SOME ASPECTS OF GROUPS WITH UNIQUE ROOTS

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Contents

Introduction

1. The multiplicative notation employed in the study of groups leads, in a natural way to the notion of roots in groups. Thus if n is a positive integer and g is an element of a group G then a solution x of the equation

$$x^n = g$$

is called an *n-th root* of g. In general g may not have an nth root; on the other hand it may have more than one. If every element in G has an nth root for every positive integer n then G is called a *divisible* or *complete* group.

Divisible groups appeared first in the theory of abelian groups; one of the classical theorems in this connection asserts that every abelian group can be embedded in a divisible group, which is also abelian. In recent years a large number of Russian mathematicians have carried out investigations of particular classes of divisible groups which have certain commutativity properties. Thus Černikov [7] has studied divisible groups with an ascending central series which sweeps out the whole group, and Mal'cev [25], [26] has studied locally nilpotent groups which are divisible. In particular Mal'cev [25] proved the beautiful theorem that every torsion-free locally nilpotent group can be embedded in a torsion-free locally nilpotent divisible group. Mal'cev [25] proved also that the extraction of roots is unique in a torsion-free locally nilpotent group G; in other words for any $x, y \in G$ and any non-zero integer n, the equation

$$x^n = y^n$$

implies $x = y$. Groups with this property were given the name of R-groups by Kontorovič [18], [19] who extended some earlier work by Baer [1] on torsion-free abelian groups to the larger class of R-groups.

We shall be concerned here with three kinds of groups which contain divisible groups, R-groups and divisible R-groups as special cases: For each non-empty set of primes ω we define 3 associated classes of groups. Thus E_ω denotes the class of groups in which pth roots exist for all $p \in \omega$, and U_ω denotes the class of groups in which pth roots are unique for all $p \in \omega$; consequently $E_\omega \cap U_\omega$ is the class of those groups in which pth roots not only exist, but are unique—we shall henceforth denote the class $E_\omega \cap U_\omega$ by D_ω. If $G \in E_\omega$ we call G an E_ω-group; on the other hand, if $G \in U_\omega$ we call G a U_ω-group; if $G \in D_\omega$ we call G a D_ω-group. So in the particular case where ω coincides with the set of all primes, an E_ω-group is a divisible group, a U_ω-group is an R-group and a D_ω-group is a divisible R-group.

2. Part I of this work is concerned with miscellaneous properties of E_ω-groups, U_ω-groups and D_ω-groups. Here (and throughout this paper) we are motivated by the concepts of universal algebra to fix some of our attention on certain subgroups of E_ω-groups and D_ω-groups. One of our results in this connection is that the derived group of a locally nilpotent D_ω-group is itself a D_ω-group. We concern ourselves also with various "extension" problems. Thus we prove that an extension of a ZA-group in E_ω by a periodic group in U_ω belongs to E_ω. A similar, although unrelated, result is the following: A locally nilpotent

group which is an extension of a D_ω-group by a D_ω-group is a D_ω-group. Another topic treated in Part I is the construction of U_ω-groups, E_ω-groups and D_ω-groups.

3. The class of groups D_ω forms a "variety" of algebras in the sense of P. Hall; equivalently, it is equationally definable. To see this we introduce a set Ω_ω of unitary operators in a fixed one-to-one correspondence with ω, $\pi \in \Omega_\omega$ corresponding to $p \in \omega$; these, together with the group operations, are to be the operators of the variety D_ω. The laws of D_ω are the group laws together with the further laws, 2 for each $p \in \omega$:

$$(x\pi)^p = x, \quad x^p\pi = x.$$

It is easy to see that a group which admits the operators π (in the sense of Higgins [14]) is a D_ω-group and conversely.

Now it follows from general results on varieties of algebras (Birkhoff [5]) that there are free algebras in the variety D_ω; we call these free algebras D_ω-free groups. Although the existence of D_ω-free groups is thus taken care of there are still a large number of important questions that the bare knowledge of the existence will not answer.

The notion of a free group is of vital importance in the theory of groups. It seems likely that a D_ω-free group will play as important a role in the study of D_ω-groups. A simple "normal form" with reference to a fixed set of free generators is available for the elements of a free group. This normal form facilitates the proof of a large number of theorems about free groups. One of the difficulties involved in the study of D_ω-free groups is the absence, at first sight at least, of a useful simple normal form. This makes difficult the proof of such a seemingly obvious, and in fact true, assertion as: A D_ω-free group is torsion-free. The most important part of this paper seems to be Part II in which a D_ω-free group is constructed as the union of an ascending sequence of U_ω-groups; these U_ω-groups are themselves generalised free products of U_ω-groups with a single amalgamation. The complications involved in the study of generalised free products therefore occur here and so this construction serves also to illustrate the difficulties inherent in such groups. However we are able to utilise it to investigate the structure of D_ω-free groups. In particular we prove the surprising result that every E_ω-group is the homomorphic image of a D_ω-free group.

4. B. H. Neumann [27] has shown that a free product of groups can be defined by means of a homomorphism property. We define, analogously, a D_ω-free product of D_ω-groups. The existence of this product is established by an actual construction which enables us to derive some of the properties of D_ω-free products.

5. Only a small number of the questions that present themselves in connection with D_ω-free groups and D_ω-free products of D_ω-groups have been answered here. The solution of some of these questions seem, by the very nature of the groups involved, to offer more than token resistance. However the methods developed here make possible, at least theoretically, the solution of most of these questions; we hope to deal with further questions in later papers.

6. *Acknowledgments.* I take this opportunity to express my gratitude and appreciation to my Parents, without whose help and encouragement this opportunity for further study would have been both unwanted and impossible.

It is a very great pleasure to acknowledge the help of Dr. B. H. Neumann who, with his ever-ready advice, criticism and creative remarks, has made it a privilege and a delight to have him as a supervisor.

7. *Notation.* For the reader's convenience we list some of the notations used.

ω	a non-empty set of primes.
$\mathrm{gp}(X, R)$	the group generated by the set X with defining relations R.
$[g, h]$	the commutator $g^{-1}h^{-1}gh$ of g and h.
$[G, H]$	the group generated by the commutators $[g, h]$, $g \in G$, $h \in H$.
$G' = \Gamma(G) = \Gamma_1(G)$	$[G, G]$, the derived group of G.
$\Gamma_{i+1}(G)$	$[\Gamma_i(G), \Gamma_i(G)]$, the $i+1$st derived group of G.
g^h	the transform $h^{-1}gh$ of g by h, $g, h \in G$.
$g^{h_1+h_2+\cdots+h_n}$	$g^{h_1} \cdot g^{h_2} \ldots g^{h_n}$ ($g, h_1, h_2, \ldots, h_n$ in the group G).
g^{mh}	$\underbrace{g^h \cdot g^h \ldots g^h}_{m}$, with $m > 0$ an integer.
g^{φ}	the image of g under the homomorphism φ of the group G.
$g^{\varphi+\psi}$	$g^{\varphi} \cdot g^{\psi}$, where here φ and ψ are homomorphisms of the group G into the group H.
$\zeta(G)$	the centre of the group G.
$\lvert G \rvert$	the order of the group G.
$\lvert S \rvert$	the cardinality of the set S.
$\mathrm{C}(S, G)$	the centraliser of the subset S in the group G.
$\mathrm{C}(S)$	the centraliser of the subset S in the group G, where G here is understood.
$\mathrm{C}(s, G)$	$\mathrm{C}(\{s\}, G)$.
$\mathrm{C}(s)$	$\mathrm{C}(\{s\})$.
$\mathrm{N}(S, G)$	the normaliser of the subset S in the group G.
$\mathrm{N}(S)$	the normaliser of the subset S in the group G, where here G is understood.
$\mathrm{nm}(S)$	the normal closure of the subset S in the group G, i.e. the intersection of all normal subgroups of G containing S.

$\prod^*_{\lambda\in\Lambda} A_\lambda$	the free product of the groups A_λ $(\lambda\in\Lambda)$.
$F_1 * F_2 * \cdots * F_n$	the free product of $F_1, F_2, \ldots, F_n$.
$\{A * B; H\}$	the free product of A and B with amalgamated subgroup H.
$\prod_{\lambda\in\Lambda} A_\lambda$	the (restricted) direct product of the A_λ $(\lambda\in\Lambda)$.
$A\times B$	the direct product of A and B.
$\{A\times B; H\}$	the direct product of A and B with amalgamated subgroup H.
$H\leqslant G$	H is a subgroup of G.
$H<G$	H is a proper subgroup of G (here we *do not* exclude the possibility $H=1$).
$H\lhd G$	H is a normal subgroup of G.
$Z(p^\infty)$	the multiplicative group of all p^nth roots of unity, where p is a fixed prime and n ranges over the non-negative integers.
(m, n)	the greatest common divisor of the integers m and n.
$S-T$	the set-theoretical difference between S and T, where T is a subset of S.

8. *Preliminaries.*

Suppose that f, g and h are elements of a group G. Then the following relations between commutators hold:

$$[gf, h]=[g, h]^f[f, h] \quad\text{and}\quad [g, fh]=[g, h][g, f]^h.$$

We refer the reader to Kurosh [21] vol. 2 for the definitions of a free group, a free product, upper central series of a group, lower central series of a group, nilpotent group, ZA-group, locally nilpotent group, soluble group and derived series of a group.

We say that G is an extension of A by B if $A\lhd G$ and $G/A\cong B$; an extension is called *central* if $A\leqslant\zeta(G)$.

PART I

CHAPTER I

Definitions and generalities

9. Let H be a subgroup of an arbitrary group G. Then we call H an ω-subgroup(1) of G if the relation $g^p \in H$ implies $g \in H$ for any pair g and p, with $g \in G$ and $p \in \omega$. If $H \lhd G$ we call H an *ω-ideal*(2) of G if $G/H \in U_\omega$. These two concepts are relative; however, if there is no ambiguity involved we shall simply call an *ω-subgroup of G* an *ω-subgroup* and an *ω-ideal of G* an *ω-ideal.*

LEMMA 9.1 *The intersection of ω-subgroups is an ω-subgroup.*

Proof. Let $\{H_\alpha\}$ be a set of ω-subgroups of G, α ranging over an index set $\mathcal{A}$. If g in G, p in ω are such that $g^p \in \bigcap_{\alpha \in \mathcal{A}} H_\alpha$, then $g^p \in H_\alpha$ for all $\alpha \in \mathcal{A}$, so $g \in H_\alpha$ for all $\alpha \in \mathcal{A}$. Thus $\bigcap_{\alpha \in \mathcal{A}} H_\alpha$ is an ω-subgroup.

LEMMA 9.2. *The intersection of ω-ideals is an ω-ideal.*

Proof. Let $\{H_\alpha\}$ be a set of ω-ideals of G, α ranging over an index set $\mathcal{A}$, and let H be their intersection. Then $H \lhd G$. Furthermore $G/H \in U_\omega$. For suppose $p \in \omega$, $g_1, g_2 \in G$ and $(g_1H)^p = (g_2H)^p$. Then $(g_1H_\alpha)^p = (g_2H_\alpha)^p$ and hence $g_1H_\alpha = g_2H_\alpha$ for every $\alpha \in \mathcal{A}$. In other words $g_1g_2^{-1} \in H_\alpha$ for every $\alpha \in \mathcal{A}$ and hence $g_1g_2^{-1} \in H$. Consequently $g_1H = g_2H$.

Suppose that H is a normal ω-subgroup of G. It does not follow, in general, that H is an ω-ideal of G, even if we presuppose that $G \in U_\omega$. For let

$$C = \mathrm{gp}\,(a, b;\ a^2 = b^2);$$

it can be verified directly that C contains no elements of order 2 (and in fact no elements of finite order). Next present C as a factor group of a free group G by a normal subgroup H:

$$C \cong G/H.$$

Now a free group is a $U_{\{2\}}$-group (it is in fact an R-group, see e.g. Kontorovič [18] or Theorem 17.2 of this paper). It is clear that H is a $\{2\}$-subgroup of G because C contains no elements of order 2. So here we have a normal ω-subgroup of a U_ω-group which is not an

(1) In the case where ω coincides with the set of all primes and $G \in U_\omega$ an ω-subgroup of G is called isolated by Kurosh [21] vol. 2, p. 243.

(2) In the case where $G \in D_\omega$ the ω-ideals defined here coincide with the ideals defined by Higgins [14].

ω-ideal. For D_ω-groups it is more difficult to make an example of such a situation; we shall however give an example of this kind in 39.

Now let S be a subset of an arbitrary group G. By Lemma 9.1 there is a unique minimal ω-subgroup of G containing S, namely the intersection of the ω-subgroups of G containing S; we call this ω-subgroup of G the *ω-closure of S in G* and we shall denote it by $\mathrm{cl}_\omega(S, G)$ or by $\mathrm{cl}_\omega(S)$, if there is no consequent ambiguity. It is useful to have an alternative characterisation of $\mathrm{cl}_\omega(S)$. This is provided by Theorem 9.3.

THEOREM 9.3. *Let S be a subset of a group G. Put $S_1 = S$ and $H_1 = \mathrm{gp}(S_1)$. Define H_{i+1} inductively by putting*

$$S_{i+1} = \{g \mid g^p \in H_i,\ g \in G,\ p \in \omega\}$$

and

$$H_{i+1} = \mathrm{gp}(S_{i+1}).$$

Then

$$\mathrm{cl}_\omega(S) = \bigcup_{i=1}^{\infty} H_i.$$

Proof. Let $H^* = \bigcup_{i=1}^{\infty} H_i$ and suppose $g^p \in H^*$, where $g \in G$ and $p \in \omega$. Thus $g^p \in H_i$ for some i and hence, by definition, $g \in S_{i+1}$ and so $g \in H^*$. Consequently H^* is an ω-subgroup of G containing S and therefore

$$H^* \geqslant \mathrm{cl}_\omega(S). \tag{9.31}$$

On the other hand, it is clear that $\mathrm{cl}_\omega(S) \geqslant H_1$. Suppose in fact that $\mathrm{cl}_\omega(S) \geqslant H_i$. Obviously then $\mathrm{cl}_\omega(S) \geqslant S_{i+1}$ and so $\mathrm{cl}_\omega(S) \geqslant H_{i+1}$. It follows by induction that $\mathrm{cl}_\omega(S) \geqslant H_j$ for all j and so

$$\mathrm{cl}_\omega(S) \geqslant H^*. \tag{9.32}$$

Putting (9.31) and (9.32) together we have the required result.

COROLLARY 9.4. *The ω-closure of a normal subset S of a group G is a normal subgroup of G.*

Proof. We make use here of the representation of $\mathrm{cl}_\omega(S)$ afforded by Theorem 9.3; consequently we adopt the notation used there. The proof is by induction. Suppose we have proved $H_i \lhd G$. Then S_{i+1} is also normal in G. For let $g \in S_{i+1}$ and $x \in G$. Since $g^p \in H_i \lhd G$ we have $x^{-1}g^p x = (x^{-1}gx)^p \in H_i$ and so $x^{-1}gx \in S_{i+1}$. Consequently $H_{i+1} \lhd G$. It follows that $H_i \lhd G$ for all i and so

$$\mathrm{cl}_\omega(S) = \bigcup_{i=1}^{\infty} H_i \lhd G.$$

In a similar way one can prove that the ω-closure of a characteristic subset is characteristic and that the ω-closure of a fully invariant subset is fully invariant.

10. If $p \in \omega$ and $G \in U_\omega$ then pth roots are unique in G. Thus we may speak of "the" pth root of $g \in G$ whenever g has a pth root. We shall sometimes denote the pth root of g, if it exists, by $g\pi$.

In U_ω-groups there is a certain interaction between elements and their pth roots. The corollaries to the following lemma illustrate this interaction.

LEMMA 10.1. *Let G, $H \in U_\omega$ and let θ be a homomorphism of G into H. Then*

$$(g\pi)\theta = (g\theta)\pi; \tag{10.11}$$

this is to be interpreted as stating that $g\theta$ has a p-th root if g does and in this case (10.11) *holds.*

Proof. By applying θ to the equation $(g\pi)^p = g$ we obtain

$$(g\pi)^p\theta = ((g\pi)\theta)^p = g\theta$$

It follows that $g\theta$ has a pth root and that

$$((g\pi)\theta)^p\pi = (g\pi)\theta = (g\theta)\pi.$$

COROLLARY 10.2. *Let g, $h \in G$, $G \in U_\omega$. Then*

$$g^{-1}h\pi g = (g^{-1}hg)\pi.$$

Proof. The mapping $\hat{g}$ defined by $x\hat{g} = g^{-1}xg$, where g is fixed and x ranges over the elements of G, is an automorphism; hence Lemma 10.1 applies and the result follows.

COROLLARY 10.3. (Kontorovič) *Suppose $G \in U_\omega$, g, $h \in G$, p, $q \in \omega$ and that k and l are non-negative integers. Then g^{p^k} and h^{q^l} are permutable if and only if g and h are permutable.*

Proof. We may assume $k = 1$, $l = 0$. Corollary 10.2 can now be applied:

$$g = g^p\pi = (h^{-1}g^p h)\pi = h^{-1}(g^p\pi)h = h^{-1}gh,$$

and so g and h are permutable if g^p and h are permutable. The converse is immediate.

COROLLARY 10.4. *Let g, h be two permutable elements in a U_ω-group G. Then*

$$(gh)\pi = g\pi h\pi; \tag{10.41}$$

this is to be interpreted as stating that if gh, g and h have p-th roots then (10.41) *holds.*

Proof. Since g and h commute so do $g\pi$ and $h\pi$ (by Corollary 10.3). Hence

$$(g\pi h\pi)^p = (g\pi)^p(h\pi)^p = gh,$$

and the result follows on applying π to this equation.

11. We shall adopt the following notational conventions: Instead of $U_{\{p\}}$, $E_{\{p\}}$ and $D_{\{p\}}$ we shall write U_p, E_p and D_p respectively. A D_2-group will be called a σ-group and a D_3-group a τ-group. The square root of an element g in a σ-group G will be denoted by $g\sigma$; the cube root of an element g in a τ-group G will be denoted by $g\tau$.

We have seen in Corollary 10.4 that if g and h are permutable elements of a D_p-group then $(gh)\pi = g\pi h\pi$. The converse is not true in general. However, we shall prove this converse in some special cases.

THEOREM 11.1. *Two elements g and h in a σ-group G are permutable if and only if*

$$(gh)\sigma = g\sigma h\sigma.$$

Proof. Suppose $(gh)\sigma = g\sigma h\sigma$. Squaring both sides of this equation yields

$$gh = g\sigma h\sigma g\sigma h\sigma;$$

on cancellation of $g\sigma$ on the left and $h\sigma$ on the right this equation reduces to

$$g\sigma h\sigma = h\sigma g\sigma.$$

Consequently $$gh = (g\sigma)^2(h\sigma)^2 = (h\sigma)^2(g\sigma)^2 = hg.$$

On the other hand, if $gh = hg$ then $(gh)\sigma = g\sigma h\sigma$, by Corollary 10.4.

COROLLARY 11.2. *In a σ-group G the mapping which takes each element into its square root is an automorphism if and only if G is abelian.*

We need to digress for the moment to record a few properties, connected with elements of finite order, of U_ω-groups and D_ω-groups.

We shall call a group G ω-free if it does not contain elements of order p if $p \in \omega$.

LEMMA 11.3. *If $G \in U_\omega$, then G is ω-free.*

Proof. Let $g \in G$, $p \in \omega$ and suppose $g^p = 1$. Then

$$g^p = 1 = 1^p,$$

and as $G \in U_\omega$ we have $g = 1$.

LEMMA 11.4. *Let G be an extension of a U_ω-group A by a U_ω-group B. Let $f, g \in G$ have finite order modulo A and suppose*

$$f^p = g^p (p \in \omega). \tag{11.41}$$

Then $$f = g.$$

Proof. By hypothesis there exists m prime to p such that $f^m, g^m \in A$; hence (cf. (11.41))

$$(f^m)^p = (g^m)^p$$

is an equation in the U_ω-group A and so

$$f^m = g^m. \tag{11.42}$$

We choose λ, μ so that $$\lambda m + \mu p = 1;$$

this is possible since $(m, p) = 1$. Then, making use of (11.41) and (11.42) we have

$$g = g^{\lambda m + \mu p} = g^{\lambda m} g^{\mu p} = f^{\lambda m} f^{\mu p} = f^{\lambda m + \mu p} = f,$$

which completes the proof.

The following lemma is due, in part, to P. Hall.

THEOREM 11.5. *Let G be an extension of a U_ω-group A by a periodic ω-free group B. Then $G \in U_\omega$. If, in addition, $A \in E_\omega$ then $G \in D_\omega$.*

Proof. The first part of the theorem is an immediate consequence of Lemma 11.4; thus we are left only with the second part. Suppose then that $g \in G$ and $p \in \omega$. There exists an integer m prime to p such that $g^m \in A$. Therefore we can find $a \in A$ such that

$$g^m = a^p. \tag{11.51}$$

Since $A \lhd G$, $g^{-1}ag \in A$ and so the resulting equation (cf. (11.51))

$$(g^{-1}ag)^p = a^p$$

is an equation involving only elements of A, therefore

$$g^{-1}ag = a. \tag{11.52}$$

Since $(m, p) = 1$ we can choose λ, μ such that $\lambda m + \mu p = 1$. Then (cf. (11.51) and (11.52))

$$g = g^{\lambda m + \mu p} = a^{\lambda p} g^{\mu p} = (a^\lambda g^\mu)^p$$

and so g has a pth root, and this completes the proof.

COROLLARY 11.6. *A periodic group G is a D_ω-group if and only if it is ω-free.*

Proof. If G is ω-free we can apply Theorem 11.5 to deduce that $G \in D_\omega$—we may take $A = 1$. On the other hand, if $G \in D_\omega$ then Lemma 11.3 applies and G is ω-free.

We make use of Corollary 11.6 to give an example of a situation in which $G \in D_p$ and

g, h in G are such that $(gh)\pi = g\pi h\pi$ although $gh \neq hg$. This is given in order to show that Theorem 11.1 cannot be generalised in the obvious way. We take

$$G = \mathrm{gp}(a, b;\ a^4 = b^5 = 1,\ a^{-1}ba = b^3).$$

Now $|G| = 20$ and so by Corollary 11.6 G is a τ-group. It can easily be verified that

$$(ab)^3 = a^3 b^3.$$

Hence
$$(a^3 b^3)\tau = (ab)^3\tau = ab = a^3\tau b^3\tau.$$

However, it follows from the defining relations of G that

$$a^3 b^3 \neq b^3 a^3.$$

In this example G is not even nilpotent; for locally nilpotent groups however the result for τ-groups corresponding to Theorem 11.1 does hold:

THEOREM 11.7. *Two elements g and h in a locally nilpotent τ-group G are permutable if and only if*

$$(gh)\tau = g\tau h\tau.$$

Proof. If $gh = hg$ then $(gh)\tau = g\tau h\tau$ by Corollary 10.4.

It remains to show that $gh = hg$ whenever $(gh)\tau = g\tau h\tau$. To this end let H be any nilpotent subgroup of G containing $g\tau$ and $h\tau$—the existence of such a subgroup H is taken care of by Theorem 15.1. The proof that g and h commute will be by induction over the class c of H. If $c = 1$ the result is immediate. If $c > 1$ then the factor group H/Z, with Z the centre of H, is a τ-group of class c-1 (see e.g. Corollary 14.4). Now

$$(gZhZ)\tau = (ghZ)\tau = (gh)\tau Z = (g\tau h\tau)Z = g\tau Z h\tau Z = (gZ)\tau(hZ)\tau.$$

Thus inductively

$$gZhZ = hZgZ.$$

consequently

$$g\tau Z h\tau Z = (gZ)\tau(hZ)\tau = (hZ)\tau(gZ)\tau = h\tau Z g\tau Z.$$

Hence
$$[h\tau, g\tau] = z \in Z. \tag{11.71}$$

We have $(gh)\tau = g\tau h\tau$, by the hypothesis; on cubing this equation we obtain

$$\begin{aligned} gh &= g\tau h\tau g\tau h\tau g\tau h\tau = (g\tau)^2 h\tau[h\tau, g\tau]h\tau g\tau h\tau = \\ &= (g\tau)^2(h\tau)^2 g\tau h\tau z \text{ (since } z \in Z) = (g\tau)^3(h\tau)^2[(h\tau)^2, g\tau]h\tau z \\ &= (g\tau)^3(h\tau)^3[h\tau, g\tau]^{h\tau}[h\tau, g\tau]z = ghz^3. \end{aligned}$$

It follows that
$$z^3 = 1.$$

But $G \in D_\omega$; therefore $z = 1$. Hence $g\tau h\tau = h\tau g\tau$ (by 11.71) and so

$$gh = hg.$$

This completes the proof of the theorem.

It follows from this theorem that if a locally nilpotent τ-group G has the property that the mapping which takes every element into its cube root is an automorphism, then G is abelian. However, the restriction that G be locally nilpotent is redundant.

THEOREM 11.8. *Let G be a τ-group in which the mapping which takes every element into its cube root is an automorphism. Then G is abelian.*

Proof. Let $g, h \in G$. Then

$$(gh)\tau = g\tau h\tau,\ (hg)\tau = h\tau g\tau.$$

The result of cubing both sides of the equation $(gh)\tau = g\tau h\tau$ and then cancelling $g\tau$ on the left and $h\tau$ on the right is the further equation

$$(g\tau)^2 (h\tau)^2 = (h\tau g\tau)^2.$$

Starting instead from the equation $(hg)\tau = h\tau g\tau$ and proceeding similarly the corresponding equation

$$(h\tau)^2 (g\tau)^2 = (g\tau h\tau)^2$$

can be obtained. Hence

$$\begin{aligned}(g\tau)^2 h &= (g\tau)^2 (h\tau)^3 = (g\tau)^2 (h\tau)^2 h\tau = (h\tau g\tau)^2 h\tau \\ &= h\tau (g\tau h\tau g\tau h\tau) = h\tau (g\tau h\tau)^2 = h\tau (h\tau)^2 (g\tau)^2 = h (g\tau)^2.\end{aligned}$$

Thus h commutes with $(g\tau)^2$ and therefore so also does $h\tau$. Consequently

$$(h\tau g\tau)^2 = (g\tau)^2 (h\tau)^2 = (h\tau)^2 (g\tau)^2.$$

This leads after cancellation of $h\tau$ on the left and $g\tau$ on the right to

$$g\tau h\tau = h\tau g\tau,$$

and so g and h also commute and G is abelian. This completes the proof.

