ALMOST PERIODIC FUNCTIONS ON SEMIGROUPS

BY

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1. Introduction

This paper is devoted to the extension, to certain commutative topological semigroups, of the fundamental approximation theorem for almost periodic functions on groups. Several definitions are necessary before we are able to state our results.

If S is a commutative topological semigroup,(2) we shall denote by $C(S)$ the Banach algebra of all bounded continuous complex valued functions on S supplied with the norm $\|\cdot\|$ defined by

$$\|f\| = \sup_{\sigma \in S} |f(\sigma)|.$$

For each σ in S and f in $C(S)$, the translated function σf in $C(S)$ is defined by

$$\sigma f(\tau) = f(\tau + \sigma), \quad \tau \in S.$$

A function f in $C(S)$ is called *almost periodic* if the set $\{\sigma f : \sigma \in S\}$ of translates of f is conditionally compact(3) in $C(S)$. We shall denote by $A(S)$ the collection of all almost periodic functions on S. It is simple to check that $A(S)$ is a closed translation invariant subalgebra of $C(S)$.

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(2) We call S a *commutative topological semigroup* if it is a commutative semigroup having an identity element, supplied with a topology in which the map $(\sigma, \tau) \to \sigma + \tau$ from $S \times S$ to S is continuous. In the terminology of [6], S would be a commutative topological semigroup with jointly continuous addition. Subsemigroups need *not* have identities.

(3) $C(S)$ will always be considered to be topologized with the norm topology, that is, the topology of uniform convergence. We shall use "conditionally compact" to mean "having compact closure". Our definition of almost periodicity is weaker than that used by Maak in [11] and our results are disjoint from his.

A function χ in $C(S)$ is called a *semicharacter* if

$$\chi(\sigma+\tau)=\chi(\sigma)\chi(\tau), \quad \sigma \in S, \ \tau \in S,$$

and

$$|\chi(\sigma)| \leqslant 1, \quad \sigma \in S.$$

Each semicharacter of S is almost periodic; for χ is a semicharacter, then

$$\sigma\chi=\chi(\sigma)\chi, \quad \sigma \in S,$$

so $\{\sigma\chi: \ \sigma \in S\}$ is contained in the compact subset $\{\lambda\chi: \ |\lambda| \leqslant 1\}$ of $C(S)$. Note that the set of semicharacters of S is closed under complex conjugation.

We shall say that the *approximation theorem* holds for S if linear combinations of semicharacters of S are dense in $A(S)$; that is, if each function in $A(S)$ can be approximated uniformly on S by linear combinations of semicharacters of S. That the approximation theorem holds if S is a commutative topological group is the fundamental result of the theory of almost periodic functions on commutative groups (see Chapter 7 of [10]). We show in the next section that the approximation theorem does not hold in general for commutative semigroups, and that it may fail even for a subsemigroup of a discrete group.

The remainder of the paper is devoted to establishing the approximation theorem for certain classes of topological semigroups. The most important of these are

1. Cones in finite dimensional Euclidean spaces in the usual topology.
2. Finitely generated subsemigroups of commutative groups in the discrete topology.
3. Semigroups consisting of the non-negative elements of a totally ordered commutative group in any "reasonable" topology.

We also show that the class of commutative topological semigroups for which the approximation theorem holds is closed under the formation of products.

The main tool that we use is the almost periodic compactification introduced in [6]. For groups the approximation theorem is an immediate consequence (see Chapter 7 of [10]) of the existence of a compactification and the Peter–Weyl theorem. No such direct method is available for semigroups, as there is no analogue of the Peter–Weyl theorem for compact commutative semigroups. On the contrary, such semigroups may have few semicharacters, as the first example of the next section shows. For this reason we are forced to carry out a detailed analysis of the structure of the compactification in each of the cases that we consider.

Some comments on notation: In the following, the letters S and T (with or without subscript or superscripts) will be used to denote commutative topological semigroups. If $\varphi: S \to T$ is a continuous mapping, we shall always denote by $\tilde{\varphi}$ the induced mapping $\tilde{\varphi}: C(T) \to C(S)$ defined by

$$\tilde{\varphi}(f) = f \circ \varphi, \quad f \in C(T).$$

If X and Y are any sets,

$$\{x: x \text{ in } X, \ x \text{ not in } Y\}$$

is denoted by $X \setminus Y$.

The results that follow were announced in part in Section 4 of [5].

2. Some Examples

The approximation theorem may fail for a variety of reasons, some of which become apparent in the following examples.

Let S be the commutative topological semigroup formed by the closed unit interval supplied with the usual topology and the composition

$$(\sigma, \tau) \to \sup(\sigma, \tau).$$

It is simple to check that each function in $C(S)$ is almost periodic. Nevertheless the only semicharacters of S are the constant functions equal to 0 and to 1.

Even with many semicharacters, indeed sufficiently many to separate points, approximation may fail due to the lack of homogeneity in topological semigroups. For let S be the non-negative reals under addition, supplied with the usual topology on $[1, \infty]$ but with the discrete topology on $[0, 1]$. If f is a character of the discrete reals and

$$h(\sigma) = \begin{cases} 1 - \sigma, & 0 < \sigma < 1 \\ 0, & 1 \leqslant \sigma, \end{cases}$$

the fh is in $A(S)$. However, the only semicharacters of S are exponentials and a pair of functions equal to 0 for all $\sigma > 0$, so the approximation theorem fails for S.

Another possible reason for the failure of the approximation theorem is that a non-trivial function may have only trivial translates. For let S be the semigroup of lattice points

$$\{(m, n): \ m > 0 \ \text{ or } \ m = n = 0\}$$

in the plane. Then if f is any function in $C(S)$ that satisfies $f(m, n) = 0$ for $m > 1$, $\{\sigma f: \sigma \in S\}$ consists only of f and functions that are zero except at $(0, 0)$, so f is almost periodic. On the other hand, the semicharacters of S, with two exceptions, are of the form

$$\chi(m, n) = e^{\lambda m + i\mu n},$$

where μ is real and $\operatorname{Re}(\lambda) \leqslant 0$, so the approximation theorem fails for S. Note that S is not a finitely generated subsemigroup of the lattice point group; because of Theorem 10.1 no finitely generated example can be found.

3. The Compactification

The following lemma lists three of the basic properties of almost periodic functions that we shall need. The results are simple to establish and are contained in Theorems 5.1 and 6.1 and Lemma 5.2 of [6].

LEMMA 3.1. *Let S and T be commutative topological semigroups.*

(i) *If $\varphi: S \to T$ is a continuous homomorphism, then the induced map $\tilde{\varphi}: C(T) \to C(S)$ satisfies $\tilde{\varphi}(A(T)) \subset A(S)$.*

(ii) *If T is compact, then $A(T) = C(T)$.*

(iii) *If f is in $A(S)$, the map $\sigma \to \sigma f$ of S into $A(S)$ is continuous.*

Note that as a consequence of (i) and (ii) if S is a commutative topological semigroup, $\varphi: S \to T$ a continuous homomorphism of S into a *compact* commutative topological semigroup T and f a function in $C(T)$, then the composite $f \circ \varphi$ is almost periodic on S. As we shall see in Lemma 3.2, all of $A(S)$ can be obtained in this way.

The main tool in our study of the space $A(S)$ of almost periodic functions on a commutative topological semigroup S is an associated *compact* commutative topological semigroup S^a and a mapping $I_S: S \to S^a$. S^a was introduced in [6] and called the *almost periodic compactification* (1) of S. It is defined to be the closure in the strong operator topology of the semigroup $\{R_\sigma: \sigma \in S\}$ of operators on $A(S)$, where R_σ is the translation operator

$$R_\sigma(f) = \sigma f, \quad f \in A(S);$$

$I_S: S \to S^a$ is defined by

$$I_S(\sigma) = R_\sigma, \quad \sigma \in S.$$

(1) If S is a group, S^a is of course isomorphic to the usual almost periodic compactification, for which see Chapter 7 of [10].

For our purposes these definitions have no relevance; we shall need only those facts about the compactification that are listed in the following three lemmas (see Section 6 of [6]).

The first describes the basic property of the compactification, the second shows that this property characterizes the compactification, while the third states the existence of induced homomorphisms for compactifications.

LEMMA 3.2. *Let S be a commutative topological semigroup. Then $I_S: S \to S^a$ is a continuous homomorphism with $I_S(S)$ dense in S^a. The induced mapping $\tilde{I}_S: C(S^a) \to C(S)$ is an isometry and an algebra isomorphism of $C(S^a)$ onto $A(S)$.*

LEMMA 3.3. *Let S and S^* be commutative topological semigroups with S^* compact. Suppose that $\varphi: S \to S^*$ is a continuous homomorphism, with $\varphi(S)$ dense in S^*, which is such that the induced map $\tilde{\varphi}: C(S^*) \to C(S)$ satisfies $A(S) = \tilde{\varphi}(C(S^*))$. Then there is a topological isomorphism $\psi: S^a \to S^*$ so that the diagram*

$$\begin{tikzcd} & S^* \\ S \arrow[ur, "\varphi"] \arrow[dr, "I_S"'] & \\ & S^a \arrow[uu, "\psi"'] \end{tikzcd}$$

is commutative, i.e. $\psi \circ I_S = \varphi$.

LEMMA 3.4. *Let S and T be commutative topological semigroups. If $\varphi: S \to T$ is a continuous homomorphism, then there is an associated continuous homomorphism $\varphi^a: S^a \to T^a$ which is such that the diagram*

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "I_S"] & T \arrow[d, "I_T"] \\ S^a \arrow[r, "\varphi^a"] & T^a \end{tikzcd}$$

is commutative, i.e. $\varphi^a \circ I_S = I_T \circ \varphi$.

Let S be a commutative topological semigroup. If f is in $A(S)$, by Lemma 3.2 there is a unique function h in $C(S^a)$ that satisfies $f = h \circ I_S$. We shall denote this function h by $\hat{f}$.

Thus if f is an almost periodic function on S, $\hat{f}$ is its "extension" to the compactification S^a. The mapping (1) $f \rightarrow \hat{f}$ of $A(S)$ onto $C(S^a)$ is inverse of the isomorphism $\tilde{I}_S$ of Lemma 3.2.

The following is an immediate consequence of Lemma 3.2 and the fact that semicharacters are almost periodic and "extend" so as to remain multiplicative.

LEMMA 3.5. *Let S be a commutative topological semigroup. Then $\chi \rightarrow \hat{\chi}$ is a $1-1$ correspondence between the semicharacters of S and the semicharacters of S^a.*

This leads to the following result, which reduces the approximation problem for $A(S)$ to a question about the existence of semicharacters of S^a.

THEOREM 3.6. *Let S be a commutative topological semigroup. Then the following are equivalent:*

(i) *The approximation theorem holds for S.*

(ii) *S^a has sufficiently many semicharacters to separate points, i.e. if x and y are distinct elements of S^a, then there is a semicharacter χ of S^a that satisfies $\chi(x) \neq \chi(y)$.*

Proof. (i) implies (ii). For if linear combinations of semicharacters of S are dense in $A(S)$, then by Lemmas 3.2 and 3.5, linear combinations of semicharacters of S^a are dense in $C(S^a)$, so there must be sufficiently many to separate points of S^a. (ii) implies (i). For if S^a has sufficiently many semicharacters, by the Stone-Weierstrass Theorem, linear combinations of these semicharacters are dense in $C(S^a)$, so by Lemmas 3.2 and 3.5, linear combinations of semicharacters of S are dense in $A(S)$.

In what follows we shall, in many cases, be dealing with commutative topological semigroups S which have sufficiently many almost periodic functions to separate points. In these situations the map $I_S: S \rightarrow S^a$ is $1-1$ and we shall identify S with its image in S^a. For each f in $A(S)$, $\hat{f}$ will then simply be the unique continuous extension of f to S^a. And the restriction of a function in $C(S^a)$ to S will be a function in $A(S)$.

If S is identified with its image in S^a and Q is a subset of S, we shall denote by Q^- the closure of Q in S^a (and not the closure of Q in S).

4. The Product Theorem

Let S and T be commutative topological semigroups. Then the product semigroup $S \times T$ is defined to have coordinatewise addition

(1) $f \rightarrow \hat{f}$ is the Gelfand representation of $A(S)$, in the sense of Chapter IV of [10].

$$(\sigma_1, \tau_1) + (\sigma_2, \tau_2) = (\sigma_1 + \sigma_2, \tau_1 + \tau_2)$$

and the product topology. If $\{S_\gamma\}_{\gamma \in J}$ is any collection of commutative topological semigroups, the product semigroup $\prod_{\gamma \in J} S_\gamma$ is defined in the same manner.

We establish in this section the isomorphism (1)

$$(\prod_{\gamma \in J} S_\gamma)^a \cong \prod_{\gamma \in J} S_\gamma^a \tag{4.1}$$

and obtain as a consequence the fact that the approximation theorem holds for $\prod_{\gamma \in J} S_\gamma$ if and only if it holds for each of the S_γ. This result is used later in an induction step in our proof of the approximation theorem for cones.

