CONCERNING A THEOREM OF PALEY ON LACUNARY POWER SERIES

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1. The main purpose of this paper is to prove the following theorem:

THEOREM 1. *If a lacunary power series*

$$\sum_{k=1}^{\infty} c_k z^n \qquad (n_{k+1}/n_k > q > 1) \tag{1.1}$$

satisfies the conditions

$$c_k \to 0, \ \sum |c_k| = \infty \tag{1.2}$$

and if ζ is any point in the complex plane, then there exists a point ξ on the unit circle such that $\sum c_k \xi^{n_k}$ converges to ζ.

This theorem was stated by Paley in a note [1]. Since the note was in the nature of a research announcement no proof was given for the theorem. However, in a letter to Prof. Zygmund dated Oct. 7, 1932, Paley gave an outline of the proof, as he saw it, for his theorem. The argument is made to depend on a lemma (Theorem 2 of the present paper). Paley believed that the proof of the lemma would follow the reasoning given in Zygmund's article [4]. However, attempts to reconstruct the proof along these lines have not succeeded. Taking this lemma for granted, Paley next sketches how his theorem may be deduced from it. He presents an ingenious idea how this part of the argument is to be carried out. Paley's idea here is, as it turns out, completely successful, but the details which need to be supplied are lengthy.

The purpose of the present paper is, therefore, twofold. First, a proof is given for theorem 2. The argument, which is contained in sections 1–4, is rather complex, and seems to indicate that the simpler idea envisaged by Paley could not succeed. It should be noted that when $q > 3$, Theorem 2 can be given a very simple proof.

This line of reasoning (which is still different) cannot be made to work in the general case. Secondly, in sections 5–7 we give in detail the argument which reduces Theorem 1 to Theorem 2.

We return to Theorem 1. This theorem implies in particular that the set E of points of convergence of the series on the unit circle is of the power of the continuum, and in this connection it is interesting to observe that if $\sum |c_k|^2 = \infty$ (for example if $c_k = k^{-\frac{1}{2}}$) then E is of measure 0 (see e.g. [5_I] p. 203). The proof of Theorem 1 actually gives a litte more than stated since it shows that the sums of $\sum c_k z^{n_k}$ cover the whole plane even if we restrict ourselves to an arbitrarily small arc of $|z|=1$.

Considering the real part

$$\sum_1^\infty (a_k \cos n_k \theta + b_k \sin n_k \theta) \tag{1.3}$$

of $\sum c_k z^{n_k}$ we deduce from Theorem 1 that if

$$a_k,\ b_k \to 0,\ \sum (|a_k| + |b_k|) = \infty \tag{1.4}$$

then the series (1.3) converges to any prescribed number in a set which is of the power of the continuum in every interval.(1)

In what follows we denote by $A_q, A'_q, B_q, \ldots$ etc. positive constants depending on q only, not necessarily the same at each occurrence. By A we denote absolute constants.

THEOREM 2. *There exist constants A_q and A'_q with the following properties: If a lacunary polynomial*

$$T(\theta) = \sum_{k=1}^{N} (a_k \cos n_k \theta + b_k \sin n_k \theta) \tag{1.5}$$

satisfies

$$T(\theta) \leqslant M$$

in an interval $a \leqslant \theta \leqslant b$, and if $b - a \geqslant A_q/n_1$, then

$$\sum (|a_k| + |b_k|) \leqslant A'_q M. \tag{1.6}$$

It will often be convenient to write T in the complex form

$$T(\theta) = \sum_{k=-N}^{N} \gamma_k e^{in_k\theta}, \tag{1.7}$$

where

(1) That under the hypothesis (1.4) the series (1.2) converges to any prescribed sum in a set which is dense in $(0, 2\pi)$ is an earlier result, (see [3]), and is a simple consequence of properties of smooth functions.

$$\gamma_0 = 0,\ \gamma_k = \tfrac{1}{2}(a_k - i\,b_k),\ \gamma_{-k} = \bar{\gamma}_k,\ n_{-k} = -n_k,\ n_{k+1}/n_k > q > 1 \text{ for } k > 0. \tag{1.8}$$

Then (1.6) can be written

$$\sum |\gamma_k| \leqslant A'_q M. \tag{1.9}$$

In what follows we systematically denote by T a polynomial (1.5) or (1.7) satisfying (1.9).