An example of a non-abelian D_p-group in which π acts as an automorphism is the quaternion group Q of order 8:

$$Q = \mathrm{gp}(a, b;\ a^4 = b^4 = 1,\ [a, b] = a^2 = b^2).$$

We may take p here to be the prime 5. Then

$$(cd)\pi = c\pi d\pi$$

for all $c, d \in Q$ since each element in Q coincides with its fifth root. This example shows Theorem 11.8 cannot be extended to include all D_p-groups.

CHAPTER II

Locally nilpotent groups, ZA-groups and nilpotent groups

12. We shall be concerned here with groups satisfying certain commutativity conditions. For example we prove that the terms of the upper central series of a ZA-group in E_ω are ω-ideals. The main result in this chapter is of a different kind: The ω-closure of a nilpotent subgroup of class c of a U_ω-group is nilpotent of class c. We deduce a number of related results and, in particular, a sort of dual to the result mentioned at the outset: The first ω terms of the lower central series of a locally nilpotent D_ω-group are ω-ideals.

Some of the results stated, and sometimes proved here, are, per se, similar to known results for R-groups and divisible groups; in particular there is a certain amount of overlapping between our results and those of Kontorovič [18] and Černikov [7].

13. A set of groups $\{H_i\}$, where i ranges over a well-ordered index set I, is said to form an *ascending sequence* if $H_i \leqslant H_j$ whenever $i \leqslant j$. Suppose now that each H_i is a ω-subgroup of some fixed supergroup(¹) G and let $H = \bigcup_{i \in I} H_i$. Then H is itself a ω-subgroup of G. For if $g \in G$, $p \in \omega$ and $g^p \in H$, then $g^p \in H_i$ for some $i \in I$. Consequently $g \in H_i$ and so $g \in H$. Thus we have proved

LEMMA 13.1. *The union of an ascending sequence of ω-subgroups of an arbitrary group is an ω-subgroup.*

In a similar way we can prove:

LEMMA 13.2. *The union of an ascending sequence of ω-ideals of an arbitrary group is an ω-ideal.*

The following theorem and Corollary 13.4 are immediate generalisations of a theorem of Kontorovič [18] on R-groups—the proofs are similar and are therefore omitted.

THEOREM 13.3. *The centraliser of an arbitrary set of elements of a U_ω-group is an ω-subgroup.*

(¹) G is a *supergroup* of H_i if $H_i \leqslant G$.

COROLLARY 13.4. *The terms of the upper central series of a U_ω-group are ω-ideals.*

COROLLARY 13.5. *An ω-subgroup H of a U_ω-group G which is contained in the centre is an ω-ideal of G.*

Proof. Suppose $p\in\omega$, $x, y\in G$ and $(xH)^p = (yH)^p$. Then

$$x^p = y^p h, \quad h\in H. \tag{13.51}$$

The centre $\zeta(G)$ is an ω-ideal of G (Corollary 13.4); furthermore $\zeta(G)$ contains H. Consequently $(x\zeta(G))^p = (y\zeta(G))^p$ and so $x\zeta(G) = y\zeta(G)$. Thus $x = yz$, $z\in\zeta(G)$ and

$$x^p = y^p z^p. \tag{13.52}$$

On comparing (13.51) and (13.52) we see that $h = z^p$. But as H is an ω-subgroup of G, $z\in H$. Consequently $xH = yH$ and H is therefore an ω-ideal of G.

We saw in *9* that a normal ω-subgroup of a group is not necessarily an ω-ideal; however, this is always true whenever the factor group is locally nilpotent. This follows immediately from the following theorem, which is due to Mal'cev [26] and Černikov (see Kurosh [21] vol. 2, p. 247).

THEOREM 13.6. *A locally nilpotent group is a U_ω-group if and only if it is ω-free.*

COROLLARY 13.7. *A normal ω-subgroup N of a group G is an ω-ideal of G if G/N is locally nilpotent.*

Proof. By Theorem 13.6 $G/N\in U_\omega$, since it is ω-free.

14. We shall consider in this section some properties of various kinds of E_ω-groups. First we prove:

THEOREM 14.1. *Let G be a ZA-group, let $p\in\omega$ and let G be an E_ω-group. Then the set of elements of G whose orders are a power of p forms a subgroup contained in the centre of G.*

The proof of Theorem 14.1 depends on the following lemma (cf. Kurosh [21] vol. 2, p. 234).

LEMMA 14.2. *Let G be a ZA-group in the class E_ω. Then the centre $Z = \zeta(G)$ is an ω-ideal of G.*

Proof. Let $1 \neq g\in G$, let $p\in\omega$ and suppose $g^p\in Z$. Choose α so that[1]

[1] We define $\zeta^1(G) = \zeta(G)$, $\zeta^\alpha(G) = \bigcup_{\beta<\alpha} \zeta^\beta(G)$ if α has no predecessor, and $\zeta^{\alpha+1}$ by $\zeta(G/\zeta^\alpha(G)) = \zeta^{\alpha+1}(G)/\zeta^\alpha(G)$.

$$g \notin \zeta^{\alpha}(G),\ g \in \zeta^{\alpha+1}(G).$$

If $\alpha = 0$, then $g \in Z$, as required. Suppose, if possible, that $\alpha = 1$ and let x be an arbitrary element of G. Let x_0 be a pth root of x. Then

$$1 = [g^p, x] = [g, x_0^p] = [g, x]$$

and so $\alpha \neq 1$. Let us now suppose that any element of $\zeta^{\beta}(G)$ which has a pth power in Z, for every β satisfying $1 \leqslant \beta \leqslant \alpha$, is contained in Z. Then

$$[x, g] = x^{-1}g^{-1}x \cdot g \in \zeta^{\alpha}(G),$$

and as both g and $x^{-1}g^{-1}x$ have order p modulo Z, $[x, g]$ itself has order modulo Z a power of p(1) and hence belongs to Z. This holds for all $x \in G$ and so $g \in \zeta^2(G)$, which, as we showed above, implies $g \in Z$. This completes the proof since a normal ω-subgroup of a locally nilpotent group is an ω-ideal (Corollary 13.7).

The proof of Theorem 14.1 now follows easily. For if $g^{p^r} = 1$ then $g^{p^r} \in \zeta(G)$, and so, by Lemma 14.2, $g \in \zeta(G)$.

The first of the following two corollaries is due to Černikov [1]; the proof of Theorem 14.1 is based on his original proof.

COROLLARY 14.3. *In a divisible ZA-group the elements of finite order form a subgroup of the centre.*

COROLLARY 14.4. *The terms of the upper central series of a ZA-group G in E_ω are ω-ideals.*

Proof. By Corollary 14.2 $Z = \zeta(G)$ is a ω-ideal. Thus G/Z is a U_ω-group and so by Corollary 13.4 the terms $\zeta^{\alpha}(G)/Z$ of the upper central series of G/Z are ω-ideals of G/Z. It follows easily that the terms $\zeta^{\alpha}(G)$ of the upper central series of G are ω-ideals of G.

We remark that the centre of a locally nilpotent E_ω-group is not necessarily an ω-ideal since every locally nilpotent p-group can be embedded in a divisible locally nilpotent p-group with a trivial centre (see Baumslag [3]) (cf. Corollary 14.3).

THEOREM 14.5. *The terms of the lower central series of a nilpotent group G in E_ω are themselves E_ω-groups.*

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(1) In a locally nilpotent group the elements of order a power of p form a subgroup (see Kurosh [21] vol. 2, p. 215).

Proof. The proof is by induction on the class c of G. In the case $c = 1$ the result is immediate, so we have a basis for induction. Now $\Gamma_c(G)$ is generated by the commutators:

$$[g, h], \quad g \in G, \quad h \in \Gamma_{c-1}(G).$$

If $p \in \omega$ then g has a pth root, say g_0. Hence

$$[g, h] = [g_0^p, h] = [g_0, h]^p$$

and so $[g, h]$ has a pth root in $\Gamma_c(G)$. Consequently every element of the abelian group $\Gamma_c(G)$ has a pth root and therefore $\Gamma_c(G) \in E_\omega$. Consider now the factor group $H = G/\Gamma_c(G)$. Then $H \in E_\omega$ and is nilpotent of class $c - 1$ and we may apply induction to deduce that $\Gamma_i(H) = \Gamma_i(G)/\Gamma_c(G)$ is an E_ω-group for $i = 1, 2, \ldots, c$. It follows immediately that $\Gamma_i(G) \in E_\omega$ for $i = 1, 2, \ldots, c$ and the theorem is proved.

It is easy to make examples of nilpotent E_ω-groups in which the terms of the lower central series are not ω-ideals. On the other hand, for nilpotent D_ω-groups such examples cannot exist.

COROLLARY 14.6. *The terms of the lower central series of a nilpotent group G in D_ω are ω-ideals.*

Proof. A normal ω-subgroup of a (locally) nilpotent U_ω-group is an ω-ideal (Corollary 13.7) and so the result follows from Theorem 14.5, because an E_ω-group which is a subgroup of a U_ω-group is an ω-subgroup of that group.

By Corollary 14.6 the derived group of a nilpotent D_ω-group is an ω-ideal. This result is, however, not true for D_ω-groups in general. An example of a D_ω-group in which the derived group is not an ω-ideal will be given in Part II (see Theorem 37.2).

If we consider the more general class U_ω, then again Corollary 14.6 is no longer true. For let

$$G = \mathrm{gp}(a, b, c;\ [a, b] = c^2,\ ca = ac,\ cb = bc).$$

It is easy to verify that $G \in U_2$ and that G is nilpotent of class two; but the derived group of G is generated by c^2 and so is not an ω-ideal of G.

15. Mal'cev [25] has proved that if G is a torsion-free locally nilpotent group then G can be embedded in a torsion-free locally nilpotent divisible group G^*. We shall prove here, as an application of the main theorem of this section, that if G is a torsion-free locally nilpotent group and if K is a supergroup of G, which is a divisible R-group, then G^*, the

ϱ-closure of G in K (ϱ being the set of *all* primes) has no option other than to be a locally nilpotent divisible group.

The main theorem is the following.

THEOREM 15.1. *Let G be a subgroup of a U_ω-group K. Then, if G is nilpotent of class c, so is* $\mathrm{cl}_\omega(G, K)$.

Proof. Let

$$1 < Z_1 < Z_2 < \cdots < Z_c = G$$

be the upper central series of G. It follows, on applying Theorem 13.3 that[1]

$$[\mathrm{cl}(Z), G] = 1,$$

and similarly that

$$[\mathrm{cl}(G), \mathrm{cl}(Z)] = 1. \tag{15.11}$$

Thus we see from (15.11) that $\mathrm{cl}(Z)$ is a subgroup of the centre of $\mathrm{cl}(G)$ and so, by Corollary 13.5, $\mathrm{cl}(Z)$ is an ω-ideal of $\mathrm{cl}(G)$; we shall make use of this fact in the sequel.

Now if $c = 1$ then, by (15.11), $[\mathrm{cl}(G), \mathrm{cl}(G)] = 1$ and so $\mathrm{cl}(G)$ is abelian—thus we have proved the theorem in the case $c = 1$. We shall use this fact as the basis for an induction: if L is a subgroup of a U_ω-group M and if L is nilpotent of class d ($d \leqslant c - 1$) we shall assume $\mathrm{cl}_\omega(L, M)$ is nilpotent of class d.

We now make use of Theorem 9.3. We put

$$S_1 = G \cdot \mathrm{cl}(Z_1);$$

according to Theorem 9.3 if $H_1 = \mathrm{gp}(S_1)$, $S_{i+1} = \{g \mid g^p \in H_i,\ g \in K,\ p \in \omega\}$, $H_{i+1} = \mathrm{gp}(S_{i+1})$, then $\mathrm{cl}(S_1) = \bigcup_{i=1}^{\infty} H_i$. We shall show that H_i is nilpotent of class c for $i = 1, 2, \ldots$ and consequently that $\mathrm{cl}(G \cdot \mathrm{cl}(Z_1))$ is nilpotent of class c. It follows from (15.11) that S_1 is a subgroup; therefore $H_1 = G\,\mathrm{cl}(Z_1)$. Now

$$H_1/\mathrm{cl}(Z_1) \cong G/G \cap \mathrm{cl}(Z_1) = G/Z_1;$$

thus H_1 is nilpotent of class c since $\mathrm{cl}(Z_1) \leqslant \zeta(H_1)$.

Suppose now that we have proved H_i nilpotent of class c for all $i \leqslant j$, for some $j \geqslant 1$. We now well-order the elements of S_{j+1}:

$$S_{j+1} = \{a_1, a_2, \ldots, a_\alpha, \ldots\}.$$

(1) Throughout the proof of this theorem we shall simply write $\mathrm{cl}(H)$ for the ω-closure of H in K.

Define $$F_\beta = \mathrm{gp}(H_j, a_1, a_2, \ldots, a_\alpha, \ldots; \alpha < \beta).$$

Now F_1 is nilpotent of class c. To see this we consider $H_j/\mathrm{cl}(Z_1)$ as a nilpotent subgroup (of class $c-1$) of $F_1/\mathrm{cl}(Z_1)$. There is a prime $p \in \omega$ for which $(a_1 \mathrm{cl}(Z))^p \in H_j/\mathrm{cl}(Z_1)$ and so, by induction, $F_1/\mathrm{cl}(Z_1) = \mathrm{cl}(H_j/\mathrm{cl}(Z_1),\ F_1/\mathrm{cl}(Z_1))$ is nilpotent of class $c-1$. But $\mathrm{cl}(Z_1) \leqslant \zeta(F_1)$ and so F_1 is nilpotent of class c. It follows by transfinite induction that

$$H_{j+1} = \bigcup_\beta F_\beta$$

is the union of an ascending sequence of subgroups, each nilpotent of class c and so H_{j+1} is nilpotent of class c. Therefore, by induction, each H_i is nilpotent of class c and hence $\mathrm{cl}(G \cdot \mathrm{cl}(Z)) = \bigcup_{i=1}^{\infty} H_i$ is nilpotent of class c. In particular $\mathrm{cl}(G)$ is nilpotent of class c and this completes the proof of the theorem.

COROLLARY 15.2. *Let G be a subgroup of a U_ω-group K. If G is locally nilpotent then its ω-closure in K is also locally nilpotent.*

Proof. Let $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n$ be n elements taken arbitrarily from $\mathrm{cl}(G)$ and let

$$H = \mathrm{gp}(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n).$$

We have to show that H is nilpotent. We make use of Theorem 9.3. and write

$$\mathrm{cl}(G) = \bigcup_{i=1}^{\infty} H_i,$$

where $$H_1 = G_1 = G,$$

$$H_{i+1} = \mathrm{gp}(G_{i+1})$$

and $$G_{i+1} = \{g \mid g \in K,\ g^p \in H_i \text{ for some } p \in \omega\}.$$

This expression for $\mathrm{cl}(G)$ enables us to find a finite set

$$\{g_1, g_2, \ldots, g_m\} \quad (g_i \in G)$$

such that $$H \leqslant \mathrm{cl}(\{g_1, g_2, \ldots, g_m\}).$$

To see this let us consider the case of a single element $g \in \mathrm{cl}(G)$. Since g belongs to the union of the H_i it must belong to one of them, say to H_j. Therefore we can write

$$g = h_1 h_2 \ldots h_\lambda,$$

where $h_\mu \in G_j$ and has therefore a prime $p_\mu \in \omega$ associated with it such that

$$h_\mu^{p_\mu} \in H_{j-1}.$$

Now, inductively, corresponding to each element in H_{j-1} we can find a finite set of elements in G whose ω-closure contains that element. Thus we can find

$$\{g_{\mu,1}, g_{\mu,2}, \ldots, g_{\mu,n_\mu}\} \quad (g_{\mu,i} \in G),$$

such that
$$h_\mu^{p_\mu} \in \mathrm{cl}(\{g_{\mu,1}, g_{\mu,2}, \ldots, g_{\mu,n_\mu}\}).$$

Then, since $p_\mu \in \omega$, we have also

$$h_\mu \in \mathrm{cl}(\{g_{\mu,1}, g_{\mu,2}, \ldots, g_{\mu,n_\mu}\}).$$

Hence
$$g \in \mathrm{cl}(g_{1,1}, g_{1,2}, \ldots, g_{\lambda,n_\lambda}).$$

It follows that we can, in a similar way, find a set

$$\mathcal{G} = \{g_1, g_2, \ldots, g_m\} \quad (g_i \in G),$$

such that
$$\hat{g}_i \in \mathrm{cl}(\mathcal{G}) \quad (i = 1, 2, \ldots, n);$$

consequently
$$H \leqslant \mathrm{cl}(\mathcal{G}).$$

The group generated by a finite subset $\mathcal{G}$ of a locally nilpotent group G is necessarily nilpotent. Furthermore, by Theorem 15.1, the ω-closure of a nilpotent subgroup of a U_ω-group is again nilpotent. Now $\mathrm{cl}(\mathcal{G}) = \mathrm{cl}(\mathrm{gp}(\mathcal{G}))$; thus H is a subgroup of a nilpotent group and so is itself nilpotent. This completes the proof of the corollary.

COROLLARY 15.3. *Let $K \in U_\omega$, let G be nilpotent of class c and suppose G is a subgroup of K. Suppose, furthermore, that every element $k \in K$ has a power $k^n \in G$, where all the prime divisors of n belong to ω. Then K is nilpotent of class c.*

Proof. The ω-closure of G is K, so the theorem follows on applying Theorem 15.1.

It may be of interest to note a connection between Corollary 15.3 and a theorem of Duguid and Maclane [8] which states that a subgroup of finite index in a torsion-free nilpotent group of class c is also of class c. Since a torsion-free nilpotent group is an R-group (Theorem 13.6) their theorem is an immediate consequence of Corollary 15.3.

Corollary 15.3 has an analogue for locally nilpotent groups, for it follows easily from Corollary 15.2 that if $K \in U_\omega$ and if G is a locally nilpotent subgroup of K such that every

$k \in K$ has a power $k^n \in G$, where all the prime divisors of n belong to ω, then K is itself locally nilpotent. One may also prove a similar result for ZA-groups.

THEOREM 15.4. *The first ω terms of the lower central series of a locally nilpotent D_ω-group G are ω-ideals of G.*

Proof. Let

$$G = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_i \geqslant \cdots \geqslant 1$$

be the lower central series of G. Suppose $p \in \omega$, $g \in G$ and $g^p \in H_i$ for some finite integer i; then g^p can be written as a product of (say) n i-fold commutators involving the $n(i+1)$ elements

$$g_{1,1}, g_{1,2}, \ldots, g_{1,i+1}, \ldots, g_{n,\,i+1};$$

a 1-fold commutator is a simple commutator $[x, y]$ and, inductively, an i-fold commutator is the commutator of an $(i-1)$-fold commutator and an element of G. We put

$$H = \mathrm{gp}(g_{1,1}, g_{1,2}, \ldots, g_{n,\,i+1}).$$

Then H is nilpotent and so therefore $\mathrm{cl}(H)$ is also nilpotent, by Theorem 15.1. The terms of the lower central series of $\mathrm{cl}(H)$ are ω-ideals, by Corollary 14.6; thus since g^p belongs to the ith member of the lower central series of $\mathrm{cl}(H)$ (by the choice of H), g belongs also to the ith member of the lower central series of $\mathrm{cl}(H)$. Consequently $g \in H_i$ and hence H_i is an ω-subgroup of G. Furthermore H_i is normal in G and so on applying Corollary 13.7 we see that H_i is an ω-ideal of G. The same analysis holds for every i and so we have H_i is an ω-ideal of G for $i = 1, 2, \ldots$. Hence $H_\omega = \bigcap_{i=1}^{\infty} H_i$ is also an ω-ideal of G (Lemma 9.2). This completes the proof of the theorem.

I do not know whether it is possible to extend this theorem so as to include *all* the members of the lower central series of a locally nilpotent D_ω-group.

COROLLARY 15.5. *The terms of the derived series of a locally nilpotent D_ω-group are ω-ideals.*

Proof. By Lemma 9.2 the intersection of ω-ideals is an ω-ideal and the corollary follows immediately from this remark and Theorem 15.4.

We complete this section with the proof of the following theorem and its corollary.

THEOREM 15.6. *Let G be a nilpotent group and suppose that G is generated by its p-th powers, for each prime $p \in \omega$. Then G is an E_ω-group.*

Proof. Suppose

$$G = H_0 > H_1 > \cdots > H_c = 1$$

is the lower central series of G. The theorem is obviously true when $c = 1$; we shall use this fact as a basis for an induction over the class c of G.

We prove first that H_{c-1} is an E_ω-group. The elements of H_{c-1} can be written as products of commutators of the form

$$h = [g, g'], \quad g \in G, \; g' \in H_{c-2}. \tag{15.61}$$

Now G is generated by its pth powers and so g can be written in the form

$$g = g_1^p g_2^p \ldots g_n^p,$$

where the g_i are elements of G. We substitute this expression for g in equation (15.61) and make use of the equation

$$[fg, h] = [f, h]^g [g, h]$$

to show that h is a pth power of an element in H_{c-1}. Explicitly

$$\begin{aligned} h &= [g_1^p g_2^p \ldots g_n^p, g'] = [g_1^p, g']^{g_2^p g_3^p \cdots g_n^p} \cdot [g_2^p g_3^p \ldots g_n^p, g'] \\ &= [g_1^p, g'] [g_2^p, g']^{g_3^p \cdots g_n^p} \cdot [g_3^p \ldots g_n^p, g'] = [g_1, g']^p [g_2^p, g'] \ldots = \ldots \\ &= [g_1, g']^p [g_2, g']^p \ldots [g_n, g']^p = ([g_1, g'] [g_2, g'] \ldots [g_n, g'])^p. \end{aligned}$$

Thus h is a pth power of an element in H_{c-1} and it follows that every element of H_{c-1} is a pth power of an element in H_{c-1}. This procedure applies for all $p \in \omega$ and since H_{c-1} is abelian, it is an E_ω-group. Now G is generated by its pth powers for each p in ω; therefore so also is G/H_{c-1}. Now G/H_{c-1} is nilpotent of class $c - 1$ and so we are able to apply the induction hypothesis to assert that G/H_{c-1} is in fact an E_ω-group. Thus G is a central extension of an E_ω-group by an E_ω-group and so by Theorem 21.2 (see Chapter IV) G is itself an E_ω-group. This completes the proof of the theorem.

CHAPTER III

Construction of U_ω-groups, E_ω-groups and D_ω-groups

16. In this chapter we shall make use of the *free product* and the *second nilpotent product*. We shall later also require the *free product with an amalgamated subgroup*, and, more generally, the *generalised free product*. It is convenient therefore to give here a short exposition on such products and to record some of their properties which we shall need in subsequent work. As far as generalised free products are concerned, we shall take this opportunity to draw heavily from B. H. Neumann's "An essay on free products of groups with amalgamations"; we even go so far as to take the liberty of quoting from this essay (B. H. Neumann [27]).

Let F be a group and let F_λ be subgroups of F, where λ ranges over an index set Λ. We call F the *generalised free product of the* F_λ if i) F is generated by its subgroups F_λ, and ii) for every group G and every set of homomorphic mappings φ_λ of each F_λ into G, every two φ_λ, φ_μ of which agree where both are defined, there exists a homomorphic mapping φ of F into G that coincides with φ_λ on each F_λ (see B. H. Neumann [27], Theorem 1.1). Now suppose F is the generalised free product of its subgroups $F_\lambda (\lambda \in \Lambda)$ and put

$$F_\lambda \cap F_\mu = H_{\lambda\mu} (= H_{\mu\lambda}), \quad \lambda, \mu \in \Lambda.$$

If all the intersections $H_{\lambda\mu}$ coincide to form a single subgroup H:

$$F_\lambda \cap F_\mu = H, \ (\lambda \neq \mu)$$

then F is called *the (generalised) free product of the* F_λ *with an amalgamated subgroup* H. In the case where $H = 1$, the trivial subgroup, then F is called simply the *free product*, or, to emphasise the distinction, *the ordinary free product of the* F_λ.

Let now groups F_λ be given, where λ runs over a suitable non-empty index set Λ. In every F_λ and to every index $\mu \in \Lambda$ let a subgroup $H_{\lambda\mu}$ be distinguished; $H_{\lambda\lambda}$ is always to be the whole group F_λ. If there exists a group F which is the generalised free product of groups $\hat{F}_\lambda$ with intersections

$$\hat{H}_{\lambda\mu} = \hat{F}_\lambda \cap \hat{F}_\mu = \hat{H}_{\mu\lambda},$$

and if there are isomorphic mappings φ_λ of F_λ onto $\hat{F}_\lambda$,

$$\hat{F}_\lambda = F_\lambda \varphi_\lambda$$

such that always $$\hat{H}_{\lambda\mu} = H_{\lambda\mu}\varphi_\lambda$$

then we say that *the generalised free product of the* F_λ *with amalgamated* $H_{\lambda\mu}$ *exists*, or simply the generalised free product of the F_λ exists. The generalised free product does not always exist; however, in the special case of the generalised free product with a single subgroup amalgamated, the generalised free product does always exist (Schreier [32]).

It is often convenient when dealing with generalised free products not to distinguish between a group and an isomorphic copy of it; we shall adopt this convention whenever it is both convenient and unambiguous.

Let us suppose for the moment that F is the free product of the groups F_λ with amalgamated subgroup H $(\lambda \in \Lambda)$. The elements in F can be represented by a certain *normal form*: We choose in every group F_λ a system S_λ of left coset representatives modulo H containing the unit element; thus every element $f \in F_\lambda$ can be uniquely represented in the form

$$f = sh \quad (s \in S_\lambda, h \in H).$$

Now we distinguish certain words in elements of the F_λ; specifically we call

$$w = s_1 s_2 \dots s_n h$$

a *normal* word if it satisfies the following three conditions:

(i) Every *component* $s_i (1 \leqslant i \leqslant n)$ is a representative $\neq 1$ belonging to one of the S_λ.

