MAXIMAL ALGEBRAS OF CONTINUOUS FUNCTIONS

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1. Introduction

Let X be a compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions on X. Let A be a uniformly closed complex linear subalgebra of $C(X)$. Our interest centers about such algebras A which are maximal among all proper closed subalgebras of $C(X)$. In this paper we gather together most of the known facts concerning maximal algebras, give some new results, and some new proofs of known theorems.

A major motivation for the study of maximal algebras stems from an attempt to generalize the Stone-Weierstrass approximation theorem to non-self-adjoint algebras. This theorem states that if A is a self-adjoint closed subalgebra of $C(X)$ ($f \in A$ implies $\bar{f} \in A$), and if A separates points and contains the constant function 1, then $A = C(X)$. See [11; p. 8] for a proof. This result can be restated as follows: (i) every proper self-adjoint closed algebra A is contained in a self-adjoint maximal algebra and (ii) the self-adjoint maximal algebras, B, are of two kinds; either $B = [f \in C(X), f(x_0) = 0]$ for a fixed $x_0 \in X$, or $B = [f \in C(X); f(x_1) = f(x_2)]$ for fixed $x_1, x_2 \in X$. The condition, A contains the function 1, says that A is not in a maximal algebra of the first kind. The condition, A separates points, says that A is not in a maximal algebra of the second kind. Thus A is not contained in any self-adjoint maximal algebra and consequently, from (i), A is not a proper subalgebra, i.e., $A = C(X)$. A refinement of the Stone-Weierstrass theorem classifies all self-adjoint closed subalgebras of $C(X)$ and says that such an algebra A is the algebra of all continuous functions on an identification space of X, with the common zeros of the functions in A deleted. This result can be reinterpreted as saying that A is the intersection of the self-adjoint maximal algebras which contain it.

Let us drop the self-adjointness condition on A. One might now hope that the way to generalize the Stone-Weierstrass theorem would be to show that (i) holds (with self-

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adjointness deleted) and then to classify all maximal subalgebras of $C(X)$. (To avoid trivialities we now make the assumption that all subalgebras under consideration separate points, contain 1, and are closed). It turns out, however, that (i) fails; in section 7 we exhibit a proper algebra A not contained in any maximal algebra. The example is easy to describe, but the proof that it is not contained in a maximal algebra depends on several results of earlier sections. Even if A is contained in a maximal algebra, it is not necessarily the intersection of the maximal algebras containing it. Specific examples are given in section 6.

Despite these negative results, the study of maximal algebras does give approximation theorems. In particular, if A is a maximal subalgebra of $C(X)$, then of course the algebra generated by A and any $f \in C(X) - A$ is all of $C(X)$. For example, [16] the fact that the algebra of continuous functions on the circle which are boundary values of analytic functions on the disc is maximal, implies that every continuous function on the circle can be approximated by polynomials in z and f, where f is not the boundary value of an analytic function on the disc. This is a generalization of Fejer's theorem (the case $f = \bar{z}$). For some special spaces X, one knows enough about maximal algebras so that if A lies in a restricted class of algebras (just as one restricts ones attention to self-adjoint algebras in the Stone-Weierstrass theorem), then the possible proper subalgebras B containing A can be classified. If A contains functions not in these algebras B, A must be $C(X)$. In section 6, this situation is analyzed when X is the circle and A contains a separating subalgebra of analytic functions. Wermer's results [18; 19; 20] give enough information about maximal algebras to give a strong approximation theorem.

The study of maximal algebras has one natural reduction which we now discuss. Suppose A is a maximal subalgebra of $C(X)$ and suppose S is a closed subset of X. Let A_S denote the closure of A restricted to S, and let $A_0 = [f \in C(X);\ f|_S \in A_S]$. Then A_0 is closed and $A \subset A_0 \subset C(X)$. Since A is maximal, either $A_0 = C(X)$ so that $A_S = C(S)$ or else $A = A_0$ so that A is actually a maximal algebra on S extended continuously in all possible ways to X. Among the closed sets S such that $A_S \neq C(S)$ there exists a unique minimal one $E = \bigcap_{A_S \neq C(S)} S$ which we call the *essential set* for A [3]. Thus A consists of a maximal algebra of $C(E)$ extended in all possible ways to X in a continuous fashion. Furthermore, if S is a proper closed subset of E, then $A_S = C(S)$. If $E = X$, then A is said to be an *essential maximal subalgebra* of $C(X)$. The study of maximal algebras of X is thus reduced to the study of essential maximal algebras of X and its closed subsets.

In section 2, we list the known examples of essential maximal algebras. Some new ones are exhibited in section 4. One observes that these examples all stem from algebras of analytic functions. In [10], Helson and Quigley show that essential maximal algebras display a number of properties enjoyed by analytic functions. To our mind, the reason for this is that an essential maximal algebra A is *pervasive,* that is, $A_S = C(S)$ for any proper closed subset S of X. In sections 3 and 4, pervasive algebras and their properties are analyzed. These results together with an idea of Rudin [14] show that any pervasive algebra on a disconnected space is contained in a maximal algebra (Section 4). This leads to some new essential maximal algebras on the circle and an example of an essential maximal algebra on the unit interval.

Section 5 contains a discussion of the representation of complex homomorphisms of an algebra by positive measures on the Šilov boundary, emphasizing the usefulness of such representations in studying maximal algebras. This measure representation is playing an important role in the study of function algebras. It seems clear that it will play an increasingly important role.

2. Examples

We shall list the examples of essential maximal algebras known to us.

1. (Wermer [16]). Let X be the unit circle in the complex plane and let A be the algebra of continuous functions on X which can be analytically continued to the interior of the unit disc. That is, f is in A if and only if

$$\int_0^{2\pi} e^{in\theta} f(e^{i\theta})\, d\theta = 0, \; n = 1, 2, 3, \ldots$$

2. (Wermer [21]). If F is a Riemann surface and X is an analytic curve on F which bounds a compact subset K of F, let A be the algebra of continuous functions on X which can be analytically continued to $K - X$.

3. (Bishop [6]). Let X be the topological boundary of any simply connected plane domain and let A be the algebra of functions on X which are uniform limits of polynomials:

4. (Hoffman and Singer [8]). Let X be a compact abelian group whose character group $\hat{X}$ is a subgroup of the additive group of real numbers. Let A be the algebra of all continuous functions on X whose Fourier transforms vanish on the negative half of the group $\hat{X}$.

5. Rudin [14], has proved the existence of essential maximal subalgebras of $C(X)$ where X is a certain totally disconnected set in the complex plane. This is described in section 4.

As mentioned in the introduction, we shall add to this list in section 4.

3. The essential set

Suppose that A is a subalgebra of $C(X)$, which we remind the reader means A separates the points of X, contains the constant functions, and is closed. We consider those closed subsets K of X such that A contains every continuous function on X which vanishes on K. Among such sets K there is a unique minimal one E, which we call the *essential set* for A (relative to X). The algebra A_E, obtained by restricting A to the set E, is a closed subalgebra of $C(E)$ and A consists of the algebra of all functions which are continuous extensions to X of functions in A_E. The minimality of E is characterized by saying that the algebra A_E contains no non-zero ideal of $C(E)$. If $E = X$, we say that E is an *essential subalgebra* of $C(X)$. The terminology here is due to Bear [3].

The study of function algebras A is "reduced" to the study of essential algebras, and the purely topological problem of describing closed subsets of X. We should point out that when A is a maximal subalgebra of $C(X)$ this reduction agrees with that carried out in section 1; that is, a maximal algebra is essential if and only if it is pervasive (see introduction).