LEMMA 4.1. *Let S be a commutative topological semigroup and Q a dense subset of S. Let f be a function in $C(S)$ with $\{\sigma f: \sigma \in Q\}$ conditionally compact. Then $\{\sigma f: \sigma \in S\}$ is conditionally compact, i.e. f is almost periodic.*

Proof. It suffices to show that the map $\sigma \to \sigma f$ from S to $C(S)$ is continuous. For if it is continuous, then $\{\sigma f: \sigma \in S\}$ is contained in the closure of the conditionally compact set $\{\sigma f: \sigma \in Q\}$ and is thus itself conditionally compact. As a first step we show that if τ is any element of S, the map $\sigma \to \sigma f$ is continuous when restricted to $Q \cup \{\tau\}$. $\{\sigma f: \sigma \in Q \cup \{\tau\}\}$ is a conditionally compact subset of $C(S)$ and thus its topology, which is the topology of uniform convergence on S, coincides with the topology of pointwise convergence on S. But since f is continuous, the mapping $\sigma \to \sigma f$ is continuous from S to $C(S)$ supplied with the topology of pointwise convergence on S. Thus $\sigma \to \sigma f$ is continuous when restricted to $Q \cup \{\tau\}$ as was claimed. Suppose now that the mapping $\sigma \to \sigma f$ from S to $C(S)$ is not continuous. Then there is a σ_0 in S, an $\varepsilon > 0$, and, for each neighborhood U of σ_0, an element σ_U in U with $\|\sigma_U f - \sigma_0 f\| > 2\varepsilon$. Furthermore, for each neighborhood U of σ_0, since $\sigma \to \sigma f$ is continuous on $Q \cup \{\sigma_U\}$, there is a τ_U in $Q \cap U$ with $\|\tau_U f - \sigma_U f\| < \varepsilon$ and thus with $\|\tau_U f - \sigma_0 f\| > \varepsilon$. But this contradicts the continuity of $\sigma \to \sigma f$ on $Q \cup \{\sigma_0\}$, so that our assumption that $\sigma \to \sigma f$ is not continuous on S is false and the proof is complete.

If h is a function defined on a produkt $S \times T$ and σ is in S we denote by h_σ the function defined on T by

$$h_\sigma(\tau) = h(\sigma, \tau), \quad \tau \in T.$$

(1) The corresponding result for the weakly almost periodic compactification introduced in [6] is false.

LEMMA 4.2. *Let S and T be commutative topological semigroups, and f a function in $A(S\times T)$. Then*

$$\{\tau(f_\sigma): \sigma \in S,\ \tau \in T\} \tag{4.2}$$

is a conditionally compact subset of $C(T)$. Furthermore, the map $\sigma \to f_\sigma$ from S to $C(T)$ is continuous.

Proof. Let e be the identity element of S and define $\Phi: S(S\times T)\to C(T)$ by

$$\Phi(h) = h_e, \quad h \in C(S\times T).$$

Then

$$\Phi((\sigma, \tau) f) = \tau(f_\sigma), \quad \sigma \in S,\ \tau \in T,$$

so (4.2) is the image under the continuous map Φ of the conditionally compact subset $\{(\sigma, \tau) f: (\sigma, \tau) \in S\times T\}$ of $C(S\times T)$ and must thus itself be conditionally compact. Since f is continuous on $S\times T$, the map $\sigma\to f_\sigma$ is continuous from S to $C(T)$ supplied with the topology of pointwise convergence. But since T has an identity element, $\{f_\sigma: \sigma \in S\}$ is contained in (4.2) and is thus conditionally compact, so its topology must agree with that of pointwise convergence. Therefore $\sigma\to f_\sigma$ is continuous and the proof is complete.

LEMMA 4.3. *Let S and T be commutative topological semigroups, and*

$$\varphi: S\times T \to S\times T^a$$

be defined by

$$\varphi(\sigma, \tau) = (\sigma, I_T(\tau)), \quad (\sigma, \tau) \in S\times T.$$

Then the induced mapping

$$\tilde{\varphi}: C(S\times T^a)\to C(S\times T)$$

satisfies $\tilde{\varphi}(A(S\times T^a)) = A(S\times T)$.

Proof. φ is a continuous homomorphism, so by (i) of Lemma 3.1 it suffices to show that $A(S\times T)\subset\tilde{\varphi}(A(S\times T^a))$. Let f be any function in $A(S\times T)$. By Lemma 4.2, for each σ in S, f_σ is in $A(T)$ and thus $\hat{f}_\sigma$ is a function in $C(T^a)$. Define the function h on $S\times T^a$ by

$$h(\sigma, x) = \hat{f}_\sigma(x), \quad \sigma \in S,\ x \in T^a.$$

h has been defined so that it satisfies

$$h(\varphi(\sigma, \tau)) = f(\sigma, \tau), \quad (\sigma, \tau) \in S\times T.$$

Thus if it could be shown that h is in $A(S\times T^a)$, f would be $\tilde{\varphi}(h)$ and the proof would be complete.

First we show that h is in $C(S\times T^a)$. By Lemma 4.2, $\sigma\to f_\sigma$ is continuous from S to $A(T)$, so because of Lemma 3.2, $\sigma\to\hat{f}_\sigma$ is continuous from S to $C(T^a)$. If (σ, x) is an element of $S\times T^a$ and (τ, y) in $S\times T^a$ is chosen so that $\|\hat{f}_\sigma-\hat{f}_\tau\|<\frac{1}{2}\varepsilon$ and $|\hat{f}_\sigma(x)-\hat{f}_\sigma(y)|<\frac{1}{2}\varepsilon$, then

$$|h(\sigma, x)-h(\tau, y)|=|\hat{f}_\sigma(x)-\hat{f}_\tau(y)|\leqslant|\hat{f}_\sigma(x)-\hat{f}_\sigma(y)|+|\hat{f}_\sigma(y)-\hat{f}_\tau(y)|<\varepsilon.$$

But since $\sigma\to\hat{f}_\sigma$ is continuous and $\hat{f}_\sigma$ is in $C(T^a)$, this shows that h is in $C(S\times T^a)$.

We now complete the proof by showing that h is in $A(S\times T^a)$. It is simple to check that

$$\tilde{\varphi}(\varphi(\sigma, \tau)h)=(\sigma, \tau)f, \quad (\sigma, \tau)\in S\times T.$$

Thus since $\tilde{\varphi}$: $C(S\times T^a)\to C(S\times T)$ is an isometry and

$$\{(\sigma, \tau)f\colon\ (\sigma, \tau)\in S\times T\}$$

is conditionally compact in $C(S\times T)$,

$$\{\varphi(\sigma, \tau)h\colon\ (\sigma, \tau)\in S\times T\}$$

is conditionally compact in $C(S\times T^a)$. $\varphi(S\times T)$ is a dense subset of $S\times T^a$, so by Lemma 4.1 applied to h, $S\times T^a$ and $\varphi(S\times T)$, h is in $A(S\times T^a)$.

We can now establish the isomorphism (4.1) for the case of two factors.

Corollary 4.4. *Let S and T be commutative topological semigroups. Then there is a topological isomorphism between $(S\times T)^a$ and $S^a\times T^a$.*

Proof. Using Lemma 4.3 on the map $(\sigma, \tau)\to(\sigma, I_T(\tau))$ of $S\times T$ into $S\times T^a$ and also on the map $(\sigma, x)\to(I_S(\sigma), x)$ of $S\times T^a$ into $S^a\times T^a$, we see that if φ: $S\times T\to S^a\times T^a$ is their composite, i.e.

$$\varphi(\sigma, \tau)=(I_S(\sigma),\ I_T(\tau)), \quad (\sigma, \tau)\in S\times T,$$

then the induced map $\tilde{\varphi}$: $C(S^a\times T^a)\to C(S\times T)$ satisfies $\tilde{\varphi}(A(S^a\times T^a))=A(S\times T)$. Thus by Lemma 3.3 and (ii) of Lemma 3.1, $S^a\times T^a$ is topologically isomorphic to $(S\times T)^a$.

By Lemma 4.4 and induction, the isomorphism (4.1) holds for any finite number of factors. We now proceed to show that it holds in general.

Let $\{S_\gamma\}_{\gamma\in J}$ be a collection of commutative topological semigroups, e_γ the identity element of S_γ, $S=\prod_{\gamma\in J} S_\gamma$, $S^*=\prod_{\gamma\in J} S_\gamma^a$, and φ: $S\to S^*$ the homomorphism induced by the I_{S_γ}: $S_\gamma\to S_\gamma^a$. If σ is an element of S, its coordinate in S_γ will be denoted by σ_γ. Let K be a finite subset of J. We denote by $A_K(S)$ the subspace of $A(S)$ consisting of those f that depend only on the coordinates in K, i.e. that satisfy $f(\sigma)=f(\tau)$ whenever $\sigma_\gamma=\tau_\gamma$ for all γ in K.

LEMMA 4.5. *Let $\tilde{\varphi}$: $C(S^*)\to C(S)$ be the mapping induced by $\varphi\to S\to S^*$. If K is any finite subset of J, then $A_K(S)\subset\tilde{\varphi}(C(S^*))$.*

Proof. Let $S_1=\prod_{\gamma\in K}S_\gamma^a$, $S_2=\prod_{\gamma\in J\setminus K}S_\gamma$ and φ_K: $S\to S_1\times S_2$ be the homomorphism induced by the I_{S_γ} for γ in K and the identity mappings for γ in $J\setminus K$. By successive application of Lemma 4.3, we see that the induced mapping $\tilde{\varphi}_K$: $C(S_1\times S_2)\to C(S)$ satisfies $\tilde{\varphi}(A(S_1\times S_2))=A(S)$. Now let f be any function in $A_K(S)$ and let $\tilde{\varphi}_K(h)=f$. Since f is in $A_K(S)$,

$$h(x,\sigma)=h(x,\tau),\ x\in S_1,\ \sigma\in S_2,\ \tau\in S_2.$$

Let $S_3=\prod_{\gamma\in J\setminus K}S_\gamma^a$ and define k on $S^*=S_1\times S_3$ by

$$k(x,y)=h(x,\sigma),\ x\in S_1,\ y\in S_3,$$

where σ is any element of S_2. It is clear that k is in $C(S^*)$ and that $f=\tilde{\varphi}(k)$, so since f was an arbitrary function in $A_K(S)$, the lemma is proved.

LEMMA 4.6. *Let F be the collection of all finite subsets of J. Then $\bigcup_{K\in F}A_K(S)$ is dense in $A(S)$.*

Proof. Let f be a function in $A(S)$. By (iii) of Lemma 3.1 the map $\sigma\to\sigma f$ of S into $A(S)$ is continuous. Thus there is a neighborhood U of the identity element of S which is such that

$$\|\sigma f-f\|<\varepsilon,\quad \sigma\in U. \tag{4.3}$$

As a consequence of the definition of the product topology, there is a finite subset K of J so that any σ in S, with $\sigma_\gamma=e_\gamma$ for all γ in K, will be in U. Let ψ: $S\to S$ be the homomorphism defined by

$$(\psi(\tau))_\gamma=\begin{cases}\tau_\gamma, & \gamma\in K\\ e_\gamma, & \gamma\in J\setminus K.\end{cases}$$

Then for each τ in S there is a σ in U with $\tau=\sigma+\psi(\tau)$ (we need only set $\sigma_\gamma=\tau_\gamma$ for $\gamma\in J\setminus K$, $\sigma_\gamma=e_\gamma$ for $\gamma\in K$); thus

$$|f(\tau)-\tilde{\psi}f(\tau)|=|\sigma f(\psi(\tau))-f(\psi(\tau))|<\varepsilon$$

by (4.3), and $\|f-\tilde{\psi}f\|\leqslant\varepsilon$. But by (i) of Lemma 3.1, $\tilde{\psi}f$ is in $A(S)$, and therefore clearly in $A_K(S)$. This completes the proof.

We can now finally establish the isomorphism (4.1).

THEOREM 4.7. *Let $\{S_\gamma\}_{\gamma\in J}$ be a collection of commutative topological semigroups. Then $\prod_{\gamma\in J} S_\gamma^a$ and $(\prod_{\gamma\in J} S_\gamma)^a$ are topologically isomorphic.*

Proof. We use the notation introduced before Lemma 4.5. The mapping $\tilde{\varphi}: C(S^*)\to C(S)$ induced by $\varphi: S\to S^*$ is an isometry since $\varphi(S)$ is dense in S^*. Thus $\tilde{\varphi}(C(S^*))$ is a closed linear subspace of $C(S)$, which is contained in $A(S)$ by (i) of Lemma 3.1. But by Lemmas 4.5 and 4.6 it must be all of $A(S)$, so our result follows from Lemma 3.3.

LEMMA 4.8. *Let $\{T_\gamma\}_{\gamma\in S}$ be a collection of commutative topological semigroups. Then $\prod_{\gamma\in J} T_\gamma$ has sufficiently many semicharacters to separate points if and only if each T_γ has sufficiently many semicharacters to separate points.*

Proof. If $\alpha\in J$, there is a natural isomorphism of T_α into $\prod_{\gamma\in J} T_\gamma$, so T_α will have sufficiently many semicharacters if $\prod_{\gamma\in J} T_\gamma$ does. Conversely if T_α has sufficiently many semicharacters, any two elements of $\prod_{\gamma\in J} T_\gamma$ having different coordinates in T_α can be separated by the natural projection $\prod_{\gamma\in J} T_\gamma\to T_\alpha$ followed by some semicharacter of T_α.

The following is now an immediate consequence of Lemma 4.8, applied to $T_\gamma = S_\gamma^a$, Theorem 3.6 and Theorem 4.7.

THEOREM 4.9. *Let $\{S_\gamma\}_{\gamma\in J}$ be a collection of commutative topological semigroups. Then the approximation theorem holds for $\prod_{\gamma\in J} S_\gamma$ if and only if it holds for each of the S_γ.*

5. The Kernel

Let S be a commutative semigroup. In all that follows we shall use the standard notation

$$\sigma + U = \{\sigma+\tau:\ \tau\in U\},$$

$$U + V = \{\sigma+\tau:\ \sigma\in U,\ \tau\in V\},$$

if U and V are subsets of S and σ is an element of S. If T is a subset of S, T is called a *subsemigroup* if $T+T\subset T$ and is called an *ideal* if it is non-empty and $S+T\subset T$. The *kernel* of S is defined to be $\bigcap_{\sigma\in S}(\sigma+S)$ and denoted by $K(S)$. If non-empty, it is clearly the smallest ideal of S.

