POLYHEDRAL SECTIONS OF CONVEX BODIES

BY

VICTOR KLEE

Copenhagen and Seattle(1)

§ 1. Introduction

Let us begin by repeating (in a somewhat more elaborate form) some definitions due to C. Bessaga [1]. A normed linear space E is *universal* for a class of normed linear spaces provided every member of the class is linearly isometric with some linear subspace of E. A finite-dimensional convex body K is *a-universal* for a class $\mathcal{K}$ of convex bodies provided each member of $\mathcal{K}$ is affinely equivalent to some proper section of K; and K is *centrally a-universal* for $\mathcal{K}$ provided K is centered and every centered member of $\mathcal{K}$ is affinely equivalent to some central section of K. Replacing affine equivalence by similarity leads to the notions of *s-universality* and *central s-universality*. (K is *centered* at p provided $K-p=p-K$. A *section* of K is the intersection of K with some flat. The section is *proper* provided it includes a relatively interior point of K and *central* provided it includes the center of K.)

In Problem 41 (1935) of The Scottish Book [17], S. Mazur asked whether there is a 3-dimensional Banach space which is universal for all 2-dimensional Banach spaces, or, equivalently, whether there is a 3-dimensional convex body which is centrally a-universal for all 2-dimensional convex bodies. More generally, given an integer $n \geqslant 2$, is there a finite-dimensional convex body which is centrally a-universal for all n-dimensional convex bodies? (By *convex body* we mean here a bounded closed convex set.) These problems have been studied independently by B. Grünbaum, C. Bessaga, and Z. Melzak. By very simple reasoning, Grünbaum [6] established a negative answer to Mazur's first question and obtained some information on the

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general problem. Bessaga's reasoning [1] was more complicated, but he solved (negatively) the general problem, and showed in fact that no n-dimensional Banach space is universal for all the 2-dimensional Banach spaces whose unit spheres are $(2n+2)$-gons. Melzak [12] mentioned Mazur's problem but did not attack it directly. Instead, he solved affirmatively a related problem in which sections are replaced by "limit sections". He stated Mazur's problem as follows: Is there a 3-dimensional convex body K such that every 2-dimensional convex body is affinely equivalent to some plane section of K?

In the present paper, we study some problems concerning universality of convex bodies by a method similar to Bessaga's in that Lipschitzian transformations play an important role. Our machinery is more elaborate than his, but we are repaid by sharper results. We obtain a negative solution of Melzak's version of Mazur's problem and are able to establish some other conjectures of Melzak [13]. We are interested especially in four functions $\xi^{a,v}$, $\xi^{a,f}$, $\xi^{s,v}$, and $\xi^{s,f}$ connected with universality of convex bodies, and two others $\eta^{a,v}$ and $\eta^{a,f}$ connected with central universality. These are defined as follows (for $2 \leqslant n \leqslant r$, and $x=a$ or $x=s$):

$\xi^{x,v}(n,r)$ respectively $\xi^{x,f}(n,r)$ is the smallest integer k such that some k-dimensional convex body is x-universal for all n-dimensional convex polyhedra having $r+1$ vertices respectively maximal faces;

$\eta^{a,v}(n,r)$ respectively $\eta^{a,f}(r,n)$ is the smallest integer k such that some k-dimensional convex body is centrally a-universal for all n-dimensional (centered) convex polyhedra having $2r$ vertices respectively maximal faces.

We are able to prove that

$$\infty > \xi^{a,v}(n,r) \geqslant \frac{n}{n+1}(r+1) \leqslant \xi^{a,f}(n,r) \leqslant r,$$

$$\infty > \eta^{a,v}(n,r) \geqslant \eta^{a,f}(n,r) = r,$$

$$\text{and} \quad \infty > \xi^{s,v}(n,r) \geqslant \frac{n}{n+1}(r+2) \leqslant \xi^{s,f}(n,r) < \infty .$$

Sharper results are obtained for special values of n and r, but many unsolved problems remain.

In § 2 below, we establish the Lipschitzian nature of certain transformations involving convex bodies, while § 3 studies the Hausdorff dimension of certain spaces of convex bodies. In the concluding § 4, results from §§ 2–3 are combined to yield our principal theorems, and some unsolved problems are mentioned.

§ 2. Some Lipschitzian transformations

A transformation φ of a metric space (M, ϱ) into another metric space (M', ϱ') will be called *Lipschitzian* (with *associated constant* B) provided there exists $B<\infty$ such that $\varrho'(\varphi x, \varphi y) \leqslant B\varrho(x, y)$ for all $x, y \in M$; and φ is *locally Lipschitzian* at a point $z \in M$ provided φ is Lipschitzian on some neighborhood of z.

2.1. PROPOSITION. *Suppose (M, ϱ) is a compact metric space and φ is a transformation of M into a metric space (M', ϱ'). Then φ is Lipschitzian if it is locally Lipschitzian at each point of M.*

Proof. For each point $z \in M$ there are a neighborhood V_z of z and a number $B_z < \infty$ such that $\varrho'(\varphi x, \varphi y) \leqslant B_z \varrho(x, y)$ whenever $x, y \in V_z$. Since M is compact, there are points $z_1, \ldots, z_n$ of M and a number $\varepsilon > 0$ such that for each $z \in M$, the ε-neighborhood of z lies in at least one of the sets V_{z_i}. With $B' = \max_i B_{z_i}$, we have $\varrho'(\varphi x, \varphi y) \leqslant B' \varrho(x, y)$ whenever $\varrho(x, y) < \varepsilon$. Since φ is continuous, the set φM must be compact and hence of finite diameter δ; whenever $\varrho(x, y) \geqslant \varepsilon$, we have $\varrho'(\varphi x, \varphi y) \leqslant (\delta/\varepsilon)\varrho(x, y)$. Then for $B = \max(B', \delta/\varepsilon)$, it is clear that φ is Lipschitzian with associated constant B.

For two subsets X and Y of a metric space M, the *Hausdorff distance* $D(X, Y)$ is the greatest lower bound of all numbers ε such that X lies in the ε-neighborhood of Y and Y in the ε-neighborhood of X. It is evident that if φ is a Lipschitzian transformation of M with associated constant B, then $D(\varphi X, \varphi Y) \leqslant BD(X, Y)$ for all $X, Y \subset M$.

2.2. LEMMA. *Suppose C_1 and C_2 are convex bodies in a normed linear space, having a common interior point p. Let $\mathfrak{F}$ be the family of all flats F through p, S_i the space of all sections $\{C_i \cap F : F \in \mathfrak{F}\}$, metrized by the Hausdorff metric. For each $F \in \mathfrak{F}$, set $\varphi(C_1 \cap F) = C_2 \cap F$. Then φ is a Lipschitzian transformation of S_1 onto S_2.*

Proof. We may assume without loss of generality that p is the origin 0. Let g denote the radial map of C_1 onto C_2 — for each ray r emanating from 0, g maps the segment $C_1 \cap r$ linearly onto the segment $C_2 \cap r$. It is proved in [9] that g is Lipschitzian. It is evident that (with a slight abuse of notation) $\varphi S = gS$ for each $S \in S_1$, so the desired conclusion follows from the remark just preceding the statement of 2.2.

Lemma 2.3 below extends the fact that a convex function is locally Lipschitzian at each point interior to its domain, while Theorem 2.4 generalizes both 2.2 and 2.3.

2.3. LEMMA. *Suppose C is a convex body in a normed linear space E, L_0 is a linear subspace of E, and $\mathfrak{L}$ is the set of all translates of L_0 which intersect the interior of C. For each $L \in \mathfrak{L}$, let $\varphi L = L \cap C$. Then if $\mathfrak{L}$ and $\varphi \mathfrak{L}$ are both metrized by the Hausdorff metric, the transformation φ is locally Lipschitzian at each point of $\mathfrak{L}$.*

Proof. Let U denote the unit cell of E, and for each $\varepsilon > 0$ let $\mathfrak{L}_\varepsilon$ denote the set of all $L \in \mathfrak{L}$ such that $x + \varepsilon U \subset C$ for some $x \in L$. Then $\mathfrak{L} = \bigcup_{\varepsilon > 0} \mathfrak{L}_\varepsilon$; and if $L \in \mathfrak{L}_{\varepsilon_1}$, $M \in \mathfrak{L}$, and $D(L, M) < \varepsilon_2 < \varepsilon_1$, then $M \in \mathfrak{L}_{\varepsilon_1 - \varepsilon_2}$. Thus to show that φ is locally Lipschitzian at each point of $\mathfrak{L}$ it suffices to prove that φ is Lipschitzian on each set $\mathfrak{L}_\varepsilon$. To establish the latter fact (with associated constant δ/ε where δ is the diameter of C) we show that if $L, M \in \mathfrak{L}_\varepsilon$, $d > D(L, M)$, and $x \in L \cap C$, then there exists $y \in M \cap C$ with $\|x - y\| < (\delta/\varepsilon) d$. We may assume without loss of generality that $x = 0$, whence $L = L_0$ and $M = L_0 + w$ for some point w with $\|w\| < d$. Since $M \in \mathfrak{L}_\varepsilon$, there exists $p \in M$ with $p + \varepsilon U \subset C$, and then, since $\|w\| < d$, we have $p + (\varepsilon/d) w \in C$ and $\|p + (\varepsilon/d) w\| < \delta$. Since C is convex and $0 = x \in C$, C must include the point $(d/(d + \varepsilon))(p + (\varepsilon/d) w)$, whose norm is of course less than $(\delta/\varepsilon) d$. But $p = v + w$ for some $v \in L_0$ and then

$$\frac{d}{d+\varepsilon}\left(p + \frac{\varepsilon}{d} w\right) = \frac{d}{d+\varepsilon} v + w \in M,$$

completing the proof of 2.3.