In [10], Helson and Quigley proved that every essential maximal algebra is *antisymmetric*, i.e., contains no non-constant real-valued functions, and is *analytic*, i.e., any function in the algebra which vanishes on a non-empty open subset of X is identically zero. They were motivated of course by an interest in proving that any maximal (essential) algebra has many properties in common with the algebra of analytic functions on the unit circle (example 1, section 2). They did not specifically mention the pervasive property which such algebras share with the analytic functions. What we should like to point out in this section is that the pervasive property seems to be the fundamental one. By this we mean that any proper pervasive subalgebra of $C(X)$ is analytic and antisymmetric.

THEOREM 3.1. *A proper pervasive subalgebra of $C(X)$ is analytic.*

Proof. Let f be a function in A which vanishes on a non-empty open set U in X. Choose a non-empty open set V such that $\overline{V} \subset U$. The assumption that A is pervasive tells us that if $g \in C(X)$ then there is a sequence $[f_n]$ of functions in A such that f_n converges to g uniformly on the complement of V. Then the sequence $[ff_n]$ converges uniformly to fg on all of X. Thus fg is in A for each g, or $f \cdot C(X)$ is contained in A. So A contains the closed ideal in $C(X)$ generated by f, i.e., A contains every continuous function on X which vanishes on the null set K of f. This means that when we restrict A to K we get a closed subalgebra of $C(K)$. Clearly then f must vanish on all of X; for, if K were a proper closed subset of X the restriction of A to K would be at once dense in $C(K)$ and closed and A would contain all of $C(X)$.

THEOREM 3.2. *Let A be a closed subalgebra of $C(X)$.*

(i) *If A is analytic, A is an integral domain.*

(ii) *If A is an integral domain, A is antisymmetric.*

(iii) *If A is antisymmetric, A is an essential subalgebra of $C(X)$.*

Proof. (i) is obvious. (ii) Let R be the self-adjoint part of A, i.e., the set of all functions f in A whose complex conjugate is also in A. Then R is a closed subalgebra of A, and since R is self-adjoint there is a compact Hausdorff space Y such that R is isometrically isomorphic to $C(Y)$. Since A is an integral domain, so is $C(Y)$. Clearly then Y consists of a single point and R contains only the constant functions.

(iii) It is clear that if the essential set for A is a proper closed subset of X then A contains a non-constant real-valued function.

An immediate corollary of these two theorems is that an essential maximal subalgebra of $C(X)$, being a proper pervasive subalgebra, is analytic, hence an integral domain; hence antisymmetric. In fact we see that for a maximal subalgebra of $C(X)$ the properties of being essential, pervasive, analytic, an integral domain, antisymmetric, are all equivalent.

4. Pervasive algebras

In section 3 we saw that some of the known special properties of an essential maximal algebra are possessed by every proper pervasive subalgebra of $C(X)$. There are pervasive algebras which are not maximal, a simple example being the uniformly closed algebra on the unit circle generated by $1, z^2, z^3, z^4, \dots$. Having observed this, we felt it interesting to inquire whether it is true that every proper pervasive subalgebra of $C(X)$ is contained in a maximal subalgebra of $C(X)$. Motivated by a result of Rudin [14], we did prove the somewhat strange fact that, when the underlying space X is not connected, this is true. This then is a mild existence theorem for maximal algebras. It can be used to construct a new class of essential maximal algebras, and in particular to construct a new essential maximal algebra on the unit circle.

Let us first observe the following.

LEMMA 4.1. *Let A be a subalgebra of $C(X)$ such that for each $f \in A$ the real part of f has connected range. If X is not connected, then A is contained in a subalgebra of $C(X)$ which is maximal with this property.*

Proof. The proof is essentially that of Rudin [14; theorem 2]. Let F be the class of all proper closed subalgebras B of $C(X)$ which contain A and are such that for each $f \in B$ the real part of f maps X onto a connected set. If $[B_\alpha]$ is a linearly ordered subset of F, the closure of the union of the B_α contains only functions whose real part has connected range, and this closure is a proper subalgebra of $C(X)$ since X is not connected. By Zorn's lemma, F contains a maximal element.

LEMMA 4.2. *If A is an antisymmetric subalgebra of $C(X)$, then for each $f \in A$ the real part of f has connected range.*

Proof. Again see Rudin [14]. Let $f \in A$ and suppose the real part of f does not have connected range. Then the range of f is the union of two non-empty compact sets K_0 and K_1 which are separated by a vertical line. We can find a sequence of polynomials p_n (in one complex variable) which converges uniformly to 0 on K_0 and to 1 on K_1. Then $p_n(f)$ is a sequence of elements in A which converges to a non-trivial idempotent function in A, i.e., a non-constant real-valued function in A.

It is clear from the above argument that the statement that the real part of each f in A maps X onto a connected set is equivalent to the statement that A contains no non-trivial idempotent functions. This in turn is equivalent (by a theorem of Šilov) to the maximal ideal space of A being connected.

THEOREM 4.3. *Let A be a proper pervasive subalgebra of $C(X)$, and suppose that X is not connected. Then A is contained in an essential maximal subalgebra of $C(X)$.*

Proof. Since A is proper and pervasive, A is antisymmetric, so lemmas 4.1 and 4.2 tell us that A is contained in a subalgebra B which is maximal with the "connected range" property. But then B is a maximal subalgebra of $C(X)$; for any proper subalgebra B_1 which contains B contains A and is therefore pervasive. So B_1 is antisymmetric, hence has the "connected range" property and must be equal to B.

We shall now combine theorem 4.3 with some work of Wermer [17] to prove the existence of new essential maximal algebras.

Let X be a compact set in the complex plane with these properties:

(i) X has no interior.

(ii) X does not separate the plane.

(iii) X is not connected.

(iv) X has positive Lebesque measure at each of its points, i.e., if $x \in X$ then for any neighborhood U of x the set $U \cap X$ has positive plane measure.

Let A_X be the algebra of all continuous functions on the Riemann sphere S which are analytic on $S - X$. The functions in A_X separate the points of S [17].

Each function in A_X assumes its maximum modulus on the set X, so that we can identify A_X with a proper closed subalgebra of $C(X)$. These properties of A_X require only that X have no interior and positive measure.

We now observe that properties (ii) and (iv) imply that A_X is a pervasive subalgebra of $C(X)$. Let K be a proper closed subset of X. Choose a point $x_0 \in X - K$. For simplicity let us assume that $x_0 = 0$. Choose $\delta > 0$ such that the disc $[|z| \leq \delta]$ does not intersect the set

K. When $1/n < \delta$, let Δ_n be the intersection of X and the open disc $|z| < 1/n$. By condition (iv), the measure of Δ_n, $|\Delta_n|$, is positive. Define

$$\Phi_n(z) = -\frac{1}{|\Delta_n|}\iint_{\Delta_n} \frac{dx\,dy}{x+iy-z}.$$

The functions Φ_n are (can be extended to) functions in A_X. A routine verification shows that $\Phi_n(z)$ converges to $1/z$ uniformly on the compact set K. Since X does not separate the plane (and has no interior) a theorem of Mergelyan [12] tells us that polynomials in $1/z$ are dense in the continuous functions on K. Thus we see that the restriction of A_X to K is dense in $C(K)$.

Using theorem 4.3 we then have

THEOREM 4.4. *If X is a compact set in the plane which satisfies conditions* (i)–(iv) *above, then the algebra A_X is contained in an essential maximal subalgebra of $C(X)$.*

When X is totally disconnected, this result was obtained by Rudin [14].