The following will be useful later.

LEMMA 5.1. *Let S be a compact commutative topological semigroup. If Q is a dense subset of S, then*

$$\bigcap_{\sigma \in Q} (\sigma + S) \tag{5.1}$$

is identical with the kernel $K(S)$.

Proof. Let τ be an element of (5.1). Choose any σ in S and let $\{\sigma_\gamma\}$ be a net in Q with $\sigma_\gamma \to \sigma$. Since τ is in (5.1), for each γ, there is some η_γ in S with $\sigma_\gamma + \eta_\gamma = \tau$. By the compactness of S, the net $\{\eta_\gamma\}$ has a cluster point η, which satisfies $\sigma + \eta = \tau$ since the multiplication of S is jointly continuous. But σ was an arbitrary element of S, so τ is in $\bigcap_{\sigma \in S} (\sigma + S) = K(S)$.

The next lemma, which is well known (see [8] or § 1 of [13]) is the basic result concerning the kernel.

LEMMA 5.2. *Let S be a commutative semigroup with a compact topology in which addition is jointly continuous.*(1) *Then the kernel $K(S)$ is non-empty and is a compact topological group.*

As a consequence we obtain the fact that a compact commutative topological semigroup always has sufficiently many semicharacters to separate points of the kernel.

COROLLARY 5.3. *Let S be a compact commutative topological semigroup. Suppose that σ and τ are distinct elements of the kernel $K(S)$. Then there is a semicharacter χ of S satisfying $\chi(\sigma) \neq \chi(\tau)$.*

Proof. By Lemma 5.2, $K(S)$ is a compact commutative topological group, so by the Peter–Weyl Theorem, there is a character χ_0 of $K(S)$ with $\chi_0(\sigma) \neq \chi_0(\tau)$. Let e be the identity element of $K(S)$. Then the function χ defined on S by

$$\chi(\sigma) = \chi_0(\sigma + e), \quad \sigma \in S,$$

is a semicharacter of S and satisfies

$$\chi(\sigma) = \chi_0(\sigma) \neq \chi_0(\tau) = \chi(\tau).$$

6. The Half-line and Half-integers

The proof of the approximation theorem for cones which we give in Section 9 proceeds by induction on dimension. In dimension 1 a cone is either the full line, the half-line $[0, \infty)$ or the isomorphic half-line $(-\infty, 0]$. The approximation theorem

(1) In particular, S may be a compact subsemigroup of a commutative topological semigroup.

is known for the full line. In this section we obtain, as a relatively simple consequence of the results established thus far, the validity of the approximation theorem for the half-line. The same proof shows that the approximation theorem holds for the half-integers $\{n: n=0, 1, 2, \ldots\}$.

THEOREM 6.1. *The approximation theorem holds for*

(i) *The semigroup of the half-line* $[0, \infty)$ *under addition with the usual topology.*

(ii) *The semigroup of the half-integers* $\{n: n=0, 1, 2, \ldots\}$ *under addition with the discrete topology.*

Proof. Let S be either of the semigroups described. The function χ defined on S by

$$\chi(\sigma)=e^{-\sigma}, \quad \sigma \in S, \tag{6.1}$$

is a semicharacter and separates points of S. Thus $I_S: S \to S^a$ is $1-1$ and we identify S with its image in S^a.

We show first that $S^a = S \cup K(S^a)$. Let x be a point of S^a not in S. For each σ in S, $\{\tau: \tau \in S, 0 \leqslant \tau \leqslant \sigma\}$ is closed in S^a since it is compact in the topology of S and $I_S: S \to S^a$ is continuous. Thus, since S is dense in S^a, x must be in

$$\{\tau: \tau \in S, \ \tau \geqslant \sigma\}^- = (\sigma+S)^- = \sigma + S^a$$

for each σ in S, so x is in $\bigcap_{\sigma \in S} (\sigma + S^a)$, which by Lemma 5.1 is the kernel $K(S^a)$.

To complete the proof, because of Theorem 3.6, it is only necessary to show that two distinct points x and y of S^a can be separated by a semicharacter of S^a. We have shown that $S^a = S \cup K(S^a)$, so there are the following three cases to consider.

Case I. x and y in S. If χ is defined by (6.1), then $\hat{\chi}$ is a semicharacter of S^a and satisfies

$$\hat{\chi}(x)=\chi(x) \neq \chi(y)=\hat{\chi}(y).$$

Case II. x in S, y in $K(S^a)$. For each $\sigma \in S$, $K(S^a) \subset \sigma + S^a$, so if χ is defined by (6.1), then

$$|\hat{\chi}(y)| \leqslant \inf_{\sigma \in S} |\hat{\chi}(\sigma)| = 0.$$

On the other hand, $\hat{\chi}(x)=\chi(x) \neq 0$, so $\hat{\chi}(x) \neq \hat{\chi}(y)$.

Case III. x and y in $K(S^a)$. Corollary 5.3 guarantees that in this case there will be a semicharacter of S^a separating x and y. This completes the proof of Theorem 6.1.

If S is the half-line, it is simple to show using Theorem 6.1 that $A(S)$ consists of all f in $C(S)$ of the form f_1+f_2, where f_1 is the restriction of a function almost periodic on the full line and f_2 satisfies $\lim_{\sigma\to\infty} f_2(\sigma)=0$. This is slightly stronger than a result in [7].

Theorem 6.1 together with Theorem 4.9 shows that the approximation theorem holds for certain cones; in particular it holds for closed sectors in the plane. Thus a function on a close sector is almost periodic if and only if it can be approximated uniformly by linear combinations of the semicharacters of the sector, which will be exponential functions. It can further be shown that if such a function is analytic at interior points of the sector, approximation by linear combinations of analytic exponential functions is possible. When the sector is a half-plane, this is the approximation theorem of [1].

Finally let us point out that the approximation theorem for the discrete half-line cannot be obtained as simply as Theorem 6.1. The result is contained in Theorem 11.1.

7. The Main Lemma

In this section we isolate a technical lemma that will be used later in sections 9, 10 and 12.

Let G be a commutative topological group and S a subsemigroup of G containing its identity element. S supplied with the topology induced by G is a commutative topological semigroup and the lemma we prove concerns the separation by semicharacters of points in the compactification S^a.

We assume that S itself has sufficiently many semicharacters to separate points so that $I_S\colon S\to S^a$ is $1-1$ and as before, we can identify S with its image in S^a.

Let Q be a subsemigroup of S and Q^- its closure in S^a. Q^- is a subsemigroup of S^a and we shall denote its kernel $K(Q^-)$ simply by K. The subgroup $\{\sigma-\tau\colon \sigma\in Q,\ \tau\in Q\}$ of G generated by Q will be denoted by H. We assume that H is a closed subgroup of G. The image of S under the natural projection $G\to G/H$ is a subsemigroup of G/H which will be denoted by T. T is a topological semigroup if supplied with the topology induced by the quotient topology on G/H.

LEMMA 7.1. *Let x and y be distinct elements of S^a. Suppose that x and y are in $e+S^a$, where e is the identity element of K.*

(i) *If $(x+K)\cap(y+K)$ is non-empty and if in addition each character*[1] *of Q extends to a semicharacter of S, then x and y can be separated by a semicharacter of S^a.*

[1] A character is a semicharacter that is of modulus one everywhere.

(ii) *If* $(x+K)\cap(y+K)$ *is empty and if in addition the approximation theorem holds for* T, *then* x *and* y *ban be separated by a semicharacter of* S^a.

Proof of (i). Let $x+u=y+v$ for u and v in K. By Lemma 5.2, K is a compact commutative topological group, so by the Peter–Weyl Theorem, there is a character χ_1 of K with $\chi_1(u)\neq\chi_1(v)$. The function χ_2 defined on Q^- by

$$\chi_2(w)=\chi_1(e+w), \quad w\in Q^-,$$

is a character of Q^-. Let χ_3 be the restriction of χ_2 to Q. Since the injection map of S into S^a is continuous, χ_3 is continuous in the topology of Q and is thus a character of Q. By hypothesis, χ_3 extends to a semicharacter χ_4 of S, i.e. there is a semicharacter χ_4 of S that satisfies

$$\chi_4(\sigma)=\chi_3(\sigma), \quad \sigma\in Q.$$

We will show that the semicharacter $\hat{\chi}_4$ of S^a satisfies $\hat{\chi}_4(x)\neq\hat{\chi}_4(y)$. By its definition, $\hat{\chi}_4$ when restricted to Q agrees with χ_2 and since both $\hat{\chi}_4$ and χ_2 are continuous on Q^-, they agree on u and v. Thus

$$\hat{\chi}_4(u)=\chi_2(u)=\chi_1(e+u)=\chi_1(u)\neq\chi_1(v)=\chi_1(e+v)=\chi_2(v)=\hat{\chi}_4(v)$$

and as a consequence, since $x+u=y+v$,

$$\hat{\chi}_4(x)=\frac{\hat{\chi}_4(x+u)}{\hat{\chi}_4(u)}\neq\frac{\hat{\chi}_4(y+v)}{\hat{\chi}_4(v)}=\hat{\chi}_4(y).$$

This completes the proof of (i).

Proof of (ii). The natural projection $G\to G/H$ when restricted to S yields the continuous homomorphism $\varphi: S\to T$ defined by

$$\varphi(\sigma)=\sigma+H, \quad \sigma\in S.$$

By Lemma 3.4, φ induces a continuous homomorphism $\varphi^a: S^a\to T^a$ that satisfies

$$\varphi^a(\sigma)=I_T(\varphi(\sigma)), \quad \sigma\in S, \tag{7.1}$$

since S has been identified with its image in S^a. Suppose that we could show that $\varphi^a(x)$ and $\varphi^a(y)$ were distinct. Then, since we have assumed that the approximation theorem holds for T, by Theorem 3.6 there would be a semicharacter χ of T^a with $\chi(\varphi^a(x))\neq\chi(\varphi^a(y))$ and thus the composite $\chi\circ\varphi^a$ would be a semicharacter of S^a separating x and y. So to complete the proof of (ii) it suffices to show that $\varphi^a(x)\neq\varphi^a(y)$. Since $x+K$ and $y+K$ are disjoint compact subsets of S^a, there is a function f in $C(S^a)$ satisfying

$$f(u) = \begin{cases} 0, & u \in x + K \\ 1, & u \in y + K. \end{cases} \tag{7.2}$$

Let μ be normalized Haar measure on K. By Lemma 3.1 the map $v \to v f$ of S^a into $C(S^a)$ is continuous, so the vector valued integral (1)

$$\int_K v f \, d\mu(v)$$

exists and represents a function h in $C(S^a)$. Because of (7.2), $h(x) = 0$ and $h(y) = 1$. Furthermore, because of the invariance of μ, $v h = h$ for all v in K and thus since $e + Q^- \subset K$,

$$u h = u(e h) = (u + e) h = h, \quad u \in Q^-. \tag{7.3}$$

Let k be the restriction of h to S, so that k is in $A(S)$ and $h = \hat{k}$. Because of (7.3),

$$k(\sigma + \tau) = k(\sigma), \ \ \sigma \in S, \ \ \tau \in Q,$$

and as a consequence, if $\varphi(\sigma_1) = \varphi(\sigma_2)$, then $k(\sigma_1) = k(\sigma_2)$. Thus we can define the function g on T by

$$g(\varphi(\sigma)) = k(\sigma), \quad \sigma \in S. \tag{7.4}$$

It is simple to check that g is in $C(T)$. Let $\tilde{\varphi}\colon C(T) \to C(S)$ be the map induced by $\varphi\colon S \to T$. Because of (7.4), $\tilde{\varphi}(g) = k$.

Furthermore,
$$\tilde{\varphi}(\varphi(\sigma) g)) = \sigma k, \quad \sigma \in S,$$
so $\tilde{\varphi}$ maps
$$\{\tau g\colon \tau \in T\} \tag{7.5}$$
into
$$\{\sigma k\colon \sigma \in S\}. \tag{7.6}$$

But since φ is onto, $\tilde{\varphi}$ is an isometry and thus (7.5) must be conditionally compact since (7.6) is conditionally compact. As a consequence g is in $A(T)$. For each σ in S, because of (7.1),

$$\hat{g}(\varphi^a(\sigma)) = \hat{g}(I_T(\varphi(\sigma)) = g(\varphi(\sigma)) = k(\sigma) = h(\sigma).$$

Thus since S is dense in S^a,

$$\hat{g}(\varphi^a(u)) = h(u), \quad u \in S^a,$$

so in particular
$$\hat{g}(\varphi^a(x)) = h(x) = 0 \neq 1 = h(y) = \hat{g}(\varphi^a(y)).$$

But this shows that $\varphi^a(x) \neq \varphi^a(y)$ so the proof of (ii) is complete.

---

(1) We shall use the standard properties of vector valued integration found for example in [3].

8. More Lemmas

In this section we establish several lemmas for later use. Most of the results are of little independent interest and occur here simply because they will be used more than once in the following.

Until Lemma 8.5 we assume that any semigroup S which is discussed has sufficiently many almost periodic functions to separate points. In these cases the map $I_S\colon S\to S^a$ is $1-1$ and we identify S with its image in S^a.

LEMMA 8.1. *Let G be a commutative group, S a subsemigroup and H a subgroup, with $H\subset S\subset G$. Let T be the image of S under the natural projection $G\to G/H$. If the approximation theorem holds for T supplied with the discrete topology, then the approximation theorem holds for S supplied with the discrete topology.*

Proof. We apply Lemma 7.1, taking Q to be the subgroup H and the topology of G to be discrete. Since Q is a group, $\sigma+Q=Q$ for all σ in Q, $\sigma+Q^-=(\sigma+Q)^-=Q^-$, and so the kernel K of Q^- is $\bigcap_{\sigma\in Q}(\sigma+Q^-)=Q^-$ by Lemma 5.1. Thus $Q^-=K$ is a group. In particular, the identity element e of K is the identity element of G, so $e+S^a=S^a$. Since G is a discrete group, any character of Q extends to a character of G and thus of S. Furthermore, we have assumed that the approximation theorem holds for T. Thus the hypotheses of Lemma 7.1 are satisfied, so if x and y are two distinct elements of S^a, there is a semicharacter χ of S^a with $\chi(x)\neq\chi(y)$. That the approximation theorem holds for S is then a consequence of Theorem 3.6.