2.4. THEOREM. *Suppose C and K are convex bodies in a normed linear space E and $\mathfrak{F}$ is the family of all flats in E which intersect both the interior of C and the interior of K. For each $F \in \mathfrak{F}$, let $\xi F = F \cap C$ and $\eta F = F \cap K$. Then if $\xi \mathfrak{F}$ and $\eta \mathfrak{F}$ are both metrized by the Hausdorff metric, the transformation $\eta \xi^{-1}$ (of $\xi \mathfrak{F}$ onto $\eta \mathfrak{F}$) is locally Lipschitzian at each point of $\xi \mathfrak{F}$.*

Proof. For each $\varepsilon > 0$, let $\mathfrak{F}_\varepsilon$ be the set of all flats F which include points x and y such that $x + \varepsilon U \subset C$ and $y + \varepsilon U \subset K$ (U being the unit cell of E). We shall prove that the transformation $\eta \xi^{-1}$ is Lipschitzian on $\xi \mathfrak{F}_\varepsilon$, and from this the desired conclusion follows. Since $\eta \mathfrak{F}_\varepsilon$ is bounded, it suffices (as in the proof of 2.1) to produce numbers $B < \infty$ and $d > 0$ such that

$$D(\eta F, \eta F') \leqslant B D(\xi F, \xi F') \text{ whenever } F, F' \in \mathfrak{F}_\varepsilon \text{ and } D(\xi F, \xi F') < d.$$

In proving 2.2 we appealed to a theorem on radial mappings, established in [9], which asserted the Lipschitzian nature of a transformation associated with a pair of convex bodies. Examination of [9] shows its reasoning to be of a "uniform" nature in that it actually establishes the following: For each pair of positive numbers r and s there is a number $J_{r,s} < \infty$ such that whenever V and W are convex bodies in a normed linear space with unit cell U, and $rU \subset V \cap W \subset V \cup W \subset sU$, then the radial transformations of V onto W and of W onto V are both Lipschitzian with associated constant $J_{r,s}$. This makes possible a uniformized version of 2.2. Similarly, in 2.3 the simple form of the associated constant (for the restriction of φ to $\mathfrak{C}_\varepsilon$) leads to a stronger result of uniform nature. Now let Z be a convex body containing $C \cup K$, δ the diameter of Z, $J = J_{\frac{1}{2}\varepsilon, \delta}$, and $A = \delta/(\frac{1}{2}\varepsilon)$. We shall show that if $F, F' \in \mathfrak{F}_\varepsilon$ with $D(\xi F, \xi F') < \alpha < \frac{1}{2}\varepsilon$, then $D(\eta F, \eta F') \leqslant (A + 3AJ + J^2)\alpha$.

With $F' \in \mathfrak{F}_\varepsilon$, there are points x and y of F' such that $x + \varepsilon U \subset C$ and $y + \varepsilon U \subset K$. And $D(F, F') \leqslant D(\xi F, \xi F') < \alpha < \frac{1}{2}\varepsilon$, so there are points $p \in F \cap (x + \alpha U)$ and $q \in F \cap (y + \alpha U)$; we have $p + \frac{1}{2}\varepsilon U \subset C$ and $q + \frac{1}{2}\varepsilon U \subset K$. Let $F_1 = F' + (p - x)$ and $F_2 = F' + (q - x)$. For each $G \in \mathfrak{F}$, let $\zeta G = G \cap Z$. Then employing the triangle inequality for D and the uniform versions of 2.2 and 2.3 we see that

$$D(\eta F, \eta F') \leqslant D(\eta F, \eta F_2) + D(\eta F_2, \eta F'),$$

$$D(\eta F_2, \eta F') \leqslant A\,D(F_2, F') < A\alpha,$$

$$D(\eta F, \eta F_2) \leqslant J\,D(\zeta F, \zeta F_2),$$

$$D(\zeta F, \zeta F_2) \leqslant D(\zeta F, \zeta F_1) + D(\zeta F_1, \zeta F_2),$$

$$D(\zeta F_1, \zeta F_2) \leqslant A\,D(F_1, F_2) < 2A\alpha,$$

$$D(\zeta F, \zeta F_1) \leqslant J\,D(\xi F, \xi F_1) \leqslant J\,D(\xi F, \xi F') + J\,D(\xi F', \xi F_1),$$

$$D(\xi F, \xi F') < \alpha,$$

and $D(\xi F', \xi F_1) \leqslant A\,D(F', F_1) < A\alpha.$

It follows that $D(\eta F, \eta F') \leqslant (A + 3AJ + J^2)\alpha$, and Theorem 2.4 has been proved.

The next two lemmas (which will be employed in proving Theorem 2.7) can be improved in quantitative aspects, but for our present purposes they are adequate as they stand. (1)

2.5. LEMMA. *Suppose $x_1, \ldots, x_k$ is an orthonormal basis for E^k, $0 < \varepsilon < 1/2k$, y is a unit vector, and $|\,\|y - x_j\| - \sqrt{2}\,| < \varepsilon$ for $j = 1, 2, \ldots, k-1$. Then either $\|y - x_k\| < \sqrt{2k}\,\varepsilon$ or $\|y + x_k\| < \sqrt{2k}\,\varepsilon$.*

Proof. Let $y = \sum_1^k b_i x_i$, so that

$$\sum_1^k b_i^2 = 1. \tag{1}$$

(1) See the footnote on page 251.

For $1 \leqslant j \leqslant k-1$, define η_j by the equation

$$\eta_j = [\sum_{i \neq j} b_i^2 + (b_j - 1)^2]^{\frac{1}{2}} - \sqrt{2}, \tag{2}$$

so that

$$|\eta_j| < \varepsilon. \tag{3}$$

For $1 \leqslant j \leqslant k-1$, substitution of (1) in (2) shows that

$$(2 - 2 b_j)^{\frac{1}{2}} = \sqrt{2} + \eta_j, \text{ or } -b_j = \sqrt{2}\,\eta_i + \tfrac{1}{2}\eta_j^2,$$

whence from (3) and the fact that $\varepsilon < \frac{1}{2}k < 1$ we have

$$|b_j| < \sqrt{2}\,\varepsilon + \tfrac{1}{2}\varepsilon^2 = (\sqrt{2} + \tfrac{1}{2}\varepsilon)\,\varepsilon < 2\varepsilon. \tag{4}$$

Now by (1) and (4),

$$1 \geqslant |b_k| = [1 - \sum_1^{k-1} b_i^2]^{\frac{1}{2}} > B_\varepsilon = [1 - (k-1)\,4\,\varepsilon^2]^{\frac{1}{2}}. \tag{5}$$

Assuming that $b_k \geqslant 0$ (for the other case is handled similarly), we see from (4) and (5) that

$$\|y - x_k\|^2 \leqslant (k-1)\,4\,\varepsilon^2 + (1 - B_\varepsilon)^2 = 2 - 2 B_\varepsilon.$$

Now whenever $|\varepsilon| < 1/(2\sqrt{k-1})$, define

$$f\varepsilon = 2 k \varepsilon^2 - 2 + 2 B_\varepsilon.$$

To prove the lemma it suffices to show that $f\varepsilon > 0$ whenever $0 < \varepsilon < \frac{1}{2}k$. Now the function f is differentiable on the interval $]-1/(2\sqrt{k-1}),\ 1/(2\sqrt{k-1}[\supset [-1/(2k),\ 1/(2k)]$, and of course $f0 = 0$. Since for $|\varepsilon| < \frac{1}{2}k$ we have

$$B_\varepsilon > \left[1 - 4(k-1)\left(\frac{1}{2k}\right)^2\right]^{\frac{1}{2}} = \frac{(k^2 - k + 1)^{\frac{1}{2}}}{k} > \frac{(k^2 - 2k + 1)^{\frac{1}{2}}}{k} = \frac{(k-1)}{k},$$

and since

$$f'\varepsilon = 4 k \varepsilon - \frac{4(k-1)\varepsilon}{B_\varepsilon},$$

we conclude that

$$f'\varepsilon > 0 \text{ for } 0 < \varepsilon < \tfrac{1}{2}k,$$

and the desired conclusion is then a consequence of the mean-value theorem.

2.6. Lemma. (1) *For each positive integer k there is a number A_k which has the following property:*

(1) See the footnote on page 251.

whenever F and G are k-dimensional subspaces of a Euclidean space with unit cell C, $F \cap C \subset G + \varepsilon C$, and $x_1, \ldots, x_k$ is an orthonormal basis for F, then there is an orthonormal basis $y_1, \ldots, y_k$ for G such that always $\|x_i - y_i\| \leqslant A_k \varepsilon$.

Proof. For $n = 1, 2, \ldots$, let $a_n = 4(1+\sqrt{2k})^{n-1}$. We shall prove below that if $0 < \varepsilon < 1/(2ka_k)$, then the subspace G admits an orthonormal basis $y_1, \ldots, y_k$ with always $\|x_i - y_i\| < a_k \varepsilon$. And of course if $y_1, \ldots, y_k$ is an arbitrary orthonormal basis for G, then always $\|x_i - y_i\| \leqslant 2$, and hence $\|x_i - y_i\| \leqslant 4ka_k\varepsilon$ provided $\varepsilon \geqslant 1/(2ka_k)$. Thus it will follow that the constant $A_k = 4ka_k$ has the stated property.