(ii) Successive components s_i belong to different systems of representatives; in other words, if $1 \leqslant i \leqslant n$, $s_i \in S_\lambda$, $s_{i+1} \in S_\mu$, then $\lambda \neq \mu$.

(iii) The last component belongs to the common subgroup $h \in H$.

We call n the *length* of the normal word. Note that a word is a string of symbols; if we interpret it as a product (which is written in the same way) we obtain an element of the group, and we say the word *represents* the element. Then every element is represented by one and only one normal word (cf. B. H. Neumann [21] Theorem 2.4). The uniquely determined normal word representing f we call *the normal form of* f and we call the length of the normal word representing f the *length of the element* f; we write $\lambda(f) = n$ if f is of length n. The following lemma is due to B. H. Neumann [27]:

LEMMA 16.1. *If* $n > 1$, *if*

$$f = f_1 f_2 \cdots f_n,$$

and if no two successive factors f_i, f_{i+1} *are elements of the same group* F_λ, *then* n *is the length of* f. *If* $n = 1$, *the length of* f *is* 0 *or* 1 *according as* f *lies in* H *or not.*

We call the element $f \in F$ *cyclically reduced* if none of its conjugates in F has smaller length than itself. The following lemma is due to B. H. Neumann [27].

LEMMA 16.2. *If* f *is cyclically reduced and if it has the normal form*

$$f = s_1 s_2 \dots s_n h$$

of length $n > 1$, *then* s_1 *and* s_n *belong to different groups* $F_\lambda \neq F_\mu$.

LEMMA 16.3. (B. H. Neumann [27].) *If f has length $n > 1$ and if in its normal form*

$$f = s_1 s_2 \dots s_n h$$

the components s_1 and s_n belong to different groups F_λ and F_μ, then f is cyclically reduced.

THEOREM 16.4. (B. H. Neumann [27].)

Let F be the free product of groups F_λ with amalgamated subgroup H; if f is an element of finite order in F, then f is conjugate to an element in (at least) one of the F_λ.

COROLLARY 16.5. (B. H. Neumann [27].) *The free product of locally infinite groups with an amalgamated subgroup is locally infinite.*

Suppose now that F is a group generated by its subgroups F_λ, where λ ranges over an *ordered* index set Λ. Then F is a *regular product* of the groups F_λ (Golovin [11], [12]) if every element $f \in F$ has a unique *regular representation* of the form

$$f = f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(n)} u,$$

where $$f_{\lambda(i)} \in F_{\lambda(i)},\ \lambda(1) < \lambda(2) < \dots < \lambda(n),\ u \in \mathrm{nm}([F_\lambda])$$

and $$[F_\lambda] = \mathrm{gp}([f_\lambda, f_\mu];\ \ f_\lambda \in F_\lambda,\ f_\mu \in F_\mu,\ \lambda \neq \mu,\ \ \lambda, \mu \in \Lambda).$$

We revert to the case where F is the free product of the groups F_λ, $\lambda \in \Lambda$, Λ an ordered index set. Let

$${}_1[F_\lambda] = \mathrm{nm}([F_\lambda]), \quad {}_k[F_\lambda] = [F, {}_{k-1}[F_\lambda]],$$

where $k = 2, 3, \dots$. Then Golovin [11] calls the factor group

$$F/{}_k[F_\lambda]$$

the *k-th nilpotent product* of the groups F_λ, $\lambda \in \Lambda$; furthermore he proves that the kth nilpotent product of groups is in fact a regular product.

17. We consider now methods of constructing 'new' U_ω-groups, E_ω-groups and D_ω-groups from given U_ω-groups, E_ω-groups and D_ω-groups.

The proof of the following lemma is straightforward and is omitted.

LEMMA 17.1. *The restricted direct product and the unrestricted direct product of X_ω-groups is an X_ω-group where X here stands for any of the three letters U, E and D.*

Kontorovič [8] has proved that the free product of R-groups is an R-group; we generalise this result to the following

THEOREM 17.2. *The free product F of U_ω-groups F_λ, where λ ranges over an index set Λ, is a U_ω-group.*

Proof. Suppose $p \in \omega$, $f, g \in F$ and

$$f^p = g^p \ (f \neq 1). \tag{17.21}$$

We may assume f is cyclically reduced; then the proof that $f = g$ falls naturally into two parts:

(i) $\lambda(f) = 1$. We have $g^{-1} f^p g = f^p$ and since $f \neq 1$, $f^p \neq 1$ and so

$$1 = \lambda(g^{-1} f^p g) = \lambda(g^p) \geqslant \lambda(g).$$

Thus $\lambda(g) \leqslant 1$ and therefore, by (17.21), $\lambda(g)$ is precisely 1. It follows then with the aid of (17.21) that f and g belong to the same subgroup, say to F_λ. But F_λ is a U_ω-group and therefore (17.21) yields $f = g$.

(ii) $\lambda(f) > 1$. Let us suppose that the normal form for f is

$$f = f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(m)}, \quad f_{\lambda(i)} \in F_{\lambda(i)};$$

f is cyclically reduced and consequently $\lambda(1) \neq \lambda(m)$ (Lemma 16.2); therefore the normal form of f^p is

$$f^p = f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(m)} f_{\lambda(1)} \dots f_{\lambda(m)}.$$

Now, by (i), the length of g is at least 2. Let then

$$g = g_{\mu(1)} g_{\mu(2)} \dots g_{\mu(n)} \ (n > 1)$$

be the normal form of g. We assert that g is itself cyclically reduced. For it follows from (17.21) that g and f^p are permutable i.e.

$$g_{\mu(1)} g_{\mu(2)} \dots g_{\mu(n)} \underbrace{f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(m)}}_{mp} = \underbrace{f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(m)}}_{mp} g_{\mu(1)} g_{\mu(2)} \dots g_{\mu(n)}. \tag{17.22}$$

If $\mu(n) \neq \lambda(1)$ and $\mu(1) \neq \lambda(m)$ then we see immediately from (17.22) that $\mu(1) = \lambda(1)$ and $\mu(n) = \lambda(m)$ and so g is cyclically reduced since $\lambda(1) \neq \lambda(m)$ (Lemma 16.3). If $\mu(n) \neq \lambda(1)$ and $\mu(1) = \lambda(m)$ then

$$\boldsymbol{\lambda}(g_{\mu(1)} g_{\mu(2)} \dots g_{\mu(n)} f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(m)}) = n + pm > \boldsymbol{\lambda}(f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(m)} g_{\mu(1)} \dots g_{\mu(n)}),$$

which is incompatible with (17.22) and so this case does not arise. Finally if $\mu(n) = \lambda(1)$ and $\mu(1) = \lambda(m)$ then we have, once more, g cyclically reduced. Hence we must always have

$$g^p = g_{\mu(1)} g_{\mu(2)} \dots g_{\mu(n)} g_{\mu(1)} \dots g_{\mu(n)},$$

which is the normal form of g^p; and so it follows from (17.21) that f and g are in fact identical. This completes the proof.

It is clear that the free product of more than one nontrivial E_ω-group is not an E_ω-group since extraction of pth roots is not always possible. One may ask whether or not certain regular products of U_ω-groups are again U_ω-groups and, in particular, whether such regular products when applied to D_ω-groups result in D_ω-groups.

LEMMA 17.3. *The second nilpotent product F of U_ω-groups $F_\lambda (\lambda \in \Lambda)$ is a U_ω-group if and only if $[F_\lambda]$ is w-free.*

Proof. If $[F_\lambda]$ contains elements of order p then F is not a U_ω-group. Suppose then that this is not the case. Let $p \in \omega$ and let $f, g \in F$ be such that

$$f^p = g^p. \tag{17.31}$$

Let
$$f = f_{\lambda(1)} f_{\lambda(2)} \dots f_{\lambda(n)} u, \quad \lambda(1) < \lambda(2) < \dots < \lambda(n), \quad u \in nm([F_\lambda]) \tag{17.32}$$

$$g = g_{\mu(1)} g_{\mu(2)} \dots g_{\mu(m)} v, \quad \mu(1) < \mu(2) < \dots < \mu(m), \quad v \in nm([F_\lambda]) \tag{17.33}$$

be the regular representations for f and g respectively, where $f_{\lambda(i)} \in F_{\lambda(i)}$ and $g_{\mu(j)} \in F_{\mu(j)}$. In the second nilpotent product $[F_\lambda]$ lies in the centre (cf. Golovin [12]) and so in this case $nm([F_\lambda]) = [F_\lambda]$. Thus we can easily deduce the regular representations for f^p and g^p from those of f and g (cf. Golovin [12]):

$$f^p = f^p_{\lambda(1)} f^p_{\lambda(2)} \dots f^p_{\lambda(n)} \cdot u^p \cdot \Big(\prod_{1 \leq i < j \leq n} [f_{\lambda(i)}, f_{\lambda(j)}] \Big)^{\frac{1}{2}p(p-1)} \tag{17.34}$$

$$g^p = g^p_{\mu(1)} g^p_{\mu(2)} \dots g^p_{\mu(m)} \cdot v^p \cdot \Big(\prod_{1 \leq i < j \leq m} [g_{\mu(i)}, g_{\mu(j)}] \Big)^{\frac{1}{2}p(p+1)}. \tag{17.35}$$

Now by (17.31) f^p and g^p are equal; hence the regular representations (17.34) and (17.35) coincide. Thus $m = n$ and, in particular

$$f^p_{\lambda(1)} = g^p_{\mu(1)}, \; \dots, \; f^p_{\lambda(m)} = g^p_{\mu(m)}.$$

But each F_λ is a U_ω-group and therefore we have

$$f_{\lambda(1)} = g_{\mu(1)}, \quad f_{\lambda(2)} = g_{\mu(2)}, \; \dots, \; f_{\mu(m)} = g_{\mu(m)}.$$

It follows thence that $$u^p = v^p.$$

Now u and v both lie in the centre of F; therefore $(uv^{-1})^p = 1$, i.e. $u = v$ and so (cf. 17.32) and (17.33)) $f = g$. This completes the proof.

We give now an example of a second nilpotent product of U_ω-groups which is not a U_ω-group. We take

$$G = \mathrm{gp}(a, b, c;\ [a, b] = c^2,\ [a, c] = [b, c] = 1),$$

$$H = \mathrm{gp}(d, e, f;\ [d, e] = f^2,\ [d, f] = [e, f] = 1).$$

It is easy to show that G and H are U_2-groups. But F, the second nilpotent product of G and H, is not a U_2-group. For let

$$\hat{F} = G^* H.$$

Then $$F = \hat{F}/[\hat{F}, [G, H]].$$

Now a commutator in G always commutes with every element $h \in H$:

$$[x^{-1}y^{-1}xy, h] = [x, h]^{-1}[y, h]^{-1}[x, h][y, h] = 1\ (x, y \in G).$$

So $$[c, d]^2 = [c^2, d] = 1,$$

although $$[c, d] \neq 1.$$

THEOREM 17.4. *The second nilpotent product F of E_ω-groups F_λ $(\lambda \in \Lambda)$ is an E_ω-group.*

Proof. We have only to prove the existence of pth roots in F for every $p \in \omega$. Let then $g \in F$ and let

$$g = g_{\lambda(1)}\, g_{\lambda(2)} \ldots g_{\lambda(m)}\, u, \quad \lambda(1) < \lambda(2) < \cdots < \lambda(m), \quad u \in [F_\lambda].$$

Now every simple cross-commutator has a pth root; for

$$[f_\lambda \pi, f_\mu]^p = [(f_\lambda \pi)^p, f_\mu] = [f_\lambda, f_\mu]$$

since $[F_\lambda] \leqslant \zeta(F)$. Consequently $[F_\lambda]$ is itself an E_ω-group. Choose $u_1 \in [F_\lambda]$ so that

$$u_1^p = u \cdot [(\prod_{1 \leqslant i < j \leqslant m} [g_{\lambda(j)}\pi, g_{\lambda(i)}\pi]^{\frac{1}{2}p(p-1)})]^{-1}.$$

Then it follows that

$$g_0 = g_{\lambda(1)}\pi \cdot g_{\lambda(2)}\pi \cdots g_{\lambda(m)}\pi \cdot u_1$$

is a pth root of g. This completes the proof of the theorem.

COROLLARY 17.5. *The second nilpotent product F of locally nilpotent D_ω-groups F_λ is a D_ω-group.*

Proof. It follows from a theorem of F. W. Levi that if F is the second nilpotent product of its subgroups F_λ and if the factor group of each F_λ by its commutator subgroup is ω-free then so is $[F_\lambda]$. Thus it follows in this case, from Lemma 17.3, that F is a U_ω-group since the derived group of a locally nilpotent D_ω-group is an ω-subgroup (Corollary 15.5); furthermore F is, by Theorem 17.4, an E_ω-group and so the corollary follows.

18. The wreath product W of two groups A and B (cf. e.g. Hall [13], Kaloujnine & Krasner [16]) is a useful method of constructing examples. It can be defined as follows: Let K denote the direct product of groups A_b which are isomorphic copies of A indexed by the elements of B:

$$K = \prod_{b \in B} A_b;$$

put

$$W = \mathrm{gp}(K, B;\ b_1^{-1} a_b b_1 = a_{bb_1},\ b, b_1 \in B,\ a \in A).$$

Then W is called the *wreath product* of A by B.

We shall prove that the wreath product of a U_ω-group by a U_ω-group is a U_ω-group; further connections between roots in groups and wreath products are found in Baumslag [3] and [4].

THEOREM 18.1. *The wreath product W of a U_ω-group A by a U_ω-group B is a U_ω-group.*

Proof. Suppose $p \in \omega$, $g_1, g_2 \in W$ and

$$g_1^p = g_2^p. \tag{18.11}$$

Now

$$W = \mathrm{gp}(K = \prod_{b \in B} A_b,\ B;\ b_1^{-1} a_b b_1 = a_{bb_1}),$$

and so we can write $g_1 = b_1 k_1$, $g_2 = b_2 k_2$, where $k_1, k_2 \in K$ and $b_1, b_2 \in B$. Making use of these expressions and equation (18.11) we see that

$$g_1^p = b_1^p k_1^u = b_2^p k_2^v = g_2^p, \tag{18.12}$$

where

$$u = b_1^{p-1} + b_1^{p-2} + \cdots + b_1 + 1,\ v = b_2^{p-1} + b_2^{p-2} + \cdots + b_2 + 1.$$

Hence

$$b_1^p = b_2^p.$$

But B is a U_ω-group and hence $b_1 = b_2 = b$ (say).

Thus $$g_1 = b k_1,\ g_2 = b k_2.$$

Now if b is of finite order then g_1 and g_2 have finite order modulo K; therefore we can apply Lemma 11.4 to deduce that $g_1 = g_2$. We may therefore concern ourselves only with the case where b is of infinite order.

Suppose now that $g_1 \neq g_2$; then $k_1 \neq k_2$ and hence

$$k_2^{-1} k_1 \neq 1. \tag{18.13}$$

Furthermore, it follows from (18.12) that we can write

$$g_2^p = b^p k^*, \quad k^* \in K.$$

Now $g_1^p = g_2^p$ and hence g_2^p commutes with g_1; it follows that g_2^p commutes with $g_2^{-1} g_1 = k_2^{-1} k_1$. Thus

$$k_2^{-1} k_1 = g_2^p (k_2^{-1} k_1) g_2^{-p} = b^p k^* (k_2^{-1} k_1) k^{*-1} b^{-p} = b^p (k^* (k_2^{-1} k_1) k^{*-1}) b^{-p}. \tag{18.14}$$

Let now $k_2^{-1} k_1$ have a non-trivial component in the groups $A_{b_1}, A_{b_2}, \ldots, A_{b_n} (b_i \in B)$, and trivial components in all the other groups A_b; it follows from (18.13) that $k_2^{-1} k_1$ has non-trivial components and, furthermore, that $k^* (k_2^{-1} k_1) k^{*-1}$ has non-trivial components in exactly the same groups $A_{b_1}, A_{b_2}, \ldots, A_{b_n}$ and trivial components otherwise. Therefore, by (18.14), b^p transforms, by right multiplication, the set of suffixes $b_1, b_2, \ldots, b_n$ into itself. More precisely

$$\{b_1 b^p, b_2 b^p, \ldots, b_n b^p\} = \{b_1, b_2, \ldots, b_n\}. \tag{18.15}$$

Consequently $$b_1 b^p,\ b_1 b^{2p}, \ldots, b_1 b^{(n+1)p}$$

must all belong to the set

$$\{b_1, b_2, \ldots, b_n\}.$$

But no two of these elements $b_1 b^{ip}$, $b_1 b^{jp}$ $(i \neq j)$ coincide since b is of infinite order. Thus our assumption (18.13) was not a valid one and so in fact $k_2^{-1} k_1 = 1$, i.e. $k_1 = k_2$. Hence

$$g_1 = b k_1 = b k_2 = g_2,$$

and this completes the proof of the theorem.

19. One can define, analogously to the wreath product, certain allied products. It turns out that if one starts with U_ω-groups then these products give rise to U_ω-groups. However, we shall not consider them here but prefer to discuss them in a separate work.

CHAPTER IV

Extensions of U_ω-groups, E_ω-groups and D_ω-groups

20. In *11* we we proved that an extension of a U_ω-group by a periodic U_ω-group is a U_ω-group, and that an extension of a D_ω-group by a periodic D_ω-group is again a D_ω-group. These results on extensions lead to a number of related questions: Is an extension of a U_ω-group by an arbitrary U_ω-group a U_ω-group? Is an extension of an E_ω-group by an arbitrary E_ω-group an E_ω-group? Is an extension of a D_ω-group by an arbitrary D_ω-group a D_ω-group? All of these questions have negative answers and appropriate counter-examples will be given in this chapter.

The main results in this chapter are concerned with special kinds of extensions. Thus we prove that if G is locally nilpotent and contains a normal subgroup A, which is itself a D_ω-group, and, furthermore, if G/A is a D_ω-group, then G is a D_ω-group. The corresponding theorem is, however, not true for E_ω-groups (see Baumslag [4]). A similar, although unrelated, result for E_ω-groups is the following: Let A be an E_ω-group and suppose A is a ZA-group, and let B be a periodic D_ω-group. Then every extension of A by B is an E_ω-group.

21. We begin by considering central extensions.

LEMMA 21.1. *A central extension G of a U_ω-group A by a U_ω-group B is a U_ω-group.*

Proof. Let $g,\ h \in G$, let $p \in \omega$ and suppose

$$g^p = h^p. \tag{21.11}$$

Making use of the isomorphism between G/A and B it follows that g and h are equal modulo A, i.e.

$$q = ha, \quad a \in A. \tag{21.12}$$

Now $a \in \zeta(G)$ and so by (21.12) we have

$$g^p = h^p a^p. \tag{21.13}$$

Comparing the right-hand sides of (21.11) and (21.13) we see that $a^p = 1$. But since $A \in U_\omega$, $a = 1$, and hence $g = h$; this completes the proof of the lemma.

We prove now the following theorem for E_ω-groups.

THEOREM 21.2. *A central extension G of an E_ω-group A by an E_ω-group B is an E_ω-group.*

Proof. Suppose $p \in \omega$. The isomorphism $G/A \cong B$ enables every $g \in G$ to be written in the form

$$g = h^p a \quad (h \in G, a \in A).$$

Now $A \in E_\omega$ and so we can find $a_0 \in A$ such that $a_0^p = a$; hence

$$g = h^p a_0^p = (h a_0)^p.$$

Thus extraction of pth roots is always possible in G; this completes the proof of the theorem.

COROLLARY 21.3. *A central extension of a D_ω-group by a D_ω-group is a D_ω-group.*

22. It is not true that every extension of a U_ω-group by a U_ω-group is a U_ω-group. An example of an extension G of a U_ω-group A by a U_ω-group B, which is not a U_ω-group will now be given.

Let B be a multiplicatively written group which is isomorphic to the additive groups of dyadic rationals:

$$B = \mathrm{gp}(b_1, b_2, \ldots; b_{i+1}^2 = b_i, i = 1, 2, \ldots).$$

Let A denote the unrestricted direct product of $|B|$ copies of B indexed by the elements of B. Then A is a σ-group since B is a σ-group (Lemma 17.1). It is now possible to define an extension G of A by B by defining transformation of an element $\bar{a}$ of A by an element $b' \in B$ by defining the coordinate of $b'^{-1} a b'$ in the group $A_{bb'}$, to be $a_{bb'}$, if a_b is the coordinate of $\bar{a}$ in A_b. This enables us to define the group G generated by A and B in this way:

$$G = \mathrm{gp}(A, B\colon \quad b' a_b b' = a_{bb'}, \quad a, b, b' \in B);$$

G is called the unrestricted wreath product (Hall [13]) of B by B. Now both A and B are, of course, U_2-groups, A is normal in G and

$$G/A \cong B.$$

We choose now an element $b (\neq 1)$ in B and consider the element $a^* \in A$ whose component in B_{b^n} is $a^{(-1)^n}$, where a is some element of B different from 1, and whose components in all other $B_{b'}$ $(b' \in B)$ is the identity. But $g = ba^*$. Now it can be verified that

$$a^{*b} a^* = 1;$$

consequently $$g^2 = b^2 a^{*b} a^*, = b^2,$$

but $g = ba^* \neq b$ since $a^* \neq 1$ and therefore square roots are not unique in G. Thus this example shows also that an extension of a D_ω-group by a D_ω-group is not always a D_ω-group.

In the group G above every element has at least one square root (see Baumslag [4]); however it is possible to make extensions of σ-groups by σ-groups in which extraction of square roots is unique but not always possible. This is accomplished by making use of the wreath product. Explicitly the procedure is as follows:

Let B be a multiplicatively written group which is isomorphic to the additive group of dyadic rationals (as in the first example above). Put

$$A = \prod_{b \in B} B_b$$

and define

$$G = \mathrm{gp}\,(A,\ B;\ b'^{-1} a_b\, b' = a_{bb'},\ a,\ b,\ b' \in B);$$

G is, as we saw earlier, the wreath product of A by B. Hence it is, by Theorem 18.1, a U_2-group. Moreover, $G/A \cong B$ and so G is an extension of a σ-group A by a σ-group B. However, the element g in G:

$$g = b^2 a', \quad a' \neq 1,\ a' \in B_1,$$

does not have a square root in G. For suppose that $G \ni h$ and $h^2 = g$. Then h must be of the form

$$h = ba^*,\ b \in B,\ a^* \in A,$$

and hence

$$a^{*b} a^* = a'; \tag{22.1}$$

Now a^* can itself be written in the form

$$a^* = a\,(1)_{b(1)}\, a\,(2)_{b(2)} \ldots a\,(n)_{b(n)}, \quad a\,(i),\ b\,(i) \in B, \quad b\,(i) \neq b\,(j) \quad \text{if} \quad i \neq j; \tag{22.2}$$

this representation is unique. Now, by (22.1)

$$(a\,(1)_{b(1)b}\, a\,(2)_{b(2)b} \ldots a\,(n)_{b(n)b})\,(a\,(1)_{b(1)}\, a\,(2)_{b(2)} \ldots a\,(n)_{b(n)}) = a'. \tag{22.3}$$

At least one of the lower suffixes involved in the left-hand side of (22.3) must be the identity. Suppose $b(1) = 1$. Then it follows that some other $b(i)$ say $b(2)$ must coincide with b. Further, it follows also, possibly after renaming, that for $i = 1, 2, \ldots$

$$b\,(i) = b^{i-1}. \tag{22.4}$$

Therefore b^n is not equal to any $b(i)$ for $i = 1, 2, \ldots$, since b is of infinite order. Hence the equation (22.3) yields, using the commutativity of B,

$$a(1) = a', \; a(2)a(3) = 1, \; \ldots \;, a(n-1)a(n) = 1, \; a(n) = 1. \tag{22.5}$$

It follows by (22.5) that $a(1) = 1$ and hence also $a' = 1$. We have here assumed that $b(1) = 1$; there is a second possibility, namely $bb(1) = 1$; it follows similarly in this case that $a' = 1$. We are therefore always led to the same conclusion, namely that a' is the identity. However, this contradicts the choice of a' at the outset and we can only conclude that g does not have a square root in G.

We may as well make the remark here that the first example given in *22* is an example of a metabelian group in which no element has order 2 but it is not a U_2-group; this shows that Theorem 13.6 cannot be generalised to include even soluble groups.

23. We shall need the following lemma.

LEMMA 23.1. *Let* $p \in \omega$, let $H \in U_\omega$ *and let* $K \in D_\omega$. *Furthermore, suppose* H *is nilpotent and that* K *is a normal subgroup of* H. *Then*

$$h = g^p a \quad (g \in H, \, a \in K)$$

has a p-th root in H.

Proof. Put

$$K_i = K \cap \zeta^i(H).$$

There is a least integer d satisfying $K_d = K$. The proof that h has a pth root will be by induction on d. Suppose firstly that $d = 1$. Then $K \leqslant \zeta(G)$ and so if a_0 is the pth root of a then

$$h = g^p a_0^p = (g a_0)^p.$$

Therefore we have the first step of a proof by induction. Consider next H/K_1. Now K_1 is the intersection of two ω-subgroups of H, since, on the one hand, $\zeta(H)$ is an ω-subgroup by Corollary 13.4, and, on the other hand, K is an ω-subgroup since it belongs to D_ω and H belongs to U_ω; consequently, by Lemma 9.1, K_1 is an ω-subgroup of H and therefore also an ω-subgroup of K. Thus K_1 must be a D_ω-group. We apply the induction hypothesis to H/K_1 and deduce thereby that hK_1 has a pth root. Hence we can write

$$h = f^p a_1 \quad (f \in H, \, a_1 \in K_1).$$

But $K_1 \in D_\omega$ and therefore a_1 has a pth root a_2, say, which belongs also to K_1. Since $K_1 \leqslant \zeta(H)$ we therefore have

$$h = f^p a_2^p = (f a_2)^p$$

and this completes the proof of the lemma.

We can now prove the following "extension theorem" for locally nilpotent D_ω-groups.

