertex, say $P_l$, is not in $K_2$. It follows that, since inner points of $\lambda$ are by definition in the interior of $K^*$, $\lambda$ contains an inner point in each of $P_r P_l$ and $P_{r+1} P_l$. Hence $\Theta P_i \Theta P_j$

contains an inner point of $\Theta P_r \Theta P_l$ and an inner point of $\Theta P_{r+1} \Theta P_l$. If therefore, $\Theta P_r$ lies in $L$ then $\Theta P_r \Theta P_l$ lies in $L$ and in particular $\Theta P_l$ lies in $L$ and conversely. Similarly, if $\Theta P_l$ lies in $L$ so also does $\Theta P_{r+1}$. Considering each of the sides of $K_2$ we see that if one of the vertices $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$ lies in $L$ so do they all and in addition $L$ contains the images of the vertices of $K_1$, which are end points of the $\Theta$-diagonals the other end points of which are $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$. If, on the other hand, $\Theta P_r$ does not lie in $L$ then $\Theta P_l$ does not lie in $L$; indeed $\Theta P_r$ lies in one of the open half-planes bounded by $L$ and $\Theta P_l$ in the other. Similarly, if $L$ does not contain $\Theta P_l$ then $\Theta P_{r+1}$ does not lie in $L$ but in the open half-plane bounded by $L$ which does not contain $\Theta P_l$ namely the same one as contains $\Theta P_r$. Again considering each of the sides of $K_2$ we see that if one of the vertices $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$ does not lie in $L$ they all lie in one of the open half-planes bounded by $L$ while the other contains the images of those vertices of $K_1$ which are end points of the $\Theta$-diagonals with end points $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$.

What we have just shown implies that amongst $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$ there is a vertex, say $\Theta P_r$, at which the angle is not greater than $\pi$. For if $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$ all lie in one of the open half-planes bounded by $L$ it suffices to choose $\Theta P_r$ to be one which is furthest from $L$.

Let us suppose that $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$ all lie in $L$, then the angles, $\angle \Theta P_{i+1}, \ldots, \angle \Theta P_{j-1}$ can have the values $0, \pi$, or $2\pi$ and and we must show that they are not all $2\pi$. We choose a direction in $L$ according to which $\Theta P_j$ is to the right of $\Theta P_i$. Let $\Theta P_{i+1}$, be to the left of $\Theta P_i$. If $\angle \Theta P_{i+1}$ were equal to $2\pi$ there would be a $\Theta$-diagonal with end point $P_{i+1}$, say $P_{i+1} P_a$, such that $\Theta P_a$ is to the left of $\Theta P_{i+1}$. Thus $\Theta P_{i+1} \Theta P_a$ would have no point in common with $\Theta P_i \Theta P_j$ which is not possible since $P_{i+1} P_a$ contains points of $\lambda$. Hence if $\Theta P_{i+1}$ is to the left of $\Theta P_i$ then $\angle \Theta P_{i+1}$ is equal to 0 or $\pi$. Let $\Theta P_{i+1}$ be to the right of $\Theta P_i$ and let $\angle \Theta P_{i+1} = 2\pi$. Then there is a $\Theta$-diagonal, say $P_{i+1} P_b$, with $P_{i+1}$ as an end point such that $\Theta P_b$ is to the right of $\Theta P_{i+1}$. Since, moreover, $\Theta P_{i+1} \Theta P_b$ and $\Theta P_i \Theta P_j$ have an interior point in common, $\Theta P_{i+1}$ is to the left of $\Theta P_j$ and so $\Theta P_{i+2}$ is distinct from $\Theta P_j$. If $\angle \Theta P_{i+2} = 2\pi$ there would be a $\Theta$-diagonal, say $P_{i+2} P_c$, with $P_{i+2}$ as an end point such that $\Theta P_c$ is to the left of $\Theta P_{i+2}$. Since, however, $P_{i+2} P_c$ contains a point of $\lambda$ between $P_j$ and those points of $\lambda$ in $P_{i+1} P_b$ it follows that $\Theta P_{i+2} \Theta P_c$ cannot lie wholly to the left of $\Theta P_{i+1} \Theta P_b$. Hence $\Theta P_c$ is to the right of $\Theta P_{i+2}$ and $\angle \Theta P_{i+2}$ cannot be equal to $2\pi$. Thus under all circumstances there is a vertex amongst $\Theta P_{i+1}, \ldots, \Theta P_{j-1}$ say $\Theta P_r$, such that $\angle \Theta P_r \leqslant \pi$.

We fix our attention on $P_r$ and the union of those triangles in the vertex triangulation of $K^*$ which have $P_r$ as a vertex. Let $S_r$ denote the polygon bounding the domain defined in this way. Amongst the vertices of $S_r$ other than $P_{r-1}$, $P_r$ and $P_{r+1}$ there is one, say $P_l$, with the following property. The triangles, say $P_r P_l P_a$ and $P_r P_l P_b$, in the vertex triangulation of $K^*$ which have $P_r P_l$ as a common side are mapped by $\Theta$ onto a quadrilateral no angle of which exceeds $\pi$. Thus, if $\Theta P_r$ does not lie in $L$, then no vertex of $\Theta S_r$ lies in $L$ and we can choose $\Theta P_l$ to be that which is furthest from $L$. For then $\Theta P_r \Theta P_a \Theta P_l \Theta P_b$ lies in the strip bounded by the two lines parallel to $L$ one of which contains $\Theta P_r$ and the other $\Theta P_l$. If $\Theta P_r$ is contained in $L$ then so are all of the vertices of $\Theta S_r$ and from amongst those of $S_r$ other than $P_{r-1}$, $P_r$ and $P_{r+1}$ we choose $P_l$ to be such that $\Theta P_r \Theta P_l$ is longest.

We shall show that $\Theta P_r \Theta P_a \Theta P_l \Theta P_b$ then has the desired property. Let $P_r P_l P_a$ be on the same side of $P_r P_l$ as $P_i$ and $P_r P_l P_b$ on the same side as $P_j$. It suffices to show that at least one of the angles at $\Theta P_l$ in $\Theta P_r \Theta P_l \Theta P_a$ and $\Theta P_r \Theta P_l \Theta P_b$ is 0. Let $\Theta P_j$ be to the right of $\Theta P_i$ as before and suppose that $\Theta P_l$ is to the right of $\Theta P_r$. We must show that $\Theta P_a$ is to the left of $\Theta P_l$. Were it otherwise then $\Theta P_r \Theta P_a$ would be longer than $\Theta P_r \Theta P_l$ hence $P_a \equiv P_{r-1}$. However, $P_{r-1} P_l$ contains a point of $\lambda$ between $P_i$ and the points of $\lambda$ which $P_r P_l$ contains. Hence $\Theta P_{r-1} \Theta P_l$ cannot lie to the right of $\Theta P_r \Theta P_l$. Thus $\Theta P_a$ cannot be to the right of $\Theta P_l$ and the angle at $\Theta P_l$ in $\Theta P_r \Theta P_l \Theta P_a$ is 0. Similarly, if $\Theta P_r$ is to the right of $\Theta P_l$ then the angle at $\Theta P_l$ in $\Theta P_r \Theta P_l \Theta P_b$ is 0.

The angles in the quadrilateral $\Theta P_r \Theta P_a \Theta P_l \Theta P_b$ being each not greater than $\pi$, there is an orientation preserving mapping which takes $P_r P_a P_l$ and $P_r P_b P_l$ barycentrically onto $\Theta P_r \Theta P_a \Theta P_l$ and $\Theta P_r \Theta P_b \Theta P_l$ respectively. Thus we can retriangulate the domain bounded by $P_r P_a P_l P_b$ and modify $\Theta$ accordingly. As a result the number of vertices of $S_r$ is decreased. Repeating the process sufficiently many times $P_{r-1} P_{r+1}$ becomes a $\Theta$-diagonal and we again have the situation dealt with in (a).

By giving similar consideration to the vertices of $K_1$ as we have given to those of $K_2$ we obtain the existence of $\Theta_2$ also.

COROLLARY. *Let $K = P_1 \ldots P_n P_{n+1}$, $P_{n+1} = P_1$ be a convex Jordan polygon and $\Theta K$ a polygon in $\mathfrak{A}(\mu)$. Let $\lambda$ be a path in $K^*$ one of whose end points is $P_i$ and the other an interior point, $Q$, of the side $P_j P_{j+1}$ $(i \neq j, j+1)$ while the inner points of $\lambda$ are in the interior of $K^*$. Furthermore, let $\Theta \lambda = \Theta P_i$.*

*There exists a mapping, $\Theta_1$, which agrees with $\Theta$ on $P_i P_{i+1} \ldots P_{j-1} P_j$ such that $\Theta_1 (P_i P_{i+1} \ldots P_{j-1} P_j P_i)$ is in $\mathfrak{A}(\mu)$ provided that $j \neq i+1$; also a mapping, $\Theta_2$, exists which agrees with $\Theta$ on $P_{j+1} P_{j+2} \ldots P_{i-1} P_i$ such that $\Theta_2 (P_i P_{j+1} P_{j+2} \ldots P_{i-1} P_i)$ is in $\mathfrak{A}(\mu)$ provided that $j+1 \neq i-1$.*

*Proof.* In the vertex triangulation of $K^*$ associated with $\Theta$ let the other sides of the triangle with one side $P_j P_{j+1}$ be $P_j P_k$ and $P_{j+1} P_k$. The path, $\lambda$, contains a point, say $A$, in one of these two sides. Substituting for the section of $\lambda$ between $A$ and $Q$ by $A P_j$ we get a path with end points $P_i$ and $P_j$ which is mapped by $\Theta$ onto $\Theta P_i \Theta P_j$. Thus, if $j \neq i + 1$,

$\Theta P_i \Theta P_j$ is a diagonal and applying the theorem we obtain the existence of $\Theta_1$. Similarly, substituting $AP_{j+1}$ for the section of $\lambda$ between $A$ and $Q$ we find that $\Theta P_i \Theta P_{j+1}$ is a diagonal provided $j+1 \neq i-1$. Again applying the theorem we obtain the existence of $\Theta_2$, which proves the corollary.

We shall say that $\Theta P_i$ is an inner point of $\Theta P_j \Theta P_{j+1}$ ($i \neq j, j+1$) when, as in the above corollary, there exists a path, $\lambda$, whose end points are $P_i$ and an inner point of $P_j P_{j+1}$, whose remaining points are in the interior of $K^*$ and which is mapped by $\Theta$ onto the single point, $\Theta P_i$.