2. The proof of Theorem 2 is based on several lemmas.

LEMMA 1. *There exist constants B_q and B'_q such that if*

$$b - a \geqslant B_q/n_1 \tag{2.1}$$

then

$$\frac{1}{b-a}\int_a^b |T(\theta)|\,d\theta \geqslant B'_q \left(\sum |\gamma_k|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

We write

$$\sum_{k=1}^{N} |\gamma_k|^2 = \Gamma^2.$$

It is enough to show that, under the hypothesis (2.1),

$$\int_a^b T^2\,d\theta \geqslant (b-a)\,\Gamma^2, \tag{2.3}$$

and that for $b - a \geqslant K/n_1$, where K is any positive constant (independent of q this time)

$$\int_a^b T^4\,d\theta \leqslant C_{q,K}\,(b-a)\,\Gamma^4 \tag{2.4}$$

since then, by Hölder's inequality,

$$\int_a^b T^2\,d\theta \leqslant \left(\int_a^b |T|\,d\theta\right)^{\frac{2}{3}} \left(\int_a^b T^4\,d\theta\right)^{\frac{1}{3}},$$

and (2.2) is a corollary of (2.3) and (2.4).

The proof of (2.3) is straightforward. We have

$$\int_a^b T^2\,d\theta = \int_a^b \left(\sum \bar{\gamma}_j e^{-in_j\theta}\right)\left(\sum \gamma_k e^{in_k\theta}\right)d\theta = 2\,(b-a)\,\Gamma^2 + \sum_{j\neq k} \bar{\gamma}_j \gamma_k \int_a^b e^{i(n_k - n_j)\theta}\,d\theta. \tag{2.5}$$

Denote the last sum by R. Then

$$|R| \leqslant \sum_{j \neq k} \tfrac{1}{2} (|\gamma_j|^2 + |\gamma_k|^2) \frac{2}{|n_k - n_j|} \leqslant 4 \sum_{k=2}^{N} \sum_{j=1}^{k-1} (|\gamma_j|^2 + |\gamma_k|^2) \frac{1}{n_k - n_j}$$

$$\leqslant A_q \sum_{k=2}^{N} \frac{1}{n_k} \sum_{j=1}^{k-1} (|\gamma_j|^2 + |\gamma_k|^2) \leqslant A_q \sum_{k=2}^{N} \frac{1}{n_k} (\Gamma^2 + k\,|\gamma_k|^2)$$

$$\leqslant A_q \Gamma^2 \left\{ \sum_{k=2}^{N} \frac{1}{n_k} \right\} + A_q \Gamma^2 \left\{ \max \frac{k}{n_k} \right\}.$$

In view of the hypothesis $n_{k+1}/n_k > q > 1$, the two expressions in curly brackets are majorized by A_q/n_1, and so, if B_q is large enough, $b-a \geqslant B_q/n_1$ implies that $|R| \leqslant (b-a)\Gamma^2$, which in conjunction with (2.5) gives (2.3).

We now pass to (2.4) and assume first that $q \geqslant 4$.

We call two positive integers s and t *distant* if either $s/t \geqslant 2$ or $t/s \geqslant 2$; in the first case $s-t \geqslant s/2$. The hypothesis $q \geqslant 4$ implies that if $j<k$ then $n_k > 2n_j$ and the numbers n_k and $2n_j$ are distant.

We write
$$T = \sum_{-N}^{N} \gamma_k e^{in_k\theta} = \sum_{1}^{N} + \sum_{-N}^{-1} = T_1 + T_2.$$

Since $|T_1| = |T_2|$ it is enough to prove (2.4) with T_1 for T. Now

$$|T_1|^2 = \Gamma^2 + 2\,\Re \sum_{1 \leqslant j < k \leqslant N} \bar{\gamma}_j \gamma_k e^{i(n_k - n_j)\theta}$$

and (2.4) will be proved if we show that the complex-valued polynomial

$$S(\theta) = \sum_{1 \leqslant j < k \leqslant N} \bar{\gamma}_j \gamma_k e^{i(n_k - n_j)\theta}$$

satisfies
$$\int_a^b |S|^2 \, d\theta \leqslant C_K (b-a) \Gamma^4. \tag{2.6}$$

Clearly,

$$\int_a^b |S|^2 \, d\theta = \sum \bar{\gamma}_j \gamma_k \bar{\gamma}_{j'} \gamma_{k'} \int_a^b e^{i[(n_k - n_{k'}) - (n_j - n_{j'})]\theta} \, d\theta,$$

where $1 \leqslant j < k$, $1 \leqslant j' < k'$. We split the terms of the series into three separate groups;

(i) $k = k'$, $j = j'$;

(ii) $k = k'$, $j \neq j'$;

(iii) $k \neq k'$,

and evaluate the contribution of each group.