We suppose, then, that $x_1, \ldots, x_k$ is an orthonormal basis for F and that $\varepsilon < 1/(2ka_k)$. For each i there is a point y_i'' of V such that $\|x_i - y_i''\| \leqslant \varepsilon$. Then, of course,

$$1 \geqslant \|y_i''\| \geqslant \|x_i\| - \|x_i - y_i''\| \geqslant 1 - \varepsilon,$$

so with $y_i' = y_i''/\|y_i''\|$ we have

$$\|y_i'\| = 1, \; \|x_i - y_i'\| \leqslant 2\varepsilon.$$

Let $y_1 = y_1'$. Then

$$\|x_1 - y_1\| \leqslant (a_1 - 2)\varepsilon,$$

and for $1 < j \leqslant k$,

$$|\,\|y_j' - y_1\| - \sqrt{2}\,| = |\,\|y_j' - y_1\| - \|x_j - x_1\|\,| \leqslant \|y_j' - x_j\| + \|x_1 - y_1\| \leqslant 4\varepsilon = a_1\varepsilon.$$

Now suppose the orthonormal set $y_1, \ldots, y_m$ has been constructed so that

$$\|x_i - y_i\| \leqslant (a_i - 2)\varepsilon \quad \text{for } i = 1, \ldots, m,$$

and

$$|\,\|y_j' - y_i\| - \sqrt{2}\,| \leqslant a_i \varepsilon \quad \text{for } 1 \leqslant i \leqslant m < j \leqslant k.$$

(Such a construction has already been effected for $m = 1$.) In determining y_{m+1}, we first note that since

$$a_i\varepsilon < a_m\varepsilon \leqslant a_k\varepsilon < \tfrac{1}{2}k,$$

there follows from 2.5 the existence of a unit vector $y_{m+1} \in G$ such that y_{m+1} is orthogonal to y_i $(1 \leqslant i \leqslant m)$ and

$$\|y_{m+1}' - y_{m+1}\| < \sqrt{2k}\, a_m \varepsilon.$$

Then

$$\|x_{m+1} - y_{m+1}\| \leqslant \|x_{m+1} - y_{m+1}'\| + \|y_{m+1}' - y_{m+1}\| < 2\varepsilon + \sqrt{2k}\, a_m\varepsilon \leqslant (a_{m+1} - 2)\varepsilon.$$

And for $m + 1 < j \leqslant k$,

$$| \|y_j' - y_{m+1}\| - \sqrt{2} | = | \|y_j' - y_{m+1}\| - \|x_j - x_{m+1}\| |$$
$$\leqslant \|y_j' - x_j\| + \|x_{m+1} - y_{m+1}\| \leqslant 2\varepsilon + (a_{m+1} - 2)\varepsilon = a_{m+1}\varepsilon.$$

Thus we proceed by mathematical induction to construct the orthonormal sequence $y_1, \ldots, y_k$ with always

$$\|x_i - y_i\| \leqslant (a_i - 2)\varepsilon < a_k \varepsilon,$$

and the proof of 2.6 is complete.

We wish now to describe certain spaces of equivalence-classes of convex sets which will play a fundamental role in the sequel. For $n \geqslant 2$, let $\mathcal{B}^n$ denote the class of all n-dimensional convex bodies in E^n, $\mathcal{A}^n$ the group of all nonsingular affine transformations of E^n onto itself, and $\mathcal{S}^n$ the group of all similarity transformations of E^n onto itself. (Neither the members of $\mathcal{A}^n$ nor those of $\mathcal{S}^n$ need preserve orientation.) Let $G = \mathcal{A}^n$ or $G = \mathcal{S}^n$. Two members K and K' of $\mathcal{B}^n$ are said to be *G-equivalent* provided $K = \sigma K'$ for some $\sigma \in G$; the set of equivalence-classes so obtained will be denoted by $\mathcal{E}_G$. Now for $K, K' \in \mathcal{B}^n$, let

$$\psi(K, K') = \inf_{\sigma \in G,\, \sigma K \supset K'} V(\sigma K)/V(K'),$$

where V is the n-dimensional volume function. Then $1 \leqslant \psi$ and ψ is affine-invariant — that is, $\psi(\sigma K, \tau K') = \psi(K, K')$ for all $\sigma, \tau \in G$, $K, K' \in \mathcal{B}^n$. For $\mathfrak{K}, \mathfrak{K}' \in \mathcal{E}_G$ choose $K \in \mathfrak{K}$, $K' \in \mathfrak{K}'$, and define

$$\Delta(\mathfrak{K}, \mathfrak{K}') = \log \psi(K, K') + \log \psi(K', K).$$

The argument employed by Macbeath [11] for the case $G = \mathcal{A}^n$ shows that Δ is a metric for $\mathcal{E}_G$. We shall henceforth regard $\mathcal{E}_G$ as a metric space with distance-function Δ.

The above definitions can be paraphrased for the class $\mathcal{B}_0^n$ of all members of $\mathcal{B}^n$ which are centered at the origin 0, and we denote by $\mathcal{E}_G^0$ the resulting set of equivalence classes. Let G_0 denote the set of all linear members of G (those which map 0 into 0). Then two members K and K' of $\mathcal{B}_0^n$ are G-equivalent if and only if they are G_0-equivalent, and the number $\psi(K, K')$ defined above is equal to $\inf_{\sigma \in G_0,\, \sigma K \supset K'} V(\sigma K)/V(K')$. Thus the metric on $\mathcal{E}_G^0$ induced by that of $\mathcal{E}_G$ agrees with the metric on $\mathcal{E}_G^0$ obtained by dealing only with $\mathcal{B}_0^n$ and G_0. This renders permissible certain "identifications" which we shall employ without further comment.

2.7. THEOREM. *Suppose* $0 \leqslant k \leqslant n$, E_k^n *is the space of all k-dimensional convex bodies in* E^n *(metrized by the Hausdorff metric),* $\mathcal{A}^k$ *is the group of all nonsingular affine transformations of* E^k *onto* E^k, *and* $\mathcal{S}^k$ *is the group of all similarity transformations of* E^k *onto* E^k. *Let* $G = \mathcal{A}^k$ *or* $G = \mathcal{S}^k$ *and let* φ *denote the natural map of* E_k^n *onto the space* $\mathcal{E}_G$ *of all G-equivalence classes of k-dimensional convex bodies in* E^k *(metrized by Macbeath's metric). Then* φ *is locally Lipschitzian at each point of* E_k^n.

Proof. Consider an arbitrary $K \in E_k^n$ and let L denote the flat determined by K. We may assume without loss of generality that the origin is interior to K relative to L. Then if U is the unit cell of the subspace L, there exist positive numbers m and M such that $m < \frac{1}{2}$ and $5mU \subset K \subset \frac{1}{2}MU$. We shall prove that φ is Lipschitzian on the m-neighborhood of K.

Consider $X, Y \in E_k^n$ with $D(X, K) < m > D(Y, K)$ and $D(X, Y) = \varepsilon$. There exist $p \in X$ with $\|p\| < m$ and $q \in Y$ with $\|p-q\| \leqslant \varepsilon < 2m$. Let $X' = X - p$ and $Y' = Y - q$. Then

$$D(X', K) \leqslant D(X', X) + D(X, K) < \|p\| + m < 2m,$$

$$D(Y', K) \leqslant D(Y', Y) + D(Y, K) < \|q\| + m < 4m,$$

and $D(X', Y') \leqslant D(X-p, Y-p) + D(Y-p, Y-q) \leqslant D(X, Y) + \|p-q\| \leqslant 2\varepsilon$.

Let π denote the orthogonal projection of E^n onto L; let $X'' = \pi X'$ and $Y'' = \pi Y'$. Then $\pi K = K$ and π is Lipschitzian with associated constant 1, so

$$D(X'', K) \leqslant D(X', K) < 4m > D(Y', K) \geqslant D(Y'', K).$$

Since $5mU \subset K$, it follows that $mU \subset X'' \cap Y''$. (For example, if there exists $z \in L \sim X''$ with $\|z\| \leqslant m$, then by the separation theorem for convex sets there exists $u \in L$ with $\|u\| = 1$ and $(u, z) \geqslant \sup_{x \in X''}(u, x)$ where $(\,,)$ denotes the inner product. But then of course $\sup_{x \in X''}(u, x) \leqslant m$ and it follows that the minimum distance from the point $5mu$ to the set X'' is at least $4m$, contradicting the fact that $D(X'', K) < 4m$ and $5mu \in 5mU \subset K$.)

Now let F and G denote, respectively, the linear subspaces determined by X' and by Y' in E^n. From the fact that $mU \subset X'' \cap Y''$ it can be deduced that the projection π is biunique on both F and G, and that $X' \supset F \cap mC$ and $Y' \supset G \cap mC$, where C is the unit cell of E^n. Now since $F \cap mC \subset X'$ and $D(X', Y') \leqslant 2\varepsilon$, it follows that $F \cap C \subset G + (2\varepsilon/m)C$, whence by Lemma 2.6 [1] there are orthonormal bases $x_1, \ldots, x_k$

[1] Professor R. Kadison has remarked that if f and g are othogonal projections of E^n onto linear subspaces F and G of the same dimension, $\|f-g\| < \delta < 1$, v is the partial isometry determined by the polar decomposition of fg $(fg = v(gfg)^{\frac{1}{2}})$, and τ is the restriction to F of the adjoint of v, then τ is a linear isometry of F onto G and $\|\tau - f\| < \delta$. This fact can be used to eliminate Lemma 2.6 (and hence also 2.5) from the proof of Theorem 2.7; it leads also to a stronger form of 2.6.