THEOREM 23.2. *Let G be a locally nilpotent group and let A be a normal subgroup of G. Suppose that both A and G/A belong to D_ω. Then G belongs to D_ω.*

Proof. Since A and G/A belong to U_ω they are both ω-free (Lemma 11.3). Thus the locally nilpotent group G is ω-free; consequently G belongs to U_ω (Theorem 13.6). It remains to prove that G belongs to E_ω.

Suppose then that $p \in \omega$, $h \in G$. Now modulo A the element h has a pth root and so we can write

$$h = g^p a \quad (g \in G, a \in A). \tag{23.21}$$

Put

$$H = \mathrm{cl}_\omega(\{g, a\}, G).$$

By Theorem 15.1 H is nilpotent since it is the ω-closure of a finitely generated (and hence nilpotent) subgroup of a locally nilpotent U_ω-group G. Let J be the normal closure in H of the group generated by a and put

$$K = \mathrm{cl}_\omega(J, H).$$

Then, by Corollary 9.4, K is normal in H. Furthermore, K is a ω-subgroup of A (it is clearly a subgroup of A since $A \lhd G$); for if $c \in A$ and $c^p \in K$, then $c \in H$ as H is a ω-subgroup of G; and so $c \in K$, since K is a ω-subgroup of H. Thus we have a nilpotent group H in U_ω with a normal subgroup K which belongs to D_ω. Consequently the element h of equation (23.21) is the product of a pth power in H and an element in K:

$$h = g^p a \quad (g \in H, a \in K);$$

we are therefore entitled to apply Lemma 23.1—so h has a pth root and the proof of the theorem is complete.

24. We shall now prove that an extension of a ZA-group in the class E_ω by a periodic D_ω-group is an E_ω-group. We need the following lemma.

LEMMA 24.1(1). *Let H be any group, let K be an abelian normal subgroup of H. Suppose*

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(1) This lemma is due to the referee. It greatly simplifies my original proof of Theorem 24.2.

that $g \in H$, that $g^m \in K$ and that g^m has a p-th root in K. Then there exists $b \in K$ such that

$$[b, g] = 1 \text{ and } b^p = g^{m^2}.$$

Proof. By hypothesis there exists an $a \in K$ such that

$$g^m = a^p. \tag{24.11}$$

Put
$$b = a \cdot a^g \cdot a^{g^2} \dots a^{g^{m-1}}.$$

Now K is an abelian normal subgroup of H, so $b \in K$ and

$$b^g = a^g \cdot a^{g^2} \dots a^{g^{m-1}} \cdot a = a \cdot a^g \cdot a^{g^2} \dots a^{g^{m-1}} = b;$$

thus $[b, g] = 1$, as required. Furthermore,

$$b^p = a^p \cdot a^{pg} \dots a^{pg^{m-1}} = a^{pm}, \tag{24.12}$$

since by (24.11) g commutes with a^p. But by (24.11) $g^m = a^p$ and so by (24.12)

$$b^p = g^{m^2}$$

and this completes the proof of the lemma.

Suppose now that A is a ZA-group in the class E_ω. Then $\zeta(A)$ is an ω-ideal of A (Corollary 14.4). Consequently if $x, y \in A$, $p \in \omega$ and $x^p = y^p$, then

$$(x\zeta(A))^p = (y\zeta(A))^p$$

is an equation in the D_ω-group $A/\zeta(A)$ and so

$$x\zeta(A) = y\zeta(A).$$

Thus x and y differ by an element in $\zeta(A)$ and consequently are permutable. So we have proved

LEMMA 24.2. *If A is a ZA-group in the class E_ω and if*

$$x^p = y^p \quad (x, y \in A),$$

where $p \in \omega$, then x and y commute.

We can now proceed to the proof of the following theorem.

THEOREM 24.3. *An extension G of a ZA-group A in the class E_ω by a periodic D_ω-group is an E_ω-group.*

Proof. Let p be any prime in ω and let $g \in G$. Then there exists a positive integer m prime to p such that

$$g^m \in A.$$

Since A is an E_ω-group g^m has a pth root c, say, in A:

$$g^m = c^p.$$

Put
$$K = \mathrm{gp}(c, c^g, c^{g^2}, \dots, c^{g^{m-1}}).$$

Since any pair of the given generators of K have equal pth powers, they commute (Lemma 24.2). Therefore K is abelian. Now put

$$H = \mathrm{gp}(g, K).$$

Then it follows from the definition of K that K is a normal subgroup of H. Furthermore, g^m has a pth root in K. We can now apply Lemma 24.1; thus there exists $b \in K$ such that

$$[b, g] = 1,\ b^p = g^{m^2}. \tag{24.31}$$

Since m and p are coprime there exist λ, μ such that

$$\lambda m^2 + \mu p = 1.$$

Hence, using (24.31),
$$g = g^{\lambda m^2 + \mu p} = b^{\lambda p} g^{\mu p} = (b^\lambda g^\mu)^p.$$

So g has a pth root and this completes the proof of the theorem.

PART II

CHAPTER V

The abstract D_ω-free group

25. In this chapter the D_ω-free group is defined abstractly and some consequences of this definition are derived. We prove that the factor group of a D_ω-free group by its commutator ideal (i.e. the ω-closure of its commutator subgroup) is a direct product of groups isomorphic to Γ_ω, where Γ_ω is the additive group of those rationals whose denominators are products of primes in ω only. Analogous notions to those occurring in the theory of free groups are also defined here for D_ω-free groups.

We say that a D_ω-group F is ω-generated by a set X, or, alternatively, X ω-generates F, if $\mathrm{cl}_\omega(X) = F$. Then a set X is called a free ω-generating set of a D_ω-group F if i) X ω-generates F and ii) for every D_ω-group H and every mapping θ of X into H there exists a homomorphism φ of F into H that coincides with θ on X. A D_ω-group F is called a D_ω-free group if it is freely ω-generated by some set X; the existence of such groups (cf. *3*) is ensured by a theorem on abstract algebra due to Birkhoff [5].

It is well known that the factor group of a free group of rank n by its commutator subgroup is a direct product of n infinite cyclic groups (cf. e.g. Kurosh [21]). We prove an analogous result for D_ω-free groups.

THEOREM 25.1. *Let F be a D_ω-free group which is freely ω-generated by the set X. Then the factor group of F by its commutator ideal is a direct product of $|X|$ groups isomorphic to Γ_ω.*

Proof. Let H be a direct product of $|X|$ copies of Γ_ω:

$$H = \prod_{x\in X} D_x;$$

here D is a multiplicative group isomorphic to Γ_ω and $D_x \cong D$. Let θ be a mapping of X into H defined as follows:

$$x\theta = d_x \quad (1 \neq d \in D,\ x \in X,\ \mathrm{cl}_\omega(\{d\}, D) = D).$$

This mapping θ can be extended to a homomorphism φ of F into H since X is a free ω-generating set of F. Now it can easily be verified that $\{d_x \mid x \in X\}$ is a ω-generating set of H and thus φ is in fact an epimorphism. Hence it follows that if N is the kernel of φ then

$$F/N \cong H.$$

Now by Lemma 13.7, $\mathrm{cl}_\omega(F')$ is a w-ideal of F and therefore

$$N \geqslant \mathrm{cl}_\omega(F') \tag{25.11}$$

since any ω-ideal containing F' obviously also contains $\mathrm{cl}_\omega(F')$.

Consider on the other hand the factor group $F/\mathrm{cl}_\omega(F')$; $F/\mathrm{cl}_\omega(F')$ is an abelian D_ω-group which is ω-generated by the set $\{x\,\mathrm{cl}_\omega(F')\}_{x\in X}$. We remark that the pth root of the product of two elements in $F/\mathrm{cl}_\omega(F')$ is the product of the pth roots of each of the elements (Corollary 10.4). Therefore we may write any element $f \in N$ (and, of course, any element of F) in the form

$$f = y_{i(1)}^{m_1} y_{i(2)}^{m_2} \cdots y_{i(k)}^{m_k} \cdot f' \quad (f' \in \mathrm{cl}_\omega(F')), \tag{25.12}$$

where $y_{i(j)}$ is the n_jth root of $x_{i(j)} \in X$, with n_j a product of primes in ω. Then on applying φ to f wee see that

$$f\varphi = (y_{i(1)}^{m_1}\varphi)(y_{i(2)}^{m_2}\varphi) \cdots (y_{i(k)}^{m_k}\varphi) = 1, \tag{25.13}$$

since, by (25.11), $\mathrm{cl}_\omega(F') \leqslant N$. The elements

$$y_{i(1)}\varphi,\ y_{i(2)}\varphi,\ \ldots,\ y_{i(k)}\varphi$$

lie in distinct factors D_x of the direct product H since that is true of

$$d_{xi(1)},\ d_{xi(2)},\ \ldots,\ d_{xi(k)}.$$

Thus (25.13) can hold only if the exponents m_i are all zero:

$$m_1 = m_2 \cdots = m_k = 0.$$

Consequently
$$f = f' \in \mathrm{cl}_\omega(F')$$
and so we have proved
$$N \leqslant \mathrm{cl}_\omega(F'). \tag{25.14}$$

Putting the inequalities (25.11) and (25.14) together we see

$$N = \mathrm{cl}_\omega(F'),$$

and this completes the proof of the theorem.

It is not true that the factor group of a D_ω-free group F by its commutator subgroup F' is a direct product of groups isomorphic to Γ_ω; the exact structure of F/F' will be determined later in 37 (Theorem 37.3). Consequently Theorem 25.1 is a 'best' analogue of the corresponding result for free groups.

It is convenient to investigate abelian D_ω-groups by considering the corresponding modules over a principal ideal ring (for these concepts, cf. eg. Jacobson [15]). Let us take R to be that subring of the ring of rational fractions consisting of those rationals whose denominators are products of primes in ω only. Then an abelian D_ω-group corresponds to an R-module in the natural way. Explicitly, if A is an abelian D_ω-group, $m/n \in R$ and $a \in A$, then we define

$$a^{m/n} = b,$$

if

$$b^n = a^m;$$

this definition is not ambiguous since b is uniquely defined because pth roots are unique in A for all $p \in \omega$. It is easy to verify that this definition turns A into an R-module. An

R-module is termed *free* if it is a direct sum of copies of R. It can then be verified that the number of copies of R that go into the formation of a free R-module M is an invariant of the isomorphism class determined by M (see Kaplansky [17]). Thus we can define unambiguously, the *ω-rank* of a D_ω-free group to be the cardinal of any one of its free ω-generating sets; for, by the preceding remark and Theorem 25.1, this cardinal is uniquely determined by the given D_ω-free group.

The ω-rank of a D_ω-free group uniquely determines the isomorphism class to which this D_ω-free group belongs, i.e. two D_ω-free groups are isomorphic if, and only if, they have the same ω-rank. This result is a particular case of a theorem due to Birkhoff (cf. e.g. Słomiński [35]) on abstract algebras. We shall, however, prove it here, in a group theoretical way, by making use of Theorem 25.1.

THEOREM 25.2. *Two D_ω-free groups are isomorphic if, and only if, they have the same ω-rank m.*

Proof. Suppose F and F^* are isomorphic D_ω-free-groups. Then there is a natural isomorphism between $F/\mathrm{cl}_\omega(F')$ and $F^*/\mathrm{cl}_\omega(F^*)'$ induced by the isomorphism between F and F^*. It follows from Theorem 25.1 and the remark as to the number of copies of R in a free R-module that F and F^* have the same ω-rank.

Conversely if F and F^* are D_ω-free groups of the same ω-rank, then they have free ω-generating sets X and X^* of the same cardinality. Thus there is a one-to-one mapping η of X onto X^*; this mapping can now be extended to a homomorphism θ of F into F^*. We note that θ is necessarily an epimorphism since $F\theta \geqslant X^*$.

Now let φ denote the epimorphism of F^* to F that extends the mapping η^{-1} of X^* onto X.

Consider the composite mapping $\theta\varphi$; this mapping is an automorphism of F. For in the first place $\theta\varphi$ maps X identically onto X; therefore, as X ω-generates F, $\theta\varphi$ must in fact be the identity automorphism of F. It follows that θ is a one-to-one epimorphism of F to F^*, i.e. θ is an isomorphism and this completes the proof.

LEMMA 25.3. *Let F and F^* be isomorphic D_ω-free groups and let θ be an isomorphism of F onto F^*. Then if X is a free ω-generating set of F, $X\theta$ is a free ω-generating set of F^*.*

Proof. We note firstly that $X\theta$ ω-generates F^*; for

$$F^* = F\theta = (\mathrm{cl}_\omega(X))\theta = \mathrm{cl}_\omega(X\theta).$$

We note secondly that for every D_ω-group K and every mapping η' of $X\theta$ into K there is a homomorphism φ' of F^* into K that coincides with η' on $X\theta$. To begin with η'

leads naturally to a mapping η of X into K defined by

$$x\eta = (x\theta)\eta' \, (x \in X).$$

This mapping can then be extended to a homomorphism of F into K, say φ, that now induces a homomorphism φ' of F^* into K defined by

$$f^*\varphi' = (f^*\theta^{-1})\varphi.$$

This homomorphism φ' extends η' since

$$(x\theta)\varphi' = x\varphi = x\eta = (x\theta)\eta';$$

consequently $X\theta$ is a free ω-generating set of F^* and this completes the proof of the lemma.

Another property of a free ω-generating set X of a D_ω-free group F is that if $x \in X$, then $x \notin \mathrm{cl}_\omega(X - x)$; this follows immediately from Theorem 25.1. This property of X is an instance of a more generally definable concept. Explicitly, a subset X of an arbitrary D_ω-group F is called *ω-independent*, if, for all $x \in X$, $x \notin \mathrm{cl}_\omega(X - x)$. The remark at the beginning of this paragraph states then that a free ω-generating set of a D_ω-group is ω-independent.

Chapter VI

The fundamental embedding theorem

26. This chapter is concerned with a particular class of groups $\mathfrak{D}_\omega$, corresponding to a given non-empty set of primes ω. This class contains, in particular, all D_ω-groups. It contains also U_ω-groups which are not D_ω-groups. For such groups there are elements in them without pth roots, for some $p \in \omega$. The fundamental embedding theorem proved in this chapter enables us to embed such U_ω-groups (in the class $\mathfrak{D}_\omega$) in D_ω-groups, which belong also to the class $\mathfrak{D}_\omega$.

27. We define the class $\mathfrak{D}_\omega$ by stipulating that its members are those groups G which satisfy the following 4 conditions:

27.1. G is a U_ω-group.

27.2. If $p \in \omega$ and if the element g in G has no pth root, then the centraliser $C(g, G)$ of g in G is isomorphic to a subgroup of Γ_ω.

27.3. If $p \in \omega$ and if the element g in G has no pth root, then the centraliser of any non-trivial power of g coincides with the centraliser of g.

27.4. If $p \in \omega$, if the element g in G has no pth root and if $h^{-1}g^m h = g^n$, for some $h \in G$ and some integers m and n, then $m = n$.

Thus a group G belongs to the class $\mathfrak{D}_\omega$ if, and only if, it satisfies the four conditions above. We see therefore that all D_ω-groups belong to $\mathfrak{D}_\omega$. It is convenient to note that if $p \in \omega$, if $g \in G \in \mathfrak{D}_\omega$, and g has no pth root, then, by 27.2, g has infinite order. In this case the centraliser $\mathrm{C}(g)$ is isomorphic to a *proper* subgroup of Γ_ω.

For groups in $\mathfrak{D}_\omega$ the following lemma holds; we shall make use of this lemma in the sequel.

LEMMA 27.5. *Let G be a group in the class $\mathfrak{D}_\omega$ and let g be an element in G which does not have a p-th root for some $p \in \omega$. Let $a \in \mathrm{C}(g)$ and suppose $h^{-1}ah \in \mathrm{C}(g)$ for some $h \in G$. Then $h^{-1}ah = a$ and $h \in \mathrm{C}(g)$.*

Proof. Put
$$h^{-1}ah = a', \quad a' \in \mathrm{C}(g). \tag{27.51}$$

Now $\mathrm{C}(g)$ is isomorphic to a subgroup of Γ_ω and therefore it is locally cyclic; in other words, every finitely generated subgroup of $\mathrm{C}(g)$ is cyclic. Thus we can find $b \in \mathrm{C}(g)$ such that

$$\mathrm{gp}(a, a', g) = \mathrm{gp}(b).$$

This means that we can write

$$a = b^k, \quad a' = b^m, \quad g = b^n, \tag{27.52}$$

where k, m, n, are integers. Now g does not have a pth root and so b does not have one either; hence condition 27.4 applies to b. Making use of (27.52) we can rewrite (27.51) in the form

$$h^{-1}b^k n = b^m.$$

We see then by 27.4 that $k = m$ and so h commutes with $b^k = a$. Furthermore, by 27.3,

$$\mathrm{C}(a) = \mathrm{C}(b^k) = \mathrm{C}(b) = \mathrm{C}(b^n) = \mathrm{C}(g),$$

and so $h \in \mathrm{C}(g)$. This completes the proof of the lemma.

28. Let now G be a group in the class $\mathfrak{D}_\omega$. We are interested in embedding G in a D_ω-group. Thus if G is already a D_ω-group we have no further interest in G. Let us assume, therefore, that G is not a D_ω-group. Then there exists a prime $p \in \omega$ and an element $g \in G$ which does not have a pth root in G. Hence, by 27.2, $\mathrm{C}(g, G)$ is isomorphic to a proper sub-

group of Γ_ω. We take then P to be a supergroup[1] of $C(g, G)$ which is isomorphic to Γ_ω and whose intersection with G is simply $C(g, G)$. We then form the generalised free product F of G and P (with $C(g, G)$ amalgamated):

$$F = \{G*P;\ C(g, G)\}.$$

Then g has a pth root in F since it has a pth root already in P. We shall prove that in fact g has a unique pth root in F, and, even more, that F belongs also to the class $\mathfrak{D}_\omega$. We remark that the generalised free product of two locally infinite groups is locally infinite (Corollary 16.5); thus if G is locally infinite then so also is F.

The proof that F belongs to $\mathfrak{D}_\omega$ will be accomplished with the aid of a number of lemmas. It is this string of lemmas that we now prove.

LEMMA 28.1. *For every* $a \in P$

$$C(a, F) = P.$$

Proof. It is clear that
$$C(a, F) \geqslant P. \tag{28.11}$$

We want the reverse inequality. Let $b \in C(a, F)$ and suppose b is written in the form

$$b = c_1 c_2 \dots c_n, \tag{28.12}$$

where no two successive factors c_i, c_{i+1} are elements of the same constituent.[2] Put $Z = C(g, G)$. Now $b \in C(a, F)$; therefore utilising (28.12),

$$a^b = c_n^{-1} c_{n-1} \dots c_2^{-1} c_1^{-1} a c_1 c_2 \dots c_{n-1} c_n = a. \tag{28.13}$$

Suppose, firstly, that $a \in P - Z$ and $c_1 \in G$. If $n > 1$, $c_2 \in P - Z$ since no two successive factors c_i, c_{i+1} lie in the same constituent. This is however impossible for then $\lambda(a^b) = 2n + 1$, by Lemma 16.1; this contradicts equation (28.13) and so $n \leqslant 1$, i.e. $n = 1$. If $c_1 \in G - Z$, then $\lambda(a^b) = 3$ but $\lambda(a)$ is either 0 or 1, again a contradiction. Hence the only possibility is that $c_1 \in P$ and thus $b = c_1 \in P$.

Suppose, secondly, that $a \in P - Z$ and $c_1 \in P$. If $n > 1$ then $c_2 \in G - Z$. Consequently $\lambda(a^b) = 2n - 1$ and since $n > 1$ this contradicts (28.13). Therefore $n = 1$ and again $b = c_1 \in P$.

Suppose, thirdly, that $a \in Z$ and $c_1 \in G$. If $n > 1$, then by Lemma 16.1, $c_1^{-1} a c_1$ must lie in Z. It follows then by Lemma 27.5 that $c_1 \in Z \leqslant P$; but $c_2 \in P$ and so we have a contradiction

(1) We say A is a supergroup of B if A contains a distinguished isomorphic copy of B; note that B may be a subgroup of A.

(2) By a constituent we mean either G or P.

since c_1 and c_2 were assumed to lie in different constituents. Consequently $n=1$ and $b=c_1 \in P$.

Finally suppose $a \in Z$ and $c_1 \in P-Z$. Then if $n>1$, $c_2 \notin P$ and so $\boldsymbol{\lambda}(a^b) \neq 0$, contradicting (28.13). Hence $n=1$ and $b=c_1 \in P$.

The four paragraphs above have shown that if an element in F commutes with an element in P then it must belong to P. Therefore

$$\mathrm{C}(a, F) \leqslant P. \tag{28.14}$$

Putting (28.11) and (28.14) together then yields $\mathrm{C}(a, F)=P$ and this is just what is required.

LEMMA 28.2. *Let $a \in G$ and suppose a is not conjugate to an element in Z. Then*

$$\mathrm{C}(a, F)=\mathrm{C}(a, G).$$

Proof. Let $b \in \mathrm{C}(a, F)$ be written in the form

$$b=c_1 c_2 \ldots c_n,$$

where no two successive factors c_i, c_{i+1} belong to the same constituent.

Suppose $c_1 \in P$. If $n>1$, then $c_1 \in P-Z$. Consequently, by Lemma 16.1. $\boldsymbol{\lambda}(a^b)+1=\boldsymbol{\lambda}(a)$, a contradiction. Hence $n=1$ and so $b=c_1 \in C(a, G)$.

On the other hand, suppose $c_1 \notin P$, i.e. suppose $c_1 \in G-Z$. If $n>1$, then $c_2 \in P-Z$. If the equation (28.13) is to remain valid then c_1 must transform a into an element in $P \cap Z=Z$; however, by hypothesis a is not conjugate to an element in Z and so this case cannot arise. Hence $n=1$ and $b=c_1 \in \mathrm{C}(a, G)$.

The two paragraphs above have shown that $\mathrm{C}(a, F) \leqslant \mathrm{C}(a, G)$; the reverse inequality is obvious and hence the lemma follows.

LEMMA 28.3. *Let $a \in F$ be a cyclically reduced element of length at least two. Then if $b \in \mathrm{C}(a, F)$ $(b \neq 1)$, b is cyclically reduced of length at least two.*

Proof. Let S_1 be a set of representatives of the left cosets of G modulo Z containing the identity, and let S_2 be a set of representatives of the left cosets of P modulo Z containing the identity element.

We write a in normal form (cf. 16):

$$a=s_1 s_2 \ldots s_m z_a. \tag{28.31}$$

We write b also in normal form

$$b = t_1 t_2 \dots t_n z_b. \tag{28.32}$$

Observe that z_a and z_b both belong to Z.

We notice, to begin with, that $b \notin Z$; for if this were the case then $b = z_a$ and $a \in \mathrm{C}(z_b, F) = P$ (by Lemma 28.1) which contradicts the hypothesis of the lemma. Hence $n \geqslant 1$.

We can rewrite the equation $ab = ba$, using (28.31) and (28.32), in the form

$$ab = s_1 s_2 \dots s_m z_a t_1 t_2 \dots t_n z_b = t_1 t_2 \dots t_n z_b s_1 s_2 \dots s_m z_a = ba. \tag{28.33}$$

Now, by hypothesis, a is cyclically reduced of length at least two; consequently s_1 and s_m lie in different constituents (cf. Lemma 16.2), say $s_1 \in G$ and $s_m \in P$. Suppose that both t_1 and t_n lie in the same constituent, say $t_1, t_n \in G$. In this case there will be no cancellation in the left-hand side of equation (28.33) but at least one in the right-hand side. Hence

$$m + n = \boldsymbol{\lambda}(ab) = \boldsymbol{\lambda}(ba) \leqslant m + n - 1.$$

This equation is clearly impossible; hence both t_1 and t_n cannot lie in G. Similarly they cannot both lie in P. It follows that t_1 and t_n belong to different constituents and so (cf. Lemma 16.3) b is a cyclically reduced element of length at least two.

For $s_1 \in P$ and $s_m \in G$, a similar argument again shows that t_1 and t_n lie in different constituents. This completes the proof of the lemma.

Lemma 28.4. *Let $a \in F$ be cyclically reduced, $\boldsymbol{\lambda}(a) = m \geqslant 2$. Let $b \in \mathrm{C}(a, F)$ be such that $\boldsymbol{\lambda}(b) = n \geqslant m$. Put $n = \alpha m + \beta$, $0 \leqslant \beta < m$. Then*

$$\beta = 0 \text{ or } \beta \geqslant 2,$$

and b can be written in the form

$$b = a^{\pm\alpha} b^*,$$

where $$\boldsymbol{\lambda}(b^*) = \beta;$$

furthermore, if $\beta = 0$, then b^* is trivial and if $\beta \neq 0$, then b^* is cyclically reduced of length at least two.

Proof. We shall assume here the notation employed in Lemma 28.3. Let us assume also that $s_1 \in G$ and $s_m \in P$. Now, by Lemma 28.3, b is cyclically reduced of length at least two, hence t_1 and t_n belong to different constituents, say $t_1 \in G$ and $t_n \in P$.