We next prove that certain sequences of polygons in $\mathfrak{A}(\mu)$ have a limit which is again a polygon in $\mathfrak{A}(\mu)$. Thus

THEOREM 4. *Let $K = P_1 \dots P_n P_{n+1}$, $P_{n+1} = P_1$ be a convex Jordan polygon and $\mathfrak{L}$ a vertex triangulation of $K^*$, the domain bounded by $K$.*

*Let $\Theta_1, \Theta_2, \dots$ be a sequence of mappings of $K^*$ such that*

(i) *$\Theta_r K$ is in $\mathfrak{A}(\mu)$, $r = 1, 2, \dots$;*
(ii) *The vertex triangulation of $K^*$ associated with $\Theta_r$ is $\mathfrak{L}$, $r = 1, 2, \dots$;*
(iii) *$\lim_{r\to\infty} \Theta_r P_i$ exists for $i = 1, 2, \dots, n$.*

*The mapping, $\varphi$, of $K^*$ which takes $P_i$ onto $\lim_{r\to\infty} \Theta_r P_i$, $i = 1, \dots, n$, and is barycentric on the triangles of $\mathfrak{L}$ is such that $\varphi K$ belongs to $\mathfrak{A}(\mu)$.*

*Proof.* Let $P_u P_v P_w$ be a triangle in $\mathfrak{L}$ and in a cartesian coordinate system let

$$|P_u, P_v, P_w| = \begin{vmatrix} x_u & y_u & 1 \\ x_v & y_v & 1 \\ x_w & y_w & 1 \end{vmatrix}$$

where $(x_u, y_u)$, $(x_v, y_v)$ and $(x_w, y_w)$ are coordinates of $P_u$, $P_v$ and $P_w$ respectively. That $\Theta_r$ $(r = 1, 2, \dots)$ preserves the orientation of $P_u P_v P_w$ is equivalent to

$$|P_u, P_v, P_w| \cdot |\Theta_r P_u, \Theta_r P_v, \Theta_r P_w| \geqslant 0.$$

But

$$\lim_{r\to\infty} |\Theta_r P_u, \Theta_r P_v, \Theta_r P_w| = |\lim Q_r P_u, \lim \Theta_r P_v, \lim \Theta_r P_w| = |\varphi P_u, \varphi P_v, \varphi P_w|.$$

Hence
$$|P_u, P_v, P_w| \cdot |\varphi P_u, {}_\varphi P_v, \varphi P_w| \geqslant 0$$

and $\varphi$ preserves the orientation of $P_u P_v P_w$.

We next prove that

$$\mu(\varphi P_i, \varphi P_{i+1}) \geqslant 1 \quad (i = 1, \dots, n)$$

and
$$\mu(\varphi P_i, \varphi P_j) \geqslant 1$$

whenever $\varphi P_i \varphi P_j$ is a diagonal.

Since

$$\mu(\varphi P_i, \varphi P_{i+1}) = \mu(\lim_{r\to\infty} \Theta_r P_i, \lim_{r\to\infty} \Theta_r P_{i+1}) = \lim_{r\to\infty} \mu(\Theta_r P_i, \Theta_r P_{i+1})$$

and $\mu(\Theta_r P_i, \Theta_r P_{i+1}) \geqslant 1$ $(r = 1, 2, \ldots)$ it follows that $\mu(\varphi P_i, \varphi P_{i+1}) \geqslant 1$.

Suppose that $\varphi P_i \varphi P_j$ is a diagonal, the image under $\varphi$ of a polygonal path, say $\lambda = Q_0 Q_1 \ldots Q_s Q_{s+1}$ where $Q_0 = P_i$, $Q_{s+1} = P_j$ and $Q_1, \ldots, Q_s$ are interior points of $\varphi$-diagonals, $P_{k_1} P_{l_1} \ldots, P_{k_s} P_{l_s}$ respectively. In order to show that $\mu(\varphi P_i, \varphi P_j) \geqslant 1$ we consider the sequence $\Theta_1 \lambda$, $\Theta_2 \lambda$, .... Since

$$\lim_{r\to\infty} \Theta_r Q_h = \varphi Q_h (h = 0, 1, \ldots s, s+1)$$

it follows that

$$\lim_{r\to\infty} \mu(\Theta_r Q_h, \Theta_r Q_{h+1}) = \mu(\varphi Q_h, \varphi Q_{h+1}) (h = 0, 1, \ldots s).$$

Recalling that, by definition, $\varphi$ preserves the sense of $\lambda$ in mapping it onto $\varphi P_i \varphi P_j$ we have that

$$\mu(\varphi P_i, \varphi P_j) = \sum_{h=0}^{s} \mu(\varphi Q_h, \varphi Q_{h+1}) = \sum_{h=0}^{s} \lim_{r\to\infty} \mu(\Theta_r Q_h, \Theta_r Q_{h+1}) = \lim_{r\to\infty} \sum_{h=0}^{s} \mu(\Theta_r Q_h, \Theta_r Q_{h+1}.$$

That $\mu(\varphi P_i, \varphi P_j) \geqslant 1$ will follow therefore if we can show that, for each $r$, $M(\Theta_r \lambda) = \sum_{h=0}^{s} \mu(\Theta_r Q_h, \Theta_r Q_{h+1}) \geqslant 1$.

In order to do so we consider the set, $\Lambda$, of all the polygonal paths with end points $P_i$ and $P_j$ which have vertices which correspond to those of $\lambda$ and which lie in the closed segments $P_{k_1} P_{l_1}, \ldots, P_{k_s} P_{l_s}$. Considering the images of these paths under $\Theta_r$, the set, $\Lambda$, as defined, being closed, it contains a path, say $\bar{\lambda} = Q_0 \bar{Q}_1 \ldots \bar{Q}_s Q_{s+1}$, such that $M(\Theta_r \bar{\lambda})$ is a minimum. It suffices to show that $M(\Theta_r \bar{\lambda}) \geqslant 1$.

Let $t$, $0 < t \leqslant s+1$, be the smallest index for which $\bar{Q}_t = P_{k_t}$ or $\bar{Q}_t = P_{l_t}$, say $\bar{Q}_t = P_{k_t}$, where $\bar{Q}_{s+1} = P_{k_{s+1}} = P_j$. If $t = 1$ then $\Theta_r \bar{\lambda}$ contains $\Theta_r P_i \Theta_r P_{k_1}$. Noting that $P_i P_{k_1} P_{l_1}$ is a triangle in the vertex triangulation, $\mathfrak{L}$, it follows that $\Theta_r P_i \Theta_r P_{k_1}$ is either a side or a diagonal of $\Theta_r K$ and therefore has $\mu$-length not less than 1. Hence $M(\Theta_r \bar{\lambda}) \geqslant 1$. Suppose that $t > 1$. If for some $u$, $0 < u < t$, $\angle \Theta_r \bar{Q}_{u-1} \Theta_r \bar{Q}_u \Theta_r \bar{Q}_{u+1}$ is different from $\pi$, then $\bar{Q}_u$ being an interior point of $P_{k_u} P_{l_u}$ we can vary it along $P_{k_u} P_{l_u}$ in such a way that $\mu(\Theta_r \bar{Q}_{u-1}, \Theta_r \bar{Q}_u) + \mu(\Theta_r \bar{Q}_u, \Theta_r \bar{Q}_{u+1})$ is decreased. Indeed we obtain thereby a path, say $\bar{\lambda}'$ in $\Lambda$ such that $M(\Theta_r \bar{\lambda}') < M(\Theta_r \bar{\lambda})$ which is a contradiction. Hence, $\angle \Theta_r \bar{Q}_{u-1} \Theta_r \bar{Q}_u \Theta_r \bar{Q}_{u+1} = \pi$ $(0 < u < t)$

implying that $\Theta_r P_i \Theta_r P_{k_t}$ is a diagonal with $P_i Q_1 \ldots Q_{t-1} P_{k_t}$ as antecedent or $\Theta_r P_i \Theta_r P_{k_t}$ is a side of $\Theta_r K$. In either case $\mu(\Theta_r P_i, \Theta_r P_{k_t}) \geqslant 1$. Hence $M(\Theta_r \bar{\lambda}) \geqslant 1$.

It follows that, in particular, the sides of the triangles of $\mathfrak{C}$ are mapped by $\varphi$ onto segments of $\mu$-length not less than 1. Thus the images under $\varphi$ of these triangles have distinct vertices and the angles at these vertices are therefore well defined.

It remains for us to show that if $T_{i_1}$ and $T_{i_2}$ are a pair of triangles in $\mathfrak{C}$ which have $P_i$ as a common vertex then the angles at $\varphi P_i$ in $\varphi T_{i_1}$ and $\varphi T_{i_2}$ have no common interior point. Let us suppose, on the contrary, that $A$ is such a common interior point. There exists $r_1$ such that, if $r > r_1$, $A$ is an inner point of the angle at $\Theta_r P_i$ in $\Theta_r T_{i_1}$ and $r_2$ such that, if $r > r_2$, $A$ is an inner point of the angle at $\Theta_r P_i$ in $\Theta_r T_{i_2}$. Thus, if $r > \max(r_1, r_2)$, $A$ is an inner point of the angles at $\Theta_r P_i$ in both $\Theta_r T_{i_1}$ and $\Theta_r T_{i_2}$ which contradicts the hypothesis that $\Theta_r K$ is in $\mathfrak{A}(\mu)$. Hence the angles at $\varphi P_i$ in $\varphi T_{i_1}$ and $\varphi T_{i_2}$ have no inner points in common. We have therefore shown that $\varphi K$ satisfies the conditions necessary for it to belong to $\mathfrak{A}(\mu)$.

In addition to the above we shall have need of

Theorem 5. *Let $K = P_1 \ldots P_n P_{n+1}$, $P_{n+1} = P_1$ be a convex Jordan polygon and $\Theta K$ a polygon in $\mathfrak{A}(\mu)$. If no vertex of $\Theta K$ is an interior point of a side there exists a mapping, $\Theta'$, of $K^*$ which agrees with $\Theta$ on $K$, $\Theta' K$ is in $\mathfrak{A}(\mu)$ and $\Theta'$ maps the triangles in the vertex triangulation of $K^*$ associated with it onto proper triangles.*

*Proof.* If $\Theta$ maps the triangles in the vertex triangulation of $K^*$ associated with it onto proper triangles the theorem is true with $\Theta' = \Theta$ throughout $K^*$.

Let $m(\Theta)$ denote the number of triangles which $\Theta$ maps onto improper triangles and suppose $m(\Theta) > 0$. It suffices to show that a mapping, $\varphi$, of $K^*$ exists which agrees with $\Theta$ on $K$, such that $\varphi K$ is in $\mathfrak{A}(\mu)$ and $m(\varphi) < m(\Theta)$.

Let $P_r P_s P_t$ be a triangle which $\Theta$ maps onto an improper triangle the angle, say at $\Theta P_t$, being equal to $\pi$. We can assume without loss of generality that $r < s$. Since, according to our hypothesis, $P_r P_s$ cannot be a side of $K$ let $P_r P_u P_s$ be the other triangle with $P_r P_s$ as a side. We note that $r < u < s$ if we assume, as we may, that $t < r$ or $t > s$.

The triangle $\Theta(P_r P_u P_s)$ can be distinguished according to whether

(1) $\Theta(P_r P_u P_s)$ is a proper triangle,

(2) $\Theta(P_r P_u P_s)$ is improper and the angle at $\Theta P_u$ is 0,

or (3) $\Theta(P_r P_u P_s)$ is improper and the angle at $\Theta P_u$ is $\pi$.