(i) the contribution of this group is

$$(b-a)\sum_{k=2}^{N}|\gamma_k|^2\sum_{j=1}^{k-1}|\gamma_j|^2 \leqslant (b-a)\,\Gamma^4.$$

(ii) It is enough to consider here the terms with $j' < j < k$. Their sum is majorized by

$$A\sum_{k=3}^{N}\sum_{j=2}^{k-1}\sum_{j'=1}^{j-1}\frac{|\gamma_k^2\gamma_j\gamma_{j'}|}{n_j-n_{j'}} \leqslant A\sum_{k=3}^{N}|\gamma_k|^2\sum_{j=2}^{k-1}\frac{1}{n_j}\sum_{j'=1}^{j-1}(|\gamma_{j'}|^2+|\gamma_j|^2)$$

$$\leqslant A\sum_{k=3}^{N}|\gamma_k|^2\sum_{j=2}^{k-1}\frac{1}{n_j}(\Gamma^2+j\,|\gamma_j|^2)$$

$$\leqslant A\,\Gamma^4\left\{\sum_{j=2}^{N}\frac{1}{n_j}\right\}+A\,\Gamma^4\left\{\max\frac{1}{n_j}\right\}.$$

Since the expressions in curly brackets are majorized by A/n_1, the contribution of the terms in (ii) does not exceed $A\,\Gamma^4/n_1$.

(iii) In estimating the terms we may suppose that $k' < k$. Their sum does not exceed

$$\sum|\gamma_j\gamma_k\gamma_{j'}\gamma_{k'}|\frac{2}{|(n_k-n_j)-(n_{k'}-n_{j'})|},$$

where $1\leqslant j<k$, $1\leqslant j'<k'$, and $k'<k$. Clearly,

$$n_k-n_j-(n_{k'}-n_{j'}) \geqslant n_k-2\,n_{k-1}$$

and since $q\geqslant 4$ the numbers n_k and $2\,n_{k-1}$ are distant, $n_k-2\,n_{k-1}\geqslant n_k/2$, and the last sum is not greater than

$$A\sum_{k=3}^{N}\sum_{k'=2}^{k-1}\sum_{j=1}^{k-1}\sum_{j'=1}^{k'-1}\frac{(|\gamma_j|^2+|\gamma_{j'}|^2)(|\gamma_k|^2+|\gamma_{k'}|^2)}{n_k}$$

$$=A\sum_{k=3}^{N}\frac{1}{n_k}\sum_{k'=2}^{k-1}(|\gamma_k|^2+|\gamma_{k'}|^2)\left\{\sum_{j=1}^{k-1}\sum_{j'=1}^{k'-1}(|\gamma_j|^2+|\gamma_{j'}|^2)\right\}.$$

Since the sum in curly brackets does not exceed $(k+k')\,\Gamma^2\leqslant 2\,k\,\Gamma^2$, the whole sum is majorized by

$$A\,\Gamma^2\sum_{k=3}^{N}\frac{k}{n_k}\sum_{k'=2}^{k-1}(|\gamma_k|^2+|\gamma_{k'}|^2)\leqslant A\,\Gamma^2\sum_{k=3}^{N}\frac{k}{n_k}(k\,|\gamma_k|^2+\Gamma^2)$$

$$\leqslant A\,\Gamma^4\left\{\mathrm{Max}\,\frac{k^2}{n_k}\right\}+A\,\Gamma^4\sum_{k=3}^{N}\frac{k}{n_k}\leqslant A\,\Gamma^4/n_1.$$

Collecting results we see that if $q\geqslant 4$, then

$$\int_a^b T^4\,d\theta \leqslant A\left[(b-a)+\frac{1}{n_1}\right]\Gamma^4$$

and so if we make the additional assumption that $b-a \geqslant K/n_1$ we obtain (2.4) with $C_{q,K} = C_K$.