LEMMA 8.2. *Let S be a locally compact commutative topological semigroup. Suppose that S has a semicharacter χ that is nowhere zero and vanishes at infinity. Then*

$$S=\{x\colon x\in S^a,\ \hat{\chi}(x)\neq 0\}.$$

Furthermore, S is an open subsemigroup of S^a whose complement $S^a\setminus S$ is an ideal.

Proof. Since χ vanishes at infinity, for each $\varepsilon>0$

$$\{\sigma\colon \sigma\in S,\ |\chi(\sigma)|\geqslant\varepsilon\}$$

is compact in the topology of S and thus in the topology of S^a. As a consequence, any x in $S^a\setminus S$ must be in the closure of each

$$\{\sigma\colon \sigma\in S,\ |\chi(\sigma)|\leqslant\varepsilon\}$$

and thus satisfy $\hat{\chi}(x)=0$. On the other hand, if $x\in S$, then $\hat{\chi}(x)=\chi(x)\neq 0$. This

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establishes the first assertion of the lemma. The second is immediate since $\hat{\chi}$ is a semicharacter of S^a.

LEMMA 8.3. *Let S be a commutative topological semigroup. Suppose that T is an open and closed subsemigroup of S whose complement $S \setminus T$ is an ideal. Then the closure T^- of T in S^a is an open and closed subsemigroup of S^a, whose complement $S^a \setminus T^-$ is an ideal of S^a. Furthermore, if $j: T \to S$ is the injection mapping, the induced homomorphism*(1) *$j^a: T^a \to S^a$ is a homeomorphism of T^a onto T^-.*

Proof. Let χ be the characteristic function of T in S, i.e.

$$\chi(\sigma) = \begin{cases} 1, & \sigma \in T \\ 0, & \sigma \in S \setminus T. \end{cases}$$

χ is a semicharacter of S. The semicharacter $\hat{\chi}$ of S^a satisfies

$$\hat{\chi}(x) = \begin{cases} 1, & x \in T^- \\ 0, & x \in S^a \setminus T^-, \end{cases}$$

so T^- has the properties claimed. Because of the commutativity of the diagram in Lemma 3.4, T is a dense subset of $j^a(T^a)$. Furthermore $j^a(T^a)$ is compact, since T^a is compact and j^a continuous, so $j^a(T^a) = T^-$. It remains only to show that j^a is $1-1$. Let x and y be distinct elements of T^a. Choose f in $A(T)$ so that $\hat{f}(x) \neq \hat{f}(y)$ and define h on S by

$$h(\sigma) = \begin{cases} f(\sigma), & \sigma \in T \\ 0, & \sigma \in S \setminus T. \end{cases}$$

It is simple to check that h is in $A(S)$. Moreover, for σ in T,

$$\hat{h}(j^a(\sigma)) = \hat{h}(j(\sigma)) = h(\sigma) = f(\sigma) = \hat{f}(\sigma),$$

so that $\hat{h}(j^a(x)) = \hat{f}(x) \neq \hat{f}(y) = \hat{h}(j^a(y))$, and $j^a(x) \neq j^a(y)$. Thus j^a is $1-1$ and the proof is complete.

Let S be a commutative semigroup and T a subsemigroup of S. A collection W of subsets of T is called an *initial family* for T if it satisfies the following conditions.

(i) W does not contain the empty set.
(ii) If U and V are in W, then $U \cap V$ is in W.
(iii) For each U in W, there is a V in W with $V + V \subset U$.
(iv) For each τ in T there is a U in W so that $\tau \in \sigma + S$ for each σ in U.

(1) The induced homomorphism j^a is that given by Lemma 3.4.

LEMMA 8.4. *Let S be a commutative semigroup supplied with the discrete topology. Suppose that T is a subsemigroup of S and W an initial family for T. Then*[1]

$$H = \bigcap_{U \in W} U^-$$

is a compact topological group whose identity is an identity for T^-.

Proof. Because W satisfies conditions (i) and (ii), any finite number of the U^-, for U in W, have a non-void intersection. Thus by the compactness of S^a, H is compact and non-empty. It is a subsemigroup of S^a because W satisfies condition (iii). The kernel $K(H)$ is a compact topological group by Lemma 5.2. Let e be its identity element and let τ be any element of T. By condition (iv) there is a U in W so that $\tau \in \sigma + S$ for each σ in U. Since e is in H, there is a net $\{\sigma_\gamma\}$ in U with $\sigma_\gamma \to e$. If $\{\eta_\gamma\}$ is chosen so that $\tau = \sigma_\gamma + \eta_\gamma$ for each γ, then $\tau = e + x$, where x is any cluster point of the net $\{\eta_\gamma\}$ in S^a. As a consequence, $e + \tau = e + (e + x) = e + x = \tau$ and since τ was an arbitrary element of T, e is an identity element for T^-. In particular, $e + H = H$. But since e is the identity element of $K(H)$, $e + H = K(H)$, so $H = K(H)$, which is a compact topological group.

If S is a subset of a commutative topological group G, a complex valued function f defined on S will be called *uniformly continuous* if it is uniformly continuous with respect to the uniform structure on S induced by that of G, i.e. if for each $\varepsilon > 0$ there is a neighborhood U of the identity element 0 in G so that $|f(\sigma+\tau) - f(\sigma)| < \varepsilon$ if σ and $\sigma + \tau$ are in S and τ is in U. We shall need the following "one-sided" criterion for uniform continuity.

LEMMA 8.5. *Let G be a commutative topological group and S a subsemigroup of G containing 0. Suppose that the interior T of S is dense in S. Let f be a complex valued function on S for which, for each $\varepsilon > 0$, there is a neighborhood U of 0 in G so that*

$$|f(\sigma+\tau) - f(\sigma)| < \varepsilon, \quad \sigma \in S, \quad \tau \in U \cap T.$$

Then f is uniformly continuous on S.

Proof. Choose a neighborhood V of 0 in G so that

$$|f(\sigma' + \tau') - f(\sigma')| < \tfrac{1}{2}\varepsilon, \quad \sigma' \in S, \quad \tau' \in V \cap T. \tag{8.1}$$

$V \cap T$ is non-empty since 0 is a limit point of T. Choose an element η of $V \cap T$ and a symmetric neighborhood U of 0 in G so that $\eta + U \subset V \cap T$. Let σ and $\sigma + \tau$ be

[1] Recall that $^-$ means closure in S^a.

elements of S with τ in U. We shall complete the proof by showing that

$$|f(\sigma+\tau)-f(\sigma)|<\varepsilon.$$

Since $\eta+U\subset V\cap T$ and U is symmetric, $\eta-\tau$ is in $V\cap T$. Then by (8.1),

$$\begin{aligned}|f(\sigma+\tau)-f(\sigma)| &\leq |f(\sigma+\tau)-f(\sigma+\eta)|+|f(\sigma+\eta)-f(\sigma)|\\ &=|f((\sigma+\tau)+(\eta-\tau))-f(\sigma+\tau)|+|f(\sigma+\eta)-f(\sigma)|<\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon.\end{aligned}$$

LEMMA 8.6. *Let G be a commutative topological group and S a subsemigroup of G containing 0 whose interior is dense in S. Let S_d be the semigroup S supplied with the discrete topology. Then there is a projection*[1] *Φ of $A(S_d)$ onto $A(S)$ that satisfies*

$$\|\Phi f\|\leq\|f\|,\quad f\in A(S_d), \tag{8.2}$$

and $\Phi\chi=0$ for each semicharacter χ of S_d that is not continuous on S.

Proof. Let T be the interior of S and W the collection of all subsets of T of the form $V\cap T$ for V a neighborhood of 0 in G. We show first that W is an initial family for T in the sense of the definition given before Lemma 8.4. Condition (i) holds since 0 is a limit point of T. (ii) and (iii) are clear. If τ is in T and V is chosen to be a symmetric neighborhood of 0 in G with $\tau+V\subset T$, then $\tau\in\sigma+T$ for each σ in V and thus in particular for each σ in $V\cap T$. Therefore condition (iv) is satisfied and W is an initial family for T as claimed. S_d is a subsemigroup of a discrete commutative group and thus has sufficiently many characters to separate points, so the map $I_{S_d}\colon S_d\to S_d^a$ is $1-1$. We shall identify S_d with its image in S_d^a and consider T and its subsets as subsets of S_d^a. Since W is an initial family for T, by Lemma 8.4 applied to S_d,

$$H=\bigcap_{U\in W}U^-$$

is a subsemigroup of S_d^a which is a compact topological group. Let μ be normalized Haar measure on H. If h is in $C(S_d^a)$, by Lemma 3.1 the map $x\to xh$ is continuous from S_d^a to $C(S_d^a)$ so the vector valued integral

$$\int_H xh\,d\mu(x)$$

is defined and represents a function in $C(S_d^a)$. We define the map $\Psi\colon C(S_d^a)\to C(S_d^a)$ by

$$\Psi h=\int_H xh\,d\mu(x),\quad h\in C(S_d^a).$$

---

[1] We are identifying $A(S)$ with the subspace of $A(S_d)$ consisting of those functions that are continuous on S. Φ is called a *projection* if it is linear and satisfies $\Phi^2=\Phi$.

It is simple to check, using standard properties of vector valued integrals, and the invariance of μ, that Ψ is a projection of $C(S_d^a)$ onto the linear subspace

$$\{h: h \in C(S_d^a),\ x h = h,\ \text{all } x \in H\}$$

and that
$$\|\Psi h\| \leqslant \|h\|,\ h \in C(S_d^a). \tag{8.3}$$

We define Φ to be the composite mapping

$$A(S_d) \xrightarrow{j} C(S_d^a) \xrightarrow{\Psi} C(S_d^a) \xrightarrow{j^{-1}} A(S_d),$$

where j is the isomorphism defined by

$$j(f) = \hat{f}, \quad f \in A(S_d).$$

Φ is a projection since Ψ is a projection. Φ satisfies (8.2) since Ψ satisfies (8.3) and j is an isometry. To establish that the range of Φ is as claimed, we must show that a function f in $A(S_d)$ is in $A(S)$ if and only if it satisfies $x\hat{f} = \hat{f}$ for all x in H. So let f be in $A(S)$ and x be an element in H. Because of the definition of H, it is possible to find a net $\{\tau_\gamma\}$ in T with $\tau_\gamma \to 0$ in the topology of S and $\tau_\gamma \to x$ in the topology of S_d^a. Then for each σ in S_d,

$$x\hat{f}(\sigma) = \hat{f}(x+\sigma) = \lim_\gamma \hat{f}(\tau_\gamma + \sigma) = \lim_\gamma f(\tau_\gamma + \sigma) = f(\sigma) = \hat{f}(\sigma),$$

so since S_d is dense in S_d^a, $x\hat{f} = \hat{f}$. For the converse, let f be a function in $A(S_d)$ that is not continuous on S. Then f is not uniformly continuous on S, so by Lemma 8.5 there is an $\varepsilon > 0$ and for each neighborhood V of 0 in G a τ_V in $V \cap T$ with $\|\tau_V f - f\| > \varepsilon$, and thus $\|\tau_V \hat{f} - \hat{f}\| \geqslant \varepsilon$. Let x be a cluster point in S_d^a of the net $\{\tau_V\}$. By the definition of H, x is in H. Furthermore $\|x\hat{f} - \hat{f}\| \geqslant \varepsilon$ because of Lemma 3.1, so $x\hat{f} \neq \hat{f}$. This completes the proof that a function f in $A(S_d)$ is in $A(S)$ if and only if $x\hat{f} = \hat{f}$ for all x in H, and thus establishes the fact that Φ maps $A(S_d)$ onto $A(S)$. It remains to show that if χ is a semicharacter of S_d that is not continuous on S, then $\Phi\chi = 0$. Let χ be such a semicharacter. Then, as we have shown, there must be some y in H with $y\hat{\chi} \neq \hat{\chi}$. But since $y\hat{\chi} = \hat{\chi}(y)\hat{\chi}$, the restriction of $\hat{\chi}$ to the group H is not identically 1 so must be either identically 0 or agree with a character of H that is not identically 1. In either case,

$$\int_H \hat{\chi}(x)\, d\mu(x) = 0,$$

and as a consequence,

$$\Phi \hat{\chi} = \Psi \hat{\chi} = \int_H x \chi \, d\mu(x) = \int_H \hat{\chi}(x) \hat{\chi} \, d\mu(x) = \left(\int_H \hat{\chi}(x) \, d\mu(x) \right) \hat{\chi} = 0,$$

so $\Phi \chi = 0$. This completes the proof of Lemma 8.6.

COROLLARY 8.7. *Let G be a commutative topological group and S a subsemigroup of G containing 0. Suppose that the interior of S is dense in S. Then the approximation theorem holds for S if it holds for S supplied with the discrete topology.*

Proof. Let S_d be S supplied with the discrete topology and Φ: $A(S_d) \to A(S)$ the projection whose existence is established in Lemma 8.6. Because of the properties of Φ stated there, if f is in $A(S)$ and g is a linear combination of semicharacters of S_d with $\|f - g\| < \varepsilon$, then Φg is a linear combination of semicharacters of S with $\|f - \Phi g\| < \varepsilon$. Thus the approximation theorem will hold for S if it holds for S_d.