If (1) is the case we define $\varphi$ as a mapping which agrees with $\Theta$ on $K$ and on those triangles associated with $\Theta$ other than $P_r P_s P_t$ and $P_r P_u P_s$ while it maps $P_t P_r P_u$ and

$P_t P_u P_s$ barycentrically. Indeed $\varphi$ maps these onto proper triangles and we have $m(\varphi) = m(\Theta) - 1$, while $\varphi K$ is clearly in $\mathfrak{A}(\mu)$.

If $\Theta(P_r P_u P_s)$ is as in (2) the angle at either $\Theta P_r$ or $\Theta P_s$ is 0, say at $\Theta P_r$. Let us write $t_1$ for $s$, $r_1$ for $r$ and $s_1$ for $u$. The triangle, $P_{r_1} P_{s_1} P_{t_1}$, is mapped by $\Theta$ onto an improper triangle in which the angle at $\Theta P_{t_1}$ is $\pi$. In particular $0 < s_1 - r_1 < s - r$. If $\Theta(P_r P_u P_s)$ is as in (3) we replace $\Theta$ by $\Theta_1$ where $\Theta_1$ is defined in the same way as $\varphi$ in the preceding paragraph, although in this case $m(\Theta_1) = m(\Theta)$. However, we may replace $\Theta$ by $\Theta_1$ in the proof of the theorem. In one of the triangles $\Theta_1(P_t P_r P_u)$ and $\Theta_1(P_t P_u P_s)$ the angle at $\Theta_1 P_t$ is equal to $\pi$ and in the other 0. We choose the one in which it is $\pi$, say in $\Theta_1(P_t P_r P_u)$. Writing $t_1$ *for* $t$, $r_1$ *for* $r$ *and* $s_1$ *for* $u$, *the triangle* $P_{r_1} P_{s_1} P_{t_1}$ *is mapped by* $\Theta_1$ *onto an improper triangle* in which the angle at $\Theta_1 P_{t_1}$ is $\pi$. Furthermore, $0 < s_1 - r_1 < s - r$.

In either of the cases (2) and (3) we apply the same argument to $P_{r_1} P_{s_1} P_{t_1}$ as we have done to $P_r P_s P_t$. Either the other triangle with side $P_{r_1} P_{s_1}$ is as in (1) and we obtain the mapping $\varphi$ or we are led to a triangle $P_{r_2} P_{s_2} P_{t_2}$ which is obtained in the same manner as $P_{r_1} P_{s_1} P_{t_1}$. Continuing in this way either we come to the mapping $\varphi$ with $m(\varphi) = m(\Theta) - 1$ or to a sequence of triangles $P_{r_1} P_{s_1} P_{t_1}$, $P_{r_2} P_{s_2} P_{t_2}$, ... each of which is obtained in the same way as $P_{r_1} P_{s_1} P_{t_1}$. But since $s_1 - r_1 > s_2 - r_2 > \cdots$, for some $k$, $\Theta P_{t_k}$ is the interior point of a side, namely $\Theta P_{r_k} \Theta P_{s_k}$, which contradicts the hypothesis.

## III

We return now to the proof of Theorem 2 and in order to establish $(I_3)$ we first prove

LEMMA 1. *If $T$, a triangle in $\mathfrak{A}(\mu)$ with vertices $P$, $Q$, $R$ has a pair of sides of $\mu$-length greater than* 1 *there exists a triangle $T'$ in $\mathfrak{A}(\mu)$ such that $F(T') < F(T)$.*

*Proof.* If $T$ is an improper triangle with, say, $\angle P = \pi$, $\mu(Q, R) > 2$ and if, say, $\mu(P, R) > 1$ a point $R'$ in $PR$ such that $1 \leqslant \mu(P, R') < \mu(P, R)$ satisfies $F(PQR') < F(T)$.

Suppose that $T$ is a proper triangle and let $\mu(P, Q) > 1$ and $\mu(P, R) > 1$. Since $P$ lies outside $\Gamma(Q)$ and $\Gamma(R)$ (see Fig. 3) there exists a neighbourhood of $P$ with the same property

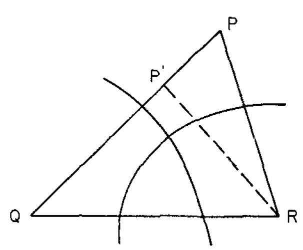



Fig. 3.

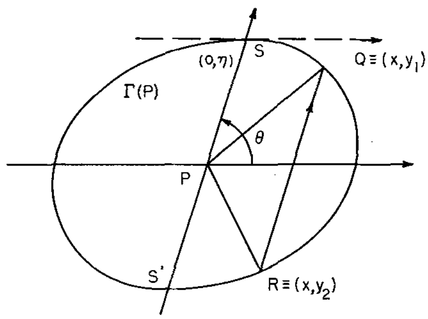



Fig. 4.

and in particular a point, $P'$, such that $\overrightarrow{QP'} = \lambda \overrightarrow{QP} (0 < \lambda < 1)$. Let $T'$ be the triangle with vertices $P', Q, R$. We have $A(T'^*) < A(T^*)$ and since $\mu(P', R) \leqslant \mu(P', P) + \mu(P, R)$

$$\mu(P', R) + \mu(P', Q) \leqslant \mu(P', P) + \mu(P', Q) + \mu(P, R)$$

and
$$M(T') \leqslant M(T).$$

Hence
$$F(T') < F(T).$$

It suffices therefore to prove $(\mathrm{I}_3)$ for a triangle having at most one side of $\mu$-length greater than 1.

In the triangle, $T$, with vertices $P, Q$ and $R$ let $\mu(P, Q) = \mu(P, R) = 1$ and $\mu(Q, R) \geqslant 1$.

We choose a coordinate system in the following manner. With $P$ as origin let the $y$-axis be the line through $P$ parallel to and having the same sense as $RQ$. This meets $\Gamma(P)$ at $S$ and $S'$, say. As $x$-axis we choose the line through $P$ parallel to the tangent [1] to $\Gamma(P)$ at $S$ and having that sense by which the $x$-coordinate of $Q$, hence also of $R$, is non-negative.

Since $\mu(Q, R) \geqslant 1$, $QR \geqslant PS$ and since $Q$ and $R$ lie on $\Gamma(P)$ they lie on opposite sides of the $x$-axis.

Let $(x, y_1)$ and $(x, y_2)$ be the coordinates of $Q$ and $R$ respectively. We have $x \geqslant 0$, $y_1 > 0$ and $y_2 < 0$. Referring to Fig. 4,

$$F(T(x)) = \frac{\sin\theta}{2\Delta} x(y_1 - y_2) + \frac{1}{2\eta}(y_1 - y_2) + 2.$$

Differentiating twice with respect to $x$:

$$F''(x) = \frac{\sin\theta}{2\Delta}[x(y_1'' - y_2'') + 2(y_1' - y_2')] + \frac{1}{2\eta}(y_1'' - y_2'').$$

[1] We shall assume throughout that $\Gamma$ is twice differentiable.

Since $y_1 > 0$, $y_1' \leqslant 0$ and $y_1'' < 0$ while $y_2 < 0$ implies that $y_2' \geqslant 0$ and $y_2'' > 0$. It follows that $F''(x) < 0$. Thus, either $\mu(Q, R) = 2$, $\mu(Q, R) = 1$ or there exists $T'$ such that $F(T') < F(T)$.

It is sufficient therefore to consider the cases: $\mu(Q, R) = 2$ and $\mu(Q, R) = 1$. If $\mu(Q, R) = 2$ then $F(T) = 3$ and $(I_3)$ is satisfied with equality. In the case $\mu(Q, R) = 1$, $(I_3)$ is a consequence of the fact (see Mahler [3], p. 693) that if $P$, $Q$ and $R$ are such that $PQ$, $PR$ and $QR$ each have $\mu$-length 1 the lattice generated by $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is admissible. For this implies that $A(T)$ in this case is not less than $\frac{1}{2}\Delta$. Hence

$$F(T) \geqslant \frac{\frac{1}{2}\Delta}{\Delta} + \frac{3}{2} + 1 = 3.$$

This completes the proof of $(I_3)$.

We shall complete the proof of Theorem 2 by means of induction on $n$. This will be based on

LEMMA 2. *Let $K$ be a convex Jordan polygon with vertices $P_1, \ldots, P_n$ and $\Theta K = \Pi$ be a polygon in $\mathfrak{A}(\mu)$. Either*

(i) *$\Pi$ has a vertex which is an interior point of a side,*

(ii) *$\Pi$ has a diagonal of $\mu$-length* 1

or (iii) *there exists a polygon, $\Pi'$, in $\mathfrak{A}(\mu)$ with $n$ vertices such that $F(\Pi') < F(\Pi)$ or $\Pi'$ has property* (i) *or* (ii) *and $F(\Pi') = F(\Pi)$.*

In proving this lemma we shall repeatedly apply

LEMMA 3. *Let $\Pi = \Theta P_1 \Theta P_2 \ldots \Theta P_n \Theta P_1$ be a polygon in $\mathfrak{A}(\mu)$ whose diagonals each have $\mu$-length greater than* 1 *and which has no vertex an interior point of a side. If it is possible to vary the vertices of $\Pi$ in such a way that $\Pi$ remains in $\mathfrak{A}$ and its sides remain of $\mu$-length not less than* 1 *then a sufficiently small such variation exists under which $\Pi$ remains in $\mathfrak{A}(\mu)$.*

*Proof.* Let us suppose that the lemma is false. There exists a sequence of polygons with $\Pi$ as limit none of which is in $a(\mu)$. Since, however, each of these, by hypothesis, is in $\mathfrak{A}$ and has sides of $\mu$-length not less than 1 there must be in each a diagonal of $\mu$-length less than 1. There is therefore a subsequence in which these diagonals are the images of paths in $K^*$ all of which have the same end points, say $P_r$ and $P_s$. Let these paths be $\lambda_1, \lambda_2, \ldots$. Recalling that each such path is polygonal with vertices lying in the diagonals of $K$ there exists a subsequence $\lambda_{i(1)}, \lambda_{i(2)}, \ldots$ with corresponding vertices which occur in the same diagonals of $K$. The limit of the sequence $\lambda_{i(1)}, \lambda_{i(2)}, \ldots$ is a path, $\lambda$, which is mapped by $\Theta$ onto $\Theta P_r \Theta P_s$. Furthermore since the images of $\lambda_{i(1)}, \lambda_{i(2)}, \ldots$ each have $\mu$-length less

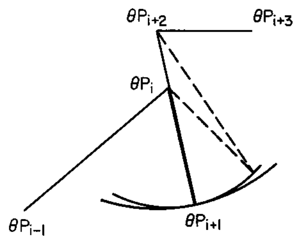



Fig. 5.

than 1 it follows that $\mu(\Theta P_r, \Theta P_s) \leqslant 1$. Either the inner points of $\lambda$ lie in the interior of $K^*$ or certain of the vertices of $\lambda$ coincide with vertices of $K$.

In the former case it follows that $\Theta P_r \Theta P_s$ is a diagonal. Since, however, $\mu(\Theta P_r, \Theta P_s) \leqslant 1$ this is a contradiction.