Suppose now that we can write b in the form

$$b = a^{\gamma} \tilde{b}, \quad 0 \leqslant \gamma < \alpha, \quad \tilde{b} \in F. \tag{28.41}$$

Now $\tilde{b}$ clearly commutes with a since b does; hence, by Lemma 28.3, $\tilde{b}$ is cyclically reduced of length at least two. We have also, by (28.41), $\boldsymbol{\lambda}(\tilde{b}) = (\alpha - \gamma)m + \beta$; we write $\tilde{b}$ in normal form

$$\tilde{b} = u_1 u_2 \ldots u_k z_{\tilde{b}}, \quad k = (\alpha - \gamma)m + \beta.$$

It follows from (28.41) that $u_1 \in G$ and $u_k \in P$. Further, the element $\tilde{b}$ commutes with a:

$$a^{\tilde{b}} = z_{\tilde{b}}^{-1} u_k^{-1} \ldots u_2^{-1} u_1^{-1} s_1 s_2 \ldots s_m z_a u_1 u_2 \ldots u_k z_{\tilde{b}} = s_1 s_2 \ldots s_m z_a. \tag{28.42}$$

Now u_1 and s_1 both belong to G. The equation (28.42) thus implies that $u_1^{-1} s_1 \in Z$. For let us suppose the contrary; then, making use of Lemma 16.1,

$$\boldsymbol{\lambda}(a^{\tilde{b}}) = 2k + m - 1;$$

furthermore,
$$\boldsymbol{\lambda}(a) = m,$$

and therefore
$$2k + m - 1 = m.$$

However, such an equation is not possible and so we have a contradiction of the assumption $u_1^{-1} s_1 \notin Z$. Thus $u_1^{-1} s_1 \in Z$ and therefore

$$u_1 = s_1. \tag{28.43}$$

Similarly we can prove

$$u_1 = s_1,\ u_2 = s_2,\ \ldots,\ u_m = s_m. \tag{28.44}$$

Thus utilising (28.44) we see that

$$\tilde{b} = a \cdot z_a^{-1} u_{m+1} \ldots u_k z_{\tilde{b}} = a\hat{b}, \text{ say.}$$

Hence
$$b = a^{\gamma+1}\hat{b},$$

where $\boldsymbol{\lambda}(\hat{b}) = (\alpha - \gamma - 1)m + \beta$. We can apply the same procedure to $\hat{b}$ as that applied to $\tilde{b}$ providing $\boldsymbol{\lambda}(\hat{b}) \leqslant m$; hence it is possible to write b in the form

$$b = a^{+\alpha} b^*, \quad \boldsymbol{\lambda}(b^*) = \beta.$$

The initial assumptions that $s_1 \in G$, $s_m \in P$, $t_1 \in G$, $t_n \in P$ may always be presupposed. It is sufficient for this purpose to note that $b \in \mathrm{C}(a^{-1}, F)$, $b^{-1} \in \mathrm{C}(a, F)$ and $b^{-1} \in \mathrm{C}(a^{-1}, F)$. This remark then completes the proof of the lemma.

LEMMA 28.5. *Let $a \in F$ be cyclically reduced of length at least two. Suppose, further, that $b \in C(a)(b \neq 1)$. Then there is a cyclically reduced element c in F of length at least two, and integers μ and ν such that*

$$a = c^{\mu},\ b = c^{\nu}.$$

Proof. We again adopt the notation of Lemma 28.3. It follows from this lemma that b is cyclically reduced of length at least two. Put $b = b_0$, $a = b_1$. We may assume, without loss of generality, that $\boldsymbol{\lambda}(b_0) \geqslant \boldsymbol{\lambda}(b_1)$. Then, by Lemma 28.4, we can write $b_0 = b_1^{\alpha_1} b_2$, where either $b_2 = 1$ or b_2 is cyclically reduced of length $2 \leqslant \boldsymbol{\lambda}(b_2) < \boldsymbol{\lambda}(b_1)$. Clearly $[b_2, b_1] = 1$ since $[b_1, b_0] = 1$. Thus if $b_2 \neq 1$ we can repeat the process, writing $b_1 = b_2^{\alpha_2} b_3$, where $b_3 = 1$ or $2 \leqslant \boldsymbol{\lambda}(b_3) < \boldsymbol{\lambda}(b_2)$. Proceeding in this way we obtain a set of elements $b_0, b_1, \ldots$ of strictly decreasing length. Consequently the process must stop, i.e. $b_r = 1$ for some r. We now put c equal to the last b in the sequence which is not the identity, say $c = b_s$. Then c is cyclically reduced of length at least 2. Furthermore,

$$b_{s-2} = c^{\alpha_{s-2}},\ b_{s-3} = b_{s-2}^{\alpha_{s-2}} c,\ \ldots,\ b_0 = b_1^{\alpha_1} b_2.$$

Consequently each b_i can be expressed as a product of succeeding $b_k - s$ and so also as a power of c. In particular

$$b = b_0 = c^{\nu} \text{ and } a = b_1 = c^{\mu},$$

and so a and b are powers of a cyclically reduced element of length at least two, as claimed. This completes the proof of the lemma.

LEMMA 28.6. *Suppose that a is neither conjugate to an element in G nor to an element in P. Then $C(a, F)$ is an infinite cyclic group.*

Proof. Since

$$x^{-1} \cdot C(a, F) \cdot x = C(x^{-1} a x, F) \quad (a, x \in F),$$

it is sufficient to prove only that $C(a, F)$ is an infinite cyclic group in the case where a is cyclically reduced. Further, we note that, by hypothesis, a is not conjugate to an element in G or P and so it is of length at least two.

We choose now an element $c^* \in F$ so that

$$a = (c^*)^{\gamma}, \quad \gamma > 0, \tag{28.61}$$

and, further, so that if $a = (c')^{\gamma'}$, $\gamma' > 0$, then $\boldsymbol{\lambda}(c') \geqslant \boldsymbol{\lambda}(c^*)$. Suppose now that $b \in C(a)$; then, by Lemma 28.5, there is a cyclically reduced element $c \in F$, of length at least two, such that

$$a = c^{\mu} \text{ and } b = c^{\nu};$$

thence, by (28.61), $$c^{\mu} = (c^*)^{\nu} \quad (= a). \tag{28.62}$$

Let
$$c \;= u_1 u_2 \ldots u_k z_c, \quad z_c \in Z$$
$$c^* = v_1 v_2 \ldots v_j z_{c^*}, \quad z_{c^*} \in Z$$

be the normal forms of c and c^* respectively. We shall, without any consequent loss af generality, suppose that $u_1 \in G$; it follows then that $v_1 \in G$. The equation (28.62) implies by the choice of c^*, that $\boldsymbol{\lambda}(c) \geqslant \boldsymbol{\lambda}(c^*)$. It follows, on substituting the normal forms of c and c^* in (28.62), that
$$u_1 = v_1, \quad u_2 = v_2, \ldots, u_j = v_j.$$

Therefore $$c = c^* \cdot z_{c^*}^{-1} u_{j+1} \ldots u_k z_c = c^* c', \text{ say}. \tag{28.63}$$

We consider two possibilities:

i) $\boldsymbol{\lambda}(c') = 0$. Both c and c^* commute with a and so their quotient $c^{*-1}c = c'$ also commutes with a; but this is, by Lemma 28.1 only possible if $c' = 1$. Hence $c = c^*$ and $b = (c^*)^{\nu}$.

ii) $\boldsymbol{\lambda}(c') > 0$. It follows from (28.63) that the first component of c' belongs to G and the last component of c' belongs to P; hence c' is cyclically reduced of length at least two. Thus we see, on substituting (28.63) in (28.62) that $(c^* c')^{\mu} = (c^*)^{\nu}$; transforming both sides of this equation by c^* then yields
$$(c' c^*)^{\mu} = (c^*)^{\nu} = (c^* c')^{\mu}. \tag{28.64}$$

Now $c'c^*$ and c^*c' are of the same length; it follows from (28.64) that they can differ only by an element in Z or, more precisely, $(c^*c')^{-1} \cdot (c'c^*) \in Z$. Furthermore, both c^* and c commute with a and thus so also does c'. Therefore both $c'c^*$ and c^*c' commute with a and $(c^*c')^{-1} \cdot (c'c^*)$ also commutes with a; but $(c^*c')^{-1} \cdot (c'c^*) \in Z$, and hence, by Lemma 28.1, $(c^*c')^{-1} \cdot (c'c^*) = 1$; thus $c'c^* = c^*c'$. Thus c' and c^* commute and are also cyclically reduced of length at least two; we can apply Lemma 28.5 to c' and c^*, yielding $c' = f^{\eta}$, $c^* = f^{\zeta}$, $f \in F$, where η and ζ are here integers. However, by the choice of c^*, ζ can only be $+1$ or -1. It follows that $f = (c^*)^{\varepsilon}$, $\varepsilon = \pm 1$, and hence c' is a power of c^*; therefore, by (28.63), c is itself a power of c^* and so b is also a power of c^*.

The two cases i) and ii) show that b is always a power of c^*. But b was an arbitrary element of $\mathrm{C}(a, F)$ and so it follows that $\mathrm{C}(a, F)$ is the infinite cyclic group generated by c^*; this completes the proof of the lemma.

We need, finally, the following lemma:

LEMMA 28.7. *Let $p \in \omega$ and let $a \in F$ not have a p-th root in F. Let m and n be integers and let $b \in F$ be such that*

$$b^{-1} a^m b = a^n. \tag{28.71}$$

Then $$m = n.$$

Proof. Since $x^{-1}bx$ transforms $(x^{-1}ax)^m$ into $(x^{-1}ax)^n$, we can assume that a is cyclically reduced. Furthermore, since G is a group in the class $\mathcal{D}_\omega$ it follows that a is in fact of infinite order (cf. Theorem 16.4). Thus if $m = 0$ then $n = 0$ and the lemma holds. Let us suppose then that $m > 0$.

Let the normal form for b be

$$b = r_1 r_2 \dots r_\varrho z_b \quad (z_b \in Z). \tag{28.72}$$

Now as a does not have a pth root, $a \notin P$ and so $\boldsymbol{\lambda}(a) = \mu \geqslant 1$. The proof that $m = n$ is split into a number of cases.

Suppose firstly that $\mu = 1$, i.e. suppose $a \in G$. Then $a^m \notin Z$; for if this were the case then condition *27.3* would apply and so a itself would belong to Z, contradicting $\mu \geqslant 1$. Now $\varrho = \boldsymbol{\lambda}(b)$ cannot exceed three. For suppose the contrary. Since $a^m \in G - Z$, r_1 must belong to G (this must be the case whenever $\varrho > 1$). It follows from the equation

$$(a^m)^b = z_b^{-1} r_\varrho^{-1} \dots r_2^{-1} r_1^{-1} a^m r_1 r_2 \dots r_\varrho z_b = a^n \tag{28.78}$$

that $$a^* = r_1^{-1} a^m r_1 \in Z. \tag{28.74}$$

Then, since r_2 leaves a^* fixed, r_3 must transform a^* into some other element of Z. This is, however, only possible if $r_3 \in Z$, which is not the case since $\varrho > 3$. Thus in fact $\varrho \leqslant 3$. Consider now the case $\varrho = 3$. Then it follows, just as in the analysis above, that

$$(r_1 r_3)^{-1} a^m (r_1 r_3) = a^n;$$

this equation involves only elements of G, and so $m = n$ since $G \in \mathcal{D}_\omega$ and a does not have a pth root in G. If $\varrho = 2$ then, by (28.74), $a^n \in Z$, which is not the case because $a \notin Z$; so this case does not arise. Finally if $\varrho = 1$, then $m = n$ since $G \in \mathcal{D}_\omega$ and condition 27.4 applies; note that if $\varrho = 0$, i.e. if $b \in Z$, then $m = n$ follows once more from the fact that $G \in \mathcal{D}_\omega$.

We consider secondly the case $\mu > 1$. Thus as a is cyclically reduced of length at least two, it follows that a^m and a^n are also cyclically reduced of lengths $m\mu$ and $n\mu$ respectively. Let

$$\left.\begin{aligned} a^m &= s_1 s_2 \ldots s_{m\mu} z_1 \quad (z_1 \in Z) \\ a^n &= t_1 t_2 \ldots t_{n\mu} z_2 \quad (z_2 \in Z) \end{aligned}\right\} \tag{28.75}$$

be the normal forms for a^m and a^n. Since s_1 and $s_{m\mu}$ belong to different constituents and

$$b^{-1}(a^{-1})^m b = (a^{-1})^n$$

we lose nothing by supposing that $s_1 \in G$ and $s_{m\mu} \in P$. Consequently $t_1 \in G$ and $t_{n\mu} \in P$. We rewrite equation (28.71) in the more convenient form

$$a^m b = s_1 s_2 \ldots s_{m\mu} z_1 r_1 r_2 \ldots r_\varrho z_b = r_1 r_2 \ldots r_\varrho z_b t_1 t_2 \ldots t_{n\mu} z_2 = b a^n. \tag{28.76}$$

If $\varrho = 0$, i.e. if $b \in Z$, then it follows from (28.76) that

$$m\mu = \boldsymbol{\lambda}(a^m b) = \boldsymbol{\lambda}(b a^n) = n\mu;$$

hence $m = n$.

Suppose now that $\varrho \geqslant 1$. Consider firstly the case $r_1 \in G$, $r_\varrho \in G$. Now $s_{m\mu} \in P$ and $t_1 \in G$; thus it follows by (28.76) that

$$m\mu + \varrho = \boldsymbol{\lambda}(a^m b) = \boldsymbol{\lambda}(b a^n) \leqslant n\mu + \varrho - 1. \tag{28.77}$$

The equation (28.76) can also be expressed in the form

$$b^{-1} a^m = z_b^{-1} r_\varrho^{-1} \ldots r_2^{-1} r_1^{-1} s_1 s_2 \ldots s_{m\mu} z_1 = t_1 t_2 \ldots t_{n\mu} z_2 z_b^{-1} r_\varrho^{-1} \ldots r_2^{-1} r_1^{-1} \tag{28.78}$$

Here $s_1 \in G$, $t_{n\mu} \in P$ and so we have

$$m\mu + \varrho - 1 \geqslant \boldsymbol{\lambda}(b^{-1} a^m) = \boldsymbol{\lambda}(a^n b^{-1}) = n\mu + \varrho. \tag{28.79}$$

It follows from (28.77) and (28.79) that

$$m\mu + \varrho \geqslant n\mu + \varrho + 1 \geqslant m\mu + \varrho + 2;$$

this equation is impossible and so the case $r_1 \in G$, $r_\varrho \in G$ does not arise. In a similar way one can show that the case $r_1 \in P$, $r_\varrho \in P$ also does not occur. We are left with the remaining possibilities $r_1 \in G$, $r_\varrho \in P$ and $r_1 \in P$, $r_\varrho \in G$—the procedure in both cases is similar and so we shall only deal with the first of these two cases here. We inspect equation (28.76):

$$m\mu + \varrho = \boldsymbol{\lambda}(a^m b) = \boldsymbol{\lambda}(b a^n) = n\mu + \varrho.$$

Thus we see that $m\mu = n\mu$ and hence $m = n$. This completes the proof of the lemma.

29. This chapter is brought to a close with the proof of the "fundamental embedding theorem" mentioned in 26.

THEOREM 29.1. *Let G be a group in the class $\mathfrak{D}_\omega$ and let p be a prime in ω such that the element g in G does not have a p-th root. Let P be a supergroup of $C(g, G)$ which is isomorphic to Γ_ω and which intersects G in $C(g, G)$. Then the generalised free product F of G and P belongs also to the class $\mathfrak{D}_\omega$.*

Proof. We have to show that F satisfies the four conditions 27.1, 27.2, 27.3 and 27.4. Let us consider firstly the condition 27.1. Let $p \in \omega$, $a, b \in F$ and suppose

$$a^p = b^p. \tag{29.11}$$

We may assume that a is cyclically reduced. Then the proof that $a = b$ falls naturally into three parts:

i) $a \in P$. Here $b \in C(a, F) = P$ (cf. Lemma 28.1) and since P is a U_ω-group and both a and b lie in P, the equation (29.11) implies that $a = b$.

ii) $a \in G - Z$. We note that a^p is not conjugate to an element in Z. For suppose, on the contrary, that $a^p = x^{-1}zx$, where $x \in F$ and $z \in Z$. Then, by Lemma 28.1,

$$C(a^p, F) = x^{-1}Px;$$

now $a \in C(a^p, F)$ and so can be written in the form

$$a = x^{-1}z_0x,\ z_0 \in P. \tag{29.12}$$

Now $\lambda(z_0) = 1$ since a is cyclically reduced, by hypothesis, of length one. It follows directly that this is not possible and, consequently, that a^p is not conjugate to an element in Z. Now b commutes with a^p and therefore we have, using Lemma 28.2,

$$b \in C(a^p, F) = C(a^p, G);$$

so $b \in G$ and thus the equation (29.11) is an equation in the U_ω-group G. Therefore $a = b$.

iii) $a \notin G \cup P$. In this case a^p is cyclically reduced of length at least two; hence its centraliser, which contains both a and b, is an infinite cyclic group (cf. Lemma 28.6) and therefore $a = b$. The cases i), ii) and iii) take care of all possibilities and so F is a U_ω-group and 27.1 is therefore satisfied.

Secondly we consider condition 27.2. Let us suppose $a \in F$ does not have a pth root for some $p \in \omega$. We may assume that a is cyclically reduced. If $a \in G$, then a is not conjugate to an element in Z and so $C(a, F) = C(a, G)$ cf. Lemma 28.2) is isomorphic to a subgroup of Γ_ω. The other possibility is $a \notin G$; in this case a is cyclically reduced of length at least two and so $C(a, F)$ is an infinite cyclic group (cf. Lemma 28.6) which is isomorphic to a subgroup of Γ_ω.

Thirdly we consider 27.3. Let $p \in \omega$ and let $a \in F$ be cyclically reduced and not have a pth root in F; let m be a non-zero integer. We consider the following possibilities:
i) $a \in G - Z$. Now a is not conjugate to an element in Z and so neither is a^m. Thus making use of Lemma 28.2 we see that

$$C(a, F) = C(a, G) = C(a^m, G) = C(a^m, F).$$

ii) $a \notin G \cup Z$. Then a^m is cyclically reduced of length at least two and so $C(a^m, F)$ is an infinite cyclic group (cf. Lemma 28.6). Furthermore $a \in C(a^m, F)$, and since $C(a^m, F)$ is abelian, $C(a^m, F) \leqslant C(a, F)$. The reverse inequality is obvious and so $C(a, F) = C(a^m, F)$. The above three cases exhaust all possibilities and so F satisfies 27.3.

We consider, finally, 27.4: Suppose $p \in \omega$, suppose m and n are integers and that $a \in F$ has no pth root. Suppose also that $b \in F$ is such that

$$b^{-1} a^m b = a^n.$$

Then Lemma 28.7 applies and so $m = n$. This completes the proof of the theorem.

CHAPTER VII

Embedding of U_ω-groups in D_ω-groups

30. It is not true that every U_ω-group can be embedded in a D_ω-group (The counterexample is due to B. H. Neumann — see Baumslag [3].) However, it is certainly true for some U_ω-groups. In fact we shall make use here of Theorem 29.1 to develop a constructive process for embedding a given group G, which belongs to the class $\mathfrak{D}_\omega$, in a D_ω-group G^*, which also belongs to the class $\mathfrak{D}_\omega$. If the constructive process is carried out in a certain way the group G^* turns out to be the "freest" D_ω-group ω-generated by G; in other words, for every D_ω-group H and every homomorphism φ of G into H there exists a homomorphism φ^* of G^* into H which coincides with φ on G.

31. The union of an ascending sequence of groups in $\mathfrak{D}_\omega$, under certain conditions, is also in $\mathfrak{D}_\omega$. In particular we have the following result.

LEMMA 31.1. *Let G_α be given groups in the class $\mathfrak{D}_\omega$, where α ranges over an ordered index set $\mathcal{A}$. Let $G_\alpha \leqslant G_\beta$ whenever $\alpha < \beta\,(\alpha, \beta \in \mathcal{A})$; suppose further that if $a \in G_\alpha$ $(a \neq 1)$ and $\mathrm{C}(a, G_\beta) > \mathrm{C}(a, G_\alpha)$, then $\mathrm{C}(a, G_\beta)$ is isomorphic to Γ_ω. Then G^*, the union of the groups G_α, also belongs to the class $\mathfrak{D}_\omega$.*

Proof. We have to show that G^* satisfies the conditions 27.1, 27.2, 27.3 and 27.4.

We begin with 27.1, i.e. we show that $G^* \in U_\omega$. Thus suppose that $p \in \omega$, that $a, b \in G^*$ and that $a^p = b^p$. Now there is an α in $\mathcal{A}$ such that both a and b belong to G_α. Since $G_\alpha \in \mathfrak{D}_\omega$ we have $a = b$.

Now suppose for the remainder of this lemma that g is an element of G^* which does not have a pth root in $G^*\,(p \in \omega)$.

We prove next that G^* satisfies 27.2. Now $g \in G_\alpha$ for some α in $\mathcal{A}$. Since $G_\alpha \in \mathfrak{D}_\omega$ it follows that $\mathrm{C}(g, G_\alpha)$ is isomorphic to a proper subgroup of Γ_ω. Now if $\mathrm{C}(g, G^*) = \mathrm{C}(g, G_\alpha)$ then obviously $\mathrm{C}(g, G^*)$ is itself isomorphic to a subgroup of Γ_ω. Alternatively $\mathrm{C}(g, G^*) > \mathrm{C}(g, G_\alpha)$. Thus we can find $\beta \in \mathcal{A}$ such that $\mathrm{C}(g, G_\beta) > \mathrm{C}(g, G_\alpha)$. Therefore by hypothesis $\mathrm{C}(g, G_\beta) \cong \Gamma_\omega$ and so g has a pth root in G_β and hence also a pth root in G^*, contrary to our initial supposition.

Next we prove that G^* satisfies the condition 27.3. Let m be a non-zero integer. It is obvious that $\mathrm{C}(g^m, G^*) \geqslant \mathrm{C}(g, G^*)$. We show that the reverse inequality also holds. Suppose $x \in \mathrm{C}(g^m, G^*)$. Then we can choose $\alpha \in \mathcal{A}$ so that $x, g \in G_\alpha$. Since $G_\alpha \in \mathfrak{D}_\omega$ it follows ttha $x \in \mathrm{C}(g, G_\alpha)$ and hence $x \in \mathrm{C}(g, G^*)$. Thus $\mathrm{C}(g^m, G^*) \leqslant C(g, G^*)$ and so $\mathrm{C}(g^m, G^*) = \mathrm{C}(g, G^*)$.

Finally we prove that G^* satisfies 27.4. Suppose that m and n are arbitrary integers, that $h \in G^*$ and that $h^{-1}g^m h = g^n$. Then g, $h \in G_\alpha$ for some $\alpha \in \mathcal{A}$ and hence $m = n$ since $G^\alpha \in \mathfrak{D}_\omega$. This completes the proof of the lemma.

Let now G be a group in the class $\mathfrak{D}_\omega$ and suppose that G is not a D_ω-group. Then there exists a prime p in ω and an element g in G which does not have a pth root. We can embed G in a group $\hat{G}$, which is also in $\mathfrak{D}_\omega$, in such a way that g now has a pth root in $\hat{G}$ (e.g. by Theorem 29.1). Even more is possible, namely:

LEMMA 31.2. *Every group G in $\mathfrak{D}_\omega$ can be embedded in a group $\hat{G}$ in $\mathfrak{D}_\omega$ such that every element of G has in $\hat{G}$ p-th roots for every p in ω.*

Proof. We proceed by the classical "tower" argument. If G is a D_ω-group then we may take $\hat{G} = G$. Hence we assume that G is not a D_ω-group and thus infinite. Let the elements of G be well-ordered by the relation $<$, the successor of the element g being denoted by g^+, the predecessor of g, if it exists, being denoted by g^-, the first element of the well-order being the unit element e,(1) and the well-order being so chosen that there is no last element. We put $G = G_e$, and define inductively, supposing that G_g is a group in $\mathcal{D}_\omega$, its successor G_{g^+} as a group in $\mathcal{D}_\omega$ in which g has a pth root for every p in ω. Specifically, if g already has all these roots in G_g, we choose $G_{g^+} = G_g$. If, on the other hand, g fails to have a pth root for some $p \in \omega$, then the centraliser $\mathrm{C}(g, G_g)$ in G_g is isomorphic to a subgroup of Γ_ω; we now choose P to be a supergroup of $\mathrm{C}(g, G_g)$ which is isomorphic to Γ_ω and which intersects G_g in $\mathrm{C}(g, G_g)$. We then define

$$G_{g^+} = \{G_g * P;\ \mathrm{C}(g, G_g)\};$$

we observe that, by Theorem 29.1, $G_{g^+} \in \mathcal{D}_\omega$. If g is an element without a predecessor in the well-order, we define

$$G_g = \bigcup_{h<g} G_h.$$

This is legitimate because the groups form a well-ordered chain by inclusion (see, e.g., Kurosh [21] vol. 1, p. 226). Now G_g is a group in the class $\mathcal{D}_\omega$ because the groups G_h $(h < g)$ satisfy the conditions of Lemma 31.1—this follows by applying the following lemma.

LEMMA 31.3. *Let G_α be given groups in $\mathcal{D}_\omega$, where α ranges over a well-ordered index set $\mathcal{A}$. Suppose that $G_\alpha \leqslant G_\beta$ whenever $\alpha < \beta$ $(\alpha, \beta \in \mathcal{A})$ and suppose that if α does not have a predecessor then*

$$G_\alpha = \bigcup_{\beta<\alpha} G_\beta \ (\alpha, \beta \in \mathcal{A}).$$

Furthermore, suppose that if α does have a predecessor β, then either

$$G_\alpha = G_\beta$$

or

$$G_\alpha = \{G_\beta * P;\ \mathrm{C}(g, G_\beta)\},$$

where g is an element of G_β which does not have root in G_β, and P is a supergroup of $\mathrm{C}(g, G_\beta)$ isomorphic to Γ_ω intersecting G_β in $\mathrm{C}(g, G_\beta)$. Suppose now that a in G_γ is such that for some $\delta \in \mathcal{A}$, $\mathrm{C}(a, G_\delta) > \mathrm{C}(a, G_\gamma)$. Then $\mathrm{C}(a, G_\delta)$ is isomorphic to Γ_ω.

(1) We find it convenient to introduce the symbol e for the unit element, which will, as has been done, also be denoted by 1.

Proof. Let γ^* ($\geqslant \gamma$) be the first member of $\mathcal{A}$ for which

$$C(a, G_\gamma) \neq C(a, G_{\gamma^+}). \tag{31.31}$$

Clearly γ^* has a predecessor γ^- and so, by hypothesis,

$$G_{\gamma^*} = \{G_{\gamma^-} * P;\ C(g, G_{\gamma^-})\},$$

where g is some element of G_{γ^-} which does not have a pth root in G_{γ^-} for some p in ω, and P is an isomorphic copy of Γ_ω intersecting G_{γ^-} in $C(g, G_{\gamma^-})$. By Lemmas 28.1, 28.2, 28.6 and equation (31.31) it follows that a is conjugate to an element in P and that its centraliser in G_{γ^*} is therefore conjugate to P and so is isomorphic to Γ_ω. It follows now from the hypothesis of this lemma and Lemmas 28.1, 28.2 that

$$C(a, G_\delta) = C(a, G_{\gamma^*})$$

for all δ in $\mathcal{A}$ which follow γ^*. This completes the proof of the lemma.