It is now easy to complete the proof of (2.4) in the general case $q > 1$. We take r so large that $q^r \geqslant 4$ and split T into a sum of r polynomials, $T = T^{(1)} + T^{(2)} + \cdots + T^{(r)}$, in each of which the indices k of the n_k form an arithmetic progression of difference r. Since

$$\int_a^b (T^{(s)})^4\,d\theta \leqslant C_K\,(b-a)\,\Gamma^4,$$

for each s, $1 \leqslant s \leqslant r$, provided $b-a \geqslant K/n_1$ and since, by Jensen's inequality,

$$\int_a^b T^4\,d\theta \leqslant r^3 \sum_{s=1}^{r} \int_a^b (T^{(s)})^4\,d\theta,$$

where r depends on q only, the proof of (2.4) is complete. This also completes the proof of Lemma 1.

3. LEMMA 2. *There exist constants C_q, C'_q and C''_q with the following property: If $b-a \geqslant C_q/n_1$, then (a, b) contains a point ξ such that*

$$T(\xi) \geqslant C'_q\,\Gamma; \tag{3.1}$$

more generally there is a whole subinterval I *of (a, b) of length $\geqslant C''_q/n_N$ where*

$$T(\theta) \geqslant C'_q\,\Gamma. \tag{3.2}$$

First of all,

$$\left|\int_a^b T\,d\theta\right| \leqslant 4\sum_1^N |\gamma_k|\,n_k^{-1} \leqslant 4\left(\sum_1^N |\gamma_k|^2\right)^{\frac{1}{2}}\left(\sum_1^N n_k^{-2}\right)^{\frac{1}{2}} \leqslant A_q\,\Gamma/n_1. \tag{3.3}$$

Next, if $u^+ = \max\,(u, 0)$, then $u^+ = \frac{1}{2}\,(|u| - u)$, and, in view of (2.2) and (3.3)

$$\int_a^b T^+\,d\theta = \tfrac{1}{2}\int_a^b |T|\,d\theta - \tfrac{1}{2}\int_a^b T\,d\theta \geqslant B'_q\,(b-a)\,\Gamma - D_q\,\Gamma/n_1 \geqslant \tfrac{1}{2}B'_q\,(b-a)\,\Gamma$$

if C_q is large enough. A comparison of the extreme terms here gives (3.1) with $C'_q = \frac{1}{2}\,B'_q$.

Let θ be any point of the interval $(\xi-\delta, \xi+\delta)$ where δ will be defined shortly. Then

$$|T(\theta)-T(\xi)| \leqslant 2\delta \sum_1^N n_k |\gamma_k| \leqslant 2\delta \left(\sum_1^N |\gamma_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^N n_k^2\right)^{\frac{1}{2}} \leqslant \delta A_q \Gamma n_N$$

so that if $\delta \leqslant C_q''/n_N$ and C_q'' is small enough, we obtain (3.2) with $\frac{1}{2} C_q'$ for C_q'. This completes the proof of Lemma 2.

4. We shall now apply Lemmas 1 and 2 to the proof of Theorem 2 and restate the latter in the following equivalent form.

THEOREM 2′. *There exist constants A_q and A_q' such that in every interval of length A_q/n_1 there is a point ξ such that*

$$T(\xi) = \sum_{-N}^{N} \gamma_k e^{in_k\xi} \geqslant A_q' \sum_{-N}^{N} |\gamma_k|.$$

We begin with the observation that for any sequence $\delta_1, \delta_2, \ldots, \delta_k$,

$$(\sum |\delta_j|^2)^{\frac{1}{2}} \leqslant \sum |\delta_j| \leqslant k^{\frac{1}{2}} (\sum |\delta_j|^2)^{\frac{1}{2}} \tag{4.1}$$

so that if k is bounded the "first norm", $\sum |\delta_j|$, and the "second norm", $(\sum |\delta_j|^2)^{\frac{1}{2}}$, are comparable.

Next we consider the constants C_q, C_q'' of Lemma 2 and suppose that $T = \sum (a_k \cos n_k\theta + b_k \sin n_k \theta)$ is split into successive blocks of terms which we call "long" and "short" and which alternate. We call the long blocks Δ_j' and the short ones Δ_j''; hence $T = \Delta_1' + \Delta_1'' + \Delta_2' + \Delta_2'' + \cdots$ where, of course, the sum on the right is finite. If T is written in the complex form then each block consists of two parts symmetric with respect to the origin. We denote the "second norms" of the coefficients γ_j of Δ_k' and Δ_k'' by Γ_k' and Γ_k'' respectively; the "first norms" will be denoted by $\overline{\Gamma}_k'$ and $\overline{\Gamma}_k''$. The norms contain only positive j's, so that, for example, $|\Delta_k'| \leqslant 2\overline{\Gamma}_k'$.