The following special case of theorem 4.4 is of particular interest. Suppose the set X consists of two disjoint homeomorphic images of the unit interval. (These two arcs can be so embedded as to satisfy condition (iv)). The algebra A_X is then included in an essential maximal subalgebra B of $C(X)$, where X consists of two disjoint copies of the unit interval. From B we wish to obtain an essential maximal subalgebra on the unit circle, by taking the subalgebra of functions which identify the respective ends of the two intervals. If we identify only one pair of endpoints we obtain an essential maximal algebra on the unit interval. We shall need the following lemma.

LEMMA 4.5. *Let B a maximal subalgebra of $C(X)$ and let x and y be two points in X. Let B_0 be the subalgebra of B of functions f for which $f(x)=f(y)$, and let Y be the compact space obtained from X be identifying x and y. Then B_0 is a maximal subalgebra of $C(Y)$.*

Proof. What we must prove is this. If g is a continuous function on X such that $g(x) = g(y)$ and $g \notin B_0$ then the closed subalgebra of $C(X)$ generated by B_0 and g contains every continuous function f for which $f(x)=f(y)$. It clearly will suffice to consider the case in which $g(x)=g(y)=0$.

Since $g(x)=g(y)$ and $g \notin B_0$, $g \notin B$. Thus, the linear algebra $[B, g]$ generated by B and g is dense in $C(X)$. This linear algebra consists of all functions of the form

$$f_0 + f_1 g + \cdots + f_n g^n$$

where $f_0, \ldots, f_n$ are in B. Let I be the set of all functions in $[B, g]$ such that $f_k(x) = f_k(y) = 0$, $k = 0, 1, \ldots, n$. Then I is an ideal in $[B, g]$, so the closure of I is a closed ideal in the closure of $[B, g]$ (which is $C(X)$). The ideal I contains every function in B which is 0 at both x and y, and since B separates points on X the set of points on which every function in I vanishes consists of the two points x and y. Thus the closure of I must contain every continuous function on X which vanishes at both x and y. But I is contained in the algebra generated by B_0 and g. Thus the closed algebra generated by B_0 and g contains every continuous function vanishing at x and y. Since B_0 contains the constants, B_0 and g generate all functions which identify x and y.

Now let us return to the algebra A_X above when $X = I_1 \cup I_2$ where I_1 and I_2 are disjoint homeomorphic images of the unit interval. Let B be an essential maximal subalgebra of $C(X)$, containing A_X. If x_i and u_i are the end points of I_i, $i = 1, 2$, we consider the subalgebra B_0 of B of functions f such that $f(x_1) = f(x_2)$, $f(u_1) = f(u_2)$. By the above lemma, B_0 is an essential maximal subalgebra of a homeomorphic image of the unit circle.

We wish to show that the algebra B_0 is not isomorphic to any of the examples cited in section 2. To do this it will suffice to show the following. If A_X^0 is the subalgebra of A_X which identifies x_1 with x_2 and u_1 with u_2, then A_X^0 cannot be isomorphic to a closed subalgebra of the algebra of boundary values of analytic functions on a Riemann surface (with boundary).

We shall content ourselves with a sketch of this proof. Let Γ be an analytic circle on a Riemann surface which bounds a compact piece K of the surface. Let A be the algebra of all continuous functions on K which are analytic on $K - \Gamma$. Suppose that the algebra A_X^0 is (isomorphic to) a subalgebra of A. Each complex homomorphism h of the algebra A gives rise to a complex homomorphism h_0 of A_X^0 by restriction. The mapping $\pi : h \to h_0$ is a continuous mapping of K into the space S_0 of complex homomorphisms of the algebra A_X^0. The space S_0 can be identified as the Riemann sphere S with the pairs of points (x_1, x_2) and (u_1, u_2) identified. This follows from a result of Arens [1] that the space of complex homomorphisms of A_X is S. The mapping π, when restricted to Γ, gives a homeomorphism of Γ onto the circle on S_0 obtained by identifying the ends of the intervals I_1 and I_2.

It is now relatively easy to argue that such a continuous mapping π cannot exist. For, let p be the point on Γ such that $q = \pi p$ is the point of S_0 which arises from identifying x_1 and x_2. A sufficiently small neighborhood V of the point q is homeomorphic to two discs with their centers identified. Thus $V - [g]$ is homeomorphic to two disjoint copies of the punctured open disc. Suppose we select a neighborhood U of the point p which is connected and for which $\pi(U) \subset V$. Then $U - [p]$ is still connected and must be mapped by π into one of the two punctured discs comprising $V - [q]$. But this is impossible, since the part of Γ which lies in the neighborhood U is mapped by π partly into one of the punctured discs and partly into the other.

We should perhaps comment that in a rough sense the algebra A_X^0 is not an algebra of analytic functions on a Riemann surface with boundary, because the circle we have on the pinched sphere S_0 does not bound. We should also note that it may well be that A_X is already a maximal subalgebra of $C(X)$. One can prove that if B is any proper subalgebra of $C(X)$ which contains A_X, then every complex homomorphism of A_X extends to a complex homomorphism of B. But whether A_X is actually maximal remains unknown.

5. Measures and the Šilov boundary

Our discussion thus far of maximal subalgebras of $C(X)$ has not involved any detailed information about the relation of the space X to the algebra A. Further discussion requires the introduction of the maximal ideal space and Šilov boundary for A.

Let A be a closed subalgebra of $C(X)$, as usual containing the constants and separating points. The *space of maximal ideals* (or complex homomorphisms) of A is the set $S(A)$ of all non-zero complex linear functionals on A which are multiplicative. Each such multiplicative functional is automatically of norm 1, and we give to $S(A)$ the weak topology which it inherits as a subset of the unit sphere in the conjugate space of A. The space $S(A)$ is the largest compact Hausdorff space on which the algebra A can be realized as a separating algebra of continuous functions. In $S(A)$ there is a unique minimal closed subset $\Gamma(A)$ on which every function in A assumes its maximum modulus. We call $\Gamma(A)$ the *Šilov boundary* for A [7].

For each point $x \in X$ we have a complex homomorphism h_x of A defined by

$$h_x(f) = f(x).$$

Since A separates points on X, the mapping $x \to h_x$ is a homeomorphism of X onto a closed subset of $S(A)$. The image of X under this mapping includes $\Gamma(A)$ because each function in A certainly assumes its maximum on X.

Since each function in A assumes its maximum on $\Gamma(A)$ we may (if we wish) regard A as a subalgebra of $C(\Gamma)$. The minimality of Γ shows that Γ is the smallest compact Hausdorff space on which A can be realized as a closed separating algebra of continuous functions.

If $p \in S(A)$, there is a (not necessarily unique) positive Baire measure μ_p on Γ such that

$$f(p) = \int_\Gamma f \, d\mu_p$$

for every f in A [see 2]. We say that μ_p "represents" p. This representation results from the fact that any continuous linear functional on $C(\Gamma)$ which has norm 1 and is 1 at the identity is positive. We are particularly interested here in the role of these measures in the study of maximal algebras.

Let us first make some simple observations. We have been discussing special types of subalgebras of $C(X)$: antisymmetric, pervasive, etc. The representation of homomorphisms by positive measures makes it clear that A is antisymmetric if and only if A is an antisymmetric subalgebra of $C(\Gamma)$. In other words, antisymmetry is independent of which space X we represent A on, because X always contains Γ. Likewise the property of being an essential subalgebra of $C(X)$ is independent of X. However, certain properties we have discussed do depend upon the underlying space X. For example, the exact description of the essential set depends heavily on X. Also the property of being pervasive depends on X. To rule out discrepancies, let us make the following conventions. The essential set for A will be the essential set for A relative to Γ. We shall call A pervasive if A is a pervasive subalgebra of $C(\Gamma)$. (It follows that if A is a pervasive subalgebra of $C(X)$ then $X=\Gamma$; but A may be pervasive on Γ and not on X.)