We now continue with the proof of Lemma 31.2 by defining

$$\hat{G} = \bigcup_{g \in G} G_g.$$

Here again Lemma 31.3 can be applied and hence also Lemma 31.1. Therefore $G \in \mathfrak{D}_\omega$. Furthermore, every element in G has pth roots in $\hat{G}$ for every $p \in \omega$ because, by our choice of well-order g has a successor g^+ in it and, at the latest in G_{g^+}, it then has the requisite roots. This completes the proof of the lemma.

We now apply Lemma 31.2 to prove the following theorem.

THEOREM 31.4. *Every group G in the class $\mathfrak{D}_\omega$ can be embedded in a group G^*, also in $\mathfrak{D}_\omega$, in which all elements have a p-th root for every $p \in \omega$; in other words G^* is a D_ω-group.*

Proof. We put $G_0 = G$ and, inductively, define $G_{i+1} = \hat{G}_i$ (cf. Lemma 31.2). Then each $G_i \in \mathfrak{D}_\omega$ and the G_i-s form, in their natural order, an ascending sequence. We put

$$G^* = \bigcup_{i=0}^{\infty} G_i.$$

Here Lemma 31.3 applies and so we can make use of Lemma 31.1. Hence G^* belongs to Γ_ω. Furthermore, every element $g \in G^*$ belongs to G_i for some i, and hence, for every $p \in \omega$, has a pth root in G_{i+1}. Consequently G^* is a D_ω-group and this completes the proof of the theorem.

32. We have shown in Theorem 31.3 that to every group G in $\mathfrak{D}_\omega$ there is a supergroup G^* of G which is simultaneously a D_ω-group and a member of $\mathfrak{D}_\omega$. We wish to construct a supergroup G^* of G in such a way that G^* turns out to be the freest D_ω-group which is ω-generated by G. This can be done by making slightly more refined usage of Theorem 29.1. The procedure is similar to that described in the proof of Theorem 31.4. Care is needed when adjoining pth roots to ensure, at each stage, and ultimately at G^*, that the freest possible D_ω-group is obtained. We will need certain facts concerning Γ_ω. In our dealings with Γ_ω we shall assume that whenever we postulate $m/n \in \Gamma_\omega$, the integers m and n are coprime.

LEMMA 32.1. *Let A be a given subgroup of Γ_ω. Then there exists a subgroup B of Γ_ω containing the integer 1 such that B is isomorphic to A.*

Proof. Let γ be the greatest common divisor of the integers in A. Now these integers generate a cyclic subgroup of A which, by the definition of γ, must in fact be generated by γ. Thus

$$\gamma \in A.$$

Suppose now that $a \in A$; then a is of the form $a = m/n$ and $(m, n) = 1$. Furthermore, since γ divides m, $(m, n\gamma) = \gamma$. Therefore we can find two integers α and β such that

$$\alpha m + \beta n\gamma = \gamma.$$

Now, using the additive notation for Γ_ω, we see that

$$A \ni \alpha\left(\frac{m}{n}\right) + \beta\gamma = \frac{\alpha m + \beta n\gamma}{n} = \frac{\gamma}{n}.$$

Thus we have shown that if $a = m/n \in A$, then γ divides m and that $a' = \gamma/n \in A$. Consequently A is generated by rationals of the form $a' = \gamma/n$, where n is a product of primes in ω, and so it is isomorphic to that subgroup B of Γ_ω generated by corresponding rationals $a = 1/n$. This group B clearly contains the integer 1 and so we have proved the lemma.

We prove next

LEMMA 32.2. *Let B be a subgroup of Γ_ω containing the integer* 1. *Then for every D_ω-group H and every homomorphism θ of B into H there exists a homomorphism φ of Γ_ω into D which coincides with θ on B.*

Proof. We put

$$C = \mathrm{cl}_\omega(1\theta, H).$$

Then C is an abelian subgroup of H (by Theorem 15.1). We have already remarked (in *25*) that if R is that subring of the ring of rational numbers which consists of those rationals whose denominators are products of primes in ω only, then we can regard abelian D_ω-groups as R-modules in the natural way. In particular Γ_ω will be a free R-module on a single generator, which we may take, for convenience, to be the integer 1. Hence the mapping

$$1 \to 1\theta$$

can be extended to a homomorphism φ of Γ_ω into C, and thus to a homomorphism of Γ_ω into H.

It follows now, from its definition, that the mapping φ extends θ. For if r/s is an arbitrary element of Γ_ω and if $(r/s)\varphi = c$, then (writing C additively) we have

$$sc = r\varphi = r(1\varphi) = r(1\theta);$$

since s is a product of primes in ω only, division in G by s (qua integer, of course) is uniquely possible and therefore

$$c = \frac{r(1\theta)}{s}.$$

Now if $r/s \in B$ and if we put $r\theta/s = d$, then a similar procedure to that carried out above yields

$$d = \frac{r(1\theta)}{s} = c.$$

This then completes the proof of the lemma.
Finally we combine Lemma 32.1 and Lemma 32.2 to prove

LEMMA 32.3. *Let A be a group which is isomorphic to a subgroup of Γ_ω. Then A can be embedded in a group P which is isomorphic to Γ_ω in such a way that for every D_ω-group H and every homomorphism θ of A into H there exists a homomorphism φ of P into H which coincides with θ on A.*

Proof. We can choose, by Lemma 32.1, an isomorphic copy B of A in Γ_ω such that B contains the integer 1. We take then P to be a supergroup of A so that there is an isomorphism of P onto Γ_ω which maps A onto B. Now by Lemma 32.2 every homomorphism θ of B into a D_ω-group H can be extended to a homomorphism $\hat{\varphi}$ of Γ_ω into H. It follows then from the definition of P that for every D_ω-group H and every homomorphism θ of A into H there exists a homomorphism φ of P into H which coincides with θ on A. This completes the proof of the lemma.

Not every supergroup P of A which is isomorphic to Γ_ω has the property that for every D_ω-group H and every homomorphism θ of A into H there exists a homomorphism φ of P into H which coincides with θ on A. For example we can take

$$P = \Gamma_{\{2\}},\ A = \mathrm{gp}(3;\ 3 \in \Gamma_{\{2\}}),\ H = \mathrm{gp}(h;\ 3h = 0)$$

(we are again using the additive notation for groups). Now $\Gamma_{\{2\}}$ and H are both D_2-groups. We define θ to be the homomorphism of A into H that takes 3 into h. Then θ cannot be extended to a homomorphism φ of P into H. For if φ is any homomorphism of P into H, then

$$3\varphi = 3(1\varphi) = 0;$$

but $3\theta = h \neq 0$. We see therefore that the condition demanded in Lemma 32.2 is in fact a necessary one.

33. We are now in a position to define a free ω-closure of a group G in $\mathfrak{D}_\omega$. The procedure is similar to that involved in the construction of the group G^* in the proof of Theorem 31.4. Precisely, we proceed as follows: If G is a D_ω-group then we put $G^* = G$. If, on the other hand, G is not a D_ω-group then it is infinite. Put $G_0 = G$. We define a supergroup G_{i+1} of G_1, which we assume is in $\mathfrak{D}_\omega$, by induction. Let the elements of G_i be well-ordered by the relation $<$, the successor of $g \in G_i$ being denoted by g^+, the predecessor of g, if it exists, being denoted by g^-, the first element being the unit element e, and the well-order being so chosen that there is no last element. We put $G_i = G_{i,e}$ and define a supergroup G_{i,g^+} of the group $G_{i,g}$ in $\mathfrak{D}_\omega$ inductively as follows. If g already has a pth root in $G_{i,g}$ for every p in ω, then we define $G_{i,g^+} = G_{i,g}$. If, on the other hand, g fails to have a pth root in $G_{i,g}$ for some p in ω, then its centraliser A in $G_{i,g}$ is necessarily isomorphic to a subgroup of Γ_ω. We can choose now, for example by Lemma 32.3, P to be a supergroup of A which is isomorphic to Γ_ω and such that for every D_ω-group H and every homomorphism θ of A into H there exists a homomorphism φ of P into H which coincides with θ on A; and further, such that P intersects $G_{i,g}$ in A. We form now the generalised free product G_{i,g^+} of $G_{i,g}$ and P. If g is an element without a predecessor in the well-order we define

$$G_{i,g} = \bigcup_{h<g} G_{i,h}.$$

Finally we put

$$G_{i+1} = \bigcup_{g \in G_i} G_{i,g}.$$

We now complete the definition of a free ω-closure G^* of G by defining

$$G^* = \bigcup_{i=0}^{\infty} G_i. \tag{33.1}$$

Then (cf. Lemma 31.1, Lemma 31.2, Lemma 31.3 and Theorem 31.4) it follows that G^* is a supergroup of G which is simultaneously a D_ω-group and a member of $\mathcal{D}_\omega$.

Note that a free ω-closure G^* of G is not uniquely defined, a certain amount of freedom being allowed in the well-orderings involved and also in the choice of the groups P. However, we shall show that a free ω-closure is unique in the sense that two free ω-closures of the same group are isomorphic.

It is convenient to have at hand an expression, other than (33.1), for a free ω-closure G^* of a group G in the class $\mathcal{D}_\omega$ as the union of an ascending sequence of groups in $\mathcal{D}_\omega$. Now, by (33.1),

$$G^* = \bigcup_{i=0}^{\infty} \Big(\bigcup_{g \in G_{i,e}}^{\infty} G_{i,g} \Big). \tag{33.2}$$

Let $\mathcal{A}$ be the set of all those suffixes (i, g) which index the groups $G_{i,g}$ that go into the construction of G^* (cf. (33.2)). We make $\mathcal{A}$ into a well-ordered index set in the natural way by defining an order relation $<$ in it as follows: We put

$$(i, g) < (j, h)$$

if either $i < j$, or $i = j$ and $g < h$; it follows that the first element of $\mathcal{A}$ is $(0, e) = \iota$ (say), that the succesoor α^+ of $\alpha = (i, g)$ is $\alpha^+ = (i, g^+)$, and that the predecessor of α, if it exists, is $\alpha^- = (i, g^-)$ (note the construction of G^*). We define now, for $\alpha = (i, g) \in \mathcal{A}$,

$$G_\alpha = G_{i,g};$$

then G^* is the union of a set of groups G_α in $\mathcal{D}_\omega$, indexed by a well-ordered set $\mathcal{A}$:

$$G^* = \bigcup_{\alpha \in \mathcal{A}} G_\alpha. \tag{33.3}$$

We shall make frequent use of this expression (33.3) in our dealings with a free ω-closure G^* of $G \in \mathcal{D}_\omega$.

We prove now the "freeness" property of a free ω-closure G^* of a group G in the class $\mathcal{D}_\omega$.

THEOREM 33.4. *Let G^* be a free ω-closure of a group G in the class $\mathcal{D}_\omega$. Then for every D_ω-group H and every homomorphism φ of G into H there exists a homomorphism φ^* of G^* into H which coincides with φ on G.*

Proof. We shall define φ^* by transfinite induction. First let us write G^* in the form given by equation (33.3):

$$G^* = \bigcup_{\alpha \in \mathcal{A}} G_\alpha.$$

Then we can start the induction by defining

$$\varphi_\iota = \varphi.$$

Suppose then that homomorphisms φ_α of G_α into H have been defined for all $\alpha < \beta\,(\alpha, \beta \in \mathcal{A})$ in such a way that φ_α is continued by $\varphi_{\alpha'}$ if $\alpha \leqslant \alpha'$. If β does not have a predecessor then $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ and we define φ_β to be the union of the homomorphisms φ_α for all $\alpha < \beta$. Specifically we define the action of φ_β on G_β as follows: If $g \in G_\beta$ then $g \in G_\alpha$ for some $\alpha < \beta$. We define then

$$g\varphi_\beta = g\varphi_\alpha;$$

this defines the effect of φ_β on g unambiguously since φ_α is continued by $\varphi_{\alpha'}$ if $\alpha < \alpha'$. It is clear that φ_β is a homomorphism of G_β into H. If β^- exists then there are two possibilities: (i) $G_\beta = G_{\beta^-}$. In this case we define φ_β to be φ_{β^-}. (ii) $G_\beta = \{G_{\beta^-} * P;\, A\}$—here P is a supergroup of A (which is the centraliser of some element in G_{β^-} which does not have a pth root in G_{β^-}) which is isomorphic to Γ_ω (cf. the definition of (33.3)). We have in this case already a homomorphism θ of A into H induced by the homomorphism φ_{β^-} of G_{β^-} into H. Now by the construction of G^* every homomorphism of A into a D_ω-group can be extended to a homomorphism of P into that D_ω-group. So, in particular, we can extend θ to a homomorphism θ' of P into H. Now φ_{β^-} and θ' agree on $P \cap G_{\beta^-} = A$ and so, by the definition of the generalised free product (see 16) they can be extended simultaneously to a homomorphism φ_β of G_β into H. We can now complete the definition of the homomorphisms φ_α for all $\alpha \in \mathcal{A}$ by transfinite induction. We then define φ^* to be the union of the homomorphisms φ_α:

$$\varphi^* = \bigcup_{\alpha \in \mathcal{A}} \varphi_\alpha.$$

This completes the proof of the theorem.

This theorem now enables us to prove that two free ω-closures of a group $G \in \mathfrak{D}_\omega$ are isomorphic. But first we need the following lemma.

LEMMA 33.5. *A free ω-closure G^* of a group G in the class $\mathfrak{D}_\omega$ is ω-generated by G.*

Proof. We have firstly the obvious inequality $G^* \geqslant \mathrm{cl}_\omega(G)$. It remains to prove the reverse inequality. The proof is by transfinite induction. We make use of equation (33.3) and begin the induction by noting that $G_\iota = G$ and hence $G_\iota \leqslant \mathrm{cl}_\omega(G)$. Let us suppose that for all $\alpha < \beta$ $G_\alpha \leqslant \mathrm{cl}_\omega(G)$. If β does not have a predecessor, then

$$G_\beta = \bigcup_{\alpha<\beta} G_\alpha,$$

and hence $G_\beta \leqslant \mathrm{cl}_\omega(G)$. Suppose, on the other hand, that β^- exists; then either $G_\beta = G_{\beta^-}$ or $G_\beta = \{G_{\beta^-} * P\,;\, A\}$. In the first case $G_\beta \leqslant \mathrm{cl}_\omega(G)$ since $G_{\beta^-} \leqslant \mathrm{cl}_\omega(G)$; in the second case we have $P \leqslant \mathrm{cl}_\omega(G)$ since $A \leqslant \mathrm{cl}_\omega(G)$ and also $G_{\beta^-} \leqslant \mathrm{cl}_\omega(G)$, by the induction hypothesis; hence $G_\beta \leqslant \mathrm{cl}_\omega(G)$. It follows now by transfinite induction that for all $a \in \mathcal{A}$; $G_\alpha \leqslant \mathrm{cl}_\omega(G)$. Hence

$$G^* = \bigcup_{\alpha\in\mathcal{A}} G_\alpha \leqslant \mathrm{cl}_\omega(G).$$

Thus $G^* = \mathrm{cl}_\omega(G)$ and so the proof of the lemma is complete.

We can now deduce the following corollary of Theorem 33.4.

COROLLARY 33.6. *Let G^* and H^* be two free ω-closures of a group G in $\mathfrak{D}_\omega$. Then any automorphism φ of G can be extended to an isomorphism of G^* onto H^*.*

Proof. The homomorphism φ of G (qua subgroup of G^*) onto G (qua subgroup of H^*) can be extended to a homomorphism φ^* of G^* into H^*, by Theorem 33.4. The image of G^* under φ^* contains $G\varphi^* = G\varphi = G$; now H^* is a free ω-closure of G and so, by Lemma 33.5, G ω-generates H^*. Thus $G^*\varphi^*$ must in fact be the whole of H^*, i.e. φ^* is an epimorphism of G^* to H^*. It follows that we can similarly extend φ^{-1}, which is a homomorphism from G (qua subgroup of H^*) onto G (qua subgroup of G^*) to an epimorphism $(\varphi^{-1})^*$ of H^* to G^*. It is clear that $\varphi^*(\varphi^{-1})^*$, when restricted to G, is the identity. But G ω-generates G^* and so $\varphi^*(\varphi^{-1})^*$ is in fact the identity automorphism of G^*. It follows that φ^* is simultaneously an epimorphism and a monomorphism; in other words φ^* is an isomorphism between G^* and H^*. It also follows, by its definition, that φ^* extends φ; this then completes the proof of the corollary.

This corollary shows that although there appears to be a certain amount of freedom in the formation of a free ω-closure of a group in $\mathfrak{D}_\omega$, a free ω-closure is unique up to isomorphism. Hence it is not ambiguous to speak of "the" free ω-closure of a group in the class $\mathfrak{D}_\omega$.

We conclude this section by proving that a free ω-closure of a group G in the class $\mathfrak{D}_\omega$ inherits some of the properties of G.

LEMMA 33.7. *Let G be a locally infinite group in the class $\mathfrak{D}_\omega$. Then any free ω-closure G^* of G is locally infinite.*

Proof. The proof is by transfinite induction. Put $G^* = \bigcup_{\alpha \in A} G_\alpha$ (cf. (33.3)). Then $G_\iota = G$ is locally infinite. Suppose G_α is locally infinite for all $\alpha < \beta$. If β does not have a predecessor then

$$G_\beta = \bigcup_{\alpha < \beta} G_\alpha.$$

and so being the union of an ascending sequence of locally infinite groups, is itself locally infinite. If β does have a predecessor β^- then there are two possibilities—either $G_\beta = G_{\beta^-}$ or

$$G_\beta = \{G_{\beta^-} * P\,;\, A\},$$

where P is an isomorphic copy of Γ_ω intersecting G_{β^-} in A. For the first of these two possibilities G_β is obviously locally infinite and for the second, G_β is locally infinite since the free product of locally infinite groups with a single amalgamation is again locally infinite (Corollary 16.5). Hence in all cases G_β must be locally infinite and the lemma follows since $G^* = \bigcup_{\alpha \in A} G_\alpha$.

If ω happens to be the set of all primes then any group G in $\mathfrak{D}_\omega$ is an R-group and so its ω-closure G^* is, of course, also an R-group. However, we shall show next that if G is an R-group to begin with, then, irrespective of whether ω is the set of all primes or not, G^* is also an R-group.

LEMMA 33.8. *For any non-empty set of primes ω a free ω-closure G^* of an R-group G in the class $\mathfrak{D}_\omega$ is itself an R-group.*

Proof. Let $G^* = \bigcup_{\alpha \in A} G_\alpha$ (cf. (33.3)). We prove that each G_α $(\alpha \in \mathcal{A})$ is an R-group and so G^*, which is the union of the ascending sequence of subgroups G_α, will itself be an R-group. Now G_ι is an R-group. Let us suppose G_α is an R-group for all $\alpha < \beta$ $(\alpha, \beta \in \mathcal{A})$. If β does not have a predecessor then

$$G_\beta = \bigcup_{\alpha < \beta} G_\alpha.$$

and so, being the union of an ascending sequence of R-groups, is itself an R-group. If β^- exists, then we have to consider two cases. Firstly if $G_\beta = G_{\beta^-}$ there is nothing to prove, since G_{β^-} is, by induction, an R-group. Let us consider the second possibility:

$$G_\beta = \{G_{\beta^-} * P\,;\, A\},$$

where P is isomorphic to Γ_ω and intersects G_{β^-} in A. Let m be a positive integer and let

$x, y \in G_\beta$ be such that

$$x^m = y^m.$$

We can clearly suppose that x, qua element of G_β, is cyclically reduced. If $x \in P$ then, by Lemma 28.1, $y \in P$. Since P is a torsion-free abelian group it is an R-group and so $x = y$. If $x \in G_{\beta^-}$ but $x \notin P$, then $\mathrm{C}(x^m, G_\beta) = \mathrm{C}(x^m, G_{\beta^-})$ (by Lemma 28.2) and so $y \in G_{\beta^-}$. Hence $x = y$ since G_{β^-} is an R-group by the induction hypothesis. Thus we may suppose that $\boldsymbol{\lambda}(x) > 1$. Since x is cyclically reduced it follows that x^m is also cyclically reduced (e.g. by Lemma 16.3) and hence $\boldsymbol{\lambda}(x^m) > 1$. Hence Lemma 28.6 applies and so $\mathrm{C}(x^m, G_\beta)$ is an infinite cyclic group. But

$$x, y \in \mathrm{C}(x^m, G_\beta);$$

$\mathrm{C}(x^m, G_\beta)$ is an R-group and so $x = y$. This completes the proof of the lemma.

Next we prove a lemma concerned with centralisers in a free ω-closure of a group in $\mathfrak{D}_\omega$.

LEMMA 33.9. *Let G be a group in the class $\mathfrak{D}_\omega$ and let G^* be a free ω-closure of G. Suppose that the centraliser, in G, of every non-trivial element of G is isomorphic to a subgroup of Γ_ω. Then the centraliser, in G^*, of every non-trivial element of G^* is isomorphic to Γ_ω.*

Proof. We write (cf. (33.3))

$$G^* = \bigcup_{\alpha \in \mathcal{A}} G_\alpha.$$

Put $G_* = G^*$ and define Λ to be the set-theoretical union of $\mathcal{A}$ and $\{*\}$:

$$\Lambda = \mathcal{A} \cup \{*\}.$$

We turn Λ into a well-ordered set by making use of the well-order relation in $\mathcal{A}$ and stipulating that $*$ follows every member of $\mathcal{A}$.

Suppose now that $1 \neq a \in G^*$. If $a \in G_\iota$ $(= G)$ then $\mathrm{C}(a, G_\iota)$ is isomorphic to a subgroup of Γ_ω. If $\mathrm{C}(a, G_*) (= \mathrm{C}(a, G^*)) > \mathrm{C}(a, G_\iota)$ then, by Lemma 31.3, $\mathrm{C}(a, G_*)$ is isomorphic to Γ_ω. On the other hand, if $\mathrm{C}(a, G_*) = \mathrm{C}(a, G_\iota)$ then a must have a pth root in G_ι for every $p \in \omega$; thus $C(a, G_\iota)$ is isomorphic to Γ_ω and so $\mathrm{C}(a, G^*)$ is likewise also isomorphic to Γ_ω.

Now suppose inductively that for every $\varkappa$ in Λ which precedes λ the centraliser, in G^*, of every non-trivial element of $G_\varkappa$ is isomorphic to Γ_ω. Let $1 \neq a \in G_\lambda$. We show that $\mathrm{C}(a, G^*)$ is isomorphic to Γ_ω. The result is immediate if a has a conjugate in $G_\varkappa$ for $\varkappa < \lambda$. Thus we suppose that this is not the case. Then

$$G_\lambda = \{G_{\lambda^-} * P\,;\, \mathrm{C}(g, G_{\lambda^-})\},$$

where λ^- is the predecessor of λ, g is an element of G_{λ^-} which does not have a pth root in G_{λ^-} for some p in ω, and P is an isomorphic copy of Γ_ω which intersects G_{λ^-} in $\mathrm{C}(g, G_{\lambda^-})$. If $a \in P$ then $\mathrm{C}(a, G_\lambda) = P$ and so is isomorphic to Γ_ω. It follows from Lemma 31.3 that in this case $\mathrm{C}(a, G^*) = \mathrm{C}(a, G_*)$ is isomorphic to Γ_ω. Hence it remains only to consider the case $a \notin P$. In this case a, qua element of G_λ, represents a cyclically reduced element of length at least two. Consequently its centraliser in G_λ is an infinite cyclic group (Lemma 28.6). Thus there is an element μ in Λ for which $\mathrm{C}(a, G_\lambda) < \mathrm{C}(a, G_\mu)$, and so, by Lemma 31.3,

$$\mathrm{C}(a, G^*) = \mathrm{C}(a, G_*) \cong \Gamma_\omega.$$

The theorem then follows by induction.

CHAPTER VIII

D_ω-free groups and D_ω-free products of D_ω-groups

34. The work done hitherto in Chapters VI and VII is the basis for an investigation of the properties of D_ω-free groups and D_ω-free products of D_ω-groups.

The existence of a D_ω-free group of arbitrary ω-rank is ensured by the results of Birkhoff [5] on abstract algebraic systems. The rather hazy form that a D_ω-free group takes on from the existence proof is replaced here by a more concrete realisation. We prove the important result that a free ω-closure of a free group of rank m is a D_ω-free group of ω-rank m. Now although a free ω-closure of a group (in the class $\mathfrak{D}_\omega$) bristles with transfinite ordinals, we have here a theoretically usable realisation of a D_ω-free group. We make use of this realisation to prove a number of results about D_ω-free groups. In particular we prove that a D_ω-free group is torsion-free and then, still more, we prove that a D_ω-free group is an R-group, independently of ω being the set of all primes or otherwise. An interesting result is that the centraliser of every non-trivial element of a D_ω-free group is isomorphic to Γ_ω. This enables us to deduce that a D_ω-group, which is not abelian, has trivial centre. We prove also that a free ω-generating set of a D_ω-free group generates (in the usual sense) a free group. This suggests the possibility that a D_ω-free group is locally free. However, it turns out that this is true only for D_ω-free groups which are abelian. We prove next, by a transfinite induction, that the derived group of a D_ω-free group is an ω-subgroup if, and only if, it is trivial. This provides the clue to the structure of the factor group of a D_ω-free group by its commutator subgroup, which we then determine.

The existence of a D_ω-free product of D_ω-groups is taken care of by a theorem of Sikorski [34] on general algebras. It is, however, still an interesting result in its own right that the free ω-closure of a free product of D_ω-groups is in fact a D_ω-free product of these groups; this statement implies simultaneously a proof of the existence of a D_ω-free product and a realisation of such a product as the union of an ascending sequence of U_ω-groups. In the same way as with D_ω-free groups we can utilise this realisation of a D_ω-free product of D_ω-groups to find some of their properties. In particular, it turns out that a D_ω-free product of groups isomorphic to Γ_ω is a D_ω-free group (which is as it "should" be). We prove some simple properties of D_ω-free products. One interesting "carry over" from the realm of free groups is an analogous theorem to that of Baer & Levi [2] for free products, namely: A D_ω-group cannot be decomposed in a non-trivial way simultaneously into a direct product of D_ω-groups and a D_ω-free product of D_ω-groups.