We assume that the lengths of the long blocks do not exceed a certain number L' and that all the short blocks are of the same length L''. Hence passing from the last element of a long block to the first element of the next long block we increase the corresponding n_k by at least $q^{L''+1}$, a number which is large with L''. We take for L'' the least integer satisfying

$$C_q/C_q'' \leqslant q^{L''+1} \tag{4.2}$$

and define L' a little later.

For the A_q of Theorem 2′ we take the number C_q of Lemma 2. Let $I_0 = (a, b)$ be any interval of length $\geqslant C_q/n_1$. Let n_s and n_t be respectively the ranks of the last term in Δ_1' and the first term in Δ_2'. By Lemma 2, there is a subinterval I_1 of I_0, of length C_q''/n_s, in which $\Delta_1 \geqslant C_q' \Gamma_1'$. By the same lemma, if $|I_1| \geqslant C_q/n_t$, there is a subinterval I_2 of I_1 in which $\Delta_2 \geqslant C_q' \Gamma_2'$ provided

$$C_q''/n_s > C_q/n_t.$$

Since $t-s=L''+1$, the condition is certainly satisfied if L'' is defined by (4.2). The n_t play for Δ_2' the same role as n_1 played for Δ_1'. Hence we can repeat the argument and we obtain a sequence of intervals $I_1 \supset I_2 \supset I_3 \dots$ such that $\Delta_j \geqslant C_q' \Gamma_j'$ in I_j. At common points ξ of the intervals I_j, which also belong to (a, b), we have (cf. (4.1))

$$\Delta_1' + \Delta_2' + \Delta_3' + \cdots \geqslant C_q' (\Gamma_1' + \Gamma_2' + \Gamma_3' + \cdots) \geqslant C_q' L'^{-\frac{1}{2}} (\overline{\Gamma}_1' + \overline{\Gamma}_2' + \overline{\Gamma}_3' + \cdots). \tag{4.3}$$

In this argument we have so far disregarded the short blocks Δ_j'' and now we will show that if we select their location properly we can control their contribution.

We have already defined L''. We now let L' be any integer divisible by $2L''$ satisfying

$$6L''/L'^{\frac{1}{2}} \leqslant \tfrac{1}{2}. \tag{4.4}$$

We divide $T(\theta) = \sum (a_k \cos n_k \theta + \sin n_k \theta)$ into successive blocks of length $\frac{1}{2}L'$ (completing T by zeros if necessary) and call these blocks $\Delta_1, \Delta_2, \Delta_3, \dots$. The short blocks Δ_j'' will be properly selected subblocks of Δ_{2j} so that the length of a long block Δ_j' will certainly be less than L'.

To define Δ_j'' we split Δ_{2j} into successive subblocks of length L'' and take for Δ_j'' the subblock for which the first norm of the coefficients is the least. The number of subblocks being $L'/2L''$, we have

$$\overline{\Gamma}_j'' \leqslant \frac{2L''}{L'} \overline{\Gamma}_{2j}, \tag{4.5}$$

where $\overline{\Gamma}_k$ stands for the first norm of Δ_k. This implies that

$$\sum |\Delta_j''(\theta)| \leqslant 2 \sum \overline{\Gamma}_j'' \leqslant \frac{4L''}{L'} \sum \overline{\Gamma}_{2j} \leqslant \frac{4L''}{L'} \sum_1^N |\gamma_k|, \tag{4.6}$$

for all θ.

From the inequality (4.5) we see that the first norm for the coefficients of $\Delta_{2j} - \Delta_j''$ is at least $(1-2L''/L')\overline{\Gamma}_{2j}$ from which we deduce that the first norm for the coefficients of $\sum \Delta_j'$ is at least $(1-2L''/L') \sum_1^N |\gamma_k|$. That is,

$$\overline{\Gamma}_1' + \overline{\Gamma}_2' + \overline{\Gamma}_3' + \cdots \geqslant \left(1 - \frac{2L''}{L'}\right) \sum_1^N |\gamma_k|.$$

This shows that at the point ξ where (4.3) holds we also have

$$|T| = |\sum (\Delta_j' + \Delta_j'')| \geq C_q' \left[L'^{-\frac{1}{2}} \left(1 - \frac{2L''}{L'}\right) - \frac{4L''}{L'}\right] \sum_1^N |\gamma_k|$$

which, in view of (4.4) and the fact that the expression in square brackets exceeds $L'^{-\frac{1}{2}}(1 - 6L''/L'^{-\frac{1}{2}})$, gives Theorem 2′ with

$$A_q' = \tfrac{1}{2} L'^{-\frac{1}{2}} C_q'.$$

5. In this section we will prove a few lemmas on which we will base the proof of Theorem 1.

Let l be any straight line not passing through the origin. We say that a point ζ is *to the right* of l if it is contained in the closed halfplane limited by l which does not contain the origin. If l passes through the origin all points in the complex plane will be considered as situated to the right of l.