We begin our consideration of measures with two facts which were proved for essential maximal algebras by Bear [4].

THEOREM 5.1. *Let A be a pervasive subalgebra of $C(\Gamma)$, let $p \in S(A) - \Gamma$ and let μ_p be any positive measure on Γ which represents p. Then the closed support of μ_p is all of Γ.*

Proof. Let K be the closed support of μ_p. Suppose K is a proper closed subset of Γ. Since $f(p) = \int_K f\,d\mu_p$,

$$|f(p)| \leq \sup_K |f|,$$

and since A is pervasive the measure μ_p defines a multiplicative linear functional on $C(K)$. Thus μ_p must be a point mass, which is absurd since $p \notin \Gamma$.

COROLLARY: *Let A be a pervasive subalgebra of $C(\Gamma)$ and let f be a function in A which has norm 1. If there is a point $p \in S(A) - \Gamma$ such that $|f(p)| = 1$, then f is constant.*

Proof. Choose a measure μ_p representing p. Since μ_p has mass 1, $|f| \leq 1$, and

$$1 = |f(p)| = \left|\int_\Gamma f\,d\mu_p\right|,$$

it is clear that $f(x) = f(p)$ for all x in Γ.

Of course Theorem 5.1 and its corollary hold for essential maximal algebras. We have stated them for pervasive algebras to emphasize once again that the pervasive property of maximal algebras is the fundamental one (among the known special properties).

We shall later need the following.

THEOREM 5.2. *Let f be a function in A which has norm 1, and let K be the subset of $S(A)$ on which $f = 1$ (assume K is not empty). Let A_K be the algebra obtained by restricting A to the set K. Then A_K is closed and*

(i) $S(A_K) = K$.

(ii) $\Gamma(A_K) \subset \Gamma \cap K$.

(iii) *If $p \in K$, then any measure μ_p on Γ which which represents p (as a homomorphism of A) is supported on $K \cap \Gamma$.*

Proof. A portion of this theorem was proved by Bear [5]. It is no loss of generality to assume that $K = [|f| = 1]$, for we may replace f by $\frac{1}{2}(1+f)$ if this is not so. Then $f = 1$ on K and $|f| < 1$ on $S(A) - K$. If I is the closed ideal of functions which vanish on K then A_K is isomorphic to A/I and thus inherits the quotient norm

$$\|g_0\| = \inf_{g \in I} \|g + g_0\|.$$

Clearly $\|g_0\| \leqslant \|g_0\|_\infty = \sup_K |g_0|$. But the opposite inequality also holds since

$$\lim_{n\to\infty} \|f^n g_0\| = \|g_0\|_\infty.$$

Thus the quotient norm is the sup norm on K so that A_K must be complete in the sup norm.

(i) $S(A_K) = K$ is well known, because K is a hull, i.e., the set of zeros of the function $(1-f)$.

(ii) is clearly implied by (iii).

(iii) Let $p \in K$ and let μ_p be a positive measure on Γ which represents p. For the function f we then have

$$1 = f(p) = \int_\Gamma f \, d\mu_p$$

and since $|f| \leqslant 1$ we must have $f = 1$ on the closed support of μ_p.

We should like to make some comments which place theorem 5.1 in what we believe is its proper setting. For an algebra A

(i) the *interior* of $S(A)$ is the set $I = S(A) - \Gamma$.

(ii) the *accessible set* is the set $L = \bar{I} - I$.

(iii) the *minimal support set* is the subset S_* of Γ obtained by intersecting the closed supports of all measures μ_p which represent points p in I.

(iv) *the maximal support set* is the set S^* which is the closure of the union of the closed supports of all measures μ_p which represent points p in I.

Of course I is an open subset of $S(A)$, while L, S_*, and S^* are closed subsets of Γ. If the interior I is non-empty, then $S_* \subset S^*$. If A is a maximal subalgebra of $C(\Gamma)$, or more generally, if the algebra A_E obtained by restricting A to its essential set E is a pervasive subalgebra of $C(E)$, then theorem 5.1 tells us that E is contained in the minimal support set S_*.

THEOREM 5.3. *For any algebra A, the accessible set L is contained in the essential set E.*

Proof. Let A_0 be the ideal in $C(\Gamma)$ of functions which vanish on E. By definition of the essential set, A_0 is contained in A. As is well-known, every non-zero complex homomorphism of A_0 is evaluation at a point of $\Gamma - E$. Thus if Φ is a complex homomorphism of A, the restriction of Φ to A_0 is either identically 0 or is evaluation at a point of $\Gamma - E$. If there is a point $\gamma_0 \in \Gamma - E$ such that $\Phi(f) = f(\gamma_0)$ for every f in A_0, then $\Phi(f) = f(\gamma_0)$ for all f in A, because a non-zero homomorphism on an ideal has a unique extension to a homomorphism of the full algebra. Thus we see that if p is a point of $S(A)$ which does not lie in $\Gamma - E$, then every function in A_0 vanishes at p. In particular this is true for each point $p \in L$. Thus, no point of L lies in $\Gamma - E$, i.e., $L \subset E$, q.e.d.

THEOREM 5.4. *For any algebra A, $S^* \subset E$.*

Proof. Let p be a point of I and let μ_p be any measure on Γ which represents p. With the notation of 5.3 (and a portion of the proof), for each f in A_0 we have

$$0 = f(p) = \int_\Gamma f\, d\mu_p = \int_U f\, d\mu_p$$

where $U = \Gamma - E$. But this says that the restriction of μ_p to U sends every continuous function on U which vanishes at infinity into 0. Thus this restriction is the 0 measure, i.e., $\mu_p(U) = 0$. This proves that the closed support of μ_p is contained in E.

If we put these results together for maximal algebras we have the following. If A is a maximal subalgebra of $C(\Gamma)$ and if $\Gamma \neq S(A)$, then

$$L \subset E = S_* = S^*.$$

It seems reasonable to us to conjecture that when A is maximal and I is non-empty then $L = E = S_* = S^*$. In fact, one might conjecture that for any algebra A with I non-empty the inclusion $S_* \subset L$ holds. The question posed by this conjecture has the following two equivalent formulations:

(i) If $p \in I$ does there exist a measure μ_p representing p whose closed support is contained in the accessible part of the Šilov boundary, $L = \bar{I} - I$?

(ii) If A_1 is the closure of the restriction of A to $\bar{I}$, is the Šilov boundary for A_1 exactly L?

When the question is stated in form (ii), we see that we are asking whether a strengthened maximum modulus principle holds for function algebras. This seems to be a very interesting question. An affirmative answer could have important consequences.

6. Tests for maximality

In our comments about pervasive algebras, we pointed out that if A is a pervasive subalgebra of $C(X)$ then $\Gamma = X$. From this it is easy to see that if A is any maximal subalgebra of $C(X)$ then $\Gamma = X$. So, in inquiring whether an algebra A is maximal we need only inquire whether A is a maximal subalgebra of $C(\Gamma)$.

How does one tell if a given algebra A is a maximal subalgebra of $C(\Gamma)$? This is, of course, a difficult question. One can attempt to check whether it is true that on the essential set for A the algebra is pervasive, an integral domain, antisymmetric, etc. In section 7, we shall use the analytic property of maximal algebras to give an example of an algebra A which is contained in no maximal subalgebra of $C(\Gamma)$. But there are algebras A which are pervasive subalgebras of $C(\Gamma)$ without being maximal, as we have seen. One is then certainly led to a search for other tests for the maximality of A. We should like to outline now another technique which, although simple, does seem to have some interesting consequences.