35. In this section we shall prove, by means of elementary cancellation arguments only, that a free product of groups which belong to the class $\mathfrak{D}_\omega$ is also in $\mathfrak{D}_\omega$. The proof follows a similar pattern to the proof of Theorem 29.1; it is, however, very much simpler. We shall effect the proof of the theorem by proving a number of lemmas and then make use of them to deduce this theorem.

We take now F to be the free product of groups F_λ, where λ ranges over an index set Λ.

LEMMA 35.1. *Let $a \in F$ be a cyclically reduced element of length at least two. Then if $b \in F$ $(b \neq 1)$ commutes with a, it is also cyclically reduced of length at least two.*

Proof. Let the normal forms of a and b be

$$\left.\begin{aligned} a &= c_{\lambda(1)}^{(1)} c_{\lambda(2)}^{(2)} \dots c_{\lambda(m)}^{(m)} \\ b &= \hat{c}_{\mu(1)}^{(1)} \hat{c}_{\mu(2)}^{(2)} \dots \hat{c}_{\mu(n)}^{(n)}. \end{aligned}\right\} \tag{35.11}$$

Now a is cyclically reduced of length at least two and so $\lambda(1) \neq \lambda(m)$. We have

$$b\,a = \hat{c}_{\mu(1)}^{(1)} \hat{c}_{\mu(2)}^{(2)} \dots \hat{c}_{\mu(n)}^{(n)} c_{\lambda(1)}^{(1)} c_{\lambda(2)}^{(2)} \dots c_{\lambda(m)}^{(m)} = c_{\lambda(1)}^{(1)} c_{\lambda(2)}^{(2)} \dots c_{\lambda(m)}^{(m)} \hat{c}_{\mu(1)}^{(1)} \hat{c}_{\mu(2)}^{(2)} \dots \hat{c}_{\mu(n)}^{(n)} = a\,b. \tag{35.12}$$

If $\mu(n) \neq \lambda(1)$ then $\mu(1) \neq \lambda(m)$ and (35.12) yields

$$\mu(1) = \lambda(1) \quad \lambda(m) = \mu(n),$$

and so

$$\mu(1) \neq \mu(n);$$

therefore b is cyclically reduced of length at least two. On the other hand, if $\mu(n) = \lambda(1)$ then $\mu(1) = \lambda(m)$ and so $\mu(1) \neq \mu(n)$ and b is again cyclically reduced of length at least two. This completes the proof of the lemma.

LEMMA 35.2. *Let $a \in F$ be cyclically reduced, with $\boldsymbol{\lambda}(a) = m \geqslant 2$. Let $b \in \mathrm{C}(a, f)$ and suppose $\boldsymbol{\lambda}(b) = n \geqslant m$. Put $n = \alpha m + \beta$ $(0 \leqslant \beta < m)$. Then*

$$\beta = 0 \text{ or } \beta \geqslant 2,$$

and b can be written in the form

$$b = a^{\pm\alpha} a^*,$$

where

$$\boldsymbol{\lambda}(a^*) = \beta;$$

hence a^* is either trivial or cyclically reduced of length at least two.

Proof. Let a and b have normal forms given by (35.11). It follows from the proof of Lemma 35.1 that either $\mu(1) = \lambda(1)$ and $\mu(n) = \lambda(m)$ or $\mu(1) = \lambda(m)$ and $\mu(n) = \lambda(1)$. Let us suppose $\mu(1) = \lambda(1)$ and $\mu(n) = \lambda(m)$. Then it follows immediately from (35.12) that

$$\begin{array}{cccc} \mu(1) = \lambda(1), & \mu(2) = \lambda(2), & \ldots, & \mu(m) = \lambda(m) \\ \hat{c}^{(1)} = c^{(1)}, & \hat{c}^{(2)} = c^{(2)}, & \ldots, & \hat{c}^{(m)} = c^{(m)}; \end{array}$$

thus

$$b = a\hat{b},$$

where $\boldsymbol{\lambda}(\hat{b}) = (\alpha - 1)m + \beta$ and b is cyclically reduced of length at least two or trivial. Furthermore, $\hat{b}$ commutes with a and so the process can be continued until

$$b = a^{\alpha} a^*,$$

$\beta = \boldsymbol{\lambda}(a^*) < \boldsymbol{\lambda}(a)$. We assumed at the outset that $\mu(1) = \lambda(1)$; if however $\mu(1) = \lambda(m)$ then we would have obtained

$$b = a^{-\alpha} a^*;$$

these are the only two possibilities and so the lemma follows.

LEMMA 35.3. *Let $a \in F$ be cyclically reduced of length at least two. Let $b \in \mathrm{C}(a, F)$ $(b \neq 1)$. Then a and b are powers of a common element $c \in F$.*

Proof. We suppose $\boldsymbol{\lambda}(b) \geqslant \boldsymbol{\lambda}(a)$. Then by Lemma 35.2 we can write

$$b = a^{\alpha_0} a_1,$$

$\lambda(a_1) < \lambda(a)$. Now $a \in C(a_1)$ and $\lambda(a_1) < \lambda(a)$; so we can apply the same lemma to a_1 and a:

$$a = a_1^{\alpha_1} a_2,$$

with $\lambda(a_2) < \lambda(a_1)$. Continuing the process we obtain in this way a set of elements $a_i \in F$ such that

$$\lambda(a) > \lambda(a_1) > \cdots > \lambda(a_i) > \cdots.$$

The process must terminate with $\lambda(a_i) = 0$ for some i, say $i = j+2$. Then

$$a_j = a_{j+1}^{\alpha_{j+2}} = c^{\alpha_{j+2}}, \text{ say.}$$

It follows that both a and b can be expressed as powers of c and this completes the proof of the lemma.

Lemma 35.4. *Let $a \in F$ be cyclically reduced of length at least two. Then $C(a)$ is an infinite cyclic group.*

Proof. Choose $c \in F$ so that $a = c^r$, with r as large as possible. Suppose $b \in C(a)$; then, by Lemma 35.3,

$$a = d^s, \quad b = d^t,$$

where s and t are integers and $d \in F$. Then

$$d^s = c^r. \tag{35.41}$$

Now, by the choice of r, $s \leqslant r$ and so, on comparing the two sides of (35.41) we see that the (cyclically reduced) element d has the form $d = cc_1$ $(c_1 \in F)$, where $\lambda(c) + \lambda(c_1) = \lambda(d)$. Hence

$$(cc_1)^s = c^r = (c_1 c)^s.$$

Taking the extreme right-hand-side and extreme left-hand-side of this equation we have

$$cc_1 = c_1 c,$$

and hence $d = cc_1$ commutes with c. Remembering how c was chosen it follows from Lemma 35.3 that $d = c^u$, for some integer u. It follows that $C(a)$ is the infinite cyclic group generated by c.

Lemma 35.5. *Let $p \in \omega$ and let m and n be integers. Suppose F is the free product of groups F_λ, each of which belongs to the class $\mathfrak{D}_\omega$. Suppose $a \in F$ does not have a p-th root and that*

$$b^{-1} a^m b = a^n, \tag{35.51}$$

for some $b \in F$. Then $m = n$.

Proof. We may assume a is cyclically reduced. Let us recast (35.51) in the form

$$a^m b = b a^n. \tag{35.52}$$

If $\boldsymbol{\lambda}(a) = 1$ then it follows from (35.52) that $\boldsymbol{\lambda}(b) = 1$, and since each factor F_λ belongs to $\mathfrak{D}_\omega$ it follows that $m = n$. If $\boldsymbol{\lambda}(a) > 1$ then it follows that b must be cyclically reduced and so we can arrange it that either

$$\boldsymbol{\lambda}(a^m b) = \boldsymbol{\lambda}(a^m) + \boldsymbol{\lambda}(b) \text{ and } \boldsymbol{\lambda}(b a^n) = \boldsymbol{\lambda}(b) + \boldsymbol{\lambda}(a^n)$$

or
$$\boldsymbol{\lambda}(a^n b^{-1}) = \boldsymbol{\lambda}(a^n) + \boldsymbol{\lambda}(b^{-1}) \text{ and } \boldsymbol{\lambda}(b^{-1} a^m) = \boldsymbol{\lambda}(b^{-1}) + \boldsymbol{\lambda}(a^m).$$

Since $a^m b = b a^n$, if the first of the above situations occurs, we have

$$\boldsymbol{\lambda}(a^m b) = m\boldsymbol{\lambda}(a) + \boldsymbol{\lambda}(b) = n\boldsymbol{\lambda}(a) + \boldsymbol{\lambda}(b) = \boldsymbol{\lambda}(b a^n)$$

and so $m = n$. A similar argument holds for the second case and the lemma then follows.

THEOREM 35.6. *The free product F of groups F_λ, each of which belongs to the class $\mathfrak{D}_\omega$ $(\lambda \in \Lambda)$ belongs also to the class $\mathfrak{D}_\omega$.*

Proof. We know already from Theorem 17.2 that a free product of U_ω-groups is a U_ω-group; therefore F satisfies 27.1.

Suppose next that $p \in \omega$ and that $a \in F$ does not have a pth root. We may suppose that a is cyclically reduced. If a lies in one of the factors F_λ then

$$\mathrm{C}(a, F) = \mathrm{C}(a, F_\lambda)$$

and so $\mathrm{C}(a, F)$ is isomorphic to a subgroup of Γ_ω. On the other hand, if a is of length two or more then, by Lemma 35.4, $\mathrm{C}(a, F)$ is an infinite cyclic group. Hence F satisfies 27.2.

Now let a be as above and let m be a non-zero integer. Then if $a \in F_\lambda$,

$$\mathrm{C}(a, F) = \mathrm{C}(a, F_\lambda) = \mathrm{C}(a^m, F_\lambda) = \mathrm{C}(a^m, F).$$

If a is cyclically reduced of length two or more then a^m is likewise cyclically reduced of length at least two. It follows, because $\mathrm{C}(a)$ and $\mathrm{C}(a^m)$ are infinite cyclic groups, and hence abelian (cf. Lemma 35.4), that

$$\mathrm{C}(a, F) = \mathrm{C}(a^m, F).$$

So F satisfies 27.3.

Finally, by Lemma 35.5, F satisfies 27.4 and this completes the proof of the theorem.

COROLLARY 35.7. *Every free group belongs to* $\mathfrak{D}_\omega$.

Proof. An infinite cyclic group belongs to $\mathfrak{D}_\omega$; hence, by Theorem 35.6, so does every free group.

We remark that had use been made of a theorem by Kurosh [22] on the subgroups of a free product, the proof of Theorem 35.6 could have been achieved more easily. The proof we have given here will be used in 38 to provide an *elementary* proof of the theorem due to Baer & Levi [2] which we cited earlier.

36. We shall prove in this section some of the simpler properties of D_ω-free groups.

THEOREM 36.1. *A free ω-closure F^* of a free group F of rank m is a D_ω-free group of ω-rank m. Moreover, every set of free generators of F is also a set of free ω-generators of F^*.*

Proof. Let F be a free group of rank m and let X be a free generating set of F. Now by Corollary 35.7 F belongs to $\mathfrak{D}_\omega$, and so its free ω-closure exists and belongs also to $\mathfrak{D}_\omega$. Let now H be a D_ω-group and let θ be a mapping of X into H. This mapping θ can be extended to a homomorphism φ of F into H, since X is a free generating set of F. Theorem 33.4 can now be applied to extend φ to a homomorphism φ^* of F^* into H.

Now X generates F and F ω-generates F^* (Lemma 33.5); hence X ω-generates F^*. X is in fact a free ω-generating set of F^*. Because, as we have seen above, for every D_ω group H and every mapping θ of X into H there exists a homomorphism φ^* of F^* into H which coincides with θ on X. Therefore F^* is a D_ω-free group freely ω-generated by the set X.

Now a free ω-closure of a group of order m is again of order m if m is infinite (cf. Theorem 31.4 and the constructive process that precedes it). It follows from this observation and Theorem 36.1 that

THEOREM 36.2. *A D_ω-free group F^* freely ω-generated by the non-empty set X is countably infinite if X is finite. If X is infinite then the order of F^* is equal to the number of elements of X.*

We shall make further use of Theorem 36.1 to prove a number of results concerning D_ω-free groups. But first we prove a sort of converse to it.

THEOREM 36.3. *Let G^* be a D_ω-free group freely ω-generated by the set Y. Then the group G generated by Y is a free group freely generated by Y.*

Proof. Let F be a free group freely generated by a set X of the same cardinality as Y. Further, let F^* be a free ω-closure of F; F^* is then, by Theorem 36.1, a D_ω-free group freely ω-generated by the set X. Now there is a one-to-one mapping θ of X onto Y; this can be extended to an isomorphism φ^* of F^* onto G^* (cf. the proof of Theorem 25.2). Now F maps under φ^* onto the group generated by Y i.e. onto G; hence G is free since F is free. Furthermore, X maps onto Y under φ^* and since X is a free generating set of F, Y is a free generating set of G. This completes the proof of the theorem.

We note at this point that it follows from the proof of Theorem 36.3 that *if G^* is a D_ω-free group freely ω-generated by a set Y, then G^* may be thought of as the free ω-closure of the free group G freely generated by Y.*

Theorem 36.3 brings to mind the question as to whether D_ω-free groups are locally free. However, this is true only in the case of a D_ω-free group of ω-rank one. We make use of the following lemma, which is interesting in its own right.

LEMMA 36.4. *Let n be an integer greater than one. Then a simple commutator in a free group is an n-th power if, and only if, it is the identity.*

Proof. Let F be a non-abelian free group and suppose there is a non-trivial element $f \in F$ such that

$$f^n = [g, h],$$

$g, h \in F$. Now F is locally infinite and so $f^n \neq 1$; hence g and h do not commute.

Consider now the subgroup G of F generated by f, g, h:

$$G = \mathrm{gp}(f, g, h).$$

The Nielson-Schreier theorem for free groups states that the subgroups of a free group are free (cf. e.g. Schreier [33]); so G is itself free. Now the factor group of G by its commutator subgroup G' is a free abelian group (cf. e.g. Kurosh [21]) of rank m, where m is the rank of G. Now $f^n \in G'$ and consequently, by the remark above, $f \in G'$. Hence G/G' is of rank two, and is generated modulo G' by g and h. Consequently G is itself of rank two. Therefore we can find two elements g^* and h^* such that they generate G and such that $g^* G' = gG'$ and $h^* G' = hG'$. Thus

$$g^* = gg', \quad h^* = hh', \quad g', h' \in G'. \tag{36.41}$$

A theorem of Magnus [24] states that if a free group of finite rank m is generated by m elements, then these m elements are in fact a free generating set; thus g^* and h^* are a free generating set of G.

Let now H be a nilpotent group of class two defined in the following way:

$$H = \mathrm{gp}(a, b;\ a^{n^2} = b^{n^2} = 1,\ [a, b] = a^n = b^n).$$

It follows from its defining relations that the centre of H coincides with its derived group and is of order n.

Let φ be the homomorphism of G into H defined by

$$g^*\varphi = a,\ h^*\varphi = b.$$

It follows then from the equations (36.41) and the fact that the derived group of H lies in the centre of H that $[g, h]\varphi = [a, b]$; for

$$[g, h]\varphi = [g^*g'^{-1}, h^*h'^{-1}]\varphi = [g^*\varphi g'^{-1}\varphi, h^*\varphi h'^{-1}\varphi] = [g^*\varphi, h^*\varphi] = [a, b].$$

Now $f^n = [g, h]$ and so we have shown that

$$f^n\varphi = [a, b] \neq 1;$$

thus $f^n\varphi$ is of order n. However, f lies in the derived group of G and consequently its image under φ lies in the derived group of H. But H' is of order n and therefore

$$f^n\varphi = (f\varphi)^n = 1.$$

Thus $f^n\varphi$ is simultaneously an element of order n and an element of order 1, which is impossible and so f^n cannot be a commutator. This completes the proof of the lemma.

THEOREM 36.5. *A D_ω-free group is locally free if, and only if, it is abelian.*

Proof. A D_ω-free group which is abelian is isomorphic to Γ_ω and so any finitely generated subgroup is necessarily cyclic, and hence free. Thus a D_ω-free group which is abelian is locally free.

On the other hand, let G^* be a D_ω-free group which is not abelian. Then there exist $g, h \in G^*$ such that $[g, h] \neq 1$. Let $p \in \omega$ and let f denote the pth root of $[g, h]$:

$$f^p = [g, h].$$

Then $H = \mathrm{gp}(f, g, h)$ is not free, by Lemma 36.4, and so G^* is not locally free.

We prove next the following theorem.

THEOREM 36.6. *A D_ω-free group is locally infinite.*

Proof. A free group is locally infinite and hence, by Lemma 33.7, so is its free ω-closure. Thus a D_ω-free group is locally infinite.

THEOREM 36.7. *The centraliser of every element different from 1 in any D_ω-free group is isomorphic to Γ_ω.*

Proof. The centraliser of every element different from 1 in a free group F is an infinite cyclic group—this follows easily by e.g. Lemma 35.4—which is isomorphic to a subgroup of Γ_ω. Hence the centraliser of every non-trivial element in its free ω-closure is isomorphic to Γ_ω (Lemma 33.9); this completes the proof of the theorem.

COROLLARY 36.8. *The centre of a D_ω-free group which is not abelian is trivial.*

Proof. Let F^* be a non-abelian D_ω-free group; then the ω-rank m of F^* is greater than 1. Let X be a free ω-generating set of F^* and choose x_1, x_2 to be distinct elements of X. Then $\mathrm{C}(x_1)$ and $\mathrm{C}(x_2)$ are, by Theorem 36.7, isomorphic to Γ_ω and hence

$$\mathrm{cl}_\omega(x_1) = \mathrm{C}(x_1) \text{ and } \mathrm{cl}_\omega(x_2) = \mathrm{C}(x_2). \tag{36.81}$$

But X is ω-independent (see the end of *25*) and therefore

$$\mathrm{cl}_\omega(x_1) \cap \mathrm{cl}_\omega(x_2) = 1.$$

It follows from (36.81) that

$$\mathrm{C}(x_1) \cap \mathrm{C}(x_2) = 1. \tag{36.82}$$

Now the centraliser of any element in a group contains the centre of that group; consequently so does the intersection of the centralisers of an arbitrary number of elements. In particular

$$\zeta(F^*) \leqslant \mathrm{C}(x_1) \cap \mathrm{C}(x_2);$$

so by (36.82) the centre of F^* is trivial.

THEOREM 36.9. *The normaliser of the centraliser of any element, different from* 1, *in a D_ω-free group F^* coincides with the centraliser, and is therefore isomorphic to Γ_ω.*

Proof. Let $1 \neq a \in F^*$ and suppose $y \in F^*$ normalises $\mathrm{C}(a)$. Since $\mathrm{C}(a)$ is isomorphic to Γ_ω (Theorem 36.7), it is locally cyclic. Thus

$$a \text{ and } y^{-1}ay$$

are powers of a common element b:

$$a = b^m,\ y^{-1}ay = b^n.$$

Thus

$$y^{-1}b^m y = b^n.$$

Since $F^* \in \mathfrak{D}_\omega$ it follows that $m = n$ and so $y \in \mathrm{C}(a)$. This completes the proof of the theorem.

Finally we generalise Theorem 36.6; two simple proofs present themselves and so we give them both.

THEOREM 36.10. *Let ω be any non-empty set of primes. Then every D_ω-free group is an R-group.*

Proof. (i) A free group is an R-group (Theorem 17.2). Furthermore, a free ω-closure of an R-group is an R-group (Lemma 33.8). The theorem now follows immediately on applying Theorem 36.1.

(ii) Let F^* be a D_ω-free group, let $f, g \in F^*$, and let n be a positive integer. Suppose that

$$f^n = g^n.$$

Now $f \in \mathrm{C}(g^n) = \mathrm{C}(g)$ (by Theorem 36.7). Hence

$$(fg^{-1})^n = 1;$$

but, by Theorem 36.6, F^* is locally infinite and so $fg^{-1} = 1$. Therefore $f = g$ and this completes the proof of the theorem.

37. Theorem 25.1 states that the factor group of a D_ω-free group of ω-rank m by the *ω-closure of its commutator subgroup* is a direct product of m isomorphic copies of Γ_ω. In this section we shall completely determine the structure of the factor group of a D_ω-free group by its *commutator subgroup.* We shall need the notion of a restricted direct product of groups with an amalgamated subgroup; this notion was introduced by B. H. Neumann and Hanna Neumann [30]. Only a particular case of this product is needed for our purpose; it is this case which we define here. Suppose A and B are given groups and $H \leqslant \zeta(A)$, $K \leqslant \zeta(B)$. Suppose further that $K \cong H \cong L$, where L is some given group. Let θ be an isomorphism of H to K. Put $M = A \times B$ and put

$$N = \mathrm{gp}(m = ab^{-1}\,;\, m \in M,\ a \in H,\ b \in K,\ a\theta = b).$$

Then N is normal in M and so we can form the factor group $D = M/N$; D is called the *direct product of A and B with L amalgamated, or the generalised direct product of A and B* (the amalgamation being understood); we shall write

$$D = \{A \times B\,;\, L\}.$$

D is generated by isomorphic copies of A and B which intersect in a group isomorphic to L. When dealing with such a product we shall usually identify groups with their isomorphic copies (as is usually done in the case of a direct product of groups).

LEMMA 37.1. *Let F be a non-abelian free group and let c be an element of F which is not a p-th power, for some p in ω. Let, further, c be a member of the second term of the lower central series of F but not a member of the third term of the lower central series of F. Let P be a supergroup of the (necessarily cyclic) centraliser of c in F such that P intersects F in $\mathrm{C}(c, F)$ and such that there is an element c_0 in P satisfying $c_0^p = c$. Finally let G be the generalised free product of F and P. Then the p-th root c_0 of c does not lie in the commutator subgroup of G.*

Proof. Let θ denote the natural homomorphism of F onto $F/{}^{(2)}F$, where ${}^{(i)}F$ here denotes the $(i+1)$st member of the lower central series of the free group F. Now ${}^{(1)}F/{}^{(2)}F$ is a direct product of infinite cyclic groups (by a theorem of Witt [37]) and, since $c \notin {}^{(2)}F$, $[\mathrm{C}(c, F)]\theta$ is an isomorphic copy of the cyclic centraliser $\mathrm{C}(c, F)$. Take P^+ to be a supergroup of $\mathrm{C}(c, F)\theta$ which is isomorphic to P in such a way that the isomorphism between them maps $\mathrm{C}(c, F)$, qua subgroup of P, onto $[\mathrm{C}(c, F)]\theta$ in the same way as θ maps $\mathrm{C}(c, F)$ onto $[\mathrm{C}(c, F)]\theta$; let, further, P^+ have intersection $[\mathrm{C}(c, F)]\theta$ with $F\theta$.

Now $\mathrm{C}(c, F)\theta \leqslant \zeta(F\theta)$ and so we can form the generalised direct product D of $F\theta$ and P^+:

$$D = \{F\theta \times P^+;\ \mathrm{C}(c, F)\theta\}.$$

By definition of the generalised free product it follows that we can extend the homomorphisms θ (of F onto $F\theta$) and φ (of P onto P^+) to a homomorphism ψ of G into D. Now if $c_0 \in G'$, then $c_0\psi \in D'$. However, $c_0\psi \neq 1$; furthermore,

$$D' = (F\psi)',$$

since P^+ is abelian. But $c_0\psi \in P^+$; and $c_0\psi \notin F\theta$ and so, in particular, $c_0\psi \notin (F\theta)'$; in other words

$$c_0\psi \notin D'.$$

Hence $c_0 \notin G'$ and so we have proved the lemma.

THEOREM 37.2.[1] *The commutator subgroup of a D_ω-free group is an ω-subgroup if and only if, it is trivial.*

Proof. Let F^* be a D_ω-free group; we shall take F^* to be a free ω-closure of a free group F. It is clear that the trivial subgroup is an ω-subgroup; so when the commutator subgroup K of F^* is trivial there is nothing to prove.

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(1) Thus we have examples of D_ω-groups whose derived groups are *not* ω-subgroups (see 14).

Suppose, on the other hand, that K is non-trivial. Then F^* is a non-abelian D_ω-free group. So if X is a free generating set of F, then $|X| > 1$. We make use now of the fact that F^* is a free ω-closure of F to write F^* in the form (using (33.3)):

$$F^* = \bigcup_{\alpha \in \mathcal{A}} F_\alpha.$$

In order to prove that K is not an ω-subgroup we have to prove that for some $p \in \omega$ there is an element $c_0 \in F^*$ such that $c_0^p = c \in K$ but $c_0 \notin K$. We choose a simple commutator $c \in^{(2)} F_\iota$ with $c \notin^{(3)} F_\iota$; then, by Lemma 36.4, c is not a pth power in the free group F_ι and so c_0, the pth root of c, does not lie in F_ι. We shall prove that $c_0 \notin K$ although, by our choice of c_0, $c_0^p = c \in K$.

The ordering in F can be chosen so that

$$\iota^+ = c.$$

Then
$$F_{\iota^+} = \{F_\iota * P\,;\, A\}.$$

We make use of Lemma 37.1 in asserting that $c_0 \notin F'_{\iota^+}$ since $c_0 \notin F'_{\iota^+}$. So we have the first step in a proof by transfinite induction. Let us now suppose that $\iota^+ < \beta \in \mathcal{A}$ and that $c_0 \notin F'_\alpha$ for all $\alpha < \beta$, $\alpha \in \mathcal{A}$. If β does not have a predecessor then

$$F_\beta = \bigcup_{\alpha<\beta} F_\alpha$$

and since
$$F'_\beta = \bigcup_{\alpha<\beta} F'_\alpha$$

$c_0 \notin F'_\beta$. If β does have a predecessor, say β^-, then there are two possibilities: Either $F_\beta = F_{\beta^-}$, in which case $c_0 \notin F'_\beta$, or

$$F_\beta = \{F_{\beta^-} * P^+\,;\, A^+\}.$$

We shall prove that there is a homomorphic image of F_β in which the image of c_0 is not in the commutator subgroup, which shows that c_0 is not in the commutator subgroup of F_β. Consider the factor group

$$G = F_{\beta^-}/F'_{\beta^-}.$$

Now G is a non-trivial abelian group since $c_0 \notin F'_{\beta^-}$. Furthermore, the p^kth roots of c_0,

$$c_0,\ c_0\pi,\ c_0\pi^2,\ \ldots,$$

where $c_0\pi^n$ denotes the p^nth root of c_0, generate modulo F'_{β^-} a group H isomorphic to $Z(p^\infty)$. Note that all the p^kth roots of c_0 lie in F_{β^-} since they lie already in F_{ι^+}, and by the choice of β, $\iota^+ \leqslant \beta^-$. Now H is a divisible subgroup of the abelian group G and consequently it splits off as a direct factor (cf. e.g. Kaplansky [17] p. 8):

$$G = H \times K.$$

It follows that F_{β^-} can be homomorphically mapped onto H by a homomorphism θ so that $c_0\theta \neq 1$.