In the latter case either there is a section of $\lambda$ which is a path with end points which are distinct vertices of $K$ and inner points which are in the interior of $K^*$ this leading to the same contradiction as before or $\lambda$ lies entirely in $K$. In this last case since $P_r$ and $P_s$ are not consecutive $\lambda$ consists of at least two consecutive sides of $K$. This implies that two or more sides of $\Pi$ have $\mu$-lengths the sum of which is less than or equal to 1 and hence there is a side of $\Pi$ whose $\mu$-length is less than 1 which is again a contradiction.

*Proof of Lemma* 2. We shall assume that $\Pi$ has neither property (i) nor (ii) and show that this implies (iii).

1. In virtue of Theorem 5 the negation of (i) allows us to assume that $\Theta$ maps the triangles associated with it onto proper triangles; in particular that $\angle \Theta P_i > 0\,(i = 1, \ldots, n)$.

2. If $\angle \Theta P_i < \pi$ we can assume that $\angle \Theta P_{i+1} < 2\pi$ and $\angle \Theta P_{i-1} < 2\pi$. For suppose that $\angle \Theta P_{i+1} = 2\pi$ (the argument for $\angle \Theta P_{i-1}$ is the same). If

$$\mu(\Theta P_{i+2}, \Theta P_{i+1}) > \mu(\Theta P_i, \Theta P_{i+1})$$

we may, in virtue of Lemma 3, vary $\Theta P_{i+1}$ along $\mu(\Theta P_i, \Theta P_{i+1})\Gamma(\Theta P_i)$ so as to decrease $\angle \Theta P_{i+1}$ (see Fig. 5). Since $\mu(\Theta P_i, \Theta P_{i+1})\Gamma(\Theta P_i)$ lies inside $\mu(\Theta P_{i+2}, \Theta P_{i+1})\Gamma(\Theta P_{i+2})$ this variation decreases $\mu(\Theta P_{i+2}, \Theta P_{i+1})$, hence also $M(\Pi)$. Since, moreover, $A(\Pi^*)$ is decreased this variation results in a decrease in $F(\Pi)$.

Let
$$\mu(\Theta P_{i+2}, \Theta P_{i+1}) = \mu(\Theta P_i, \Theta P_{i+1}).$$

We may vary $\Theta P_{i+1}$ along $\mu(\Theta P_i, \Theta P_{i+1})\Gamma(\Theta P_i)$ so as to increase $\angle \Theta P_i$. This leaves $F(\Pi)$ unchanged but continuing to vary $\Theta P_{i+1}$ in this way leads either to a polygon which satisfies (i) or (ii) or to one in which the number of angles in $\Pi$ which are less than $\pi$ is decreased. Thus, if $\angle\Theta P_{i+2} \leqslant 2\pi - \angle\Theta P_i$ we can decrease $\angle\Theta P_{i+2}$ to 0 if necessary while $\angle\Theta P_i$ remains less than $2\pi$ and if $\angle\Theta P_{i+2} > 2\pi - \angle\Theta P_i$ we can increase $\angle\Theta P_i$ to $\pi$ without $\angle\Theta P_{i+2}$ becoming less than $\pi$.

We must observe that in the course of such a continued variation if at some stage $\Pi$ is no longer in $\mathfrak{A}(\mu)$ there is an earlier point at which $\Pi$ is in $\mathfrak{A}(\mu)$ and has property (i) or (ii). To show this we denote the amount by which $\angle\Theta P_i$ is increased by $t$ and consider $\Pi$ as a function $\Pi(t)$ of $t$. If there are values of $t$, in the range considered, at which $\Pi$ is not in $\mathfrak{A}(\mu)$ let $t_0$ be the g.l.b. of these. Since Lemma 3 is applicable to $\Pi \equiv \Pi(0)$ it follows that $t_0 \neq 0$. We can therefore find an increasing sequence $\{t_i\}$ such that $\{\Pi(t_i)\}$ converges to $\Pi(t_0)$. Furthermore, each polygon in this sequence being in $\mathfrak{A}(\mu)$, it follows from Theorem 4 that $\Pi(t_0)$ is in $\mathfrak{A}(\mu)$. If, however, $\Pi(t_0)$ had neither property (i) nor (ii) we could apply Lemma 3 to contradict the fact that $t_0$ is the g.l.b. of values of $t$ for which $\Pi(t)$ is not in $\mathfrak{A}(\mu)$. Hence $\Pi(t_0)$ has property (i) or (ii).

Let
$$\mu(\Theta P_{i+2}, \Theta P_{i+1}) < \mu(\Theta P_i, \Theta P_{i+1}).$$

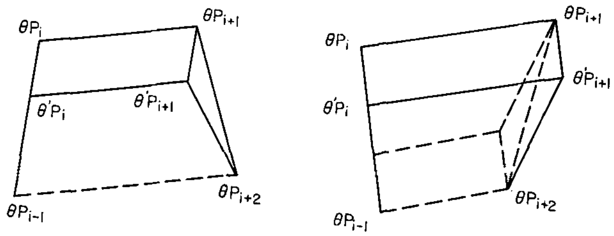
If $\angle\Theta P_{i+2} < 2\pi$ we can decrease $F(\Pi)$ by varying $\Theta P_{i+1}$ along $\mu(\Theta P_{i+2}, \Theta P_{i+1})\Gamma(\Theta P_{i+2})$ so as to decrease $\angle\Theta P_i$. If $\angle\Theta P_{i+2} = 2\pi$ we can move $\Theta P_{i+2}\Theta P_{i+1}$ along $\Theta P_i\Theta P_{i+3}$ without changing $F(\Pi)$ until $\Theta P_{i+2}$ coincides with $\Theta P_i$ or (iii) is satisfied. If at some stage in this variation a polygon is obtained which is not in $\mathfrak{A}(\mu)$ there must be one which is obtained earlier which satisfies (i) or (ii). This follows by the argument of the preceding case in which we now take the parameter, $t$, to be the amount by which $\mu(\Theta P_i, \Theta P_{i+1})$ is decreased. Finally, if $\Theta P_{i+2}$ coincides with $\Theta P_i$ then $\mu(\Theta P_{i+2}, \Theta P_{i+1}) = \mu(\Theta P_i, \Theta P_{i+1})$ and this we have already considered.

Thus, the assumption, $\angle\Theta P_{i+1} = 2\pi$ when $\angle\Theta P_i < \pi$, leads either to a decrease in $F(\Pi)$, to a polygon satisfying (i) or (ii) or to a decrease in the number of angles in $\Pi$ which are less than $\pi$. Hence we can assume that an angle in $\Pi$ which is less than $\pi$ is preceded and followed by angles which are each less than $2\pi$.

3. There are amongst the triangles in the vertex triangulation of $K$ at least two having two sides which are sides of $K$, since there are $n$ sides of $K$ and $n-2$ triangles. Let $\Theta P_{i-1}\Theta P_i\Theta P_{i+1}$ be the image of one of these. According to § 1 of this proof we may assume $\angle\Theta P_i > 0$.

If both $\mu(\Theta P_{i-1}, \Theta P_i)$ and $\mu(\Theta P_i, \Theta P_{i+1})$ are greater than 1 the method of Lemma 1 enables us to decrease $F(\Pi)$ while, according to Lemma 3, $\Pi$ remains in $\mathfrak{A}(\mu)$. We may assume therefore that at least one of $\mu(\Theta P_{i-1}, \Theta P_i)$ and $\mu(\Theta P_i, \Theta P_{i+1})$ is equal to 1. Let $\mu(\Theta P_i, \Theta P_{i+1}) = 1$.

We distinguish two cases according as $\angle\Theta P_{i+1} < \pi$ or $\angle\Theta P_{i+1} \geqslant \pi$.




*a* *b*

Fig. 6.

4. Let $\angle \Theta P_{i+1} < \pi$. According to § 2 we can thus assume that $\angle \Theta P_{i+2} < 2\pi$ and, since $\angle \Theta P_i < \pi$, that $\angle \Theta P_{i-1} < 2\pi$. This enables us to vary $\Theta P_i$ and $\Theta P_{i+1}$ locally while $\Pi$ remains in $\mathfrak{A}$. If, furthermore, the sides of $\Pi$ remain not less than 1 in $\mu$-length the negation of (ii) in addition allows us to apply Lemma 3 to ensure that $\Pi$ remains in $\mathfrak{A}(\mu)$.

If $\mu(\Theta P_{i-1}, \Theta P_i) > 1$ and $\mu(\Theta P_{i+1}, \Theta P_{i+2}) > 1$ we can decrease $F(\Pi)$ as follows: we move $\Theta P_i$ along $\Theta P_{i-1} \Theta P_i$ towards $\Theta P_{i-1}$ and $\Theta P_{i+1}$ varies in such a way as to preserve the vector $\overrightarrow{\Theta P_i \Theta P_{i+1}}$. The change in $A(\Pi^*)$ is equal to the change in $A(\Theta S^*_{i+1})$ which is equal to the change in the area of the quadrilateral $\Theta P_{i-1} \Theta P_i \Theta P_{i+1} \Theta P_{i+2}$ The change in $M(\Pi)$ is equal to that in $\mu(\Theta P_{i-1}, \Theta P_i) + \mu(\Theta P_{i+1}, \Theta P_{i+2})$. In Fig. 6a $\angle \Theta P_i + \angle \Theta P_{i+1} \geqslant \pi$; in Fig. 6b $\angle \Theta P_i + \angle \Theta P_{i+1} < \pi$. In both cases

$$\begin{aligned} &\mu(\Theta P_{i-1}, \Theta' P_i) + \mu(\Theta' P_{i+1}, \Theta P_{i+2}) \\ &\leqslant \mu(\Theta P_{i-1}, \Theta' P_i) + \mu(\Theta P_{i+1}, \Theta P_{i+2}) + \mu(\Theta P_{i+1}, \Theta' P_{i+1}) \\ &= \mu(\Theta P_{i-1}, \Theta P_i) + \mu(\Theta P_{i+1}, \Theta P_{i+2}). \end{aligned}$$

That the area of the quadrilateral is decreased is clear in Fig. 6a while in Fig. 6b it is made evident by placing the new quadrilateral onto the original with $\Theta' P_i$ on $\Theta P_i$ and $\Theta' P_{i+1}$ on $\Theta P_{i+1}$.

We shall therefore assume that $\mu(\Theta P_{i-1}, \Theta P_i)$, say, is equal to 1 and show next that if $\mu(\Theta P_{i+1}, \Theta P_{i+2}) > 1$ we can again decrease $F(\Pi)$.

Let $\Theta P_{i+1} \Theta P_{i+2}$ lie outside $\Gamma(\Theta P_i)$. By moving $\Theta P_{i+1}$ towards $\Theta P_{i+2}$ along $\Theta P_{i+1} \Theta P_{i+2}$ we increase $\mu(\Theta P_i, \Theta P_{i+1})$ which therefore remains not less than 1. However,

$$\mu(\Theta P_i, \Theta P_{i+1}) + \mu(\Theta P_{i+1}, \Theta P_{i+2})$$

is decreased. For if $X$ is a point between $\Theta P_{i+1}$ and $\Theta P_{i+2}$ we have

$$\mu(\Theta P_i, X) < \mu(\Theta P_i, \Theta P_{i+1}) + \mu(\Theta P_{i+1}, X),$$

hence

$$\mu(\Theta P_i, X) + \mu(X, \Theta P_{i+2}) < \mu(\Theta P_i, \Theta P_{i+1}) + \mu(\Theta P_{i+1}, \Theta P_{i+2}).$$

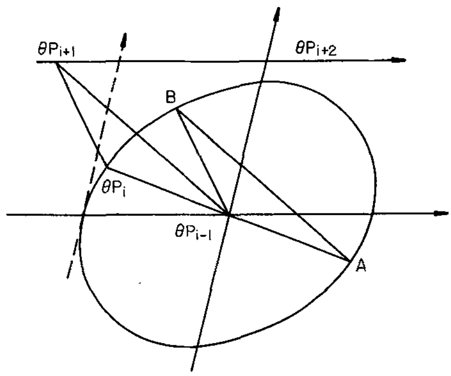



Fig. 7.