LEMMA 6.1. *Let p be a point in $S(A) - \Gamma$ and let μ_p be a representing measure for p. Then there is a proper subalgebra B of $C(\Gamma)$ which contains A and is maximal with the property that μ_p defines a multiplicative linear functional on B.*

Proof. Let F be the family of all proper closed subalgebras of $C(\Gamma)$ which contain A and on which μ_p defines a multiplicative linear functional. If $[B_\alpha]$ is a linearly ordered subset of F then μ_p is multiplicative on the closure of the union of the B_α, and this closure is not all of $C(\Gamma)$ since $p \in S(A) - \Gamma$. By Zorn's lemma, F contains a maximal element B.

By a similar argument, one can prove

LEMMA 6.2. *If Γ is a proper subset of $S(A)$ then among all closed subalgebras B of $C(S)$ such that*

(i) $A \subset B$

(ii) $\Gamma(B) = \Gamma(A)$

there is a maximal one.

Of course, if A is a maximal subalgebra of $C(\Gamma)$ then A is already a maximal closed subalgebra of $C(S)$ with the property that each function in the algebra assumes its maximum on Γ. However, the converse is not true as is shown by the following example, due largely to H. Rossi [Thesis, M.I.T., 1959].

Let $\triangle$ be the closed bicylinder $|z| \leqslant 1$, $|w| \leqslant 1$ in complex two-space, and let A be the algebra of all continuous functions on $\triangle$ which are analytic in the interior of $\triangle$. This algebra A is simply the uniform closure on $\triangle$ of the algebra of all polynomials in two complex variables. The space $S(A)$ is $\triangle$, and $\Gamma(A)$ is the torus $\Gamma = [(z, w);\ |z| = |w| = 1]$. The topological boundary T of $\triangle$ is larger than the Šilov boundary for A.

THEOREM 6.3. *Let B be a uniformly closed algebra of continuous functions on the topological boundary T such that each function in B assumes its maximum modulus on the set Γ. If B contains A, then $B = A$.*

Proof. Let (z_0, w_0) be a point in Γ, i.e., $|z_0| = |w_0| = 1$. Then the disc $K = [(z, w_0);\ |z| \leqslant 1]$ is contained in the topological boundary T. We wish to show that every function f in the algebra B is an analytic function of z on the disc K.

Let $F(z, w) = \frac{1}{2}(1 + w w_0^{-1})$. Then $F \in A$, $\|F\| = 1$ and $K = [\,|F| = 1] = [F = 1]$ (these sets relative to $\triangle$). Then, as we observed in theorem 5.2 the algebra B_K obtained by restricting B to the disc K is a closed subalgebra of $C(K)$ and $\Gamma(B_K) \subset \Gamma \cap K$. But $\Gamma \cap K$ is exactly the circumference of the disc K. Hence B_K is a uniformly closed algebra of continuous functions on K each of which assumes its maximum modulus on the circumference. Also $B_K \supset A_K$, and A_K is simply the algebra of all continuous functions on K which are analytic for $|z| < 1$. By the theorem of Wermer which we mentioned in section 2, the algebra A_K is a maximal subalgebra of the circumference of K. Thus, it must be that $A_K = B_K$, or that every function in B is analytic in z on the interior of the disc K.

Similarly we can show that each function in B is analytic in w on the disc $[(z_0, w);\ |w| \leqslant 1]$.

Since this holds for every point $(z_0, w_0) \in \Gamma$, the Fourier coefficients

$$\hat{f}(m, n) = \int_{\Gamma} e^{-im\theta} e^{-int} f(e^{i\theta}, e^{it})\, d\theta\, dt$$

vanish outside the quadrant $m \geqslant 0$, $n \geqslant 0$ for every $f \in B$. Thus $B = A$, q.e.d.

Of course the bicylinder algebra A is not a maximal subalgebra of $C(\Gamma)$. A larger algebra is for example the algebra of all continuous functions f on Γ such that

$$\int_0^{2\pi} e^{in\theta} f(e^{i\theta}, 1)\, d\theta = 0, \quad n = 1, 2, \ldots .$$

It is interesting to contrast this algebra of analytic functions in the bicylinder with with the corresponding algebra when $\triangle$ is the closed unit sphere in complex two-space. The algebra A of all continuous functions on $\triangle = [(z, w); |z| \leqslant 1, |w| \leqslant 1]$ which are analytic in the interior has $\triangle$ as its space of maximal ideals and the full topological boundary of $\triangle$ as its Šilov boundary. In this case there are larger closed subalgebras of $C(\triangle)$ in which every function assumes its maximum on the unit sphere. One such example is given in [9].

Let us return now to the consideration of our general algebra A and look more closely at lemma 6.1. In certain special cases it may happen that there is a point $p \in S(A) - \Gamma$ with these properties:

(i) if $A \subset B \subset C(\Gamma)$ and no measure representing p is multiplicative on B, then $B = C(\Gamma)$.

(ii) if $A \subset B \subset C(\Gamma)$ and any measure representing p is multiplicative on B then $B = A$. It is then clear that A is a maximal subalgebra of $C(\Gamma)$.

This is a very special situation of course, but it does arise. It was this idea which was exploited in [8] to give a very short proof of the Wermer theorem on the maximality of the analytic functions on the circle (and some generalization thereof). Royden [13] and Bishop [6] have also exploited this idea. We should like now to indicate how a simple extension of this idea leads to some examples of the "primary" algebras described in the introduction.

For each positive integer k, let A_k be the uniformly closed algebra of continuous functions on the unit circle which is generated by $1, z^k, z^{k+1}, \ldots$. Then A is (isomorphic to) the algebra of continuous functions on $|z| \leqslant 1$ which are analytic for $|z| < 1$ and have derivatives of orders $1, \ldots, k-1$ which vanish at the origin. The space of maximal ideals of A_k is the closed disc $|z| \leqslant 1$ and the Šilov boundary for A_k is the unit circle Γ. The algebra A_1 is (of course) the uniform closure on Γ of the algebra of polynomials, that is, A_1 consists of all continuous functions f on Γ such that

$$\int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0, \; n = 1, 2, 3, \ldots .$$

THEOREM 6.4. *Let B be a closed subalgebra of $C(\Gamma)$ which contains A_k (for some fixed k). Then either $B = C(\Gamma)$ or B is contained in A_1.*

Proof. The origin of the unit disc defines a complex homomorphism of the algebra A_k by

$$h(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

If this homomorphism h does not extend to B, i.e., if no measure representing the origin is multiplicative on B, then the function $f(e^{i\theta}) = e^{ik\theta}$ is in B, since the function z^k in A_k vanishes on the disc only at the origin. But then it is easily seen that B contains the functions $e^{in\theta}$ for every integer n, so that $B = C(\Gamma)$.