Consider now $A^+ : \theta$ induces a homomorphism of A^+ into H. There are two possibilities that can occur:

i) $A^+\theta = 1$;

ii) $A^+\theta \neq 1$.

In i) it is easy to see that $c_0 \notin F'_\beta$. For we can define here a homomorphism φ of P^+ into H which coincides with θ on A^+ simply by stipulating that all the members of P^+ map onto 1 under φ. Then by the definition of the generalised free product we can extend θ and φ simultaneously to a homomorphism ψ of F_β onto H. Now $c_0\psi = c_0\theta \neq 1$; but $H' = 1$ and so $c_0 \notin F'_\beta$.

In ii) we have for some $a \in A^+$, say a_0, $a_0\theta = 1$. We add this relation to the defining relations of P^+; this yields a torsion group $\hat{P}$. This group $\hat{P}$ has a subgroup isomorphic to $Z(p^\infty)$ and so H is a homomorphic image of P^+. Thus we can find a homomorphism φ of P^+ onto H which coincides with θ on A^+. The definition of the generalised free product ensures that we can extend, simultaneously, the homomorphisms θ and φ, respectively, of F_{β^-} and P^+ into H to a homomorphism ψ of F_β into H. Now again $c_0\psi \neq 1$, and so, because $H' = 1$, $c_0 \notin F'_\beta$.

We are therefore entitled, with the aid of a transfinite induction, to deduce that $c_0 \notin F'_\alpha$ for all $\alpha \in \mathcal{A}$. Hence

$$c_0 \notin \bigcup_{\alpha \in A} F'_\alpha = (F^*)' = K.$$

This completes the proof of the theorem.

Theorem 37.2 enables us to prove the following important result:

THEOREM 37.3. *The factor group of a non-abelian D_ω-free group F^* of ω-rank m by its commutator subgroup K splits into a direct product of a divisible torsion group Q and a torsion free group R:*

$$F^*/K = Q \times R.$$

When m is finite, the torsion group Q is a direct product of a countably infinite number of groups isomorphic to $Z(p^\infty)$ for each p in ω. When m is infinite then Q is a direct product of

m groups isomorphic to $Z(p^\infty)$ for each p in ω. Finally the torsion-free group R is a direct product of m groups isomorphic to Γ_ω.

Proof. The group F^*/K is abelian and, as such, it can be split into a direct product of a maximal divisible group A and a group B which contains no divisible subgroups (cf. e.g. Kaplansky [17] p. 9):

$$F^*/K = A \times B.$$

Let us suppose now that F^* is a free ω-closure of the free group F and let X' be a free generating set of F' consisting of distinct simple commutators of F (cf. Levi [23]). Choose $c \in X'$; then c is not a pth power in F, by Lemma 36.4. It follows from the method of proof of Theorem 37.2 that the p^kth roots of c:

$$c\pi,\ c\pi^2,\ \ldots$$

do not belong to K and so they generate modulo K a group isomorphic to $Z(p^\infty)$. This argument holds for every $p \in \omega$; so for each $p \in \omega$ the p^kth roots of c generate modulo K a $Z(p^\infty)$ and hence, letting p run through the whole of ω, the p^kth roots of c generate modulo K a direct product of groups isomorphic to $Z(p^\infty)$. If m is finite, then F' is of countable rank (cf. Levi [23]). Thus X' is countably infinite and so we have a countably infinite number of independent choices for c and hence, in this case, A contains a subgroup which is a direct product of a countably infinite number of groups isomorphic to $Z(p^\infty)$ for each p in ω. Similarly, in the case where m is infinite, A contains a subgroup which is a direct product of m groups isomorphic to $Z(p^\infty)$ for each $p \in \omega$. We take Q to be that subgroup of A generated by all those subgroups of A which are isomorphic to a $Z(p^\infty)$ for some $p \in \omega$; then Q is a direct product of these subgroups isomorphic to $Z(p^\infty)$ (cf. e.g. Kaplansky [17] p. 8). We can now split A into a direct product of its divisible subgroup Q and a complementary factor C. Now a direct factor of a divisible group is itself divisible (cf. Kaplansky [17]) and so C is divisible. Thus

$$F^*/K = Q \times C \times B.$$

Now $C \times B$ is a group in which no element has order p, where p is any prime in ω. For if $f \in F^*$ generates modulo K a subgroup of order p in $C \times B$ then its p^kth roots generate a subgroup of $C \times B$ isomorphic to $Z(p^\infty)$ and so in fact $fK \in Q$. Thus $C \times B$ is an abelian group without elements of order p for all $p \in \omega$. Consequently it is a U_ω-group. Therefore $(Q \times C \times B)/Q \cong C \times B$ is a U_ω-group and thus

$$F^*/\mathrm{cl}_\omega(K) \cong C \times B.$$

Theorem 25.1 then informs us that $C \times B$ is isomorphic to a direct product of m groups isomorphic to Γ_ω. We put now $R = C \times B$.

Thus we have, in both cases, a decomposition

$$F^*/K = Q \times R,$$

where Q and R are of the required form. This completes the proof of the theorem.

38. We begin this section by recasting some of the concepts connected with generalised free products (see *16*) into analogous concepts for D_ω-groups.

Let G^* be a D_ω-group and let G_λ be ω-subgroups of G^*, where λ ranges over an index set Λ; suppose that $\mathcal{G} = \bigcup_{\lambda \in \Lambda} G_\lambda$ ω-generates G^* (note that each G_λ is itself a D_ω-group). Then we call G^* the generalised D_ω-free product of its ω-subgroups G_λ (or, more simply, the generalised D_ω-free product of the G_λ) if for every D_ω-group W and every set of homomorphic mappings φ_λ of each G_λ into W, every two φ_λ, φ_μ of which agree where both are defined, there exists a homomorphic mapping φ^* of G^* into W that coincides with φ_λ on each G_λ. Now suppose G^* is the generalised D_ω-free product of its ω-subgroups G_λ $(\lambda \in \Lambda)$ and put

$$G_\lambda \cap G_\mu = H_{\lambda\mu} \quad (= H_{\mu\lambda}),$$

where λ, $\mu \in \Lambda\,(\lambda \neq \mu)$. If all the intersections $H_{\lambda\mu}$ coincide to form a single subgroup H:

$$G_\lambda \cap G_\mu = H,$$

then G^* is called the (generalised) D_ω-free product of the G_λ *with an amalgamated subgroup* H; note that H is itself a D_ω-group. In the case where $H = 1$, the trivial group, G^* is called simply the *D_ω-free product* or, to emphasise the distinction, the *ordinary D_ω-free product* of the G_λ.

Following B. H. Neumann [27], who proves a like result for the generalised free product of *groups*, we can prove the "uniqueness" of the generalised D_ω-free product. The proof of our theorem is similar to the proof of B. H. Neumann's theorem and is therefore omitted.

THEOREM 38.1. *Let G^* be the generalised D_ω-free product of its ω-subgroups G_λ $(\lambda \in \Lambda)$. Let H^* be the generalised D_ω-free product of its ω-subgroups H_λ $(\lambda \in \Lambda)$. Then, if for each $\lambda \in \Lambda$, there is an isomorphism φ_λ of G_λ onto H_λ, every two φ_λ, φ_μ of which agree where both are defined, then all the φ_λ can be extended simultaneously to an isomorphism of G^* onto H^*.*

Let now D_ω-groups G_λ be given, where λ runs over a suitable non-empty index set Λ. In every G_λ and to every index $\mu \in \Lambda$ let an ω-subgroup $H_{\lambda\mu}$ be distinguished; $H_{\lambda\lambda}$ is always taken to be the whole group G_λ. If there exists a group G^* which is the generalised D_ω-free product of groups $\hat{G}_\lambda$ with intersections

$$\hat{H}_{\lambda\mu} = \hat{G}_\lambda \cap \hat{G}_\mu = \hat{H}_{\mu\lambda},$$

and if there are isomorphic mappings φ_λ of G_λ onto $\hat{G}_\lambda$,

$$\hat{G}_\lambda = G_\lambda \varphi_\lambda,$$

such that always

$$\hat{H}_{\lambda\mu} = H_{\lambda\mu} \varphi_\lambda,$$

then we say that the *generalised D_ω-free product of the G_λ with amalgamated $H_{\lambda\mu}$ (or simply the generalised D_ω-free product of the G_λ) exists.* It is often convenient when dealing with generalised D_ω-free products to distinguish only between groups lying in different isomorphism classes; we shall adopt this procedure whenever it is convenient and also not ambiguous.

The generalised free product of groups with a single subgroup amalgamated *always* exists. However, it is not even true that the generalised D_ω-free product of two D_ω-groups with a single ω-subgroup amalgamated always exists.

For let

$$A = \mathrm{gp}(s, a, b, c\,;\, R_1),$$

where

$$R_1 = \{a^s = b,\ b^s = c,\ c^s = a,\ s^3 = a^3 = b^3 = c^3 = [a, b] = [a, c] = [b, c] = 1\}$$

and let

$$B = \mathrm{gp}(t, a, b, c;\ R_2),$$

where

$$R_2 = \{b^t = a^{-1},\ a^t = c^{-1},\ c^t = b,\ t^3 = 1\}.$$

It can easily be verified that both A and B are of order 81; hence, by Corollary 11.6, they are also both σ-groups. Now put

$$H = \mathrm{gp}(a, b).$$

Then H is clearly an ω-subgroup of both A and B (here $\omega = \{2\}$).

Suppose, if possible, that the generalised D_2-free product F of A and B with H amalgamated exists. Thus obviously

$$F \geqslant \mathrm{gp}(A, B).$$

In particular both

$$f = sta,\ g = st$$

belong to F. Now

$$f^2 = sta \cdot sta = sts \cdot bta = stst \cdot a^{-1}a = stst = g^2.$$

But $f \neq g$ and so F is not a σ-group, a contradiction. Thus this example shows that the generalised D_ω-free product of two D_ω-groups does not always exist. However, in the particular case of the ordinary D_ω-free product we can in fact prove that this product always exists.([1])

THEOREM 38.2. *Let G_λ be given D_ω-groups, where λ ranges over an index set Λ. Then the free ω-closure G^* of the free product G of the G_λ is their ordinary D_ω-free product; hence the D_ω-free product always exists.*

Proof. Every D_ω-group belongs to $\mathfrak{D}_\omega$ and the free product of groups in $\mathfrak{D}_\omega$ belongs also to $\mathfrak{D}_\omega$ (Theorem 35.6). Hence G belongs to $\mathfrak{D}_\omega$. We can form, therefore, the free ω-closure G^* of G. Now G^* is in fact the D_ω-free product of the G_λ. To see this we note first that the G_λ ω-generate G^* (Lemma 33.5). Secondly, they intersect trivially with each other. Thirdly, for every D_ω-group W and every set of homomorphic mappings φ_λ of each G_λ into W, there exists a homomorphism φ^* of G^* into W that agrees with φ_λ on G_λ ($\lambda \in \Lambda$). For the homomorphisms φ_λ of each G_λ into W can be simultaneously extended to a homomorphism φ of G into W, since G is the free product of the groups G_λ. Then we can make use of the "freeness" of the free ω-closure to extend φ to a homomorphism φ^* of G^* into W (Theorem 33.4). This completes the proof of the theorem.

THEOREM 38.3. *Let F^* be the D_ω-free product of its ω-subgroups F_λ, where λ ranges over an index set Λ. Then the groups F_λ generate in F^* their ordinary free product F.*

Proof. Let G_λ be groups isomorphic to F_λ and let φ_λ be isomorphisms of G_λ onto F_λ for each $\lambda \in \Lambda$:

$$G_\lambda \varphi_\lambda = F_\lambda.$$

Let, further, G be the free product of the groups G_λ and let G^* be a free ω-closure of G. Then G^* is the D_ω-free product of its subgroups G_λ and we can extend the isomorphisms φ_λ simultaneously to an isomorphism φ^* of G^* onto F^*. Now $G\varphi^* = F$, i.e.

$$G \cong F;$$

in other words F is the free product of its subgroups F_λ.

We remark at this point that it follows from the method of proof of Theorem 38.3 that *if F^* is the D_ω-free product of its ω-subgroups F_λ, then F^* may be thought of as the free ω-closure of the free product F of the groups F_λ.* We shall make use of this fact, *sometimes without explicit mention*, in the sequel.

([1]) See also Sikorski [34].

Corollary 38.4. *The D_ω-free product of m groups isomorphic to Γ_ω is a D_ω-free group of ω-rank m.*

Proof. Let F^* be the D_ω-free product of m groups G_λ isomorphic to Γ_ω. Take now an element x_λ $(\neq 1)$ from each of these groups and let X be the set consisting of these elements. Let now θ be any mapping of X into a D_ω-group H. The mapping θ induces a mapping θ_λ of the single element x_λ of each G_λ into H. Since $\{x_\lambda\}$ freely ω-generates G_λ these mappings can be extended to homomorphisms φ_λ of the G_λ into H. Now F^* is the D_ω-free product of the G_λ and so these homomorphisms extend to a homomorphism φ^* of F^* into H. Clearly φ^* extends θ and so F^* is a D_ω-free group of ω-rank m.

Theorem 38.5. *The D_ω-free product F^* of locally infinite D_ω-groups F^* is locally infinite.*

Proof. Let F be the free product of the F_λ $(\lambda \in \Lambda)$. Then F^* may be taken to be the free ω-closure of F. Now F is locally infinite and therefore so is F^* (Lemma 33.7). This completes the proof of the theorem.

Theorem 38.6. *Let ω be any non-empty set of primes and let F^* be the D_ω-free product of its ω-subgroups F_λ. If each F_λ is an R-group, then F^* is itself an R-group.*

Proof. The free product of R-groups is an R-group (Theorem 17.2) and the free ω-closure of an R-group is an R-group (Lemma 33.8). The theorem then follows from these remarks.

We prove next the following auxiliary lemma.

Lemma. *The free ω-closure G^* of a group G with trivial centre has trivial centre.*

Proof. We make use of (33.3) and write

$$G^* = \bigcup_{\alpha \in \mathcal{A}} G_\alpha.$$

Let $\alpha \in \mathcal{A}$ and suppose for all $\alpha < \beta$, that G_α has trivial centre. If B^- does not exist, then

$$G_\beta = \bigcup_{\alpha < \beta} G_\alpha,$$

and so G_β has trivial centre. If β^- does exist and $G_\beta = G_{\beta^-}$ then again G_β has trivial centre. We are left to consider only the case

$$G_\beta = \{G_{\beta^-} * P;\ A\}.$$

Now in this case $$\zeta(G_\beta) = \zeta(G_{\beta^-}) \cap \zeta(P) = 1$$

(cf. e.g. Kurosh [21], vol. 2, page 32). So for all cases, $\zeta(G_\alpha) = 1$, and hence by transfinite induction, $\zeta(G_\alpha) = 1$ for all $\alpha \in \mathcal{A}$; therefore $\zeta(G^*)$ is trivial. This completes the proof of the lemma.

THEOREM 38.7. *Let F^* be the D_ω-free product of its ω-subgroups F_λ $(\lambda \in \Lambda)$. If $|\Lambda| > 1$, then F^* has trivial centre.*

Proof. Let F be the free product of the F_λ. Then F has trivial centre. Hence, by the Lemma, F^* has trivial centre and so the theorem has been proved.

We complete this chapter with the following analogue of a theorem due to Baer and Levi [2].

THEOREM 38.8. *A D_ω-group cannot be decomposed simultaneously, in a non-trivial way, into both a D_ω-free product of two D_ω-groups and a direct product of two D_ω-groups.*

Proof. Suppose G is the D_ω-free product of its ω-subgroups P and Q; suppose further, that G is also the direct product of its ω-subgroups R and S.

Consider $R \cap P$; if $R \cap P \neq 1$, then $S \leqslant P$ since the centraliser of an element in P lies also in P (G is a free ω-closure of the free product of D_ω-groups and so Lemmas 28.1 and 28.2 apply). Hence $S \cap P \neq 1$ and so by a similar argument it follows that $R \leqslant P$; hence Q must be trivial, a contradiction. Thus we must have $R \cap P = 1$. It follows in like manner that the four possible intersections are trivial:

$$R \cap Q = R \cap P = 1 = S \cap Q = S \cap P.$$

Suppose now that $R \ni a \neq 1$. Now R is normal in G and so it follows that a is not conjugate to an element in P or in Q. Thus, remembering that G is the ω-closure of P^*Q we see from Lemma 28.6 that

$$\mathrm{C}(a, G) \cong \Gamma_\omega \tag{38.81}$$

Now R is an ω-subgroup; hence by (38.81) we have

$$\mathrm{C}(a, G) \leqslant R.$$

But $S \leqslant \mathrm{C}(a, G)$ and hence $S \leqslant R$; this is a contradiction and so the theorem follows.

We remark that the condition that R and S be ω-subgroups is unnecessary. For if a D_ω-group G is a direct product of its subgroups, then each of these subgroups is necessarily an ω-subgroup.

The simple proof of Theorem 38.8 can be carried over to the case of a free product. Explicitly, if G is a free product of its subgroups P and Q and also a direct product of its subgroups R and S, then at least one of these four subgroups is trivial—this is the Baer and Levi [2] theorem which we have quoted so often. For it follows, just as in the proof of Theorem 38.8, that

$$P \cap R = P \cap S = 1 = Q \cap R = Q \cap S.$$

Suppose now $R \ni a \neq 1$; then R is normal in G and so it follows that a is not conjugate to an element in P or Q. Thus, by Lemma 35.4, $\mathrm{C}(a, G)$ is an infinite cyclic group. But $S \leqslant \mathrm{C}(a, G)$ and hence

$$S \cap R \geqslant S \cap \mathrm{gp}(a) \neq 1;$$

i.e. S and R intersect non-trivially and so we have a contradiction. The theorem of Baer and Levi therefore follows.

39. *A surprising property of D_ω-free groups.*

Every D_ω-group is a homomorphic image of a (suitably chosen) D_ω-free group, and so the homomorphic images of D_ω-free groups include all D_ω-groups. However, it is clear that not all the homomorphic images of D_ω-free groups are D_ω-groups. But in any homomorphic image G of a D_ω-group the equation

$$x^p = g$$

is soluble for all $g \in G$ and all $p \in \omega$; in other words the homomorphic images of D_ω-groups are E_ω-groups. Thus, in particular, every homomorphic image of a D_ω-free group is an E_ω-group. We shall show that the converse is also true: Every E_ω-group is a homomorphic image of a (suitably chosen) D_ω-free group. Hence the homomorphic images of D_ω-free groups are precisely all the E_ω-groups.

To prove our main theorem we shall make use of a number of lemmas.

LEMMA 39.1. *Let A be a subgroup of Γ_ω containing the integer 1. Then every homomorphism θ of A into any E_ω-group B, which is abelian, can be extended to a homomorphism of Γ_ω into B.*

Proof. Let S denote the set of all pairs (X, η) where X is a subgroup of Γ_ω containing A and η is a homomorphism of X into B extending θ. We introduce an order relation $<$ into S by defining

$$(X, \eta) < (Y, \zeta)$$

if X is a subgroup of Y and ζ extends η. We now apply Zorn's Lemma to deduce the existence of a maximal element (X^*, η^*) of $\mathfrak{S}$.

If $X^* = \Gamma_\omega$ the lemma follows. Suppose the contrary. Then (see Lemma 32.1 as to the structure of the subgroups of Γ_ω) there exists an element $b \in \Gamma_\omega$ and a prime $p \in \omega$ such that

$$b \notin X^* \text{ and } pb \in X^*$$

(we are employing, in this lemma, the additive notation for groups). Suppose $(pb)\eta^* = c$. We choose $d \in B$ to be a solution of the equation

$$px = c.$$

We then put

$$X^+ = \mathrm{gp}(X^*, b),$$

and define a homomorphism η^+ of X^+ into B as follows:

$$(x^* + mb)\eta^+ = x^*\eta^* + md.$$

Then η^+ extends η^* and hence (X^*, η^*) is not a maximal element of $\mathfrak{S}$, which is a contradiction. So in fact $X^* = \Gamma_\omega$ and this completes the proof of the lemma.

LEMMA 39.2. *Every cyclic subgroup A of an E_ω-group B is contained in an abelian subgroup C of B which is itself an E_ω-group.*

Proof. It is not difficult to construct a sequence of integers

$$\alpha_1, \alpha_2, \alpha_3, \ldots \quad (\alpha_i \in \omega) \tag{39.21}$$

having the property that given any positive integer N and any p in ω, there exists an integer $M \geq N$ for which $\alpha_M = p$.

Suppose now that

$$A = \mathrm{gp}(a).$$

We then put

$$C = \mathrm{gp}(a_0, a_1, a_2, \ldots; a = a_0, a_0 = a_1^{\alpha_1}, a_1 = a_2^{\alpha_2}, \ldots),$$

the a_i being chosen subject only to the relations above. Now any pair a_i, a_j of these generators of C commute since one is always a power of the other; hence C is abelian. Further, for any non-negative integer i the property of the sequence (39.21) ensures that for all $p \in \omega$ the equation

$$x^p = a_i$$

is soluble. Since C is abelian, it follows that C is an E_ω-group and so this completes the proof of the lemma.

We prove next the following result concerning the extending of a homomorphism from a subgroup of Γ_ω into an E_ω-group, to a homomorphism from the whole of Γ_ω into that E_ω-group.

LEMMA 39.3. *Let A be a cyclic subgroup of Γ_ω containing the integer 1. Then every homomorphism θ of A into any E_ω-group B can be extended to a homomorphism θ^* of Γ_ω into B.*

Proof. We make use of Lemma 39.2 to embed $A\theta$ in an abelian subgroup C of B, with C an E_ω-group. Then we can think of θ as a homomorphism of A into an E_ω-group C, which is abelian; and so, on applying Lemma 39.1, we can extend θ to a homomorphism θ^* of Γ_ω into C; this completes the proof of the lemma.

We remark that neither of the conditions "A contains the integer 1", "A cyclic" can be omitted from the hypothesis of the lemma.

Next we state, as an immediate consequence of Lemma 39.3 and Lemma 32.1 the following lemma.

LEMMA 39.4. *Let A be isomorphic to a cyclic subgroup of Γ_ω. Then A can be embedded in an isomorphic copy P of Γ_ω in such a way that for every E_ω-group B and every homomorphism θ of A into B there exists a homomorphism θ^* of P into B which coincides with θ on A.*

The lemma places us in a position to prove the following theorem.

THEOREM 39.5. *Let G be a group in the class $\mathfrak{D}_\omega$ with the property that if $1 \neq g \in G$, then either $\mathrm{cl}_\omega(g, G)$ is cyclic, or $\mathrm{cl}_\omega(g, G)$ is isomorphic to Γ_ω. Then for every E_ω-group B, every homomorphism θ of G into B and every free ω-closure G^* of G, there exists a homomorphism θ^* of G^* into B which coincides with θ on G.*

Proof. We avail ourselves of equation (33.3):

$$G^* = \bigcup_{\alpha \in A} G_\alpha.$$

The proof of this theorem follows closely the proof of Theorem 33.4; here we make use of Lemma 39.4 instead of, as was done in the proof of Theorem 33.4, making use of Lemma 32.3. The details of the proof are left to the reader.

Let us now suppose that G^* is a D_ω-free group freely ω-generated by the set Y. Then, by Theorem 36.3, the group G generated by Y is a free group freely generated by Y. Hence any mapping η of X into an E_ω-group B can be extended to a homomorphism

θ of G into B. Now $G \in \mathfrak{D}_\omega$ and satisfies the conditions of Theorem 39.5 (cf. Lemma 35.4). Thus θ can be extended to a homomorphism θ^* of its free ω-closure G^* into B. So we have proved the following theorem.

THEOREM 39.6. *Every mapping η of a free ω-generating set Y of a D_ω-free group G^* into any E_ω-group B can be extended to a homomorphism θ^* of G^* into B.*

COROLLARY 39.7. *Every E_ω-group is a homomorphic image of a (suitably chosen) D_ω-free group.*

Theorem 39.6 enables us to give an example of a normal ω-subgroup of a D_ω-group which is not an ω-ideal (see 9). Put

$$C = \mathrm{gp}(a, b;\; a^2 = b^2).$$

Then C is torsion-free (see 9). Moreover, C can be embedded in a torsion-free E_2-group C^* (using the method of construction employed by B. H. Neumann in [28]). We make use of Corollary 39.7 to find a D_2-free group G^* which has C^* as a homomorphic image:

$$G^*/N \cong C^*.$$

Then N is a normal ω-subgroup of G^* which is not an ω-ideal, since the distinct elements a and b in C^* have equal squares.

40. In conclusion we would like to point out that many of the results and notions of free groups can be carried over into D_ω-free groups. For example the notion of an identical relation in a group, introduced by B. H. Neumann [29] can be carried over to "identical ω-relations in D_ω-groups". These identical ω-relations lead to "reduced D_ω-free groups" and a whole theory along these lines can be developed. A similar theory to the one described in this paper can be developed for E_ω-free groups. However, time and space prevent the presentation of such theories and other interesting results connected with D_ω-free and E_ω-free groups. We shall remedy this state of affairs by means of later works.

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Note. The starred references are quoted from the English Translation by K. A. Hirsch of the book by A. G. Kurosh [21].

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