Also, the area of the quadrilateral is decreased. Thus we can decrease $F(\Pi)$. So we shall assume that $\Gamma(\Theta P_i)$ intersects $\Theta P_{i+1}\Theta P_{i+2}$ or $\Theta P_{i+1}\Theta P_{i+2}$ produced beyond $\Theta P_{i+2}$ and shall show that $F(\Pi)$ can be decreased by a variation of both $\Theta P_i$ and $\Theta P_{i+1}$ under which $\mu(\Theta P_{i-1}, \Theta P_i)$ and $\mu(\Theta P_i, \Theta P_{i+1})$ remain equal to 1.

We choose a coordinate system as follows. With origin at $\Theta P_{i-1}$ we take as $x$-axis the line parallel to $\overrightarrow{\Theta P_{i+1}\Theta P_{i+2}}$ and as $y$-axis the line which is parallel to the tangents to $\Gamma(\Theta P_{i-1})$ at the points where the $x$-axis cuts $\Gamma(\Theta P_{i-1})$. Let $A$ and $B$ be points of $\Gamma(\Theta P_{i-1})$ such that $\overrightarrow{\Theta P_{i-1}A} = \overrightarrow{\Theta P_i \Theta P_{i-1}}$ and $\overrightarrow{\Theta P_{i-1}B} = \overrightarrow{\Theta P_i \Theta P_{i+1}}$ (see Fig. 7) and let the coordinates of $\Theta P_{i+1}$, $A$ and $B$ be $(x, k)$, $(x_1, y_1)$ and $(x_2, y_2)$ respectively. We may so direct the $y$-axis that $k > 0$. We observe that

$$\left.\begin{aligned} x_2 - x_1 &= x \\ y_2 - y_1 &= k. \end{aligned}\right\} \tag{1}$$

Under these constraints and requiring also that $A$ and $B$ lie on $\Gamma(\Theta P_{i-1})$ it follows that $x_1$, $y_1$, $x_2$ and $y_2$ are dependent on $x$ so that $F(\Pi)$ is also:

$$F(\Pi) = \alpha + \beta x + \gamma(x_1 y_2 - x_2 y_1),$$

where $\alpha$ and $\beta$ are constants and $\gamma$ is a positive constant.

Using (1) we find that

$$\frac{dF}{dx} = \beta + \gamma\,\{(y_2 - x_2 y_1')\,y_2' - (y_1 - x_1 y_2')y_1'\}\,(y_1' - y_2')^{-1}$$

and
$$\frac{1}{\gamma}\frac{d^2F}{dx^2} = -\frac{2\,y_1' y_2'}{y_1' - y_2'} + \frac{(y_1')^2 y_2''}{(y_1' - y_2')^3}[(x_1 - x_2)\,y_1' - (y_1 - y_2)] + \frac{(y_2')^2 y_1''}{(y_1' - y_2')^3}[(x_2 - x_1)\,y_2' - (y_2 - y_1)]$$

where $y_i'$ and $y_i''$ denote $dy_i/dx_i$ and $d^2y_i/dx_i^2$ respectively $(i = 1, 2)$.

In order to determine the sign of $d^2F/dx^2$ we need to know in particular the signs of the expressions in square brackets. Since similar expressions will occur again it is convenient for us to formulate the following simple rule.

Let $P$ be a point on a centrally symmetric closed convex curve and let $Q$ be a point distinct from $P$ which is either on the curve or in its interior. With a coordinate system as in Fig. 7 (i.e. with centre the origin and $y$-axis parallel to the tangents to the curve at its points of intersection with the $x$-axis) let $P \equiv (\xi, \eta)$ and $Q \equiv (a, b)$. Write $\eta' = d\eta/d\xi$. Then

$$\left.\begin{array}{l} \eta > 0 \text{ implies } (\xi - a)\eta' - (\eta - b) < 0 \\ \eta < 0 \text{ implies } (\xi - a)\eta' - (\eta - b) > 0. \end{array}\right\} \tag{2}$$

To show that this is so let $\eta' \neq 0$ and $(\xi_t, b)$ be the point at which the tangent to the curve at $(\xi, \eta)$ intersects the line through $(a, b)$ which is parallel to the $x$-axis. We have

$$\xi_t = \xi + \frac{b - \eta}{\eta'}$$

Let $\eta > 0$.
If $\eta' > 0$ then $\xi_t < a$, hence $\xi - a + (b - \eta)/\eta' < 0$ and $(\xi - a)\eta' - (\eta - b) < 0$
If $\eta' < 0$ then $\xi_t > a$, hence $\xi - a + (b - \eta)/\eta' > 0$ and $(\xi - a)\eta' - (\eta - b) < 0$.
If $\eta' = 0$ then $\eta$ is an absolute maximum and $-(\eta - b) < 0$.
Thus if $\eta > 0$ then $(\xi - a)\eta' - (\eta - b) < 0$ and by similar argument we find that $\eta < 0$ implies $(\xi - a)\eta' - (\eta - b) > 0$.

Returning to the expression for $d^2F/dx^2$ we note that $y_2 > 0$ since $\angle \Theta P_{i+1} < \pi$, and since $\Gamma(\Theta P_i)$ intersects $\Theta P_{i+1}\Theta P_{i+2}$ or $\Theta P_{i+1}\Theta P_{i+2}$ produced beyond $\Theta P_{i+2}$ it follows that $y_2' > 0$.

Let $y_1 > 0$. Since $y_1 < y_2$ and $\angle \Theta P_i < \pi$ it follows that $y_1' < 0$. Thus

$$y_1' - y_2' < 0 \quad \text{and} \quad -2y_1'y_2'/(y_1' - y_2') < 0.$$

Applying the above rule we find

$$[(x_1 - x_2)y_1' - (y_1 - y_2)] < 0 \quad \text{and} \quad [(x_2 - x_1)y_2' - (y_2 - y_1)] < 0.$$

Since furthermore, $y_1''$ and $y_2''$ are each negative it follows that $d^2F/dx^2 < 0$.

Let $y_1 < 0$. Since $\angle \Theta P_i < \pi$ we have $y_1' > y_2' > 0$. Thus $y_1' - y_2' > 0$ and $-2y_1'y_2'/(y_1' - y_2') < 0$. Also, $y_2'' < 0$ and $[(x_1 - x_2)y_1' - (y_1 - y_2)] > 0$ while $y_1'' > 0$ and $[(x_2 - x_1)y_2' - (y_2 - y_1)] < 0$. Again $d^2F/dx^2 < 0$.

For $y_1$ in the neighbourhood of zero we see that

$$\lim_{y_1 \to 0+} \frac{dF}{dx} = \lim_{y_1 \to 0-} \frac{dF}{dx} = \beta + \gamma(x_1 - x_2)y_2'.$$

In that $F$ is defined on an open set of values of $x$ we can choose $dx$ with either sign and in virtue of the above with such sign that $F(\Pi)$ can always be decreased.

Let us assume, therefore, that $\mu(\Theta P_{i+1}, \Theta P_{i+2}) = 1$ also and consider the change in $F(\Pi)$ resulting from a variation of $\Theta P_i$ and $\Theta P_{i+1}$ under which $\mu(\Theta P_{i-1}, \Theta P_i)$, $\mu(\Theta P_i, \Theta P_{i+1})$ and $\mu(\Theta P_{i+1}, \Theta P_{i+2})$ each remain equal to 1.

We choose a coordinate system as follows. With origin at $\Theta P_{i-1}$ we take the $x$-axis to have its positive half contain $\Theta P_{i+2}$ and the $y$-axis to be parallel to the tangents to $\Gamma(\Theta P_{i-1})$ at the points where the $x$-axis cuts $\Gamma(\Theta P_{i-1})$. Let $A$ and $B$ be points on $\Gamma(\Theta P_{i-1})$ such that $\overrightarrow{\Theta P_{i-1}A} = \overrightarrow{\Theta P_i \Theta P_{i+1}}$ and $\overrightarrow{\Theta P_{i-1}B} = \overrightarrow{\Theta P_{i+2}\Theta P_{i+1}}$ and let the coordinates of $A$, $B$, $\Theta P_i$ and $\Theta P_{i+2}$ be $(x_1, y_1)$, $(x_2, y_2)$, $(x, y)$ and $(r, 0)$ respectively (see Fig. 8).

We note that $\Theta P_i$ and $\Theta P_{i+1}$ cannot lie on opposite sides of the $x$-axis. Otherwise $\Theta P_i \Theta P_{i+1}$ would intersect the $x$-axis to the left of $\Theta P_{i-1}$ or to the right of $\Theta P_{i+2}$. In the first case we would find that $\Theta P_{i-1}$ lies inside the triangle $\Theta P_i \Theta P_{i+1} \Theta P_{i+2}$ hence also inside $\Gamma(\Theta P_{i+1})$ implying that $\mu(\Theta P_{i+1}, \Theta P_{i-1}) < 1$. Recalling, however, that we chose $P_i$ for which $P_{i-1}P_iP_{i+1}$ is a triangle in the vertex triangulation associated with $\Theta$ it follows that $\Theta P_{i+1}\Theta P_{i-1}$ is a diagonal and therefore $\mu(\Theta P_{i+1}, \Theta P_{i-1}) \geqslant 1$. Thus the first possibility is ruled out. As to the second, we would find in that case that $\Theta P_{i+2}$ lies inside the triangle $\Theta P_{i-1}\Theta P_i \Theta P_{i+1}$. This is ruled out by condition 3 of Definition 2 when we again recall that $P_{i-1}P_iP_{i+1}$ is a triangle in the vertex triangulation associated with $\Theta$. Under the circumstances we can choose the direction of the $y$-axis to be such that $\Theta P_i$ and $\Theta P_{i+1}$ both lie in the upper half-plane. Thus, in particular, $y > 0$.