Suppose the homomorphism h does extend to B. Then

$$h(f) = \int_0^{2\pi} f(e^{i\theta})\, d\mu(\theta),\ f \in B$$

for some positive measure μ on Γ. In particular then μ must evaluate every function in A_k at the origin, or

$$\int_0^{2\pi} e^{in\theta} d\mu(\theta) = 0,\ n \geqslant k.$$

Since μ is a real measure of mass 1 we must then have

$$d\mu(\theta) = \frac{1}{2\pi}\phi(\theta)\, d\theta$$

where ϕ is a trigonometric polynomial

$$\phi(\theta) = 1 + \sum_{p=1}^{k-1} [a_p e^{ip\theta} + \bar{a}_p e^{-ip\theta}].$$

Now let f be any function in the algebra B, and let $n \geqslant 0$. Then

$$0 = h(z^{n+k})\, h(f) = h(z^{n+k} f) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} e^{ik\theta} f(e^{i\theta})\, \phi(\theta)\, d\theta.$$

If we let $\psi(e^{i\theta}) = e^{ik\theta}\phi(\theta)$ the above equation says that for every $f \in B$, ψf is analytic, i.e.,

$$\int_0^{2\pi} e^{in\theta}\psi(e^{i\theta})\, f(e^{i\theta})\, d\theta = 0,\ n = 1, 2, \ldots .$$

Thus every $f \in B$ is (the boundary function of) a meromorphic function: $f = \psi f/\psi$. So f is meromorphic, ψ is analytic, and for every positive integer n the function ψf^n is analytic. Clearly f is itself analytic. Thus $B \subset A_1$.

In the case $k = 1$, the above theorem states that A_1 is a maximal subalgebra of $C(\Gamma)$. Furthermore, the theorem implies that for any k the algebra A_k is contained in precisely one maximal subalgebra of $C(\Gamma)$, namely A_1. So, when $k > 1$, the algebra A_k is contained in a maximal algebra but is not the intersection of the maximal subalgebras of $C(\Gamma)$ which contain it. Thus we might say that A_k is a primary algebra. We shall see shortly that this property of $A_k (k \geqslant 2)$ is rather easily deduced from the maximality of A_1 and the fact that A_k has finite codimension in A_1. We have given theorem 6.4 separately to show that the proof of the primary nature of A_k is not appreciably more difficult than the short proof of the maximality of A_1 given in [8].

We proceed now to enlarge our class of primary algebras. We shall need the following lemmas.

LEMMA 6.5. *Let A be a maximal subalgebra of $C(\Gamma)$ and let B be a subalgebra of $C(\Gamma)$ which contains an ideal I of A. If B contains a function f which is not in A, then B contains every continuous function on Γ which vanishes on the hull of I. If, in addition, the hull of I is finite, then $B = C(\Gamma)$.*

Proof. By the hull of I we mean the set of all points in Γ where every function in I vanishes. Since A is maximal and f is not in A, the linear algebra generated by A and f is dense in $C(\Gamma)$. This means each function h in $C(\Gamma)$ can be uniformly approximated by functions of the form

$$\sum_{k=0}^{n} g_k f^k, \; g_k \in A.$$

If g is any function in the ideal I, then the function

$$\sum_{k=0}^{n} g g_k f^k$$

belongs to B, because f is in B and each gg_k is in I, hence in B. From this it is clear that if $h \in C(\Gamma)$ and $g \in I$, then gh is in B. In other words, $J = I \cdot C(\Gamma)$ is contained in B. Now J is an ideal in $C(\Gamma)$ so that the closure $\bar{J}$ consists of all continuous functions vanishing on the hull of J. Clearly J and I have the same hull. This proves the first statement of the lemma.

If we now assume that the hull of I is a finite point set, then since B separates points of Γ and contains every continuous function vanishing on that finite point set, it is clear that $B = C[\Gamma)$.

LEMMA 6.6. *Let A be a commutative linear algebra with identity and let A_0 be a subalgebra of A which has finite codimension in A. Then A_0 contains an ideal I of the algebra A such that I has finite codimension in A.*

Proof. Let I be the set of all elements $a \in A_0$ such that $aA \subset A_0$. Then I is an ideal in A and $I \subset A_0$. For each element $a \in A_0$ let L_a be the linear transformation of A into A which is right multiplication by a. Since A_0 is a subalgebra of A, the space A_0 is invariant under L_a. Thus L_a induces a linear transformation $\tilde{L}_a$ of the linear space A/A_0 into itself. The mapping $a \to \tilde{L}_a$ is an algebra homomorphism of A_0 into the algebra of linear transformations on the finite-dimensional space A/A_0. The kernel of this homomorphism is the ideal I. Thus A_0/I is finite-dimensional, hence A/I is finite-dimensional.

THEOREM 6.7. *Let A be a maximal subalgebra of $C(\Gamma)$ and let A_0 be a subalgebra of A which has finite codimension in A. If B is a subalgebra of $C(\Gamma)$ which contains A_0, then either $B = C(\Gamma)$ or B is contained in A.*

Proof. By Lemma 6.6, A_0 contains an ideal I of the algebra A which has finite codimension in A. Thus the hull of I must be a finite point set. If B is not contained in A, then by lemma 6.5, $B = C(\Gamma)$.

Under the hypotheses of Theorem 6.7, the algebra A_0 is contained in precisely one maximal subalgebra of $C(\Gamma)$. This theorem applies (of course) to the algebras A_k of theorem 6.4. We should like to discuss now some further examples of this situation.

For the remainder of this section, let Γ denote the unit circle of the complex plane, and let A be the uniform closure on Γ of the algebra of polynomials. Suppose that $f_1, \ldots, f_n$ are functions, each analytic in a neighborhood of the unit disc, satisfying

(i) $f_1, \ldots, f_n$ separate the points of Γ

(ii) at each point of Γ, one of the functions f_j has a non-vanishing derivative.

Let A_0 be the subalgebra of $C(\Gamma)$ generated by $f_1, \ldots, f_n$. Results of Wermer [20; Lemma 3.2] and [19; Theorem 1.2] imply that there is a function

$$g(z) = (z-\lambda_1)^{p_1} \cdots (z-\lambda_k)^{p_k}$$

such that A_0 contains the ideal $I = gA$ of the algebra A. Thus A_0 is contained in precisely one maximal subalgebra of $C(\Gamma)$, namely A. As an approximation theorem, this result states that if $f_1, \ldots, f_n$ are analytic in a neighborhood of the closed disc and satisfy conditions (i) and (ii) above, then given any continuous function h on Γ which is not in A, we can approximate every continuous function on Γ by polynomials in $f_1, \ldots, f_n$ and h.

There is a theorem similar to this where one assumes that $f_1, \ldots, f_n$ are analytic in an annulus containing Γ and satisfy conditions (i) and (ii). In this case the algebra A_0 generated by $f_1, \ldots, f_n$ need not be a subalgebra of A; indeed, this algebra A_0 may be all of $C(\Gamma)$. If $A_0 \neq C(\Gamma)$, Wermer's results [19; 20; 21] state the following. There is a Riemann surface F and an analytic homeomorphism $z \to z'$ of Γ onto an analytic circle Γ' on F which bounds a domain D such that $D \cup \Gamma'$ is compact. Each of the functions $f_1, \ldots, f_n$ extends analytically from Γ' into a neighborhood of $D \cup \Gamma'$. If A' denotes the algebra of all continuous functions on Γ which extend from Γ' analytically to D then A' is a maximal subalgebra of $C(\Gamma)$, which of course contains A_0. It is once again true that A_0 contains an ideal I of A' which has a finite hull. Thus, in this case, the algebra A_0 is contained in exactly one maximal subalgebra of $C(\Gamma)$, namely A'. But the algebra A' may consist not of the functions which extend from Γ analytically to the unit disc but rather those which extend analytically into the domain D on the Riemann surface F.

If we wish to state this last result as an approximation theorem, it says that if $h \in C(\Gamma)$ and $h \notin A'$ then $f_1, \ldots, f_n$ and h generate $C(\Gamma)$. Given the function h on Γ it might be extremely difficult to determine whether or not h is in A'. But one can state a slightly weaker approximation theorem whose hypothesis is more easily verified in specific cases.