LEMMA 8.8. *Let G be a commutative topological group and S a subsemigroup of G containing 0. Suppose that the interior of S is dense in S. Let $\bar{S}$ be the closure of S in G. Then the approximation theorem holds for S if it holds for $\bar{S}$.*

Proof. Let f be a function in $A(S)$. By Lemma 3.1, the map $\sigma \to \sigma f$ from S to $A(S)$ is continuous. Since the map is continuous at $\sigma = 0$, Lemma 8.5 shows that f is uniformly continuous on S. Thus it has a continuous extension g to $\bar{S}$, i.e. there is a function g in $C(\bar{S})$ with

$$g(\sigma) = f(\sigma), \quad \sigma \in S.$$

Let r: $C(\bar{S}) \to C(S)$ be the restriction mapping. r is an isometry, so $\{\sigma g : \sigma \in S\}$, which is the inverse image under r of the conditionally compact set $\{\sigma f : \sigma \in S\}$, is conditionally compact. Thus by Lemma 4.1, g is in $A(\bar{S})$. Any approximation of g on $\bar{S}$ by a linear combination of semicharacters of $\bar{S}$, when restricted to S, yields an approximation of f on S by a linear combination of semicharacters of S. Thus the approximation theorem will hold for S if (1) it holds for $\bar{S}$.

LEMMA 8.9. *Let S be a commutative topological semigroup and F a subset of S consisting of idempotents.*(2) *Suppose that for each e in F and each u and v in $e + S$ with $u \neq v$ there is a semicharacter χ of S with $\chi(u) \neq \chi(v)$. Then for each x and y in $F + S$ with $x \neq y$ there is a semicharacter χ of S with $\chi(x) \neq \chi(y)$.*

(1) Conversely, if the approximation theorem holds for S, it holds for $\bar{S}$; for semicharacters of S extend to $\bar{S}$ continuously by the above, and thus, for f in $A(\bar{S})$, an approximation of $f | S$ extends to an approximation of f.

(2) e is called an *idempotent* if $e + e = e$.

Proof. Let x and y be distinct elements of $F+S$. Choose e in F so that $x \in e+S$.

Case I. $e+y \neq x$. Both $e+y$ and x are in $e+S$ so by hypothesis there is a semicharacter χ of S with

$$\chi(x) \neq \chi(e+y) = \chi(e)\chi(y). \tag{8.4}$$

Since e is an idempotent $\psi(e)$ is either 0 or 1. $\chi(e)$ cannot be 0, for then we would have

$$\chi(x) = \chi(e+x) = \chi(e)\chi(x) = 0 = \chi(e),$$

which contradicts (8.4). Thus $\chi(e)=1$, so by (8.4). $\chi(x) \neq \chi(y)$.

Case II. $e+y=x$. Let $y \in e'+S$ with $e' \in F$ so $y=e'+z=e'+e'+z=e'+y$. Then $x=e+y=e+e'+y$ is in $e'+S$ and $y=e'+y \neq x$, so by Case I applied to e', there is a semicharacter χ of S with $\chi(x) \neq \chi(y)$.

9. Cones

Euclidean n-space E^n is a topological group under vector addition and the usual topology. A subset S of E^n is called a *cone* if it is a subsemigroup of E^n and furthermore if, for each σ in S, the ray $\{\lambda\sigma\colon \lambda \geqslant 0\}$ is also in S. Any such cone, supplied with the induced topology from E^n, is a commutative topological semigroup. This section is devoted to the proof of the following.

THEOREM 9.1. *Let S be a cone in E^n. Then the approximation theorem holds for S.*

Even though one is interested mainly in closed cones, it is necessary for us to consider also cones that are not necessarily closed, as these will arise in the course of our proof as projections of closed cones. We shall use below without further comment standard elementary results on cones that may be found for example in [2].

The proof of Theorem 9.1 will proceed by induction on n. The only cones in E^1 are the full line E^1 and the half-lines $[0, \infty)$ and $(-\infty, 0]$. The full line is a topological group so the approximation theorem is known; the approximation theorem for the half-lines has been established in Theorem 6.1. We take as our induction hypothesis the validity of the approximation theorem for all cones in E^{n-1}. We shall show, on the basis of this, that the approximation theorem holds for all cones in E^n.

So let S be a cone in E^n. We may assume that S has non-void interior. For if that is not the case. S will be isomorphic to a cone in E^{n-1} and by the induction hypothesis, the approximation theorem holds for S. A cone with non-void interior has its interior dense. Thus Lemma 8.8 shows that the approximation theorem holds for S if it holds for the closure of S, which is also a cone. So we may assume that S is closed.

Finally, we may assume that S is *proper*, i.e. that

$$\{\sigma:\ \sigma \in S,\ -\sigma \in S\}$$

consists of 0 alone. For if S is not proper, it contains a full line $\{\lambda\sigma:\ -\infty < \lambda < +\infty\}$ and will be isomorphic to the product of E^1 and a cone in E^{n-1}, so by the approximation theorem for the line, the induction hypothesis, and Theorem 4.9, the approximation theorem holds for S.

LEMMA 9.2. *S has a semicharacter that is nowhere zero and vanishes at infinity.*

Proof. Since S is a closed proper cone in E^n, there is a linear functional h on E^n which is non-negative on S and which is such that

$$\{\sigma:\ \sigma \in S,\quad 0 \leqslant h(\sigma) \leqslant 1\}$$

is a compact subset of S. The semicharacter χ defined by

$$\chi(\sigma) = e^{-h(\sigma)},\quad \sigma \in S,$$

has the properties desired.

S has sufficiently many semicharacters to separate points, so I_S: $S \to S^a$ is $1-1$. We shall identify S with its image in S^a.

By Lemmas 8.2 and 9.2, $S^a \setminus S$ is a closed ideal in S^a. The next three lemmas lead to Corollary 9.6, which gives an identification of $S^a \setminus S$ that is sufficient for our purposes. First two definitions are necessary.

The only idempotent in S is 0. We shall denote by E the collection of all other idempotents in S^a. $S^a \setminus S$ is closed in S^a and because of the joint continuity of addition in S^a, the set of all idempotents in S^a is also closed. Thus E is a closed subset of S^a.

Let σ be an element of S, $\sigma \neq 0$. The closure in S^a of the ray $\{\lambda\sigma:\ \lambda \geqslant 0\}$ is a compact subsemigroup of S^a. By Lemma 5.2, the kernel of this compact semigroup contains a unique idempotent and this idempotent will be denoted by e_σ. We define F to be the set $\{e_\sigma:\ \sigma \in S,\ \sigma \neq 0\}$ of all idempotents of S^a obtained in this manner.

In the following, if $\sigma \in E^n$, we shall denote by $|\sigma|$ the distance from σ to 0 in the Euclidean metric.

LEMMA 9.3. *Let f be a function in $A(S)$ that satisfies $e\hat{f} = 0$ for all e in E. Then f vanishes at infinity on S.*

Proof. Choose any $\varepsilon > 0$. Let σ be a point in

$$\{\tau\colon\ \tau \in S,\quad |\tau| = 1\}. \tag{9.1}$$

Then $\lambda \to \|(\lambda\sigma)\hat{f}\|$ is a non-increasing function on the half-line $[0, \infty)$. We show first that it goes to 0 as λ increases. By Lemma 3.1, the map $\tau \to \tau\hat{f}$ from S^a to $C(S^a)$ is continuous, and since e_σ is in the closure of the ray $\{\lambda\sigma\colon \lambda \geqslant 0\}$, $0 = e_\sigma\hat{f}$ must be in the closure in $C(S^a)$ of $\{(\lambda\sigma)\hat{f}\colon \lambda \geqslant 0\}$. But this cannot happen unless

$$0 = \lim_{\lambda\to\infty} \|(\lambda\sigma)\hat{f}\| = \lim_{\lambda\to\infty} \|(\lambda\sigma)f\|.$$

In particular, there must be a λ_σ with $\|(\lambda_\sigma\sigma)f\| < \varepsilon$. Again using Lemma 3.1, the map $\tau \to \tau f$ from S to $C(S)$ is continuous, so there is a neighborhood U_σ of σ in (9.1) so that $\|(\lambda_\sigma\tau)f\| < \varepsilon$ for all τ in U_σ. (9.1) is compact and thus can be covered by a finite number of the U_σ. If λ is the maximum of the corresponding λ_σ, then $\|(\lambda\tau)f\| < \varepsilon$ for all τ in (9.1) and as a consequence $|f(\tau)| < \varepsilon$ if $|\tau| > \lambda$. Since ε was arbitrary, f vanishes at infinity as claimed.

LEMMA 9.4. $S^a \setminus S = E + S^a$.

Proof. Since $E \subset S^a \setminus S$ and $S^a \setminus S$ is an ideal, $E + S^a \subset S^a \setminus S$. Thus it remains to establish the reverse inclusion. So let x be a point in S^a not in $E + S^a$. We must show that x is in S. E is compact, so by the joint continuity of addition in S^a, $E + S^a$ is compact. Thus, by Lemma 3.2, we can choose f in $A(S)$ so that $\hat{f}(x) = 1$ and $\hat{f}$ is zero on $E + S^a$. Then $e\hat{f} = 0$ for each e in E, so by Lemma 9.3, f vanishes at infinity on S. Since S is dense in S^a and $\hat{f}(x) = 1$, x is in the closure in S^a of

$$\{\sigma\colon\ \sigma \in S,\ |f(\sigma)| \geqslant \tfrac{1}{2}\}. \tag{9.2}$$

But since f vanishes at infinity on S, (9.2) is compact in the topology of S and thus is compact in S^a. So x is in (9.2) and the proof is complete.

We must next strengthen the assertion of Lemma 9.4 to $S^a \setminus S = F + S$. The proof of Lemma 9.4 will not work with F in place of E since we do not know that F is compact. One further lemma is needed.

LEMMA 9.5. $E \subset F + S^a$.

Proof. Let S_0, S_1 and S_2 be the subsets of S consisting of those σ in S that satisfy $|\sigma| \leqslant 1$, $|\sigma| = 1$, $|\sigma| \geqslant 1$ respectively. S_0 and S_1 are compact in the topology of S and thus in the topology of S^a. Furthermore, since $S_2 = S_1 + S$ and the addition in S^a is jointly continuous, $S_2^- = (S_1 + S)^- = S_1 + S^a$. Let e be an element of E. Since e is not in S_0, it must be in $S_2^- = S_1 + S^a$, so $e \in \sigma + S^a$ for some σ in S_1. We define T_e to be

$$\{x\colon\ x \in S^a,\ e \in x + S^a\}.$$

Using the fact that e is an idempotent, it is simple to check that T_e is a closed subsemigroup of S^a, and that it contains the entire ray $\{\lambda\sigma\colon\ \lambda \geqslant 0\}$ since it contains σ. But e_σ is in the closure of $\{\lambda\sigma\colon\ \lambda \geqslant 0\}$, so e_σ is in T_e and thus $e \in e_\sigma + S^a \subset F + S^a$. Since e was an arbitrary element of E, the proof is complete.

Corollary 9.6. $S^a \setminus S = F + S^a$.

Proof. By Lemma 9.5,

$$E + S^a \subset F + S^a + S^a = F + S^a.$$

But since $F \subset E$, $F + S^a \subset E + S^a$. Thus $F + S^a = E + S^a$ and the result follows from Lemma 9.4.

We can now complete the proof of the approximation theorem for S. Let x and y be two distinct elements of S^a. By Theorem 3.6, in order to establish the approximation theorem for S it suffices to find a semicharacter of S^a that separates x and y. By Corollary 9.6, there are three cases to consider: both x and y in S; one in S and the other in $F + S^a$; both x and y in $F + S^a$.

Case I. x and y both in S. Since S is a cone in E^n, there is a character χ of S that separates x and y. Then the semicharacter $\hat{\chi}$ of S^a satisfies

$$\hat{\chi}(x) = \chi(x) \neq \chi(y) = \hat{\chi}(y).$$

Case II. x in S and y in $F + S^a$. Since by Corollary 9.6 $F + S^a = S^a \setminus S$, Lemmas 8.2 and 9.2 show that there is a semicharacter χ of S^a that satisfies $\chi(x) \neq 0$ and $\chi(y) = 0$.

Case III. x and y both in $F + S^a$. By Lemma 8.9 we may assume that x and y are both in $e_\sigma + S^a$, for some e_σ in F. Let S_σ be $\{\lambda\sigma\colon\ \lambda \geqslant 0\}$ and S_σ^- its closure in S^a. e_σ has been defined to be the identity element of the kernel $K(S_\sigma^-)$. We shall apply Lemma 7.1 to S, taking G to be E^n and Q to be S_σ, so that K is $K(S_\sigma^-)$ and e is e_σ. The subgroup H of G generated by S_σ is the line $\{\lambda\sigma\colon\ -\infty < \lambda < +\infty\}$. Each character of S_σ can be extended to a character of H which in turn extends to a character of G. Thus by (i) of Lemma 7.1, if $(x + K) \cap (y + K)$ is non-empty, x and y can be separated by a semicharacter of S^a. The image T of S under the natural projection $G \to G/H$ is isomorphic to a cone in E^{n-1} and thus because of our induction hypothesis, the approximation theorem holds for T. Therefore by (ii) of Lemma 7.1, if $(x+K) \cap (y+K)$ is empty, x and y can be separated by a semicharacter of S^a. This completes the discussion of Case III and the proof of Theorem 9.1.

10. Finitely Generated Subsemigroups of Groups

We have seen in Section 2 that the approximation theorem does not hold for all subsemigroups of discrete commutative groups. In Theorem 10.1 below we show that nonetheless the approximation theorem does hold for *finitely generated* subsemigroups of discrete commutative groups.(1)

Let S be a commutative semigroup with an identity element. If T is a subset of S, then T is said to *generate* S if the smallest subsemigroup of S containing T and the identity element is S itself. $n(S)$ is defined to be the infimum of the cardinalities of subsets that generate S and S is called *finitely generated* if $n(S) < \infty$.