and $y_1, \ldots, y_k$ for F and G respectively such that always $\|x_i - y_i\| \leqslant A_k(2\varepsilon/m)$. Let τ denote the linear isometry of F onto G for which always $\tau x_i = y_i$, and let $X_1 = \tau X'$, $Y_1 = Y'$. It is evident that $\varphi X_1 = \varphi X$ and $\varphi Y_1 = \varphi Y$. It is easy to verify that $\|x - \tau x\| \leqslant A_k(2\varepsilon/m)\|x\|$ for all $x \in F$, and hence that

$$D(X_1, Y_1) \leqslant D(X_1, X') + D(X', Y') \leqslant A_k \frac{2\varepsilon}{m} M + 2\varepsilon,$$

where the second inequality depends on the fact that $X' \subset M C$. Thus with

$$a = 2 + 2A_k M/m,$$

we have

$$D(X_1, Y_1) \leqslant a\varepsilon.$$

We shall use also the fact that if V denotes the unit cell of G, then

$$mV \subset X_1 \cap Y_1 \subset X_1 \cup Y_1 \subset MV.$$

Evidently $\varphi(1 + a\varepsilon/m)X_1 = \varphi X_1 = \varphi X$. Since $X_1 \supset mV$ and $D(X_1, Y_1) \leqslant a\varepsilon$, it follows that

$$\left(1 + \frac{a\varepsilon}{m}\right) X_1 \supset X_1 + a\varepsilon V \supset Y_1$$

and thus

$$\psi(X, Y) \leqslant \frac{\nu(1 + a\varepsilon/m)X_1}{\nu Y_1},$$

where ν denotes the k-dimensional volume. Since $X_1 \subset MV$ and $D(X_1, Y_1) \leqslant a\varepsilon$, we have

$$\left(1 + \frac{a\varepsilon}{m}\right) X_1 \subset X_1 + \frac{a\varepsilon M}{m} V \subset (Y_1 + \varepsilon V) + \frac{aM}{m}\varepsilon V = Y_1 + b\varepsilon V,$$

where the constant $b = 1 + aM/m$ is independent of X and Y (subject, of course, to the condition that $D(X, K) < m > D(Y, K)$). Now by the basic theorem on mixed volumes (or more special results on parallel bodies), it is true that

$$\nu(Y_1 + b\varepsilon V) = \nu Y_1 + \beta b\varepsilon + \gamma(b\varepsilon)^2,$$

where the non-negative coefficients β and γ are dependent on Y_1 but, since $Y_1 \subset MV$ are bounded above by the number $2^n \nu(MV)$. (For proof of the necessary inequality see, for example, pp. 84–85 of [4].) Now recalling that $\varepsilon < 2m < 1$ and $Y_1 \supset mV$, we see that

$$\nu(Y_1 + b\varepsilon V) \leqslant \nu Y_1 + 2^n \nu(MV)(b + b^2)\varepsilon$$

and hence

$$\psi(X_1, Y_1) \leqslant \nu(Y_1 + b\varepsilon V)/\nu Y_1 \leqslant 1 + \alpha\varepsilon,$$

where the constant $\alpha=(2M/m)^n(b+b^2)$ is independent of X and Y. The same argument shows that $\psi(Y_1, X_1)\leqslant 1+\alpha\varepsilon$, and we conclude that

$$\Delta(\varphi X, \varphi Y)\leqslant 2\log(1+\alpha\varepsilon)\leqslant 2\alpha\varepsilon\leqslant 2\alpha D(X, Y),$$

completing the proof of 2.7.

Observe that the lemmas 2.5 and 2.6 are unnecessary for treatment of the case $G=\mathcal{A}^n$, for then $\varphi X''=\varphi X$ and $\varphi Y''=\varphi Y$. (1)

We conclude the section with

2.8. PROPOSITION. *Suppose E is a normed linear space and $\mathcal{K}$ (resp. $\mathcal{K}^*$) is the space of all convex bodies in E (resp. E^*) whose interior includes the origin, metrized by the Hausdorff metric. For each $K\in\mathcal{K}$, let ζK denote the polar body $K^0=\{f\in E^*\colon \sup fK\leqslant 1\}\in\mathcal{K}^*$. Then ζ is a locally Lipschitzian homeomorphism of $\mathcal{K}$ into $\mathcal{K}^*$.*

Proof. It is evident that ζ is a biunique map of $\mathcal{K}$ into $\mathcal{K}^*$. We shall prove that ζ is locally Lipschitzian and hence continuous. Essentially the same argument shows that ζ^{-1} is also locally Lipschitzian, whence ζ is a homeomorphism.

For each $r>0$, let $\mathcal{K}_r$ denote the set of all $K\in\mathcal{K}$ for which $rU\subset K$, where U is the unit cell of E. We will show that ζ is Lipschitzian on $\mathcal{K}_r$ (with associated constant $1/r^2$), whence the desired conclusion follows. Consider arbitrary $C, K\in\mathcal{K}_r$, $\delta<D(C^0, K^0)$, and $\varepsilon>D(C, K)$. We wish to prove that $\varepsilon>r^2\delta$. Since $D(C^0, K^0)>\delta$, one of the sets C^0 and K^0 must include a point at distance $>\delta$ from the other — say there exists $f''\in C^0$ with $\inf_{g\in K^0}\|f-g\|>\delta$. Then the sets K^0-f'' and $\{g\in E^*\colon \|g\|\leqslant\delta\}$ are disjoint, convex, and w^*-compact, so by a known separation theorem they can be separated by a w^*-closed hyperplane — that is, there exists $x\in E$ such that $\|x\|=1$ and $\delta<\inf_{g\in K^0}(f''-g)x$. By w^*-compactness of C^0 there exists $f'\in C^0$ such that $f'x=\sup_{f\in C^0}fx$. With $a=\sup_{f\in C^0}fx$ and $b=\sup_{g\in K^0}gx$, we have

$$a-b\geqslant f''x-\sup_{g\in K^0}gx=\inf_{g\in K^0}(f''-g)x>\delta.$$

Now $C=(C^0)^0$ and $K=(K^0)^0$ under the usual duality between E and E^*, so from the definitions of a and b it follows that $tx\notin C$ for $t>1/a$, and that $(1/b)x\in K$. Since $\varepsilon>D(C, K)$ and $(1/b)x\in K$, there exist $s\in]0, \varepsilon[$ and $u\in E$ with $\|u\|=1$ such that $(1/b)x+su\in C$. And $-ru\in C$ since $C\in\mathcal{K}_r$. Then with $t=r/(r+s)$, we see by convexity of C that

$$\frac{r}{(r+s)b}x=(1-t)(-ru)+t\left(\frac{1}{b}x+su\right)\in C.$$

(1) See the footnote on page 251.

It follows that $r/(r+s)b \leqslant 1/a$, whence $r(a-b) \leqslant sb$ and we have

$$\varepsilon > s \geqslant \frac{r(a-b)}{b} \geqslant \frac{r}{b}\delta.$$

But since $K \in \mathcal{K}_r$ it is true that $1/b \geqslant r$, whence $\varepsilon > r^2\delta$ and the proof of 2.8 is complete.

§ 3. The Hausdorff dimension of certain sets

Consider a metric space M. For each $r \in [0, \infty[$, $X \subset M$, and $\varepsilon > 0$, let $m_r^\varepsilon X$ denote the greatest lower bound of all numbers of the form $\sum_{i=1}^{\infty}(\delta A_i)^r$, where A_α is a sequence of sets covering X and each set A_i is of diameter $\delta A_i < \varepsilon$. Then set $m_r X = \sup_{\varepsilon > 0} m_r^\varepsilon X$. The function m_r is the *Hausdorff r-measure* [7] for M and is a Caratheodory outer measure for the class of all subsets of M, giving rise to a regular Borel measure. If $m_r X < \infty$, then $m_s X = 0$ for all $s > r$. The *Hausdorff dimension* of X is the least upper bound of all numbers $r \in [0, \infty[$ for which $m_r X > 0$. If hdim denotes the Hausdorff dimension and tdim the topological dimension (i.e., the Menger-Urysohn dimension [8]), then from a theorem of Szpilrajn [16, 8] it is known that for each nonempty separable metric space M, hdim $M \geqslant$ tdim M and M admits a metric homeomorph M' for which hdim $M' =$ tdim M'. It is evident that if a metric space M be subjected to a Lipschitzian transformation φ with associated constant B, then $m_r \varphi X \leqslant B^r m_r X$ for all $X \subset M$ and $r \in [0, \infty[$. We shall use these facts freely without further reference, as well as the fact that a subset of E^n (with its usual metric) has finite Lebesgue outer measure if and only if its Hausdorff n-measure is finite.

In solving Mazur's problem — proving that no finite-dimensional convex body is centrally a-universal for all j-dimensional convex bodies — it is enough to know that if C is the unit cell in E^n and S_j^n is the space of all central j-sections of C, metrized by the Hausdorff metric, then the Hausdorff r-measure of S_j^n is finite for some r. But that is a very crude result, and for sharper conclusions we should like to determine the exact Hausdorff dimension of S_j^n. It is well-known that S_j^n is in fact a manifold of topological dimension $j(n-j)$ (a "Grassman manifold"), so the "best" we could hope for is that $0 < m_{j(n-j)}(S_j^n, D) < \infty$. We shall establish this inequality by using a known homeomorphism between S_j^n and a quotient space of the orthogonal group. (I am indebted to Professor W. Fenchel for suggesting this approach, and to Professor R. Kadison for a helpful suggestion concerning group representations.)