Because conditions (i) and (ii) are satisfied by $f_1, \ldots, f_n$ there will be an annulus containing Γ such that the analytic homeomorphism $z \to z'$ of Γ into the surface F can be extended to an analytic homeomorphism of this annulus onto a neighborhood of the curve Γ'. Thus one sees that if h is in A' then in the plane h must be extendable from Γ so as to be analytic in an annulus of one of the two types: $[z; 1 < |z| < p]$ or $[z; p < |z| < 1]$. That is, $f_1, \ldots, f_n$ determine a positive number p, either less than or greater than 1, such that each function in A' is extendable to a continuous function on $[z; 1 \leqslant |z| \leqslant p]$ which is analytic on $[z; 1 < |z| < p]$, or a continuous function on $[z; p \leqslant |z| \leqslant 1]$ analytic on $[z; p < |z| < 1]$. We might describe this by saying that $f_1, \ldots, f_n$ determine one side of the unit circle such that each function in A' is analytic in an annulus on that side of the circle.

Now one can certainly state the following. Suppose $f_1, \ldots, f_n$ are analytic in an annulus containing Γ and satisfy conditions (i) and (ii) above. If $h \in C(\Gamma)$ and h is not analytically extendable to an annulus on either side of Γ, then every continuous function on Γ can be uniformly approximated by polynomials in $f_1, \ldots, f_n$ and h.

7. Bounded analytic functions

Let L_∞ be the algebra of bounded measurable functions on the unit circle of the complex plane, identifying functions which are equal almost everywhere. Then L_∞ is a commutative Banach algebra with the norm

$$\|f\| = \operatorname{ess\,sup}_{\theta} |f(e^{i\theta})|.$$

Let H_∞ be the (closed) subalgebra of L_∞ consisting of the functions f such that

$$\int_0^{2\pi} e^{in\theta} f(e^{i\theta})\, d\theta = 0, \; n = 1, 1, 2, 3, \ldots .$$

For each $f \in H_\infty$ the function

$$f(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(\theta)\, d\theta,$$

where P_z is the Poisson kernel for z, is a bounded analytic function in the open unit disc, and

$$\|f\| = \sup_{|z|<1} |f(z)|.$$

Furthermore, a well-known theorem of Fatou states that every bounded analytic function in the disc arises in this way. Thus H_∞ is isometrically isomorphic to the algebra of bounded analytic functions in the open disc.

The algebra L_∞ is self-adjoint, i.e., closed under complex conjugation, and is therefore isometrically isomorphic to the algebra $C(X)$ of all continuous functions on its space of maximal ideals. This follows from the commutative Gelfand-Neumark theorem, see [11; p. 88]. The space X is also the Stone space for the Boolean algebra of measurable subsets of the circle, modulo sets of measure 0. Thus X is an extremally disconnected compact Hausdorff space. We shall need to know only that X is totally disconnected and that a basis for the topology of X is given by the open and closed sets

$$[x \in X;\ k_M(x) = 0]$$

where M is a measurable set on the circle and k_M is its characteristic function.

The algebra H_∞ is isometrically isomorphic to a closed subalgebra A of $C(X)$. For the sake of simplicity we shall sometimes write $A = H_\infty$ and/or $C(X) = L_\infty$.

We proceed now to record some facts about the space of maximal ideals of H_∞ which are contained in an unpublished paper of I. J. Schark.

Each point of the open unit disc defines a complex homomorphism of H_∞ by

$$f \to f(z)$$

and the open disc is thus "embedded" in the space $S(H_\infty)$. Also, there is a natural continuous mapping π of $S(H_\infty)$ onto the closed unit disc, as follows. The algebra H_∞ contains as a closed subalgebra the algebra A_1 of continuous "analytic" functions on the circle. The mapping π mentioned above sends a homomorphism of H_∞ into its restriction to the subalgebra A_1, and this restriction corresponds to a point in the closed disc. The mapping π is one-one over the interior of the disc; that is, if Φ is a complex homomorphism of H_∞ such that Φ evaluates each function in A_1 at some point z, $|z| < 1$, then Φ is simply evaluation at z on all of H_∞. This is evident from the fact if $f \in H_\infty$ then $f(z) = 0$ if and only if

$$g(\zeta) = \frac{f(\zeta)}{z - \zeta}$$

is in H_∞. The mapping π certainly maps $S(H_\infty)$ onto the closed unit disc since $\pi(S)$ is a compact subset of the closed disc which contains each interior point. The mapping π is (as we shall see) not one-one over the circumference. Notice also that π^{-1} actually defines a homeomorphism of $|z| < 1$ into $S(H_\infty)$ since the topology of the disc is in either case the weak topology defined by H_∞. Let us call $\triangle$ the image of the open disc in $S(H_\infty)$. One unsolved problem in this context is whether $\triangle$ is dense in $S(H_\infty)$. Since H_∞ is an algebra of continuous functions on $\triangle$, it is at least clear that the closure of $\triangle$ in $S(H_\infty)$ will contain the Šilov boundary $\Gamma(H_\infty)$.

We wish now to show that $\Gamma(H_\infty)$ is (homeomorphic to) $S(L_\infty) = X$. Since H_∞ is a subalgebra of L_∞ there is a natural continuous mapping π_1 of X into $S(H_\infty)$, obtained by restricting each homomorphism of L_∞ to the subalgebra H_∞.

LEMMA 7.1. *The mapping π_1 is a homeomorphism of X into $S(H_\infty)$ and $\pi_1(X)$ is exactly $\Gamma(H_\infty)$.*

Proof. What we must show is that the functions in H_∞, when regarded as elements of L_∞, separate points on X and that the Šilov boundary for H_∞ is all of X. We shall first show that the Šilov boundary condition is satisfied. From this proof it will be clear that H_∞ separates points on X.

Let $x \in X$ and let U be an open set in X containing x. We must show that there is a function $h \in H_\infty$ whose maximum modulus on U is greater than the modulus of h anywhere on $X - U$. It will suffice to do this when $U = [k_M = 1]$, M a measurable set on the unit circle. Define

$$\psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} k_M(e^{i\theta})\, d\theta.$$

Then ψ is analytic for $|z| < 1$ and $h = e^{\psi}$ is a bounded analytic function in the unit disc. Furthermore, on the unit circle, $|h| = \exp k_M$ almost everywhere. Thus on X, $|h| = e$ where $k_M = 1$ and $|h| = 1$ where $k_M = 0$; so h is the desired function.

Let us now regard X as a closed subset of $S(H_\infty)$. We then have the following picture of $S(H_\infty)$. This space contains the open unit disc $\triangle$. The closure of $\triangle$ contains besides $\triangle$ the Šilov boundary $\Gamma(H_\infty)$ which is the space X of maximal ideals of L_∞. It is not known whether the closure of $\triangle$ is all of $S(H_\infty)$, but it is easily seen that there are points in $\bar{\triangle} - \triangle$ which are not on the Šilov boundary. For example, the function

$$f(z) = \exp\left(\frac{z+1}{z-1}\right)$$

is in H_∞, has no zeros in $\triangle$, but must vanish somewhere on $\bar{\triangle}$ since $f(z)$ tends to 0 as z approaches 1 along the positive axis. No such 0 of f can occur on the Šilov boundary X since the function f has modulus 1 on the unit circle and is therefore invertible in L_∞. This is the extent of the information we shall need from the work of I. J. Schark.