THEOREM 10.1. *Let G be a commutative group and S a subsemigroup of G that contains the identity element of G and that is finitely generated. Then the approximation theorem holds for S supplied with the discrete topology.*

The remainder of this section is devoted to the proof of Theorem 10.1. The proof is similar in outline to that of Theorem 9.1 and proceeds by induction on $n(S)$.

So let us assume that G and S are a group and subsemigroup that satisfy the hypotheses of Theorem 10.1. There is clearly no loss of generality in assuming furthermore that the subgroup $\{\sigma - \tau\colon \sigma \in S, \tau \in S\}$ of G is G itself.

If $n(S) = 1$, then S is either a finite cyclic group, so that the approximation theorem holds for S, or S is isomorphic to the non-negative integers, in which case the approximation theorem for S is contained in Theorem 6.1.

Thus we may assume that $n(S) = m$ and that Theorem 10.1 is valid for semigroups that can be generated by fewer than m elements.

Since $n(S) = m$, it is possible to find a subset $(\sigma_1, \ldots, \sigma_m)$ of S that generates S. We shall keep this set of generators fixed throughout the course of the proof.

We consider first the case where there is some τ besides the identity element that is invertible in S, i.e. both τ and $-\tau$ are in S. Since $\{\sigma_1, \ldots, \sigma_m\}$ generates S, there are non-negative integers $n_1, \ldots, n_m$, not all zero, so that

(1) This is precisely the class of finitely generated commutative cancellation semigroups. See [9], p. 90.

$$\tau = n_1 \sigma_1 + \ldots + n_m \sigma_m.$$

By rearranging the σ_i if necessary we may assume that $n_1 > 0$. Then

$$-\sigma_1 = -\tau + (n_1 - 1)\sigma_1 + n_2 \sigma_2 + \ldots + n_m \sigma_m$$

is in S and as a consequence, the cyclic group H generated by σ_1 is contained in S. If π: $G \to G/H$ is the natural projection, then $\pi(S)$ is the subsemigroup of G/H generated by

$$\{\pi(\sigma_i): \ i = 2, 3, \ldots, m\}$$

and so by our induction hypothesis, the approximation theorem holds for $\pi(S)$. But then Lemma 8.1 shows that the approximation theorem holds for S.

Thus we may henceforth assume that S contains no element besides the identity that is invertible in S.

LEMMA 10.2. *S has a semicharacter that is nowhere zero and vanishes at infinity.*

Proof. Let H be the subgroup of G consisting of all elements of finite order and let ψ: $G \to G/H$ be the natural projection. Since $G = \{\sigma - \tau: \ \sigma \in S, \ \tau \in S\}$, G is finitely generated and thus G/H is finitely generated. Furthermore, G/H has no elements of finite order so (see section 109 of [12]) it has a basis; that is, a subset $\{\gamma_1, \ldots, \gamma_s\}$ which is such that each element of G/H has a unique representation of the form $n_1\gamma_1 + \ldots + n_s\gamma_s$, with the n_j integers. Let R^s be the linear space of all s-tuples of rational numbers supplied with the inner product $(\cdot, \cdot)$ defined by $(a, b) = \alpha_1\beta_1 + \ldots + \alpha_s\beta_s$ if $a = (\alpha_1, \ldots, \alpha_s)$ and $b = (\beta_1, \ldots, \beta_s)$. Let φ: $G/H \to R^s$ be the homomorphism defined by

$$\varphi(n_1\gamma_1 + \ldots + n_s\gamma_s) = (n_1, \ldots, n_s).$$

We define

$$b_i = \varphi(\psi(\sigma_i)), \quad i = 1, \ldots, m,$$

and denote by T the cone

$$\left\{ \sum_{i=1}^{m} \lambda_i b_i: \ \lambda_i \text{ rational}, \ \lambda_i \geq 0 \right\}$$

in R^s generated by the b_i. Suppose that none of the $-b_i$ are in T. Then by (1) Theorem 1 of [14], there must be elements a_i, $i = 1, \ldots, m$, in R^s satisfying $(a_i, b) \leq 0$ for all b in T and $(a_i, -b_i) > 0$. As a consequence, if $a = a_1 + \ldots + a_m$, the function χ defined on S by

$$\chi(\sigma) = e^{(a, \varphi(\psi(\sigma)))}, \quad \sigma \in S,$$

(1) The results of [14] are stated for real linear space but the proofs are valid for rational linear spaces.

is a semicharacter of S satisfying

$$0 < \chi(\sigma_i) < 1, \quad i = 1, \ldots, m,$$

and is thus nowhere zero and vanishes at infinity on S. So to complete the proof it remains to show that no $-b_i$ is in T. Suppose that this is not the case. By renumbering if necessary we may assume that $-b_1$ is in T. Then there are non-negative rational numbers λ_i so that

$$-b_1 = \sum_{j=1}^{m} \lambda_i b_i.$$

Multiplying through by the denominators of the λ_i and transposing we obtain

$$0 = \sum_{i=1}^{m} n_i b_i = \sum_{i=1}^{m} n_i \varphi(\psi(\sigma_i)) = \varphi\left(\psi\left(\sum_{i=1}^{m} n_i \sigma_i\right)\right),$$

where the n_i are non-negative integers and $n_1 > 0$. But the kernel of $\varphi \circ \psi$ is H, so $\sum_{i=1}^{m} n_i \sigma_i$ is in $H \cap S$. Each element of $H \cap S$ is of finite order and thus invertible in S. Since we have assumed that the only invertible element in S is the identity element 0, $\sum_{i=1}^{m} n_i \sigma_i = 0$. But this cannot occur since the n_i are non-negative, $n_1 > 0$ and σ_1 is non invertible in S. Thus we have a contradiction to our assumption that $-b_1$ is in T and the proof is complete.

S is a subsemigroup of a discrete commutative group and thus has sufficiently many characters to separate points, so I_S: $S \to S^a$ is $1-1$. We identify S with its image in S^a.

By Lemmas 8.2 and 10.2, $S^a \setminus S$ is a closed ideal in S^a. The next lemma leads to Corollary 10.4, which gives an identification of $S^a \setminus S$ that is sufficient for our purposes. First a definition is necessary.

If σ_i is one of the generators $\{\sigma_1, \ldots, \sigma_m\}$ of S, the closure of $\{n\sigma_i: n = 0, 1, 2, \ldots\}$ in S^a is a compact subsemigroup of S^a. By Lemma 5.2, the kernel of this compact semigroup contains a unique idempotent and this idempotent will be denoted by e_i. We define F to be the set $\{e_1, \ldots, e_m\}$.

LEMMA 10.3. *Let f be a function in $A(S)$ that satisfies $e\hat{f} = 0$ for all e in F. Then f vanishes at infinity on S.*

Proof Choose any $\varepsilon > 0$. The same argument as that used in the first part of the proof of Lemma 9.3 shows that there are positive integers λ_i so that

$$\|(\lambda_i \sigma_i) f\| \leq \varepsilon, \quad i = 1, \ldots, m.$$

Thus if $\sigma \in S$, $|f(\sigma)|$ can be greater than ε only if σ has a representation of the form $\sigma = n_1 \sigma_1 + \ldots + n_m \sigma_m$ with $0 \leqslant n_i < \lambda_i$ and as a consequence

$$\{\sigma\colon\ \sigma \in S,\ |f(\sigma)| > \varepsilon\}$$

is finite. Since ε was arbitrary, f vanishes at infinity and the proof is complete.

The proof of Corollary 10.4 below is identical with that of Corollary 9.4 except that Lemma 10.3 is used in place of Lemma 9.3 and F in place of E.

COROLLARY 10.4. $S^a \setminus S = F + S^a$.

We can now complete the proof of the approximation theorem for S. Let x and y be two distinct elements of S^a. By Theorem 3.6, in order to establish the approximation theorem for S it suffices to find a semicharacter of S^a that separates x and y. Corollary 10.4 shows that there are three cases to consider: both x and y in S; one in S and the other in $F + S^a$; both x and y in $F + S^a$.

Case I. x and y both in S. Since S is a subsemigroup of a discrete commutative group there is a character χ of S that separates x and y. Then the semicharacter $\hat{\chi}$ of S^a satisfies

$$\hat{\chi}(x) = \chi(x) \neq \chi(y) = \hat{\chi}(y).$$

Case II. x in S and y is $F + S$. Lemma 8.2 and 10.2 show that there is a semicharacter χ of S^a that satisfies $\chi(x) \neq 0$ and $\chi(y) = 0$.

Case III. x and y both in $F + S^a$. By Lemma 8.9 we may assume that x and y are both in $e_i + S$. Let $S_i = \{n\sigma_i\colon\ n = 0, 1, 2, \ldots\}$ and S_i^- be its closure in S^a. e_i has been defined to be the identity element of the kernel $K(S_i^-)$. We shall apply Lemma 7.1 to S, taking Q to be S_i, so that K is $K(S_i^-)$ and e is e_i. The subgroup H of G generated by S_i is the infinite cyclic group $\{n\sigma_i\colon\ -\infty < n < +\infty\}$. Each character of S_i can be extended to a character of H which in turn extends to a character of G. Thus by (i) of Lemma 7.1, if $(x + K) \cap (y + K)$ is non-empty, x and y can be separated by a semicharacter of S^a. The image T of S under the natural projection $G \to G/H$ satisfies $n(T) < m$ and because of our induction hypothesis, the approximation theorem holds for T. Therefore by (ii) of Lemma 7.1, if $(x + K) \cap (y + K)$ is empty, x and y can be separated by a semicharacter of S^a. This completes the discussion of Case III and thus the proof of Theorem 10.1.

11. Ordered Groups, Archimedian Case

In Section 6 the approximation theorem for the semigroup of non-negative integers was established. This result is extended below to the non-negative half of any subgroup of the discrete real line. The theorem obtained will be used in the next section, where we show that the approximation theorem holds for the non-negative half of any totally ordered commutative group in the discrete topology.

THEOREM 11.1. *Let G be a subgroup of the real line and S the subsemigroup $\{\sigma\colon \sigma \in G,\ \sigma \geqslant 0\}$ consisting of its non-negative elements. Then the approximation theorem holds for S supplied with the discrete topology.*

For the remainder of this section, G and S are as in the statement of Theorem 11.1. We may assume that G is dense in the real line. For if it were not, it would be isomorphic to the integers and Theorem 6.1 would apply.

One lemma is needed before we proceed to the proof of the theorem.

LEMMA 11.2. *Let τ be an element of S, $\tau \neq 0$, and h be a function in $A(S)$ with $h(\sigma)=0$ if $\sigma \leqslant \tau$ or if $\sigma \geqslant 5\tau$. Let k be the unique function in $C(S)$ satisfying $k(\sigma)=h(\sigma)$ if $0 \leqslant \sigma < 5\tau$ and $k(\sigma+5\tau)=k(\sigma)$ for all σ in S. Then k is also in $A(S)$.*

Proof. The subset

$$\{\sigma h\colon \sigma \in S,\ 0 \leqslant \sigma \leqslant \tau\} \tag{11.1}$$

of $C(S)$ is conditionally compact since h is in $A(S)$. Each of the sets

$$\{\sigma k\colon \sigma \in S,\ n\tau \leqslant \sigma \leqslant (n+1)\tau\}, \quad n=0, 1, \ldots, 4,$$

is isometric to (11.1), so their union

$$\{\sigma k\colon \sigma \in S,\ 0 \leqslant \sigma \leqslant 5\tau\} \tag{11.2}$$

is conditionally compact. But k has period 5τ, so (11.2) is all of $\{\sigma k\colon \sigma \in S\}$ and thus k is in $A(S)$.

We now proceed to the proof of Theorem 11.1. Let x and y be two distinct elements of the compactification S^a. By Theorem 3.6, the proof will be complete if we can find a semicharacter χ of S^a that separates x and y.

Let φ be the semicharacter of S defined by $\varphi(\sigma)=e^{-\sigma}$. Since φ separates points on S, $I_S\colon S \to S^a$ is $1-1$ and we identify S with its image in S^n.

If $\hat{\varphi}(x) \neq \hat{\varphi}(y)$, we are finished, so we can assume that $\hat{\varphi}(x)=\hat{\varphi}(y)$. We shall consider separately the three cases where $\hat{\varphi}(x)=0$, $0<\hat{\varphi}(x)<1$ and $\hat{\varphi}(x)=1$.

Case I. $\hat{\varphi}(x)=\hat{\varphi}(y)=0$. Let τ be an element of S. Since $\hat{\varphi}(x)=\hat{\varphi}(y)=0$ and φ is bounded away from zero on $\{\sigma\colon \sigma\in S,\ 0\leqslant\sigma\leqslant\tau\}$, x and y must be in

$$\{\sigma\colon \sigma\in S,\ \sigma\geqslant\tau\}^{-}=(\tau+S)^{-}=\tau+S^{a}.$$

But τ was an arbitrary element of S, so by Lemma 5.1, x and y are in $K(S^a)$. Thus by Corollary 5.3, there is a semicharacter χ of S^a with $\chi(x)\neq\chi(y)$.