3.1. PROPOSITION. *Suppose $\mathfrak{J}$ is the group of all linear isometries of E^n, metrized by means of the uniform norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$. Suppose J is a j-dimensional linear subspace of E^n and $\mathcal{G}$ is the subgroup of $\mathfrak{J}$ consisting of all $T \in \mathfrak{J}$ for which $TJ = J$. Then (with respect to the metric induced in $\mathfrak{J}/\mathcal{G}$, the space of left cosets, by the uniform metric in $\mathfrak{J}$) the Hausdorff $j(n-j)$ measure of $\mathfrak{J}/\mathcal{G}$ is positive and finite.*

Proof. Let $x_1, \ldots, x_n$ be an orthonormal basis for E^n such that $x_1, \ldots, x_j \in J$. Let $\mathfrak{M}_n$ be the vector space of all $n \times n$ real matrices, $\mathfrak{L}_n$ the set of all nonsingular members of $\mathfrak{M}_n$, O_n the set of all orthogonal members of $\mathfrak{M}_n$, and $O_{n,j}$ the set of all orthogonal matrices $\sigma = (\sigma_{rs})$ such that $\sigma_{rs} = 0 = \sigma_{sr}$ whenever $r < j < s$. Then employing (with respect to the orthonormal basis $x_1, \ldots, x_n$) the usual identification of matrices with linear transformations, we have $O_n = \mathfrak{J}$, $O_{n,j} = \mathcal{G}$, and for each $\sigma \in \mathfrak{M}_n$,

$$\|\sigma\| = \sup_{\Sigma_1^n t_i^2 \leq 1} \left[\textstyle\sum_{j=1}^n \left(\sum_{i=1}^n \sigma_{ji} t_i\right)^2\right]^{\frac{1}{2}}.$$

For each $\varepsilon > 0$, let U_ε denote the compact neighborhood of the origin in $\mathfrak{M}_n$, defined as follows:

$$U_\varepsilon = \{\sigma \in \mathfrak{M}_n\colon \text{ always } |\sigma_{rs}| \leq \varepsilon\}.$$

For each $\sigma \in \mathfrak{M}_n$, let $\exp \sigma$ be defined as usual:

$$\exp \sigma = \delta + \sigma + \frac{1}{2!}\sigma^2 + \cdots + \frac{1}{n!}\sigma^n + \cdots,$$

where δ is the unit matrix ($\delta_{rs} = 1$ when $r = s$, $\delta_{rs} = 0$ when $r \neq s$). We shall employ the following well-known properties of the mapping exp, which can be found, for example, on pp. 5–9 of [2] and pp. 72–73, 76–77 of [14]:

(i) exp is an analytic transformation of $\mathfrak{M}_n$ into $\mathfrak{L}_n$;

(ii) for a sufficiently small $\varepsilon > 0$ it is true that

a) U_ε is mapped topologically by the transformation exp onto a neighborhood V_0 of the unit matrix δ in $\mathfrak{L}_n$;

b) if $\mathfrak{T}_n$ is the subspace consisting of all skew-symmetric members of $\mathfrak{M}_n$, then $\exp(U_\varepsilon \cap \mathfrak{T}_n) \subset O_n$;

c) for each decomposition of $\mathfrak{M}_n$ into supplementary linear subspaces L' and L'', each element of $\mathfrak{L}_n$ near enough to δ admits a unique expression as a product $(\exp \sigma')$ $(\exp \sigma'')$ for $\sigma' \in U_\varepsilon \cap L'$ and $\sigma'' \in U_\varepsilon \cap L''$.

Now let $\mathfrak{T}_{n,j}$ denote the subspace of $\mathfrak{M}_n$ consisting of all $\sigma \in \mathfrak{T}_n$ such that $\sigma_{rs} = 0 = \sigma_{sr}$ whenever $r < j < s$. Let Q' be a subspace supplementary to $\mathfrak{T}_{n,j}$ in $\mathfrak{M}_n$, and let $Q = Q' \cap \mathfrak{J}_n$. The dimension of $\mathfrak{J}_n$ is $\frac{1}{2}n(n-1)$ and that of $\mathfrak{J}_{n,j}$ is $\frac{1}{2}j(j-1) + \frac{1}{2}(n-j)(n-j-1)$, whence it follows that the dimension of Q is $j(n-j)$. It can be verified that $\exp(U_\varepsilon \cap \mathfrak{J}_{n,j}) = V_0 \cap O_{n,j}$, which by condition (ii a) is a neigborhood of δ in $O_{n,j}$. By (ii c) there is a compact neighborhood Z of δ in $\mathfrak{L}_n$ such that each $z \in Z$ admits a unique expression in the form $z = (\exp \xi_z)(\exp \eta_z)$ for $\xi_z \in \mathfrak{J}_{n,j}$ and $\eta_z \in Q$; from (ii b) it follows that $\eta_z \in Q$ whenever $z \in Z \cap O_n$.

Since exp is analytic, it is easily seen to be Lipschitzian on compact subsets of $\mathfrak{M}_n$ — in particular, on the sets ξZ and ηZ, say with associated constants B_ξ and B_η. Denoting by ϱ the uniform metric in $\mathfrak{J}$, we see that ϱ is two-sided invariant and hence that for all $z, z' \in Z$ it is true that

$$\begin{aligned}\varrho(z, z') &= \varrho(\exp \xi_z \exp \eta_z, \exp \xi_{z'} \exp \eta_{z'})\\ &\leqslant \varrho(\exp \xi_z \exp \eta_z, \exp \xi_{z'} \exp \eta_z) + \varrho(\exp \xi_{z'} \exp \eta_z, \exp \xi_{z'} \exp \eta_{z'})\\ &= \varrho(\exp \xi_z, \exp \xi_{z'}) + \varrho(\exp \eta_z, \exp \eta_{z'})\\ &\leqslant B_\xi \varrho(\xi_z, \xi_{z'}) + B_\eta \varrho(\eta_z, \eta_{z'}).\end{aligned}$$

Now consider arbitrary $u, v \in \eta Z$, and apply the inequality just established, with $z = \eta^{-1}u$ and $z' = \eta^{-1}v$. Since $\xi_{\eta^{-1}u} = 0 = \xi_{\eta^{-1}v}$, it follows that on ηZ, the transformation $\zeta = \eta^{-1}$ is Lipschitzian with associated constant $B = B_\eta$.

Now one verifies easily that ηZ is a compact neighborhood of the origin in Q and hence has finite Hausdorff $j(n-j)$-measure. For each $v \in \eta Z$, let

$$\mu v = (\zeta v) O_{n,j} \in O_n / O_{n,j}.$$

Then $\mu \eta Z$ is a neighborhood of the "origin" (i.e., of $O_{n,j}$) in the quotient space $O_n/O_{n,j}$, and the space is covered by a finite number of isometric images of this neighborhood. Denoting by ϱ' the natural metric in the quotient space, we have

$$\varrho'(\zeta v O_{n,j}, \zeta v' O_{n,j}) = \inf_{\sigma, \tau \in O_{n,j}} \varrho((\zeta v)\sigma, (\zeta v')\tau) \leqslant \varrho(\zeta v, \zeta v') \leqslant B \varrho(v, v'),$$

so the transformation μ is Lipschitzian. Thus the $j(n-j)$-measure of $\mu(\eta Z)$ is finite and the desired conclusion follows. The proof of 3.1 is complete.

3.2. Corollary. *Suppose C is an n-dimensional convex body, p is an interior point of C, $0 \leqslant j \leqslant n$, and $\mathfrak{W}$ is the space of all j-sections of C through p, under the Hausdorff metric. Then the Hausdorff $j(n-j)$-measure of $\mathfrak{W}$ is positive and finite.*

Proof. In view of 2.2 and the behavior of Hausdorff measure under Lipschitzian transformations, we may (and shall) assume without loss of generality that C is the unit cell of E^n and p is the origin. Let J be a j-dimensional linear subspace of E^n and let $\mathfrak{J}$ and $\mathcal{G}$ be as in 3.1. For each coset $\sigma\mathcal{G}\in\mathfrak{J}/\mathcal{G}$, let $g(\sigma\mathcal{G})=(\sigma J)\cap C\in\mathcal{W}$. (Observe that if $\sigma\mathcal{G}=\tau\mathcal{G}$, then $\sigma^{-1}\tau\in\mathcal{G}$, whence $\sigma^{-1}\tau J=J$ and $\tau J=\sigma J$. Thus g is well-defined.) It is easy to verify that g maps the quotient space $\mathfrak{J}/\mathcal{G}$ biuniquely onto the space $\mathcal{W}$ of j-sections, and we wish to show that g is Lipschitzian (relative to the natural metric ϱ' in $\mathfrak{J}/\mathcal{G}$ and the Hausdorff metric D in $\mathcal{W}$). Now consider $\sigma\mathcal{G}$, $\tau\mathcal{G}\in\mathfrak{J}/\mathcal{G}$. Then

$$\varrho'(\sigma\mathcal{G},\tau\mathcal{G})=\inf_{\alpha,\beta\in\mathcal{G}}\varrho(\sigma\alpha,\tau\beta)=\inf_{\alpha,\beta\in\mathcal{G}}\sup_{x\in C}\|\sigma\alpha x-\tau\beta x\|.$$

But for arbitrary $\alpha,\beta\in\mathcal{G}$ we have $\alpha\mathcal{G}=\mathcal{G}=\beta\mathcal{G}$, so

$$D(g\sigma\mathcal{G},g\tau\mathcal{G})\leqslant\inf_{\alpha,\beta\in\mathcal{G}}D((\sigma\alpha J)\cap C,(\tau\beta J)\cap C)\leqslant\inf_{\alpha,\beta\in\mathcal{G}}\sup_{x\in C\cap J}\|\sigma\alpha x-\tau\beta x\|.$$

It follows that g is Lipschitzian with associated constant equal to 1, and thus from 3.1 that the Hausdorff $j(n-j)$-measure of $\mathcal{W}$ is finite. But of course g is a homeomorphism, so the topological dimension of $\mathcal{W}$ is equal to that of $\mathfrak{J}/\mathcal{G}$, whence the $j(n-j)$-measure of $\mathcal{W}$ must be positive. This completes the proof of 3.2.