The space $S(H_\infty^{\Gamma}) - \triangle$ and a fortiori the space X, is fibered by a natural continuous mapping π onto the unit circle, π being the map which sends each complex homomorphism into its restriction to the algebra of functions in H_∞ which are continuous on the unit circle. If ζ is a complex number, $|\zeta| = 1$, let us define the fiber of $S(H_\infty)$ over ζ [the fiber of X over ζ] to be the set $\pi^{-1}(\zeta)[\pi^{-1}(\zeta) \cap X]$. We shall denote this fiber by $S(H_\infty)_\zeta$ $[X_\zeta]$. If we consider the function $f(z) = z$, this fiber over ζ is simply the set of all points in $S(H_\infty)$ $[X]$ where $f = \zeta$.

THEOREM 7.2. *Let B be a closed subalgebra of $L_\infty = C(X)$ which contains H_∞. Then*

(i) *for each ζ, $|\zeta| = 1$, the algebra B_ζ obtained by restricting B to the fiber X_ζ is a closed subalgebra of $C(X_\zeta)$*

(ii) *if B is a proper subalgebra of $C(X)$ then for at least one ζ the algebra B_ζ is a proper subalgebra of $C(X_\zeta)$.*

Proof. (i) The algebra B contains the function $f(z) = \frac{1}{2}(1 + z\zeta^{-1})$. Also $\|f\| = 1$ and X_ζ is exactly the subset of X on which $f = 1$. By theorem 5.2, B_ζ is closed. (ii) Suppose $B_\zeta = C(X_\zeta)$ for each ζ; we shall show that $B = C(X)$. The Šilov boundary $\Gamma(B)$ is X, because B is a subalgebra of $C(X)$ and contains the algebra H_∞ whose Šilov boundary is all of X. We wish to prove that in fact X is the entire space of maximal ideals of B.

Let Φ be a complex homomorphism of B. Suppose that Φ sends the identity function z into a complex number ζ of absolute value 1. Considering once again the function $f(z) = \frac{1}{2}(1 + z\zeta^{-1})$ and applying theorem 5.2, any measure μ_Φ on $\Gamma(B) = X$ which represents Φ must be supported on the subset of X on which $f = 1$, that is, on the fiber X_ζ. But the restriction of B to X_ζ is all of $C(X_\zeta)$ and since μ_Φ is multiplicative on B_ζ, μ_Φ must be point mass and Φ must simply be evaluation at some point of X.

Suppose that the restriction of Φ to H_∞ is in $\triangle$, i.e., that Φ sends the identity function z into λ, $|\lambda| < 1$. The homomorphism of H_∞ which is evaluation at λ then extends to a homomorphism of B. But this means that for any α, $|\alpha| < 1$, the α-homomorphism of H_∞ must extend to B. If not then $(z - \alpha)^{-1}$ is in B, and since $1, z, z^2, \ldots$ are in B every continuous function on the unit circle is in B. In particular, $(z - \lambda)^{-1}$ is in B, contradicting our assumption that the λ-homomorphism extends.

Now we claim that the assumption that Φ restricted to H_∞ is evaluation at λ is contradictory. For restriction to H_∞ is a continuous mapping of $S(B)$ into $S(H_\infty)$. We just observed that if the range of this mapping contains any point of $\triangle$ it contains all of $\triangle$. But the range is compact and would therefore contain all of $\overline{\triangle}$. We noted earlier that $\overline{\triangle}$ contains points which are neither in $\triangle$ nor in X, and we proved above that every point of $S(H_\infty) - \triangle$ which is in the range is in X.

The space $S(B)$ is thus X, and since X is totally disconnected $B = C(X)$. This follows from a well-known theorem of Šilov [15], that a function algebra A must contain the characteristic function of any open and closed subset of $S(A)$.

THEOREM 7.3. *The algebra H_∞ is contained in no maximal subalgebra of $L_\infty = C(X)$.*

Proof. Suppose the contrary, and let B be a maximal subalgebra of $C(X)$ which contains H_∞. By theorem 7.2, for each ζ, $|\zeta| = 1$, the algebra B_ζ obtained by restricting B to the fiber X_ζ is closed. Since B is a proper subalgebra of $C(X)$ there is a ζ such that B_ζ is a proper subalgebra of $C(X_\zeta)$. But if B_ζ is a proper closed subalgebra of $C(X_\zeta)$, then since B is maximal, B must contain every continuous function on X which vanishes on X_ζ, because the algebra of all $f \in C(X)$ such that f restricted to X_ζ is in B_ζ is a proper subalgebra of $C(X)$ which contains B. Thus the essential set E for the algebra B lies wholly inside one fiber X_ζ.

It is no loss of generality to assume that E is contained in X_1, the fiber over 1. Now the algebra B_E obtained by restricting B to its essential set E is analytic, that is, if $f \in B$ and f vanishes on a non-empty open subset of E then f vanishes on all of E (theorem 3.1). We shall now arrive at a contradiction by showing that if E is any closed subset of the fiber X_1 which contains more than one point (as the essential set for B necessarily would) then there is a function $h \in H_\infty$ which vanishes on a non-empty open subset of E but is not identically 0 on E.

Let x_0 and x_1 be two distinct points of E. Then there is a measurable set M on the unit circle such that the characteristic function k_M is 0 at x_0 and 1 at x_1. Let

$$u(e^{i\theta}) = [1 - k_M(e^{i\theta})] \log |1 - e^{i\theta}|.$$

Then u is a Lebesgue integrable function on the unit circle which is real-valued and bounded above. If we define

$$\psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(e^{i\theta})\, d\theta$$

then ψ is analytic for $|z| < 1$, and since u is bounded above, $h = e^\psi$ is a bounded analytic function in the unit disc. Also, almost everywhere on the unit circle

$$|h| = e^u = \begin{cases} 1, & \text{on } M \\ |1 - e^{i\theta}|, & \text{off } M. \end{cases}$$

In particular, the function f defined by

$$f(e^{i\theta}) = \frac{[1 - k_M(e^{i\theta})]\, h(e^{i\theta})}{1 - e^{i\theta}}$$

is a bounded measurable function. We rewrite this equation

$$h = (1 - z) f + k_M h.$$

Let Φ be a complex homomorphism of L_∞ which lies in the fiber X_1 and for which $\Phi(k_M)=0$. Then

$$\Phi(h)=\Phi(1-z)\Phi(f)+\Phi(k_M)\Phi(h)=0\cdot\Phi(f)+0\cdot\Phi(h)=0.$$

This says that h vanishes on the non-empty open subset of the fiber X_1 on which $k_M=0$. Also, no homomorphism of L_∞ sends both h and $(1-k_M)$ into 0. For let

$$g=(1-k_M)+k_M h.$$

Then

$$g=\begin{cases} h, & \text{on } M \\ 1, & \text{off } M \end{cases}$$

so that $|g|=1$ almost everywhere. Thus g is invertible in L_∞ so that the ideal generated by $(1-k_M)$ and h contains 1.

Returning now to our two points x_0, x_1 in $E \cap X_1$, we have constructed a function $h \in H_\infty$ such that h vanishes on the non-empty open subset $E \cap [k_M=0]$ of E and does not vanish on all of $E(h(x_1) \neq 0)$.

The above argument shows that a maximal subalgebra of L_∞ containing H_∞ simply cannot exist.

We might point out one interesting fact which results from theorem 7.2. Let B be the closed subalgebra of L_∞ which is generated by H_∞ and the set of all continuous functions on the unit circle. Then B is not all of L_∞, because if $|\zeta|=1$ then the restriction of B to the fiber X_ζ is exactly the same as the restriction of H_∞ to X_ζ, because each continuous function on the circle is constant on X.

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