We observe that
$$\left.\begin{aligned} x_2 - x_1 &= x - r, \\ y_2 - y_1 &= y. \end{aligned}\right\} \tag{3}$$

We can express $F(\Pi)$ as
$$F(\Pi) = \alpha + \beta[(r + x_1)y + (r - x)y_1],$$

where $\alpha$ and $\beta$ are constants and $\beta$, in particular, is positive. Subject to (3) and the restriction of $A$, $B$ and $\Theta P_i$ to lie on $\Gamma(\Theta P_{i-1})$ we can regard $F(\Pi)$ as a function of $x$ only. Using $y_i'$ and $y_i''$ as before and $y'$ and $y''$ to denote $dy/dx$ and $d^2y/dx^2$ respectively we find
$$\frac{1}{\beta}\frac{dF}{dx} = (r + x_1)y' - y_1 + \{(r - x)y_1' + y\}\frac{y' - y_2'}{y_2' - y_1'}$$

and
$$\begin{aligned} \frac{1}{\beta}\frac{d^2F}{dx^2} = {} & 2(y' - y_1')(y' - y_2')(y_2' - y_1')^{-1} + \{y + (r + x_1)y_2' - (x + x_1)y_1'\}(y_2' - y_1')^{-1}y'' \\ & - [(x_2 - x_1)y_2' - (y_2 - y_1)](y' - y_2')^2(y_2' - y_1')^{-3}y_1'' \\ & - [(x_1 - x_2)y_1' - (y_1 - y_2)](y' - y_1')^2(y_2' - y_1')^{-3}y_2''. \end{aligned}$$

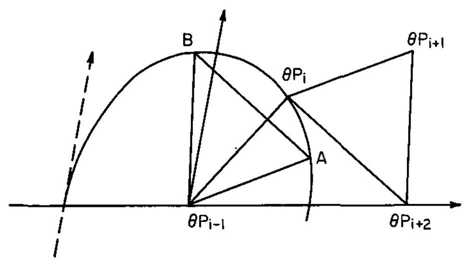



Fig. 8.

It suffices to assume that $y_1 \geqslant 0$, that is that $\angle \Theta P_i + \angle \Theta P_{i+2} \Theta P_{i-1} \Theta P_i \geqslant \pi$ for otherwise $\angle \Theta P_{i+1} + \angle \Theta P_{i-1} \Theta P_{i+2} \Theta P_{i+1} \geqslant \pi$ and we can interchange the roles of $\Theta P_{i-1}$ and $\Theta P_{i+2}$ taking the latter as origin and so on.

We observe that $\mu(\Theta P_{i-1}, \Theta P_{i+2}) > 1$. Otherwise $\Theta P_{i-1} \Theta P_{i+2}$ would not be a diagonal and there would necessarily be a $\Theta$-diagonal with $P_{i+1}$ as an end point the image of whose other end point, say $\Theta P_s$, lies inside the quadrilateral $\Theta P_{i-1} \Theta P_i \Theta P_{i+1} \Theta P_{i+2}$. Recalling condition 3 of Definition 2 and the fact that $P_{i-1} P_i P_{i+1}$ is a triangle in the vertex triangulation associated with $\Theta$ we observe that $\Theta P_s$ must in fact lie in the triangle $\Theta P_{i-1} \Theta P_{i+1} \Theta P_{i+2}$. Amongst such vertices as $\Theta P_s$ there is one, say $\Theta P_t$, for which $\angle \Theta P_t \Theta P_{i+1} \Theta P_{i+2}$ is a minimum. It follows that $\Theta P_t \Theta P_{i+2}$ is a diagonal but, since $\Theta P_{i-1} \Theta P_{i+2} \Theta P_{i+1}$ lies inside $\Gamma(\Theta P_{i+2})$, $\mu(\Theta P_t, \Theta P_{i+2}) < 1$ which is a contradiction. Thus $\mu(\Theta P_{i-1}, \Theta P_{i+2}) > 1$, implying

$$r > \max(|x|, |x_1|, |x_2|).$$

Since, therefore, $x_2 - x = x_1 - r < 0$ we have $x > x_2$ from which it follows that $y_2' > y'$. Furthermore, since $\angle \Theta P_i < \pi$, we have $y' > y_1'$. Hence $(y' - y_1')(y' - y_2')(y_2' - y_1')^{-1} < 0$.

For $y_1 > 0$ according to the rule stated earlier $[(x_1 - x_2) y_1' - (y_1 - y_2)] < 0$ and since $y_2'' < 0$ it follows that $-[(x_1 - x_2) y_1' - (y_1 - y_2)](y' - y_1')^2 (y_2' - y_1')^{-3} y_2'' < 0$.

Since $y_2 > 0$ we have $[(x_2 - x_1) y_2' - (y_2 - y_1)] < 0$ and since $y_1'' < 0$ it follows that $-[(x_2 - x_1) y_2' - (y_2 - y_1)](y' - y_2')^2 (y_2' - y_1')^{-3} y_1'' < 0$.

Since $(x_2 - x_1) y_2' - (y_2 - y_1) < 0$, $y + (r - x) y_2' > 0$; therefore $y + (r + x_1) y_2' - (x + x_1) y_1' > (x + x_1)(y_2' - y_1') = (r + x_2)(y_2' - y_1') > 0$ and since $y'' < 0$ it follows that $\{y + (r + x_1) y_2' - (x + x_1) y_1'\}(y_2' - y_1')^{-1} y'' < 0$.

Thus, for $y_1 > 0$ we have shown that $d^2F/dx^2 < 0$. For $y_1$ in the neighbourhood of 0 we observe that

$$\lim_{y_1 \to 0+} \frac{dy_1}{dx} = \lim_{y_1 \to 0-} \frac{dy_1}{dx} = y_2' - y' > 0.$$

Also $dF/dx$ exists:

$$\lim_{y_1 \to 0+} \frac{dF}{dx} = \lim_{y_1 \to 0-} \frac{dF}{dx} = (r - x) y_2' + (r + x_2) y'.$$

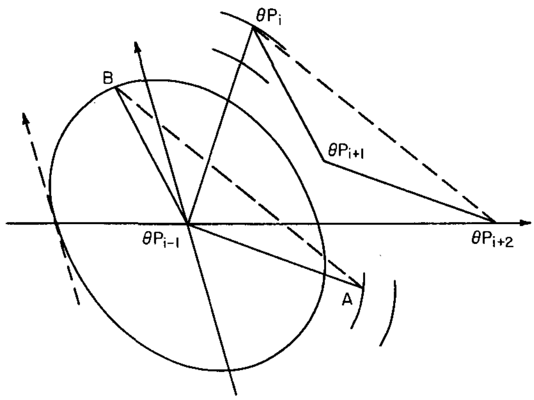



Fig. 9.

We can therefore adjoin to the values of $x$ for which $y_1 \geqslant 0$ those corresponding to $y_1$ in the neighbourhood of 0. The result is an open set and from what we have shown it follows that $F(\Pi)$ can always be decreased by choosing $dx$ with suitable sign.

5. There remains for us to consider the case in which $\angle \Theta P_i < \pi$, $\mu(\Theta P_i, \Theta P_{i+1}) = 1$ and $\angle \Theta P_{i+1} \geqslant \pi$.

We consider the variation in $F(\Pi)$ which results from varying $\Theta P_i$ and $\Theta P_{i+1}$ in such a way that $\mu(\Theta P_{i-1}, \Theta P_i)$, $\mu(\Theta P_i, \Theta P_{i+1})$ and $\mu(\Theta P_{i+1}, \Theta P_{i+2})$ remain constant.

We choose the following coordinate system. With origin at $\Theta P_{i-1}$ we take the $x$-axis to contain $\Theta P_{i+2}$ in its positive half and the $y$-axis to be parallel to the tangents to $\Gamma(\Theta P_{i-1})$ at the points where the $x$-axis cuts $\Gamma(\Theta P_{i-1})$. Let $A$ and $B$ be points such that $\overrightarrow{\Theta P_{i-1} A} = \overrightarrow{\Theta P_{i+1} \Theta P_{i+2}}$ and $\overrightarrow{\Theta P_{i-1} B} = \overrightarrow{\Theta P_{i+1} \Theta P_i}$ (see Fig. 9) and let the coordinates of $\Theta P_i$, $A$, $B$ and $\Theta P_{i+2}$ be $(x, y)$, $(x_1, y_1)$, $(x_2, y_2)$ and $(r, 0)$ respectively. Then

$$\left.\begin{aligned} x_2 - x_1 &= x - r \\ y_2 - y_1 &= y. \end{aligned}\right\} \tag{3}$$

We can write

$$F(\Pi) = \alpha + \beta\{ry - (x_1 y_2 - x_2 y_1)\} = \alpha + \beta\{(x - x_2)y - (x_1 - x_2)y_1\},$$

where $\alpha$ and $\beta$ are constants and $\beta$, in particular, is positive.

Choosing $x_1$ as independent variable we find that

$$\frac{1}{\beta}\frac{dF}{dx_1} = [(x - x_2)y' - (y - y_2)]\frac{y_2' - y_1'}{y' - y_2'} - [(x_1 - x_2)y_1' - (y_1 - y_2)].$$

Since $\mu(\Theta P_{i-1}, \Theta P_i) \geqslant 1$ and $\mu(\Theta P_{i-1}, A) \geqslant 1$ while $\mu(\Theta P_{i-1}, B) = 1$ we can determine the sign of each of the square brackets by applying the rule established earlier to $\mu(\Theta P_{i-1}, \Theta P_i)\Gamma(\Theta P_{i-1})$ and $\mu(\Theta P_{i-1}, A)\Gamma(\Theta P_{i-1})$.

We observe first that $y_2 > 0$, i.e. that $\Theta P_i$ lies above $\Theta P_{i+1}$. This is a consequence of the conditions that $0 < \angle \Theta P_i < \pi$ and $\angle \Theta P_{i+1} \geqslant \pi$. For when $\Theta P_i$ lies above the $x$-axis, $\Theta P_{i+1}$ must lie in the angle which is the intersection of the half plane containing $\Theta P_{i+2}$ and bounded by the line through $\Theta P_i$ and $\Theta P_{i-1}$ and the half plane containing $\Theta P_{i-1}$ which is bounded by the line through $\Theta P_i$ and $\Theta P_{i+2}$. When $\Theta P_i$ lies on the $x$-axis $\Theta P_{i+1}$ must lie below it since $0 < \angle \Theta P_i < \pi$. When $\Theta P_i$ lies below the $x$-axis the conditions, $0 < \angle \Theta P_i < \pi$ and $\angle \Theta P_{i+1} \geqslant \pi$ require that $\Theta P_{i+1}$ lie in the angle which is the intersection of the half plane not containing $\Theta P_{i+2}$ which is bounded by the line through $\Theta P_{i-1}$ and $\Theta P_i$ and the half plane not containing $\Theta P_{i-1}$ which is bounded by the line through $\Theta P_i$ and $\Theta P_{i+2}$.

Let $\sigma$, $\sigma_1$, $\sigma_2$ and $\tau$ denote $\angle \Theta P_i \Theta P_{i-1} X$, $\angle A \Theta P_{i-1} X$, $\angle B \Theta P_{i-1} X$ and $\angle \Theta P_i \Theta P_{i+2} X$ respectively where $X$ is any point on the $x$-axis to the right of $\Theta P_{i+2}$.

(a) Let $y_1 > 0$ then $[(x_1 - x_2) y_1' - (y_1 - y_2)] < 0$ and since $\sigma_2 = \sigma_1 + 2\pi - \angle \Theta P_{i+1} > \sigma_1 > 0$ it follows that $y_2' > y_1'$.