3.3. COROLLARY. *Suppose C is an n-dimensional convex body, $0\leqslant j\leqslant n$, and $\mathfrak{X}$ is the space of all proper j-sections of C, under the Hausdorff metric. Then the Hausdorff dimension of $\mathfrak{X}$ is equal to $(j+1)(n-j)$.*

Proof. We may regard C as lying in a hyperplane H in $E^{n+1}\sim\{0\}$. Let $r=$ $=\sup\{\|x\|:x\in C\}$ and let K be the spherical cell of radius $2r$ about 0. Let $\mathcal{S}$ denote the set of all $(j+1)$-subspaces of E^{n+1} which intersect the relative interior of C. Set $\mathcal{F}=\{S\cap K: S\in\mathcal{S}\}$, $\mathcal{G}=\{S\cap K\cap H: S\in\mathcal{S}\}$, and $\mathcal{H}=\{S\cap C: S\in\mathcal{S}\}$. The natural map of $\mathcal{G}$ onto $\mathcal{H}$ is everywhere locally Lipschitzian by 2.4. Since the natural map of $\mathcal{F}$ onto $\mathcal{G}$ is Lipschitzian (easily verified), we conclude that the natural map μ of $\mathcal{F}$ onto $\mathcal{H}$ is everywhere locally Lipschitzian and hence by 2.1 μ is Lipschitzian on every compact set. Now $\mathcal{F}$ is the union of a countable number of compact sets, and by 3.2 the Hausdorff dimension of $\mathcal{F}$ is at most $(j+1)((n+1)-(j+1))$. The desired conclusion follows easily, and the proof of 3.3 is complete.

Since 3.1 is one of our basic tools, it seemed worthwhile to give the above fairly elementary proof. We now diverge from our main attack to establish a deeper result which subsumes 3.1 but which will not be used in the sequel. In preparation for 3.4, we review the definition of Lie group in a form which, though not quite "standard", is equivalent to the usual formulations and is especially well suited to our present purpose.

A (real n-dimensional) *analytic structure* for a topological space X is a family $\mathcal{H}$ which satisfies the following three conditions: i) each member $h \in \mathcal{H}$ is a homeomorphism of a nonempty open subset D_h of X onto an open subset of E^n; ii) X is covered by the sets D_h, $h \in \mathcal{H}$; iii) whenever $h, k \in \mathcal{H}$, the open set $h(D_h \cap D_k) \subset E^n$ is mapped analytically onto the set $k(D_h \cap D_k)$ by the transformation kh^{-1}. A *Lie group* is a topological group G which admits an analytic structure $\mathcal{H}$ relative to which the transformation $xy^{-1} | (x, y)$ is everywhere analytic. (That is, whenever U and V are open subsets of D_f and D_g respectively such that $UV^{-1} \subset D_h$, then the natural map μ of $fU \times gV$ into $h(UV^{-1})$ is analytic, where

$$\mu(p, q) = h((f^{-1}p)(g^{-1}q)^{-1}).)$$

Such an $\mathcal{H}$ will be called an *admissible structure* for the Lie group G.

A left invariant metric ϱ for a Lie group G will be called *Lipschitzian* provided there is an admissible structure $\mathcal{H}$ for G such that for some $h \in \mathcal{H}$, the transformation h^{-1} is Lipschitzian as a map into (G, ϱ) of the set $hD_h \subset E^n$. It can be verified that a Lipschitzian metric must be compatible with the topology of G, and that h as described may be taken so that $e \in D_h$ (where e will denote the identity element of G). Results of Goetz [5] imply that every Lie group admits an analytic structure $\mathcal{H}$ and a left invariant metric ϱ such that for each $h \in \mathcal{H}$, both h and h^{-1} are Lipschitzian. (Let ϱ be a left invariant Riemannian metric for G.) It is evident that if a separable Lie group is metrized by a Lipschitzian metric, then its Hausdorff dimension is equal to its topological dimension. From this it is easy to construct non-Lipschitzian left-invariant metrics. In fact, suppose G is a Lie group of dimension $n \geq 1$, ϱ is a left invariant (compatible) metric for G, and $r \in]0, 1[$. Then ϱ^r is a left invariant metric for G and $m_{n/r}(G, \varrho^r) = m_n(G, \varrho) > 0$, so the Hausdorff dimension of (G, ϱ^r) is equal to n/r and ϱ^r is not Lipschitzian. For another example, consider an arbitrary infinite compact metrizable group G and let ξ be a continuous map of G onto the Hilbert parallelotope P (such a ξ must exist). Assign to the product space $G \times P$ any metric σ which produces the usual product topology and has always $\sigma((x, p), (y, q)) \geq \text{dist.}\,(p, q)$. For all $x, y \in G$, define

$$\varrho(x, y) = \sup_{a \in G} \sigma((ax, \xi ax), (ay, \xi ay)).$$

Then ϱ is a left invariant metric for G and ξ is a Lipschitzian transformation of (G, ϱ) onto the infinite-dimensional space P. Thus the Hausdorff dimension of (G, ϱ) is infinite, and ϱ cannot be Lipschitzian if G is a Lie group.

The following result should be compared with the examples of the preceding paragraph, and with the more quantitative results of Goetz [5] and Loomis [10].

3.4. THEOREM. *Suppose S is an m-dimensional closed subgroup of the n-dimensional separable Lie group G, and ϱ is a Lipschitzian metric for G. Then with respect to the metrics induced by ϱ, the Hausdorff dimension of S is equal to m and the Hausdorff dimension of G/S is equal to $n-m$.*

Proof. Let M be the subspace of E^n consisting of all points $x=(x^1, \ldots, x^n)\in E^n$ such that $x^i=0$ for $m+1\leqslant i\leqslant n$; let L be the orthogonal supplement of M. For each $a>0$, let V_a be the cube in E^n consisting of all $x\in E^n$ such that $|x^i|\leqslant a$ for $1\leqslant i\leqslant n$.

According to the hypotheses of 3.4, there are an admissible structure $\mathcal{K}$ for G and a member k of $\mathcal{K}$ with $e\in D_k$ such that k^{-1} is Lipschitzian. By the reasoning (and in the terminology) of Chevalley [2] (pp. 107–109, especially the Remark on p. 109), there is an analytic involutive distribution $\mathfrak{M}$ of dimension m on G whose maximal integral manifolds are exactly the left cosets of S in G. By additional reasoning of Chevalley (pp. 89–91, especially the statement of Theorem 1 on p. 89) there are an admissible structure $\mathcal{H}$ for G, $h\in\mathcal{H}$, and $a>0$ such that the following conditions are satisfied:

(i) $he=0\in V_a\subset hD_h$ and $e\in D_h\subset D_k$;

(ii) on the domain hD_h, the transformation kh^{-1} is analytic;

(iii) for each $p\in L\cap V_a$, the "slice" $h^{-1}((p+L)\cap V_a)$ lies in some left coset C_p of S.

Chevalley shows further (p. 110) that for sufficiently small $b\in]0, a[$, the following additional condition is satisfied:

(iv) when $p, q\in L\cap V_b$ and $p\neq q$, then $C_p=C_q$.

Now an analytic transformation must be Lipschitzian on every compact set interior to its domain, so kh^{-1} is Lipschitzian on V_b. And k^{-1} is Lipschitzian by hypothesis, so it follows that the transformation $h^{-1}=k^{-1}(kh^{-1})$, mapping $V_b\subset E^n$ into (G, ϱ), is Lipschitzian. Since $h^{-1}0=e\in S$, it follows from conditions (iii) and (iv) that $h^{-1}(V_b\cap M)=(h^{-1}V_b)\cap S$; thus the set $h^{-1}(V_b\cap M)$ is a neighborhood of e in S. This set must have positive m-measure for h^{-1} is a homeomorphism and $V_b\cap M$ is m-dimensional; it must have finite m-measure for h^{-1} is Lipschitzian and $V_b\cap M$ has finite m-measure. From separability of S we now conclude that the Hausdorff m-measure is σ-finite on S, whence $m_r S=0$ for each $r>m$ and the Hausdorff dimension of S is equal to m.

Now the function C (defined by (iii)) maps $L \cap V_b$ into G/S, and, as remarked by Chevalley (p. 110), it is in fact a homeomorphism under which the set $L \cap V_b$ is carried onto a neighborhood of S in G/S. For arbitrary p and q in $L \cap V_b$, we have

$$\varrho'(C_p, C_q) = \inf_{x \in C_p, y \in C_q} \varrho(x, y) \leqslant \varrho(h^{-1}p, h^{-1}q),$$

and since h^{-1} is Lipschitzian so is the transformation C. As in the case of S above, this yields the desired conclusion about G/S and completes the proof of 3.4.

We return now to our principal line of reasoning, to obtain one more result on Hausdorff dimension which will be used in the study of polyhedral sections. For integers $n \leqslant r$, let us denote by $P^{a,k}(n, r)$ (resp. $P^{a,f}(n, r)$) the subset of $\mathcal{E}_{A^n}$ corresponding to the class of all n-dimensional polyhedra which have $r+1$ vertices (resp. $r+1$ maximal faces). And we denote by $Q^{a,v}(n, r)$ (resp. $Q^{a,f}(n, r)$) the subset of $\mathcal{E}^0_{A^n}$ corresponding to the class of all n-dimensional centered polyhedra which have $2r$ vertices (resp. $2r$ maximal faces). We define similarly the subsets $P^{s,v}(n, r)$ and $P^{s,f}(n, r)$ of $\mathcal{E}_{S^n}$ and the subsets $Q^{s,v}(n, r)$ and $Q^{s,f}(n, r)$ of $\mathcal{E}^0_{S^n}$.

3.5. Proposition. *Under Macbeath's metric, each of the sets $P^{a,v}(n, r)$, $P^{a,f}(n, r)$, $Q^{a,v}(n, r)$, and $Q^{a,f}(n, r)$ has Hausdorff- and topological dimension equal to $(r-n)n$; while $(r-n+1)n$ is the Hausdorff- and topological dimension of each of the sets $P^{s,v}(n, r)$, $P^{s,f}(n, r)$, $Q^{s,v}(n, r)$, and $Q^{s,f}(n, r)$.*

Proof. We discuss only the cases $P^{a,v}$, $P^{a,f}$ $P^{s,v}$, since from these it will be clear how to proceed in the other cases.