Case II. $0<\hat{\varphi}(x)=\hat{\varphi}(y)<1$. Since S is dense in the non-negative reals, it is possible to find a τ in S with

$$1>\hat{\varphi}(2\tau)>\hat{\varphi}(x)=\hat{\varphi}(y)>\hat{\varphi}(3\tau)>0.$$

Then x and y must be in the closure of $\{\sigma\colon \sigma\in S,\ 2\tau\leqslant\sigma\leqslant3\tau)$ in S^a. By Theorem 3.2 there is a function f in $A(S)$ with $\hat{f}(x)\neq\hat{f}(y)$. Let g be the piecewise linear function in $C(S)$ that is equal to 0 outside of

$$\{\sigma\colon \sigma\in S,\ \tau\leqslant\sigma\leqslant4\tau\}$$

and equal to 1 on $$\{\sigma\colon \sigma\in S,\ 2\tau\leqslant\sigma\leqslant3\tau\}.$$

It is clear that g is in $A(S)$ and thus that fg is in $A(S)$. Let k be the unique function in $C(S)$ satisfying $k(\sigma)=f(\sigma)g(\sigma)$ if $0\leqslant\sigma\leqslant5\tau$ and $k(\sigma+5\tau)=k(\sigma)$ for all σ in S. By Lemma 11.2, k is in $A(S)$. Using the fact that x and y are in

$$\{\sigma\colon \sigma\in S,\ 2\tau\leqslant\sigma\leqslant3\tau\}^{-},$$

on which $\hat{g}$ is identically 1, we have

$$\hat{k}(x)=\hat{f}(x)\hat{g}(x)=\hat{f}(x)\neq\hat{f}(y)=\hat{f}(y)\hat{g}(y)=\hat{k}(y). \tag{11.3}$$

Let v be a cluster point in S^a of the net $\{5n\tau\colon n=0,1,2,\ldots\}$. Then v is in

$$\bigcap_{n=0}^{\infty}(5n\tau+S^{a})=\bigcap_{\eta\in S}(\eta+S^{a}),$$

which is $K(S^a)$ by Lemma 5.1. Furthermore, since k has period 5τ, $\hat{k}(u+v)=\hat{k}(u)$ for all u in S^a. In particular, because of (11.3),

$$\hat{k}(x+v)=\hat{k}(x)\neq\hat{k}(y)=\hat{k}(y+v),$$

so $x+v$ and $y+v$ are distinct. They are in $K(S^a)$ since $K(S^a)$ is an ideal containing v. Clearly $\hat{\varphi}$ vanishes on $K(S^a)$ and therefore, by Case I, there is a semicharacter χ of S^a that satisfies $\chi(x+v)\neq\chi(y+v)$ and thus $\chi(x)\neq\chi(y)$.

Case III. $\hat{\varphi}(x)=\hat{\varphi}(y)=1$. We may assume that neither x nor y is 0. For if one were 0, the semicharacter χ of S defined by

$$\chi(\sigma)=\begin{cases}1, & \sigma=0\\ 0, & \sigma>0,\end{cases}$$

would satisfy(1) $\hat{\chi}(x)\neq\hat{\chi}(y)$. For each τ in S with $\tau>0$ we define U_τ to be

$$\{\sigma:\ \sigma\in S,\ 0<\sigma<\tau\}.$$

$\{U_\tau\}$ is an initial family for $\{\sigma:\ \sigma\in S,\ \sigma>0\}$ in the sense of the definition given before Lemma 8.4, so that by Lemma 8.4.

$$H=\bigcap_{\substack{\tau\in S\\ \tau>0}} U_\tau^-$$

is a subgroup of S^a. For each τ in S with $\tau>0$, φ is bounded away from 1 on $\{\sigma:\ \sigma\in S,\ \sigma>\tau\}$, so since $\hat{\varphi}(x)=\hat{\varphi}(y)=1$, x and y are in U_τ^-. Thus x and y are in H. If $x+\sigma=y+\sigma$ for each σ in S with $\sigma>0$, then we would have $x=x+e=y+e=y$, where e is the identity element of H, since e is in the closure in S^a of $\{\sigma:\ \sigma\in S,\ \sigma>0\}$. Thus there is some σ in S with $\sigma>0$ and $x+\sigma\neq y+\sigma$. Since $\sigma>0$, $0<\hat{\varphi}(\sigma)<1$ and therefore

$$0<\hat{\varphi}(x+\sigma)=\hat{\varphi}(y+\sigma)<1$$

so, by Case II, there is a semicharacter χ of S^a that satisfies $\chi(x+\sigma)\neq\chi(y+\sigma)$ and thus $\chi(x)\neq\chi(y)$. This completes the proof of Theorem 11.1.

12. Ordered Groups, General Case

In this section we establish the approximation theorem for the non-negative half of any totally ordered(2) discrete commutative group.

THEOREM 12.1. *Let G be a totally ordered commutative group and S the subsemigroup $\{\sigma:\ \sigma\in G,\ \sigma\geqslant 0)$ consisting of its non-negative elements. Then the approximation theorem holds for S supplied with the discrete topology.*

For the remainder of this section G and S are as in the statement of Theorem 12.1.

G is called *Archimedian* if for each σ and τ in G with $0<\sigma<\tau$, there is a positive integer n with $\tau<n\sigma$. Since an Archimedian G is order isomorphic to a subgroup

(1) Trivially $\hat{\chi}$ vanishes except at 0.

(2) For the basic facts concerning ordered groups, see [4].

of the real line (see [4], p. 30). Theorem 11.1 is nothing but the special case of Theorem 12.1 for G Archimedian. We shall use Theorem 11.1 in our proof of Theorem 12.1.

Before we proceed to the proof, several definitions are necessary. Let σ be an element of S with $\sigma > 0$. We define the following subsets of G:

$$G_\sigma = \{\tau:\ -\sigma < n\tau < \sigma \text{ for all positive integers } n\},$$

$$G^\sigma = \{\tau:\ -n\sigma < \tau < n\sigma \text{ for some positive integer } n\},$$

$$S_\sigma = G_\sigma \cap S, \quad S^\sigma = G^\sigma \cap S,$$

$$I(\sigma) = \{\tau:\ 0 \leqslant \tau < \sigma\}.$$

It is clear that $G_\sigma \subset G^\sigma$ and that both are order subgroups(1) of G. Furthermore, it is simple to check that the quotient group G^σ/G_σ is Archimedian under the natural ordering.(2) Thus (see [4], p. 30) there is a unique order preserving isomorphism ψ_σ of G^σ/G_σ into the real line satisfying $\psi_\sigma(\sigma + G_\sigma) = 1$. We define the function φ_σ on S by

$$\varphi_\sigma(\tau) = \begin{cases} e^{-\psi_\sigma(\tau + G_\sigma)}, & \tau \in S^\sigma \\ 0, & \tau \in S \setminus S^\sigma. \end{cases}$$

It is clear that φ_σ is a semicharacter of S.

Since S is a subsemigroup of a discrete commutative group, it has sufficiently many characters so separate points. Thus I_S: $S \to S^a$ is $1-1$ and we identify S with its image in S^a.

If σ is an element of S, S_σ has been defined to consist of all elements of S that are "infinitely small" with respect to σ. It is necessary for us to extend this to a definition of S_x for x in S^a. The next lemma demonstrates the equivalence of several possible definitions.

LEMMA 12.2. *Let $x \in S^a$ and $\tau \in S$. Then the following are equivalent:*

(i) *There is no positive integer n for which $x \in I(n\tau)^-$.*

(ii) *For each positive integer n, $x \in n\tau + S^a$.*

(iii) $\hat{\varphi}_\tau(x) = 0$.

Proof. Suppose that (i) holds. Then for each n, since

$$S = I(n\tau) \cup (n\tau + S),$$

(1) H is called an *order subgroup* of G if whenever $\sigma \in H$ and $\sigma > 0$, then $\{\tau:\ -\sigma \leqslant \tau \leqslant \sigma\} \subset H$.

(2) If $\tau_1 \leqslant \tau_2$, then $\tau_1 + G_\sigma \leqslant \tau_2 + G_\sigma$.

x is in
$$(n\tau + S)^- = n\tau + S^a,$$
so (ii) follows. If (ii) holds, then for each n,
$$0 \leqslant \hat{\varphi}_\tau(x) \leqslant \varphi_\tau(n\tau) = e^{-n},$$
so $\hat{\varphi}_\tau(x) = 0$. Finally, if (iii) is valid, (i) must be also, For if x were in $I(n\tau)^-$, since φ_τ is bounded from zero on $I(n\tau)$, $\hat{\varphi}_\tau(x)$ would be non-zero.

If x is an element of S^a with $x \neq 0$, we define S_x to be the subset of S consisting of all τ that satisfy the three equivalent conditions of Lemma 12.2. G_x is defined to be
$$\{\tau\colon\ \tau \in G,\ \tau \in S_x \quad \text{or} \quad -\tau \in S_x\},$$
so that $S_x = G_x \cap S$. It is simple to check that G_x is an order subgroup of G. Furthermore, our definition of S_x agrees with that given earlier if x happens to be an element of S. For if x is in S, then τ is in S_x (second definition) if and only if $\hat{\varphi}_\tau(x) = \varphi_\tau(x) = 0$, which occurs if and only if x is not in S^τ, or equivalently, if and only if τ is in S_x (first definition).

We can now begin the proof of Theorem 12.1. Let x and y be distinct elements of S^a. By Theorem 3.6, our result will be established if we can find a semicharacter of S^a that separates x and y. The proof proceeds by a rather complicated analysis of various special cases, which are not mutually exclusive.

We may assume that neither x nor y is 0. For if one were 0, the semicharacter χ of S defined by
$$\chi(\sigma) = \begin{cases} 1, & \sigma = 0 \\ 0, & \sigma > 0, \end{cases}$$
would satisfy $\hat{\chi}(x) \neq \hat{\chi}(y)$.

Case I. $S_x \neq S_y$. In this case if σ is in, say, $S_x \setminus S_y$, then, by (iii) of Lemma 12.2, $\hat{\varphi}_\sigma(x) = 0 \neq \hat{\varphi}_\sigma(y)$.

Case II. $S_x = S_y$, x or y in $S_x^- = S_y^-$. Let χ be the semicharacter of S defined by
$$\chi(\sigma) = \begin{cases} 1, & \sigma \in S_x \\ 0, & \sigma \in S \setminus S_x. \end{cases}$$
If not both x and y are in S_x^-, then one is in $(S \setminus S_x)^-$, so $\hat{\chi}(x) \neq \hat{\chi}(y)$.

Thus it remains to consider the case that both x and y are in S_x^-. By the definition of S_x, if $\sigma \in S_x$, then $x = \sigma + x_\sigma$ for some $x_\sigma \in S^a$. If x_σ were in $(S \setminus S_x)^-$, we would have $\hat{\chi}(x_\sigma) = 0$ and thus

$$\hat{\chi}(x)=\hat{\chi}(\sigma+x_\sigma)=\hat{\chi}(\sigma)\,\hat{\chi}(x_\sigma)=0,$$

contradicting the fact that x is in S_x^-. Thus x_σ is in S_x^- and since σ was an arbitrary element of S_x, x is in

$$\bigcap_{\sigma\in S_x}(\sigma+S_x^-),$$

which is $K(S_x^-)$ by Lemma 5.1. Similarly y is in $K(S_x^-)$. Any character of S_x can be extended to a semicharacter of S simply by defining it to be 0 on $S\setminus S_x$. Thus by Corollary 5.3 there is a semicharacter χ of S^a satisfying $\chi(x)\neq\chi(y)$.

Case III. For some σ in S, $S_x=S_y=S_\sigma$ and $S^\sigma=S$. We shall apply both parts of Lemma 7.1, taking Q to be S_σ. It is only necessary to show that the hypotheses of the lemma are satisfied: first, that x and y are in $e+S^a$; second, that each character of Q extends to a semicharacter of S; and third, that the approximation theorem holds for T.

By the definition of S_x, $x\in\tau+S^a$ for each τ in S_x and thus by compactness, $x\in u+S^a$ for each u in S_x^-. In particular, $x\in e+S^a$, where e is the identity element of $K=K(Q)=K(S_x^-)$. Similarly $y\in e+S^a$.

Any semicharacter of $Q=S_\sigma$ extends to a semicharacter of S by defining it to be zero on $S\setminus S_x$.

The semigroup T that occurs in the statement of Lemma 7.1 is the image of $S=S^\sigma$ under the natural projection $G^\sigma\to G^\sigma/G_\sigma$. Thus T is the subsemigroup of non-negative elements in the archimedian ordered group G^σ/G_σ, which we know to be order isomorphic to a subgroup of the real line, so by Theorem 11.1 the approximation theorem holds for T.

We have shown that the hypotheses of Lemma 7.1 are satisfied for $Q=S_\sigma$ so as a consequence there is a semicharacter of S^a that separates x and y.

Case IV. For some σ in S, $S_x=S_y=S_\sigma$. Let $T=S^\sigma$. We shall establish Case IV by applying the result of Case III to T.

Let $j\colon T\to S$ be the injection map. By Lemma 3.4, there is an induced homomorphism $j^a\colon T^a\to S^a$, which by Lemma 8.3, maps T^a homeomorphically onto the open and closed subset T^- of S^a. x and y are both in T^-. For assume that x is not in T^-. Then, since for each positive integer n we have $I(n\sigma)\subset T$, x is in no $I(n\sigma)^-$, so by the definition of S_x, σ is in $S_x=S_\sigma$, which is a contradiction.

Let $j^a(u)=x$ and $j^a(v)=y$. If we could find a semicharacter χ of T^a that satisfied $\chi(u)\neq\chi(v)$, then the function χ_1 defined on S^a by

$$\chi_1(z)=\begin{cases}\chi(w) & \text{if } j^a(w)=z\\ 0 & \text{if } z\in S^a\setminus T^-\end{cases}$$

would satisfy $\chi_1(x) \neq \chi_1(y)$. And χ_1 would be a semicharacter of S^a since, by Lemma 8.3, T^- is an open and closed subsemigroup of S^a whose complement $S^a \setminus T^-$ is an ideal.

Thus it remains only to produce χ. But it is simple to check that $T_u = T_v = T_\sigma$ and that $T^\sigma = T$, so by the result of Case III applied to T, such a χ must exist.