Lemma 4. *Let A_q and A_q' be the constants of Theorem 2′, $P(x)$ any lacunary power polynominal*

$$P(x) = \sum_{k=1}^N c_k e^{in_k x} \qquad (n_{k+1}/n_k > q > 1)$$

and l any straight line whose distance ω from the origin satisfies

$$\omega \leq A \sum_1^N |c_k|.$$

Then any interval of length A_q'/n_1 contains a point ξ such that $P(\xi)$ lies to the right of l.

Using rotation we may suppose that l intersects the positive real axis perpendicularly, and it is now enough to observe that if $P(x) = T(x) + i\tilde{T}(x)$, $T(x) = \sum_{-N}^N \gamma_k e^{in_k x}$, then $\sum_1^N |c_k| = \sum_{-N}^N |\gamma_k|$.

Lemma 5. *If the coefficients of the lacunary series $\sum c_j e^{in_j x}$ tend to zero and if the partial sums of the series (completed by zeros) are $S_k(x)$, then $S_{n_p}(x) - S_{n_p}(x')$ tends to zero as $p \to \infty$ uniformly in x, x', provided that $|x - x'| \leq 1/n_p$.*

For
$$|S_{n_p}(x) - S_{n_p}(x')| \leq \frac{1}{n_p} \sum_{k=1}^p |c_k| n_k \leq q^{-p} \sum_{k=1}^p |c_k| q^k$$

and it is easy to see that the right-hand side tends to 0 as $p \to \infty$.

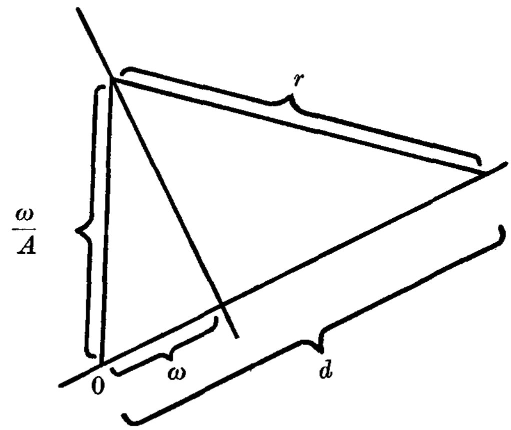

Fig. 1.

LEMMA 6. *Let ω, d, r, A be as in Fig. 1. If*

$$\frac{d}{2(1+1/A^2)} \leqslant \omega \leqslant \frac{d}{(1+1/A^2)}$$

then
$$r < d\left(1 - \frac{1}{4(1+1/A^2)}\right).$$

Since $r^2 \leqslant (d-\omega)^2 + \omega^2/A^2$, it follows that $d^2 - r^2 \geqslant 2d\omega - \omega^2 - \omega^2/A^2 > 0$. Hence

$$d - r \geqslant \frac{2d\omega - \omega^2(1+1/A^2)}{2d} \geqslant \omega\left(1 - \tfrac{1}{2}(1+1/A^2)\frac{\omega}{d}\right) \geqslant \tfrac{1}{2}\omega \geqslant \tfrac{1}{4}\cdot\frac{d}{(1+1/A^2)}$$

and the lemma follows.

6. We now pass to the actual proof of Theorem 1. The constant A_q and A_q' of Theorem 2′ will for brevity be denoted by A and A'.

We fix a complex number ζ and we want to construct a point ξ at which the given lacunary power series $\sum c_k e^{in_k x}$ converges to ζ. We first divide our series into successive blocks of terms $P_1(x)$, $P_1^*(x)$, $P_2(x)$, $P_2^*(x)$, ... $P_j(x)$, $P_j^*(x)$, ... The decomposition has some similarity to the decomposition, of a trigonometric polynomial into blocks Δ_j', Δ_j'' considered in the proof of Theorem 2′ and the blocks P_j^* (analogues of the short blocks Δ_j'') will be of constant length. The P_N will be defined inductively, as will a sequence of positive numbers δ_N tending to 0, and a nested sequence $\{I_N\}$ of intervals. Suppose we can do this in such a way that the following properties hold:

a) There exists a number $\gamma = \gamma_q$, $0 < \gamma < 1$, such that if

1) $F_N(x) = \sum_1^N P_j(x) + \sum_1^{N-1} P_j^*(x)$

2) $d_N = \min_{x \in I_N} |F_N(x) - \zeta| = |F_N(\xi_N) - \zeta|$, $(\xi_N \in I_N)$

then

3) $d_{N+1} < \gamma d_N + \delta_N$

b) $F_N(x) - F_N(x') \to 0$ for $x, x' \in I_N$

c) If $C_N = \sum |c_k|$ where the summation is taken over all coefficients of P_N^* and P_{N+1}, then $C_N \to 0$.

We will show then that if ξ is the common point of all the I_N, then

$$S_{n_j}(\xi) \to \zeta \text{ as } j \to \infty \tag{6.1}$$

that is the series $\sum c_k e^{in_k \xi}$ converges to ζ.

To see this suppose that S_{n_j} is 'between' F_N and F_{N+1}. Then

$$\begin{aligned} |S_{n_j}(\xi) - \zeta| &\leqslant |F_N(\xi) - \zeta| + |S_{n_j}(\xi) - F_N(\xi)| \\ &\leqslant |F_N(\xi_N) - F_N(\xi)| + |F_N(\xi_N) - \zeta| + |S_{n_j}(\xi) - F_N(\xi)| \end{aligned}$$

and

$$F_N(\xi_N) - F_N(\xi) \to 0 \qquad \text{by b)}$$

$$F_N(\xi_N) - \zeta \to 0 \qquad \text{by a)}$$

$$S_{n_j}(\xi) - F_N(\zeta) \to 0 \qquad \text{by c)}$$

which gives us (6.1).

We now proceed with the induction. For P_1 we choose an initial block of any length and set $I_0 = (0, 2\pi)$. Suppose that $P_1, P_1^*, P_2, P_2^*, \dots, P_N$ and intervals $I_0 \supset I_1 \supset \cdots \supset I_{N-1}$ have already been defined $(N \geqslant 1)$ and let n_{k_N} be the highest frequency of P_N. Let ξ_N be the point in I_{N-1} at which $|F_N(x) - \zeta|$ attains a minimum. We take for I_N any interval of length $1/n_{k_N}$ which contains ξ_n and is contained in I_{N-1}. Let P_N^* consist of the L terms following P_N, where

$$q^{L+1} \geqslant A' \tag{6.2}$$

and let

$$\varepsilon_N = \max_{x, x' \in I_N} |P_N(x) - P_N(x')|, \quad \eta_N = L \max_{j > k_N} |c_j|$$

$$\delta_N = \frac{6}{A} \eta_N + \varepsilon_N, \quad \gamma = 1 - \frac{1}{4(1 + 1/A^2)} \tag{6.3}$$

the constants A and A' being those of Theorem 2′.

Remarks. 1. The purpose of placing P_N^* between P_N and P_{N+1} is to make sure that n_{k_N+L+1}, the lowest frequency of P_{N+1}, is large enough to apply Theorem 2′ to P_{N+1} and the interval I_N. If L satisfies (6.2) this is certainly possible.

2. In view of Lemma 5 and the hypothesis $c_j \to 0$ we have ε_N, $\eta_N \to 0$ and so also $\delta_N \to 0$.

We proceed to define P_{N+1}, and consider two cases.

Case 1.

$$A \max_{j \geqslant k_N} |c_j| < \frac{d_N}{2(1+1/A^2)}. \tag{6.4}$$

Let P_{N+1} be chosen so that

$$\frac{d_N}{2(1+1/A^2)} \leqslant A \sum |c_j| \leqslant \frac{d_N}{(1+1/A^2)}, \tag{6.5}$$

where the summation is extended over all the coefficients of P_{N+1}.