Let $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n$ be the vertices of an n-simplex in E^n. Let X be the set of all $(r-n)$-tuples $x = (x_{n+1}, \ldots, x_r)$ of points of E^n such that the set

$$\{\bar{x}_0, \ldots, \bar{x}_n, x_{n+1}, \ldots, x_r\}$$

is convexly independent. For each $x \in X$, let

$$\xi x = \operatorname{conv}\{\bar{x}_0, \ldots, \bar{x}_n, x_{n+1}, \ldots, x_r\} \in E^n_n,$$

the space of all n-dimensional convex bodies in E^n. Let η be the natural map of E^n_n into $\mathcal{E}_{A^n}$, so that $\eta\xi X = P^{a,v}(n, r)$. Now X may be regarded as an open subset of $E^{(r-n)n}$. With respect to the usual Euclidean metric for $E^{(r-n)n}$, the Hausdorff metric for E^n_n, and Macbeath's metric for $\mathcal{E}_{A^n}$, we see directly that ξ is Lipschitzian and from 2.7 that η is locally Lipschitzian. It follows that the Hausdorff dimension of $P^{a,v}(n, r)$ is at most $(r-n)n$; to show that it and the topological dimension are both equal to $(r-n)n$, it suffices to prove that tdim $P^{a,v}(n, r) \geqslant (r-n)n$. For this we may produce directly an open subset of X which maps topologically under $\eta\xi$. Alternatively, we may observe that since X is σ-compact and $\eta\xi$ is finite-to-one, it follows from dimension theory (pp. 91–92, 30 of [8]) that tdim $\eta\xi X = \text{tdim } X$. (To establish the finite-to-oneness of $\eta\xi$, consider an arbitrary $x \in X$ and observe that to each $x' \in X$ with $\eta\xi x' = \eta\xi x$ there corresponds an affine transformation τ of E^n onto E^n taking the set $\{\bar{x}_0, \ldots, \bar{x}_n, x_{n+1}, \ldots, x_r\}$ onto the set $\{\bar{x}_0, \ldots, \bar{x}_n, x'_{n+1}, \ldots, x'_r\}$. But then τ must be one of the $r!/(n+1)!$ affine transformations which take some $n+1$ of the points $\bar{x}_0, \ldots, \bar{x}_n, x_{n+1}, \ldots, x_r$ onto the points $\bar{x}_0, \ldots, \bar{x}_n$). Thus we have disposed of $P^{a,v}(n, r)$.

Continuing the notation of the preceding paragraph, we may assume further that the origin is interior to the simplex conv $\{\bar{x}_0, \ldots, \bar{x}_n\}$. Let $\mathcal{K}$ denote the set of all convex bodies in E^n whose interior includes the origin, and for each $K \in \mathcal{K}$ let ζK denote the polar body $K^0 \in \mathcal{K}$. It is easily verified that $\eta\zeta\xi X = P^{f,v}(n, r)$, and since ζ is locally Lipschitzian by 2.8, the desired conclusion follows as in the preceding paragraph. This takes care of $P^{f,v}(n, r)$.

To handle the case of $P^{s,v}(n, r)$, we let $\bar{y}_0, \ldots, \bar{y}_{n-1}$ be the vertices of an $(n-1)$-simplex in E^n and let Y denote the set of all $(r-n+1)$-tuples $y = (y_n, \ldots, y_r)$ in E^n such that the set $\{\bar{y}_0, \ldots, \bar{y}_{n-1}, y_n, \ldots, y_r\}$ is convexly independent. For each $y \in Y$, let $\mu y = \text{conv}\{\bar{y}_0, \ldots, \bar{y}_{n-1}, y_n, \ldots, y_r\} \in E^n_n$. Then Y may be regarded as an open subset of $E^{(r-n+1)n}$ and the reasoning proceeds much as in the first paragraph.

§ 4. Principal theorems and unsolved problems

We turn finally to the functions $\xi^{\cdots}$ and $\eta^{\cdots}$ defined in § 1. The results of §§ 2–3 will be applied to establish lower bounds. For upper bounds on $\xi^{a,v}$ and $\xi^{a,f}$ we rely on the following result due to *C.* Davis [3]:

4.1. PROPOSITION (Davis). *If S is an r-dimensional simplex, then every convex polyhedron having at most $r+1$ maximal faces is affinely equivalent to some proper section of S.*

In particular, every convex plane quadrilateral is affinely equivalent to some proper plane section of the tetrahedron. This validates a conjecture of Melzak [13]. Our first principal result is

4.2. THEOREM. *For $2 \leqslant n \leqslant r$, it is true that*

$$r \geqslant \xi^{a,f}(n, r) \geqslant \frac{n}{n+1}(r+1) \leqslant \xi^{a,v}(n, r) < 2^{r+1}.$$

Proof. That $\xi^{a,f}(n, r) \leqslant r$ follows at once from 4.1. And if an n-dimensional convex polyhedron has $r+1$ vertices, then its number of maximal faces is certainly no more than $\sum_{i=n}^{r} \binom{r+1}{i} < 2^{r+1}-1$; from this crude bound(1) and the result for $\xi^{a,f}$ we conclude that $\xi^{a,v}(n, r) < 2^{r+1}$. To establish the stated lower bound for $\xi^{a,v}$ (or similarly for $\xi^{a,f}$), let us consider a k-dimensional convex body C which is a-universal for all n-dimensional convex polyhedra having $r+1$ vertices. Let $\mathfrak{X}$ denote the space of all proper n-sections of C and φ the natural map of $\mathfrak{X}$ into $\mathcal{E}_{A}{}^{n}$. Then $\varphi\mathfrak{X} \supset P^{a,v}(n, r)$, and φ is locally Lipschitzian by 2.7. We see from 3.3 and 3.5 that the Hausdorff dimensions of $\mathfrak{X}$ and of $P^{a,v}(n, r)$ are respectively equal to $(n+1)(k-n)$ and $(r-n)n$, and consequently $(r-n)n \leqslant (n+1)(k-n)$. It follows that $k \geqslant n(r+1)/(n+1)$, and the proof of 4.2 is complete.

In particular, $\xi^{a,v}(2, 4) > 3$, whence there is no 3-dimensional convex body which is a-universal for all plane convex pentagons. And of course a 3-dimensional convex body has at most countably many 2-dimensional sections which are not proper, so we conclude that no 3-dimensional convex body includes (affinely) all plane convex pentagons among its (proper or boundary) sections. This validates another conjecture of Melzak [13]. When $r < 2n+1$, the above inequality for $\xi^{a,f}$ implies that $\xi^{a,f}(n, r) = r$, but I do not know whether $\xi^{a,f}(2, 5)$ is equal to 4 or to 5. Presumably the upper bound for $\xi^{a,v}$ can be much improved. (1)

Turning to s-universality, we employ a theorem of H. Naumann [15]:

4.3. PROPOSITION (Naumann). *Each n-dimensional convex polyhedron which has m maximal faces is a proper section of some cube of dimension $2^n(n+1)m$.*

4.4. THEOREM. *For $2 \leqslant n \leqslant r$, it is true that*

$$2^n(n+1)(r+1) \geqslant \xi^{s,f}(n, r) \geqslant \frac{n}{n+1}(r+2) \leqslant \xi^{s,v}(n, r) \leqslant 2^n(n+1)2^{r+1}.$$

Proof. The proof is entirely analogous to that of 4.2, using 2.7, 3.3, and 3.5 — and 4.3 in place of 4.1.

In particular, $\xi^{s,v}(2, 3) > 3$, whence it follows that no 3-dimensional convex body includes (up to similarities) all plane convex quadrilaterals among its (proper or boundary) sections. This also validates a conjecture of Melzak [13]. From 4.4 we see that $3 \leqslant \xi^{s,f}(2, 2) \leqslant 36$, but Melzak shows that in fact $\xi^{s,f}(2, 2) = 3$ — that is, there

(1) Added in proof: A significant improvement may be achieved by applying a result stated by W. W. Jacobs and E. D. Schell, The number of vertices of a convex polyhedron, *Amer. Math. Monthly*, 66, (1959), 643.

is a 3-dimensional convex body C which is s-universal for all triangles. Melzak's set C is what he calls a "pseudopolyhedron" — that is, the convex hull of a convergent sequence together with its limit point. He conjectures that there is no 3-dimensional polyhedron which is s-universal for all triangles, and this is easy to verify, for if a triangle is a section of a 3-dimensional polyhedron then its angles are all plane sections of the (finitely many) dihedral angles determined by pairs of maximal faces of the polyhedron. Thus no triangular section can have an angle larger than the maximum of these dihedral angles and consequently not all triangles can be obtained (up to similarity) as sections. This example suggests the study of functions $\xi_p^{s,f}$, etc., defined as were $\xi^{s,f}$, etc., but with the additional condition that the universal body should be polyhedral. Although $\xi^{s,f}(2, 2)=3$, we know only that $3<\xi_p^{s,f}(2, 2)\leqslant 36$.

It would be interesting to remove the restriction to *proper* faces in 4.2, 4.4, and some of the earlier results. More generally, the following problem is of interest: Whenever C is an n-dimensional convex body and j and k are integers with

$$0\leqslant j\leqslant k\leqslant n,$$

let us denote by $\mathcal{Y}_{j,k}C$ the space of all j-dimensional sections S of C such that the facet of C determined by S is of dimension k. Let $h_{j,k}C$ denote the Hausdorff dimension of $\mathcal{Y}_{j,k}C$ (metrized by the Hausdorff metric). Then what possibilities subsist for the number-array $(h_{j,k}C)_{0\leqslant j\leqslant k\leqslant n}$? Note that $\mathcal{Y}_{0,n}C$ is isometric with the interior of C and $\bigcup_{k=0}^{n-1}\mathcal{Y}_{0,k}C$ with the boundary of C. The space of all j-sections is $\bigcup_{k=j}^{n}\mathcal{Y}_{j,k}C$, while $\mathcal{Y}_{j,n}C$ is the space of all proper j-sections. In using our results 4.2 and 4.4 to validate two conjectures of Melzak, we employed the fact that $\mathcal{Y}_{n-1,n-1}C$ is countable and hence the Hausdorff dimension of $\mathcal{Y}_{n-1,n-1}C\cup\mathcal{Y}_{n-1,n}C$ is equal to that of $\mathcal{Y}_{n-1,n}C$.