(i) If $y > 0$ then $[(x - x_2) y' - (y - y_2)] < 0$ and since $\sigma_2 = \sigma + \angle \Theta P_i > \sigma > 0$ therefore $y_2' > y'$ and $(y_2' - y_1')/(y' - y_2') > 0$.

(ii) If $y < 0$ then $[(x - x_2) y' - (y - y_2)] > 0$ and since $\sigma_2 = \sigma + \angle \Theta P_i < \sigma + \pi$ therefore $y' > y_2'$ and $(y_2' - y_1')/(y' - y_2') < 0$.

(iii) At $y = 0$,

$$\lim_{y \to 0+} [(x - x_2) y' - (y - y_2)] \frac{y_2' - y_1'}{y' - y_2'} = \lim_{y \to 0-} [(x - x_2) y' - (y - y_2)] \frac{y_2' - y_1'}{y' - y_2'} = (x - x_2)(y_2' - y_1').$$

In this expression $x$ is a maximum for $(x, y)$ on $\mu(\Theta P_{i-1}, \Theta P_i) \Gamma(\Theta P_i)$ and since $(x_2, y_2)$ lies on the latter or in its interior, $x - x_2 > 0$.

Thus for $y_1 > 0$ we find that $dF/dx_1 > 0$ and choosing $dx_1$ negative we can decrease $F(\Pi)$. Indeed this is true also for $y_1 = 0$ if with $dx_1 < 0$ we choose $dy_1 > 0$.

(b) Let $y_1 < 0$ then $[(x_1 - x_2) y_1' - (y_1 - y_2)] > 0$.

Since $y_2 > 0$, $y = y_2 - y_1 > 0$. Hence $[(x - x_2) y' - (y - y_2)] < 0$. Also $\sigma > 0$ and since $\sigma_2 = \sigma + \angle \Theta P_i > \sigma > 0$ it follows that $y_2' > y'$. Furthermore, $\sigma_2 = \sigma_1 + 2\pi - \angle \Theta P_{i+1} \leqslant \sigma_1 + \pi$ therefore $y_2' \leqslant y_1'$ and $(y_2' - y_1')/(y' - y_2') \geqslant 0$.

Thus for $y_1 < 0$ we find that $dF/dx_1 < 0$. We can therefore decrease $F(\Pi)$ when $y_1 < 0$ by choosing $dx_1 > 0$. We must however ensure that if $\angle \Theta P_{i+1} = \pi$ in choosing $dx_1 > 0$, $\angle \Theta P_{i+1}$ is increased. That this is the case we see as follows. From equations (3)

$$\frac{d}{dx_1}(y_2') = \frac{y' - y_1'}{y' - y_2'} y_2''$$

which is equal to $y_2''$ when $y_1' = y_2'$ i.e. when $\angle \Theta P_{i+1} = \pi$. But since $y_2 > 0, y_2'' < 0$ therefore if, when $\angle \Theta P_{i+1} = \pi$, we choose $dx_1 > 0$, $y_2'$ decreases i.e. $\sigma_2$ decreases. Also for $\angle \Theta P_{i+1} = \pi$, $y_1 < 0$ hence $y_1'' > 0$ and choosing $dx_1 > 0$, $y_1'$ increases, i.e. $\sigma_1$ increases. Thus $2\pi - \angle \Theta P_{i+1} = \sigma_2 - \sigma_1$ decreases and therefore $\angle \Theta P_{i+1}$ increases when $dx_1 > 0$.

Summarizing these results we see that $F(\Pi)$ can be decreased in this case by increasing the angle, $\sigma_1$, which is the same as decreasing $\angle \Theta P_{i+2}$.

This completes the proof of Lemma 2.

*Proof of Theorem* 2. We have shown that $(\mathrm{I}_3)$ holds and assume now that $(\mathrm{I}_k)$ holds for $3 < k < n$.

The values which $F$ takes at polygons in $\mathfrak{A}(\mu)$ which have $n$ vertices are clearly bounded below; let $\lambda_n$ be their greatest lower bound. There exists a sequence, $\{\Theta_r K \mid r = 1, 2, \ldots\}$, of polygons in $\mathfrak{A}(\mu)$ which have $n$ vertices such that

$$\lim_{r\to\infty} F(\Theta_r K) = \lambda_n.\text{(1)}$$

Furthermore, since there are but finitely many vertex triangulations of $K^*$ possible, there is a subsequence each member of which is defined in terms of the same triangulation of $K^*$. Let the limit of this subsequence be $\overline{\Pi} = \Theta P_1 \ldots \Theta P_n \Theta P_{n+1}$, $\Theta P_{n+1} = \Theta P_1$, then according to Theorem 4 $\overline{\Pi}$ belongs to $\mathfrak{A}(\mu)$. In virtue of the continuity of $F$ we have $F(\overline{\Pi}) = \lambda_n$.

Application of Lemma 2 to $\overline{\Pi}$ provides that $\overline{\Pi}$ has one of the properties (i) or (ii) of that lemma or, since $F(\overline{\Pi})$ cannot be decreased, there exists $\overline{\Pi}'$ in $\mathfrak{A}(\mu)$ such that $F(\overline{\Pi}') = \lambda_n$, $\overline{\Pi}'$ has $n$ vertices and (i) or (ii) is true of $\overline{\Pi}'$. We may therefore suppose that $\overline{\Pi}$ itself has one of the properties (i) or (ii) of Lemma 2.

*Case* (i). Let $\Theta P_i$ be an interior point of the side $\Theta P_j \Theta P_{j+1}$. If $j = i + 1$, then $\Pi_1 = \Theta P_i \Theta P_{j+1} \ldots \Theta P_{i-1} \Theta P_i$ is, according to the Corollary to Theorem 3, a polygon in $\mathfrak{A}(\mu)$. Since $\Pi_1$ has $n - 1$ vertices the inductional assumption is applicable and

$$\frac{A(\Pi_1^*)}{\Delta} + \frac{M(\Pi_1)}{2} + 1 \geqslant n - 1.$$

Observing that $M(\overline{\Pi}) = M(\Pi_1) + 2\mu(\Theta P_i, \Theta P_{i+1}) \geqslant M(\Pi_1) + 2$

and
$$A(\overline{\Pi}^*) = A(\Pi_1^*)$$

it follows that
$$\frac{A(\overline{\Pi}^*)}{\Delta} + \frac{M(\overline{\Pi})}{2} + 1 \geqslant n.$$

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(1) Since each polygon has $n$ vertices we may assume that $K$ is the same for all of them.

If $j+1=i-1$ we consider $\Pi_2=\Theta P_i \Theta P_{i+1} \ldots \Theta P_j \Theta P_i$ instead of $\Pi_1$ and the proof is similar.

When $\Pi_1=\Theta P_i \Theta P_{j+1} \ldots \Theta P_{i-1} \Theta P_i$ and $\Pi_2=\Theta P_i \Theta P_{i+1} \ldots \Theta P_j \Theta P_i$ each have three or more vertices both $\Pi_1$ and $\Pi_2$ are in $\mathfrak{A}(\mu)$ by the corollary to Theorem 3 and by inductional assumption

$$\frac{A(\Pi_1^*)}{\Delta}+\frac{M(\Pi_1)}{2}+1 \geqslant \nu(\Pi_1)$$

and

$$\frac{A(\Pi_2^*)}{\Delta}+\frac{M(\Pi_2)}{2}+1 \geqslant \nu(\Pi_2),$$

where $\nu(\Pi_1)$ and $\nu(\Pi_2)$ are the number of vertices of $\Pi_1$ and $\Pi_2$ respectively.

Since
$$A(\bar{\Pi}^*)=A(\Pi_1^*)+A(\Pi_2^*),$$

$$M(\bar{\Pi})=M(\Pi_1)+M(\Pi_2)-\mu(\Theta P_i, \Theta P_j)-\mu(\Theta P_i, \Theta P_{j+1})+\mu(\Theta P_j, \Theta P_{j+1})=M(\Pi_1)+M(\Pi_2)$$

and
$$\nu(\Pi_1)+\nu(\Pi_2)=n+1$$

it follows that
$$\frac{A(\bar{\Pi}^*)}{\Delta}+\frac{M(\bar{\Pi})}{2}+1 \geqslant n.$$

*Case* (ii). Let $\Theta P_i \Theta P_j$ be a diagonal of $\bar{\Pi}$ and $\mu(\Theta P_i, \Theta P_j)=1$. According to Theorem 3 $\Theta P_i \Theta P_j$ divides $\bar{\Pi}$ into two polygons, say $\Pi_1$ and $\Pi_2$, each of which is in $\mathfrak{A}(\mu)$. Again

$$\frac{A(\Pi_1^*)}{\Delta}+\frac{M(\Pi_1)}{2}+1 \geqslant \nu(\Pi_1)$$

and

$$\frac{A(\Pi_2^*)}{\Delta}+\frac{M(\Pi_2)}{2}+1 \geqslant \nu(\Pi_2).$$

In this case we have
$$A(\bar{\Pi}^*)=A(\Pi_1^*)+A(\Pi_2^*);$$

$$M(\bar{\Pi})=M(\Pi_1)+M(\Pi_2)-2\mu(\Theta P_i, \Theta P_j)=M(\Pi_1)+M(\Pi_2)-2$$

and
$$\nu(\Pi_1)+\nu(\Pi_2)=n+2.$$

Hence
$$\frac{A(\bar{\Pi}^*)}{\Delta}+\frac{M(\bar{\Pi})}{2}+1 \geqslant n.$$

Thus we have shown that $(\mathrm{I}_n)$ holds for polygons in $\mathfrak{A}(\mu)$. Since a Jordan polygon, $\Pi$, admits of a vertex triangulation it may be realised as a polygon in $\mathfrak{A}$ and if, furthermore, it is weakly admissible, as a polygon in $\mathfrak{A}(\mu)$. $(\mathrm{I}_n)$ therefore holds, in particular, for weakly admissible Jordan polygons which is Theorem 2.

## IV

*Proof of Theorem* 1. Let $(\Pi, E)$ be a weakly admissible pair. Let the vertices of $\Pi$ be $P_1, \ldots, P_{n-m}$ and the remaining points of $E$ be $Q_1, \ldots, Q_m$, these being in the interior of $\Pi$. We proceed by induction on $m$, the number of points of $E$ in the interior of $\Pi$. We have proved in Theorem 2 that Theorem 1 holds when $m = 0$. Let us assume that Theorem 1 is true for $0 < m' < m$.

We introduce a coordinate system as follows. With any point, $0$, as origin and any straight line as $y$-axis we choose the $x$-axis to be parallel to the tangents to $\Gamma(0)$ at the points where the $y$-axis cuts $\Gamma(0)$. Let the coordinates of $Q_1, \ldots, Q_m$ be $(x_1, y_1), \ldots, (x_m, y_m)$, respectively. We shall say that $Q_i$ is to the right of $Q_j$ if $x_i > x_j$; to the left if $x_i < x_j$. By moving $Q_i$ to the right (left) we shall mean moving $Q_i$ so as to increase (decrease) $x_i$ while holding $y_i$ constant. It is clear that if $Q_i$ is not to the left (right) of $Q_j$ then by moving $Q_i$ to the right (left) we do not decrease $\mu(Q_i, Q_j)$.