Case V. $S_x = S_y$ and neither x nor y in $S_x^- = S_y^-$. Let V be the collection of all open initial intervals in the ordered group G/G_x, i.e. V consists of all subsets of G/G_x of the form

$$\{\tau + G_x\colon\ 0 + G_x < \tau + G_x < \sigma + G_x\} \tag{12.1}$$

for σ in $S \setminus S_x$. Note that V contains the empty set if and only if G/G_x has a least positive element.

Let W be the collection of all subsets of G that are inverse images of the sets in V under the natural projection $G \to G/G_x$. Each set in W is a subset of $S \setminus S_x$. We define H to be

$$\bigcap_{U \in W} U^-.$$

Suppose that W does not contain the empty set. Then G/G_x does not have a least positive element and using this fact it is simple to check that W is an initial family for $S \setminus S_x$ in the sense of the definition given before Lemma 8.4. Thus, by Lemma 8.4, H is a compact topological group whose identity element is an identity for $(S \setminus S_x)^-$. If, on the other hand, W does contain the empty set, then H is empty.

Subcase Va. Not both x and y in H. We shall show that in this case there is a σ in S with $S_x = S_y = S_\sigma$, so that we are actually in Case IV which has already been settled. We may assume that x is not in H.

The inverse image of (12.1) under the natural projection $G \to G/G_x$ is

$$\{\tau\colon\ \tau \in S,\ 0 < \tau + \eta < \sigma,\ \text{all } \eta \text{ in } G_x\},$$

which will be denoted by U_σ. Since x is not in H and H is the intersection of the U_σ^-, we may assume that σ in $S \setminus S_x$ has been chosen so that x is not in U_σ^-. We show that this σ satisfies $S_x = S_\sigma$.

First, $S_x \subset S_\sigma$. For if there were a τ in S_x that were not in S_σ, then $\sigma < n\tau$ would hold for some positive integer n and σ would be in S_x. But σ has been chosen to be an element of $S \setminus S_x$.

So it remains to show that $S_\sigma \subset S_x$. Note that since each τ in S_x satisfies $\tau < \sigma$,

$$U_\sigma \cup S_x = \{\tau\colon\ \tau \in S,\ 0 \leqslant \tau + \eta < \sigma,\ \text{all } \eta \text{ in } S_x\}. \tag{12.2}$$

Let τ be an element of S_σ and n a positive integer. Then $n\tau+\eta \in S_\sigma$ for each $\eta \in S_\sigma$ and thus in particular $n\tau+\eta<\sigma$ for each η in the subset S_x of S_σ. As a consequence, $n\tau$ and thus $I(n\tau)$ is contained in (12.2). We have assumed that x is not in S_x^- and σ has been chosen so that x is not in U_σ^-. Thus x cannot be in $I(n\tau)^-$ and as a consequence τ is in S_x. Since τ was an arbitrary element of S_σ, we have $S_\sigma \subset S_x$.

This completes the proof that $S_\sigma = S_x = S_y$, which shows that we are actually in Cese IV, where we have already established the existence of a semicharacter of S^a that separates x and y.

Subcase Vb. Both x and y in H. Suppose that $x+\sigma=y+\sigma$ for all σ in $S\setminus S_x$. Then since the identity element e of the group H is in $(S\setminus S_x)^-$, we would have the contradiction

$$x=x+e=y+e=y.$$

So choose some σ in $S\setminus S_x$ with $x+\sigma \neq y+\sigma$. We show next that $S_\sigma = S_{x+\sigma}$. It is clear that $S_\sigma \subset S_{x+\sigma}$; we shall assume that equality does not hold and derive a contradiction. Choose a τ in $S_{x+\sigma}$ that is not in S_σ. Since τ is not in S_σ, there is a positive n for which $\sigma<n\tau$. Since σ is not in S_x, there is another positive integer m such that x is in $I(m\sigma)^-$. But $I(m\sigma)\subset I(mn\tau)$, so $x \in I(mn\tau)^-$ and since $\sigma \in I(n\tau)^-$, $\sigma+x$ is in $I((mn+n)\tau)^-$, which contradicts the fact that $\tau \in S_{x+\sigma}$. Thus $S_{x+\sigma}=S_\sigma$ and similarly $S_{y+\sigma}=S_\sigma$.

Since $S_{x+\sigma}=S_{y+\sigma}=S_\sigma$, we can now apply the result of Case IV to $x+\sigma$ and $y+\sigma$ to obtain a semicharacter χ of S^a that satisfies $\chi(x+\sigma)\neq\chi(y+\sigma)$ and thus $\chi(x)\neq\chi(y)$. This completes the discussion of Case V.

It is now simple to finish the proof of Theorem 12.1. By Theorem 3.6 it suffices to separate two distinct elements x and y of S^a by a semicharacter of S^a. By the argument before Case I we may assume that neither is 0. If $S_x \neq S_y$, we use Case I. If $S_x=S_y$ and either x or y is in $S_x^- = S_y^-$, we use Case II. If $S_x = S_y$ and neither x nor y is in $S_x^- = S_y^-$, we use Case V.

13. Large Subsemigroups

If S is a subset of a commutative group, we shall denote by $-S$ the set $\{-\sigma : \sigma \in S\}$. Then Theorem 12.1 can be stated as follows. If S is a subsemigroup of a commutative group G with $S \cup (-S) = G$ and $S \cap (-S) = \{0\}$, then the approximation theorem holds for S supplied with the discrete topology.(1)

(1) For if we define $\sigma \geq \tau$ to mean $\sigma-\tau \in S$, then G is a totally ordered commutative group and $S=\{\sigma\colon \sigma \in G,\ \sigma \geq 0\}$.

In this section wo obtain as a relatively simple consequence of Theorem 12.1 the following extension to a much wider class of topological semigroups.

THEOREM 13.1. *Let G be a commutative topological semigroup and S a closed subsemigroup with $S \cup (-S) = G$. Then the approximation theorem holds for S.*

We need first two lemmas.

LEMMA 13.2. *Let G be a discrete commutative group and S a subsemigroup with $S \cup (-S) = G$. Then the approximation theorem holds for S.*

Proof. Let H be the subgroup $S \cap (-S)$ of G. If T is the image of S under the natural projection $G \to G/H$, then $T \cup (-T) = G/H$ and $T \cap (-T) = \{0 + H\}$, so by Theorem 12.1, the approximation theorem holds for T. Thus by Lemma 8.1, the approximation theorem holds for S.

LEMMA 13.3. *Let G be a commutative topological group and S a closed subsemigroup with $S \cup (-S) = G$. Then the interior of S is dense in S.*

Proof. Let T be the closure of the interior of S, clearly an ideal in S. We may assume that S is not open and thus the open set $G \setminus (-S) \subset T$ is not closed. Then, since S is closed and

$$S = \{S \cap (-S)\} \cup \{G \setminus (-S)\}, \tag{13.1}$$

$S \cap (-S)$ must intersect the closure of $G \setminus (-S)$ and thus intersect T. But $S \cap (-S)$ is a subgroup of S, so since it intersects the ideal T, it must be contained in T. Therefore $S \subset T$ by (13.1).

It is now simple to complete the proof of Theorem 13.1. By Lemma 13.2, the approximation theorem holds for S in the discrete topology. But by Lemma 13.3, the interior of S is dense in S, so as a consequence of Corollary 8.7, the approximation theorem holds for S.

14. A Related Question

Let S be a commutative topological semigroup. If M is a bounded continuous matrix representation of S,

$$M(\sigma) = \begin{pmatrix} f_{11}(\sigma) & \cdots & f_{1n}(\sigma) \\ \vdots & & \vdots \\ f_{n1}(\sigma) & \cdots & f_{nn}(\sigma) \end{pmatrix}, \quad \sigma \in S,$$

it is simple to check that each of the coefficients f_{ij} is in $A(S)$. If S is a group, the functions on S that occur as such coefficients are linear combinations of characters. If S is only a semigroup there may be coefficients that are not of this form. Thus the question arises of whether the linear span of these coefficients is dense in $A(S)$, or equivalently, whether $A(S)$ is spanned by its finite dimensional translation invariant subspaces. In this section we establish that for a wide class of semigroups this occurs if and only if the approximation theorem holds.

We first show by an example that this equivalence is not universally valid. Let S be the semigroup, which was introduced in Section 2, of all lattice points (m, n) in the plane with $m=n=0$ or $m \geqslant 1$. As was noted in Section 2, if f is a bounded function on S which vanishes when $m \geqslant 2$, f is in $A(S)$ although it may not be approximable by linear combinations of semicharacters of S. Let M be the representation of S defined by

$$M(m, n)=\begin{cases}\begin{pmatrix}0 & f(1, n)\\ 0 & 0\end{pmatrix}^m, & m \geqslant 1\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & m=n=0.\end{cases} \tag{14.1}$$

f differs from a coefficient of this representation only at $(0, 0)$, and since the characteristic function χ_0 of $\{(0, 0)\}$ is a semicharacter of S, f lies in a finite dimensional invariant subspace of $A(S)$. Now let f be any function in $A(S)$. Then (14.1) defines a representation of S and subtracting from f the upper right-hand coefficient and an appropriate multiple of χ_0, we obtain a function g in $A(S)$ which vanishes for $m \leqslant 1$. Thus to show that $A(S)$ is spanned by its finite dimensional invariant subspaces, it suffices to show that g can be approximated uniformly on S by linear combinations of semicharacters of S. Let S_0 be the semigroup of all lattice points (m, n) with $m \geqslant 0$ and let g_0 be the function defined on S_0 by

$$g_0(\sigma)=\begin{cases} g(\sigma), & \sigma \in S\\ 0, & \sigma \in S_0 \setminus S.\end{cases}$$

It is simple to check that g_0 is in $A(S_0)$. Since S_0 is the product of the integers and the half-integers, the approximation theorem holds for S_0, so g_0 can be approximated by linear combinations of semicharacters of S_0. Thus, by restriction to S, g can be approximated by linear combinations of semicharacters of S. This completes the proof that $A(S)$ is spanned by its finite dimensional translation invariant subspaces.

THEOREM 14.1. *Let S be a commutative topological semigroup and n an integer, $n > 1$. Suppose that the map $\sigma \to n\sigma$ takes S onto a dense subset of itself.*(1) *Then the following are equivalent:*

(i) *The approximation theorem holds for S.*

(ii) *$A(S)$ is spanned by its finite dimensional translation invariant subspaces.*

Proof. We need only prove that (ii) implies (i). Note first that there is a $1-1$ correspondence between the bounded continuous finite dimensional representations of S and those of S^a. Thus, by the Stone–Weierstrass Theorem, (ii) holds if and only if S^a has sufficiently many continuous finite dimensional representations to separate points. As a consequence, by Theorem 3.6, we need only show that if x and y are elements of S^a separated by such a representation M, then they are also separated by a semicharacter.

It is clear that we can assume that M is indecomposable and acts on k-dimensional complex Euclidean space C^k. Choose any u in S^a and let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of $M(u)$. Then C^k is the direct sum of the r subspaces

$$\{\gamma:\ (M(u) - \lambda_i I)^n \gamma = 0 \text{ for some } n\}.$$

But these subspaces are invariant under $\{M(v):\ v \in S^a\}$ since S^a is commutative, and thus $r = 1$ since M is indecomposable.

For each u in S^a, we denote by λ_u the unique eigenvalue of $M(u)$. Let u and v be elements of S^a. Then

$$\{\gamma:\ M(u)\gamma = \lambda_u \gamma)$$

is a linear subspace of C^k which is invariant under $M(v)$ since $M(v)M(u) = M(u)M(v)$. Thus it contains a non-zero eigenvector β of $M(v)$ which satisfies

$$M(uv)\beta = M(u)\,M(v)\beta = \lambda_u \lambda_v \beta.$$

As a consequence, $\lambda_{uv} = \lambda_u \lambda_v$ for all u and v in S^a. Furthermore, $u \to \lambda_u$ is continuous, and is thus a semicharacter of S^a. If $\lambda_x \neq \lambda_y$, this semicharacter separates x and y, and we are finished. Thus we may assume that $\lambda_x = \lambda_y$.

Since $\sigma \to n\sigma$ maps S onto a dense subset of itself, $u \to nu$ maps S^a onto itself. Consequently there is an integer $m > k$ for which any u in S^a has mth roots. Suppose now that u is an element in S^a with $\lambda_u = 0$. Then, choosing v in S^a so that $mv = u$,

(1) This holds for all the examples of Section 2 besides the one that we have just discussed. Thus we have examples of S for which $A(S)$ is not spanned by its finite dimensional invariant subspaces.

we have $\lambda_v = 0$ and thus $M(u) = M(v)^m = 0$. Since $M(x) \neq M(y)$, this shows that $\lambda_x = \lambda_y \neq 0$.

Let T be the subsemigroup $\{u\colon u \in S^a, \lambda_u \neq 0\}$ of S^a. For u in T we have $M(u) = \lambda_u N(u)$, where the matrix $N(u)$ has the single eigenvalue 1 and so is non-singular. Clearly $u \to N(u)$ is continuous on T and $\{N(u)\colon u \in T\}$ generates a commutative subgroup H of the group of non-singular $k \times k$ matrices. Since $N(x) \neq N(y)$, there is a character χ_0 of H for which $\chi_0(N(x)) \neq \chi_0(N(y))$. Then the function χ defined on S^a by

$$\chi(u) = \begin{cases} \lambda_u \chi_0(N(u)), & u \in T \\ 0, & u \in S^a \setminus T, \end{cases}$$

is a semicharacter of S^a; for if the net $\{u_\gamma\}$ of elements of T converges to u in $S^a \setminus T$, then $\lambda_{u_\gamma} \to 0$ and $\chi(u_\gamma) \to 0 = \chi(u)$, so that χ is continuous. Since χ satisfies $\chi(x) \neq \chi(y)$, the proof is complete.

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