In dealing with central a-universality we employ

4.5. PROPOSITION. *If Q is an r-dimensional cube, then every centered convex polyhedron having at most $2r$ maximal faces is affinely equivalent to some central section of Q.*

Proof. We assume without loss of generality that Q is centered at the origin in E^r. We regard E^r as self-dual under the usual inner product, so that the polar body Q^0 is a subset of E^r. The body Q^0 has vertices $z_1, \ldots, z_r$ such that

$$Q^0=\{\textstyle\sum_1^r t_i z_i : \sum_1^r |t_i|\leqslant 1\}.$$

In proving 4.5, it suffices to consider a polyhedron P which is centered at the origin, has exactly $2r$ vertices, and whose affine extension is an m-dimensional linear subspace m of E^r. Let P' denote the polyhedron $\{y \in M : \sup_{x \in P}(y, x) \leqslant 1\}$, the polar of P relative to M; and let L denote the orthogonal supplement M^0 of M. There are vertices $x_1, \ldots, x_r$ of P' such that $P' = \{\sum_1^r t_i x_i : \sum_1^r |t_i| \leqslant 1\}$ and such that $x_1, \ldots, x_m$ are linearly independent. It is easy to produce points $y_{m+1}, \ldots, y_r$ of L such that $w_1, \ldots, w_r$ are linearly independent, where $w_i = x_i$ for $1 \leqslant i \leqslant m$ and $w_i = x_i + y_i$ for $m+1 \leqslant i \leqslant r$. Let $W = \{\sum_1^r t_i w_i : \sum_1^r |t_i| \leqslant 1\}$ and let π be the orthogonal projection of E^r onto M, so that always $\pi w_i = x_i$ and $\pi W = P'$. We wish to prove that $P' = W^0 \cap M$, or equivalently that $P' = (W^0 \cap M)'$; since both sets lie in M, it suffices to show that $P' + L = (W^0 \cap M)' + L$. Now using well-known properties of the polar operation ^{0}, the fact that L is the kernel of π and is supplementary to M, one can verify that

$$P' + L = \pi W + L = \text{cl conv } (W \cup L) + L$$

and

$$(W^0 \cap M)' = M \cap (W^0 \cap M)^0 = M \cap \text{cl conv } (W^{00} \cup M^0) = M \cap \text{cl conv } (W \cup L).$$

whence

$$(W^0 \cap M)' + L = M \cap \text{cl conv } (W \cup L) + L = \text{cl conv } (W \cup L) + L = P' + L.$$

It follows that $P = W^0 \cap M$.

Now let α be the linear transformation of E^r onto E^r which takes always z_i onto w_i. Then $\alpha Q^0 = W$. If β denotes the adjoint of α^{-1} —— $\beta = {}^t\alpha^{-1}$, then it can be verified that

$$\beta Q = {}^t\alpha^{-1} (Q^0)^0 = (\alpha Q^0)^0 = W^0.$$

With $P = W^0 \cap M$ and $\beta Q = W^0$, we have $P = \beta Q \cap M$ and consequently

$$\beta^{-1} P = Q \cap \beta^{-1} M,$$

whence P is affinely equivalent to a central section of Q and the proof of 4.5 is complete.

4.6. THEOREM. *For $2 \leqslant n \leqslant r$, it is true that*

$$r = \eta^{a,f}(n, r) \leqslant \eta^{a,v}(n, r) \leqslant 2^{2r}.$$

Proof. That $\eta^{a,f}(n, r) \leqslant r$ is an immediate consequence of 4.5, and the upper bound on $\eta^{a,v}$ follows from that on $\eta^{a,f}$. To establish the lower bound for $\eta^{a,f}$ (or similarly for $\eta^{a,v}$), let us consider a k-dimensional convex body C which is centrally a-universal for all n-dimensional centered convex polyhedra having $2r$ vertices. Let $\mathfrak{W}$ denote the space of all central n-sections of C and φ the natural map of $\mathfrak{W}$ into $\mathcal{E}^0_{A^n}$. Then $\varphi \mathfrak{W} \supset Q^{a,f}(n, r)$, and φ is locally Lipschitzian by 2.7. We see from 3.2 and 3.5 that the Hausdorff dimensions of $\mathfrak{W}$ and of $Q^{a,f}(n, r)$ are equal respectively to $n(k-n)$ and $(r-n)n$, whence $(r-n)n \leqslant n(k-n)$. Thus $k \geqslant r$ and the proof of 4.6 is complete.

Here again, the upper bound for $\eta^{a,v}$ is very crude and subject to much improvement. Of course $\eta^{a,v}(2, r) = \eta^{a,f}(2, r) = r$, but I do not know the value of $\zeta^{a,v}$ (3, 4). Bessaga's result [1] was that $\eta^{a,v}(2, n+1) > n$.

Now consider a family $\mathfrak{P}$ of centered polyhedra and suppose there exists a k-dimensional centered convex body C which is centrally s-universal for $\mathfrak{P}$. This is a rather restrictive assumption. For example, if r_P and R_P denote respectively (for each $P \in \mathfrak{P}$) the radii of the inscribed and circumscribed spheres of P, then the existence of C implies that $\sup_{P \in \mathfrak{P}} R_P/r_P < \infty$. Information about the possible values of k can be obtained from our present techniques in conjunction with the following theorem of Naumann [5]: Suppose P is an n-dimensional centered convex polyhedron which has $2m$ faces, that P contains the polyhedron $\{x = (x^1, \ldots, x^n) \in E^n : \sum_1^n |x^i| \leqslant r\}$ (a generalized octahedron), and that P is contained in the polyhedron $\{x : \max_i |x_i| \leqslant R\}$ (a cube). Let α be such that $\alpha \geqslant R/r$ and $m\alpha^2$ is an integer. Then P can be realized as a central section of a cube of dimension $m + m\alpha^2(n-1)$.

There remain many interesting infinite-dimensional problems concerning universal Banach spaces. If $\{B_s : s \in S\}$ is the set of all separable reflexive Banach spaces, then the l^2-product E of the spaces B_s is a reflexive Banach space which is universal for all separable reflexive Banach spaces, but of course E itself is not separable. The separable Banach space $C[0, 1]$ is universal for all separable Banach spaces, but it is not reflexive. Mazur has asked (Problem 49 (1935) of The Scottish Book [17]) whether there exists a separable reflexive Banach space which is universal for all separable reflexive Banach spaces. The problem remains open, and in fact we do not know whether there is a separable reflexive Banach space which is universal for all finite-dimensional Banach spaces. Let us call a Banach space *polyhedral* provided every finite-dimensional central section of its unit cell is polyhedral — that is, provided each of its finite-dimensional linear subspaces has a polyhedral unit cell. For each n, let J_n be an n-dimensional Banach space whose unit cell is a cube. Then the l^2-product F of the spaces J_n is a separable reflexive Banach space which is universal for all finite-dimensional polyhedral Banach spaces, but F itself is not polyhedral. We shall prove below that the space (c_0) is polyhedral and is universal for all finite-dimensional polyhedral Banach spaces, but do not know whether there exists a reflexive Banach space with both these properties or even whether there exists an infinite-dimensional Banach space which is both polyhedral and reflexive.

4.7. PROPOSITION. *If E is a subspace of (c_0) whose unit cell U has an extreme point, then E is finite-dimensional and U is polyhedral. Thus a finite-dimensional Banach space F is isometric with a linear subspace of (c_0) if and only if F is polyhedral.*

Proof. The proof is based on

(*) Suppose p is an extreme point of U, $I=\{i:|p^i|=1\}$, and $x\in E$ with $x^i=0$ for all $i\in I$. Then $x=0$.

To prove (*), suppose $x\neq 0$ and let $\delta=\sup_{j\notin I}|p^j|<1$, $\varepsilon=(1-\delta)/\|x\|>0$. Then $[p-\varepsilon x, p+\varepsilon x]\subset U$, contradicting the fact that p is an extreme point of U.

Now with p and I as in (*), for each $y\in E$ let $Ty=(y|I)\in R^I$ (R denoting the real number space). Then T is a linear transformation of E into the finite-dimensional linear space R^I. It follows from (*) that the kernel of T is $\{0\}$, and hence E must be finite-dimensional. Let $q_1, \ldots, q_r$ be a basis for E and $V=\operatorname{conv}\bigcup_{i=1}^r[-q_i, q_i]$. Then V is a convex body in E, 0 is interior to V (relative to E), and U is compact, so we have $U\subset mV$ for a sufficiently large $m<\infty$. Now there exists $N<\infty$ such that $|q_i^j|<1/m$ whenever $1\leqslant i\leqslant r$ and $j>N$, and it follows that $|v^j|<1/m$ whenever $v\in V$ and $j>N$, whence $|u^j|<1$ for all $u\in U$ and $j>N$. Thus for each extreme point z of U there must be subsets A_z and B_z of $\{1, \ldots, N\}$ such that $A_z=\{i:z^i=1\}$ and $B_z=\{i:z^i=-1\}$. If $A_z=A_{z'}$ and $B_z=B_{z'}$, then by application of (*) with $p=z$ and $x=z-z'$ we see that $z=z'$. It follows that U has only finitely many extreme points and hence is polyhedral.

We have established the "only if" part of the second assertion of 4.7. For the "if" part, if suffices to refer to 4.5. The proof of 4.7 is complete.

It would be interesting to determine all the finite-dimensional subspaces of other well-known Banach spaces (the solution being evident for l^2 and for $C[0, 1]$).

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