In what follows we shall consider variations of $Q_1, \ldots, Q_m$ under which $(\Pi, E)$ remains weakly admissible and, leaving $P_1, \ldots, P_{n-m}$ fixed, $F(\Pi)$ is unchanged. If in the course of varying $Q_1, \ldots, Q_m$ any of these points falls on $\Pi$ we shall have nothing further to prove since the number of points of $E$ in the interior of $\Pi$ is thereby decreased and the inductional assumption is immediately applicable. We may therefore omit this possibility.

Let $Q_1, \ldots, Q_m$ be so re-enumerated that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m$. If $Q_1$ is not $\mu$-distant 1 from a vertex of $\Pi$ to its right with the segment joining them in $\Pi^*$ we move it to the right until this is so. $Q_2$ is either $\mu$-distant 1 from a point amongst $P_1, \ldots, P_{n-m}, Q_1$ to its right with the segment joining them in $\Pi^*$ or we can move it to the right until this is the case. We continue in this way with $Q_3, \ldots, Q_{m-1}$. As a result each of the points $Q_1, \ldots, Q_{m-1}$ can be joined to a vertex of $\Pi$ to its right by a simple polygonal path in $\Pi^*$ whose vertices are amongst $Q_1, \ldots, Q_{m-1}$ and whose sides are of $\mu$-length 1.

We now move $Q_m$, if necessary, to the left so that it is $\mu$-distant 1 from a vertex of $\Pi$, say $P_i$, to its left and $P_iQ_m$ is in $\Pi^*$. Either $Q_m$ is $\mu$-distant 1 from some other vertex of $\Pi$ or from one of $Q_1, \ldots, Q_{m-1}$ and the segment joining them is in $\Pi^*$ or varying $Q_m$ along $\Gamma(P_i)$ this becomes so or $Q_m$ becomes a point of $\Pi$.

In this way we obtain a polygonal path of the form $P_iQ_mP_j$ or $P_iQ_mQ_{r_1} \ldots Q_{r_s} P_j$ whose inner points are in the interior of $\Pi$ and whose sides have $\mu$-length 1. In the second case $Q_{r_1} \ldots Q_{r_s}P_j$ is amongst the paths obtained earlier. Hence $P_j$ is to the right of $Q_{r_s}$ and $P_i$ is to the left. Therefore $P_i$ and $P_j$ are distinct and this is certainly so in the first case.

Let us suppose that $P_iQ_mQ_{r_1} \ldots Q_{r_s}P_j$ is not simple. Then since $Q_{r_1} \ldots Q_{r_s}P_j$ is itself a simple path one of its sides, say $Q_{r_k}Q_{r_{k+1}}$, intersects $P_iQ_m$; let the point of intersection be

$X$. $P_iQ_{r_{k+1}}$ is contained in $\Pi^*$ for otherwise there would be a vertex of $\Pi$, say $P_l$, in the interior of the triangle $P_iXQ_{r_{k+1}}$ such that $P_iP_l$ is in $\Pi^*$ and $\mu(P_i, P_l) < 1$. Thus $P_iQ_{r_{k+1}}$ is in $\Pi^*$ and $\mu(P_i, Q_{r_{k+1}}) \geqslant 1$. Similarly $\mu(Q_m, Q_{r_k}) \geqslant 1$. Noting that

$$2 = \mu(P_i, Q_m) + \mu(Q_{r_k}, Q_{r_{k+1}}) \geqslant \mu(P_i, Q_{r_{k+1}}) + \mu(Q_m, Q_{r_k})$$

it follows that $\mu(P_i, Q_{r_{k+1}}) = 1$ and we can replace $P_iQ_mQ_{r_1} \dots Q_{r_s}P_j$ by $P_iQ_{r_{k+1}} \dots Q_{r_s}P_j$ which is a simple path. Since $P_iQ_mP_j$ is simple we can obtain under all circumstances a simple polygonal path, $\lambda = P_iQ_{t_1} \dots Q_{t_w}P_j$, whose sides have $\mu$-length 1 and whose inner points lie in the interior of $\Pi$.

The polygons $\Pi_1 = P_iP_{i+1} \dots P_{j-1}P_jQ_{t_w} \dots Q_{t_1}P_i$ and $\Pi_2 = P_iQ_{t_1} \dots Q_{t_w}P_jP_{j+1} \dots P_{i-1}P_i$ contain subsets, $E_1'$ and $E_2'$ respectively, of $E$ in their interiors. Let $E_1$ denote the set consisting of $E_1'$ and the vertices of $\Pi_1$ and $E_2$ denote $E_2'$ together with the vertices of $\Pi_2$. The pairs $(\Pi_1, E_1)$ and $(\Pi_2, E_2)$ are weakly admissible and since $E_1'$ and $E_2'$ each contain fewer than $m$ points we have, by the inductional assumption,

$$\frac{A(\Pi_1^*)}{\Delta} + \frac{M(\Pi_1)}{2} + 1 \geqslant n_1$$

and

$$\frac{A(\Pi_2^*)}{\Delta} + \frac{M(\Pi_2)}{2} + 1 \geqslant n_2,$$

where $n_1$ and $n_2$ are the number of points in $E_1$ and $E_2$ respectively. Noting that

$$A(\Pi_1^*) + A(\Pi_2^*) = A(\Pi^*),$$

$$M(\Pi_1) + M(\Pi_2) = M(\Pi) + 2M(\lambda) = M(\Pi) + 2(w+1),$$

and

$$n_1 + n_2 = n + w + 2,$$

addition of the last two inequalities yields

$$\frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} + 1 \geqslant n.$$

## V

As an application of the inequality $(\mathrm{I}_n)$ we prove

THEOREM 6. *Let $\Gamma(0)$ be a plane strictly convex Jordan curve having the origin, $O$, as centre of symmetry. Let the $n$ translates, $\Gamma(P_1), \dots, \Gamma(P_n)$, of $\Gamma(0)$ be such that the domains they bound are non-overlapping and let $S$ be the boundary of the smallest convex domain which contains them. Then*

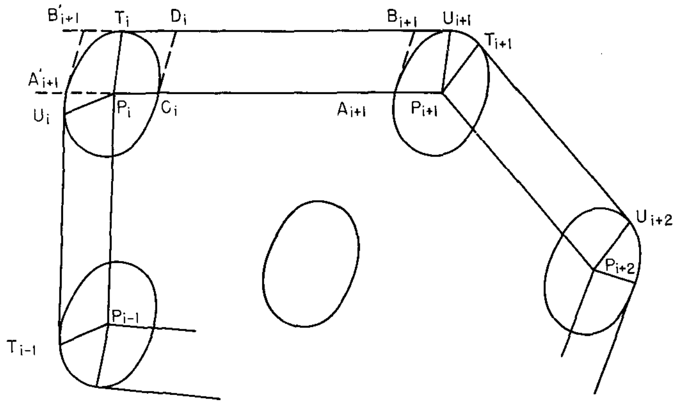


Fig. 10.

$$A(S^*) - A(\Gamma^*) - \tfrac{1}{2}[M(S) - M(\Gamma)](p - \Delta) \geqslant (n-1)\Delta \tag{4n}$$

*where $M(S)$ and $M(\Gamma)$ are the lengths of $S$ and $\Gamma$ measured by the distance function, $\mu$, determined by $2\Gamma$; $\Delta$ is the critical determinant of $2\Gamma$ and $p$ is the area of the smallest parallelogram which contains $\Gamma$.*

*Proof.* Let $\Pi$ be the boundary of the smallest convex domain which contains $P_1, \ldots, P_n$. Then

$$\frac{A(\Pi^*)}{\Delta} + \frac{M(\Pi)}{2} \geqslant n - 1. \tag{5}$$

We can assume that $P_1, \ldots, P_n$ are so enumerated that $\Pi$ is the polygon $P_1 P_2 \ldots P_m P_1$.

Referring to Fig. 10, we may describe $S^*$, the domain bounded by $S$ as the union of the following non-overlapping sets:

(i) The parallelograms $T_i P_i P_{i+1} U_{i+1}$ $(i = 1, \ldots, m-1)$ and $T_m P_m P_1 U_1$ where $T_i U_{i+1}$ is the common tangent to $\Gamma(P_i)$ and $\Gamma(P_{i+1})$ lying outside of $\Pi$.

(ii) The sector of $\Gamma(P_i)$ bounded by $U_i P_i$, $T_i P_i$ and the arc of $\Gamma(P_i)$ between $U_i$ and $T_i$ which lies outside of $\Pi$ $(i = 1, \ldots, m)$.

(iii) $\Pi^*$, the domain bounded by $\Pi$.

Let $P_i P_{i+1}$ intersect $\Gamma(P_i)$ at $C_i$ and $\Gamma(P_{i+1})$ at $A_{i+1}$. Let the points at which $T_i U_{i+1}$ intersects the tangent to $\Gamma(P_i)$ at $C_i$ and the tangent to $\Gamma(P_{i+1})$ at $A_{i+1}$ be $D_i$ and $B_{i+1}$ respectively. We note that, since $\Gamma$ is centrally symmetric, $C_i D_i$ is parallel to $A_{i+1} B_{i+1}$. Also $P_i T_i$ is parallel to $P_{i+1} U_{i+1}$.

Let us translate $A_{i+1} B_{i+1} U_{i+1} P_{i+1}$ until $P_{i+1}$ coincides with $P_i$ and $U_{i+1}$ with $T_i$. We obtain thereby a parallelogram $A'_{i+1} A_{i+1} B_{i+1} B'_{i+1}$ where $A'_{i+1}$ is the reflection of $C_i$

in $P_i$ and $B'_{i+1}$ is the point at which $U_{i+1}T_i$ produced intersects the tangent to $\Gamma(P_i)$ at $A'_{i+1}$. Furthermore, the area of $P_iP_{i+1}U_{i+1}T_i$ is the same as that of $A'_{i+1}A_{i+1}B_{i+1}B'_{i+1}$. The ratio of the latter to the area of $A'_{i+1}C_iD_iB'_{i+1}$ is the same as the ratio of $A'_{i+1}A_{i+1}$ to $A'_{i+1}C_i$ which is precisely $\mu(A'_{i+1}, A_{i+1})$ and this is equal to $\mu(P_i, P_{i+1})$. Reflecting $A'_{i+1}C_iD_iB'_{i+1}$ in $P_i$ we obtain a parallelogram which circumscribes $\Gamma(P_i)$ and whose area is therefore not less than $p$. Thus $A(A'_{i+1}C_iD_iB'_{i+1}) \geqslant \frac{1}{2}p$ hence $A(P_iP_{i+1}U_{i+1}T_i) \geqslant \mu(P_i, P_{i+1})p/2$.

We next observe that the sum of the areas of the sectors referred to in (ii) is precisely $A(\Gamma^*)$ (see [4], p. 320).

From these observations we see that

$$A(S^*) \geqslant A(\Pi^*) + M(\Pi)\frac{p}{2} + A(\Gamma^*) \tag{6}$$

while
$$M(\Pi) = M(S) - M(\Gamma). \tag{7}$$

Combining (6) and (7) with (5) we obtain (4n).

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