Consider the following geometric situation. If $\zeta' = \zeta - F_N(\xi_N)$, then the distance from O to ζ' is d_N. If we draw the line l perpendicular to $O\zeta'$ at a distance $\omega = A \sum |c_j|$ from O, then, by Lemma 4, there is a point ξ' in the interval I_N such that $P_{N+1}(\xi')$ lies to the right of l. (Consider the figure in which now $d = d_N$.) However, $P_{N+1}(\xi') \leqslant \sum |c_j| \leqslant \omega/A$. Hence the point $P_{N+1}(\xi')$ which satisfies the above conditions can be no further from ζ' than r. Since by Lemma 6,

$$r < \gamma\, d_N$$

(cf. (6.3)), it follows that

$$d_{N+1} \leqslant |F_{N+1}(\xi') - \zeta| \leqslant |F_N(\xi_N) + P_{N+1}(\xi') - \zeta| +$$
$$+ |F_N(\xi_N) - F_N(\xi')| + |P_N^*(\xi_N)| \leqslant \gamma\, d_N + \varepsilon_N + \eta_N = \gamma\, d_N + \delta_N.$$

Case 2.

$$A \max_{j > k_N} |c_j| \geqslant \frac{d_N}{2(1+1/A^2)}.$$

In this case we let P_{N+1} consist of a single term. Clearly

$$d_{N+1} \leqslant |F_N(\xi_N) + P_N^*(\xi_N) + P_{N+1}(\xi_N) - \zeta|$$
$$\leqslant |F_N(\xi_N) - \zeta| + |P_N^*(\xi)| + |P_{N+1}(\xi_N)|$$
$$\leqslant d_N + 2\eta_N \leqslant 2A(1+1/A^2) \max_{j > k_N} |c_j| + 2\eta_N \leqslant \frac{4}{A}\eta_N + 2\eta_N \leqslant \frac{6}{A}\eta_N < \delta_N$$

and point a) follows again.

Since the length of I_N is $1/n_{k_N}$, where n_{k_N} is the highest frequency of F_N, b) follows immediately from Lemma 5.

Now, since P_N is of a fixed length L, the sum of the absolute values of the coefficients of P_N tends to 0. But P_{N+1} either consists of a single term or satisfies, by (6.5)

$$\sum |c_j| \leqslant \frac{d_N}{A(1+1/A^2)} \leqslant A\, d_N \to 0.$$

This proves point c) and completes the proof of Theorem 1.

7. We conclude by a few remarks.

a) Suppose that the n_k are positive and satisfy $n_{k+1}/n_k > q > 1$ but are not necessarily integers. Theorems 1 and 2 remain valid in this case and the proofs require no change.

b) The following relationship between the Hardy-Littlewood series on the left and the lacunary series on the right has been established by Paley [2]:

$$\sum_1^N \frac{1}{\sqrt{n}\,\log^\lambda n} \exp(i\beta n \log n + i n\theta) = e^{\frac{1}{4}\pi i}(2\pi\beta^{-1})^{\lambda+\frac{1}{2}} \sum_{\nu=1}^{(\beta \log N)/2\pi} \frac{1}{\nu^\lambda} e^{i\theta' a^\nu} + \sum_1^{(\beta\log N)/2\pi} f_\nu(\theta') + o(1),$$

where $a = \exp 2\pi\beta^{-1}$, $\theta' = -\beta \exp(-1-\theta\beta^{-1})$, $0<\lambda\leqslant 1$, and $\sum f_\nu(\theta')$ is an absolutely convergent series of continuous functions. It is clear from the proof of Theorem 1 that a lacunary series plus an absolutely convergent series of continuous functions also converges to every point in the complex plane, and hence we have

THEOREM 3. *If* $0<\lambda\leqslant 1$, *then the series*

$$\sum_1^\infty \frac{1}{\sqrt{n}\,\log^\lambda n} \exp(i\beta n \log n + i n \theta)$$

converges to every point in the complex plane.

c) THEOREM 4. *Theorems* 1,2,2′, *hold under the hypothesis that* $\{n_k\}$ *is a union of a finite number of lacunary sequences.*

The proof of the above theorems hold in this more general case also except that in a few places minor modifications must be made. Typical of these is the proof of the fact that

$$\int_a^b T^4\, d\theta \leqslant C(b-a)\sum |c_k^2|. \tag{7.1}$$

In this case we write $T = T_1 + T_2 + \cdots + T_s$ where the T_i are now lacunary polynomials, and hence,

$$\int_a^b T_i^4 \, d\theta \leqslant C\,(b-a) \sum |c_k|^2,$$

where the sum is taken over all coefficients of T_i. Using Minkowski's inequality we obtain (7.1).

We note that it follows from the generalization of Theorem 2 that if a series whose frequencies are a finite union of lacunary sequences converges everywhere in an interval then the series is absolutely convergent.

d) It should be remarked that to prove that the series of Theorem 1 has even a single point of convergence seems to be no simpler than it is to prove Theorem 1 itself.

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