THE FUNCTIONS WHICH OPERATE ON FOURIER TRANSFORMS

BY

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Introduction

A classical theorem of Wiener [12] and Lévy [7] states that if $f \in A$, where A denotes the class of all functions on the unit circle which are sums of absolutely convergent trigonometric series, and if F is defined and analytic on the range of f, then $F(f) \in A$. This theorem was extended by Gelfand [2], [8; p. 78] who showed that it holds if A is replaced by any normed ring.

We are interested in the converse: which functions F have the property that $F(f) \in A$ whenever $f \in A$? We have recently announced solutions of this and of some analogous problems [6], [3], [4]; in the present paper we publish complete proofs, and we extend our results to group algebras of infinite, locally compact, abelian groups in general. Roughly speaking, we prove (Theorems 1, 2, 3 below), that the analytic functions are the only ones with the desired property.

If Γ is the dual group (or character group) of the locally compact abelian group G (with addition as group operation), we denote by $A(\Gamma)$ and $B(\Gamma)$ the algebras of all Fourier transforms and Fourier-Stieltjes transforms on Γ, respectively. That is to say, $f \in A(\Gamma)$ if there exists some $g \in L^1(G)$ (the space of all complex functions which are integrable with respect to the Haar measure of G), such that

$$f(y) = \hat{g}(y) = \int_G (-x, y)\, g(x)\, dx \qquad (y \in \Gamma).$$

The symbol (x, y) denotes the value of the character y at the point x. $A(\Gamma)$ is normed by the L^1-norm on G:

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$$\|f\| = \|\hat{g}\| = \|g\|_1 = \int_G |g(x)|\, dx \qquad (f \in A(\Gamma)).$$

Similarly, $f \in B(\Gamma)$ if there is a bounded complex Borel measure μ on G such that

$$f(y) = \hat{\mu}(y) = \int_G (-x, y)\, d\mu(x) \qquad (y \in \Gamma),$$

and we norm $B(\Gamma)$ by defining

$$\|f\| = \|\hat{\mu}\| = \|\mu\| = \text{total variation of } \mu.$$

With these norms, $A(\Gamma)$ and $B(\Gamma)$ are Banach algebras under pointwise addition and multiplication, and $A(\Gamma)$ is a closed ideal in $B(\Gamma)$. If Γ is the circle, then $A(\Gamma)$ is the algebra A mentioned in the first paragraph.

For the sake of conciseness, we make the following definition:

A function F, defined on a set E in the complex plane, is said to operate in a function algebra R if $F(f) \in R$ for all $f \in R$ whose range lies in E.

Unless the contrary is explicitly stated, we shall always assume that F is defined on the closed interval $I = [-1,1]$ of the real axis, and that

$$F(0) = 0,$$

and *we shall always assume that G and Γ are infinite, locally compact, abelian groups.*

We state our main results:

THEOREM 1. *If Γ is discrete and if F operates in $A(\Gamma)$, then F is analytic in some neighborhood of the origin.*

THEOREM 2. *If Γ is not discrete and if F operates in $A(\Gamma)$, then F is analytic on I.*

THEOREM 3. *If Γ is not compact and if F operates in $B(\Gamma)$, then F can be extended to an entire function in the complex plane.*

Note that if Γ is compact, then $B(\Gamma) = A(\Gamma)$, so that this case of F operating in $B(\Gamma)$ is covered by Theorem 2.

The conclusions of these theorems may be restated in terms of power series: in Theorem 1, there is a series $\sum_0^\infty a_n t^n$ which converges to $F(t)$ in some interval around the origin; in Theorem 2, we have such a representation in some neighborhood of each point of I; and in Theorem 3, $F(t)$ is the sum of a power series with infinite radius of convergence.

In the final section of this paper we shall indicate how these theorems have to be modified if F is defined in a plane region. Some consequences of Theorem 3 are discussed in Section VI.

The notation $A_R(\Gamma)$, $B_R(\Gamma)$ will be used to denote the subsets of $A(\Gamma)$ and $B(\Gamma)$ which consist of real-valued functions; similarly, $A_I(\Gamma)$ and $B_I(\Gamma)$ will denote the sets of functions in $A(\Gamma)$, $B(\Gamma)$, whose range lies in I.

I. The continuity of F

1.1. Before we can prove analyticity of F, we have to prove that F is continuous. If F satisfies the hypotheses of Theorem 2, this causes no difficulty: suppose $t_n \in I$ and $t_n \to t$; there exists an $f \in A(\Gamma)$ such that $f(y_n) = t_n$ for some sequence $\{y_n\}$ which has a limit point $y \in \Gamma$, and the fact that both f and $F(f)$ are continuous implies that

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} F(f(y_n)) = F(f(y)) = F(t),$$

so that F is continuous at t.

If Γ is discrete (and Theorem 3 will be reduced to this case), the above argument fails, and we shall appeal to Lemma 1.3 below. First, however, we construct certain approximate identities which will be used frequently.

1.2. Lemma. *To every finite subset E of the discrete group Γ one can associate a finite set S in Γ such that*

$$m(S+E) \leqslant 2\, m(S).$$

If $T = S + E$ and if the polynomial K is defined by

$$K(x) = \frac{1}{m(S)} \sum_{y' \in S} (-x, y') \sum_{y'' \in T} (x, y'') \qquad (x \in G), \tag{1.2.1}$$

then $\|K\|_1 < 2$, $\hat{K}(y) = 1$ for all $y \in E$, and $|\hat{K}(y)| \leqslant 1$ for all $y \in \Gamma$.

Here m denotes the Haar measure on Γ; since Γ is discrete, $m(S)$ is just the number of points in S. Note also that G is now compact.

Proof: Let Γ_1 be the smallest subgroup of Γ which contains E. Being finitely generated, Γ_1 is a direct sum of a finite group Γ_2 and a group Λ of lattice points in a euclidean space. If S is taken to be the cartesian product of Γ_2 and a large enough "cube" in Λ, it is clear that $m(S+E) \leqslant 2m(S)$.

If f_1 and f_2 are defined by

$$f_1(x) = \sum_{y \in S} (-x, y), \quad f_2(x) = \sum_{y \in T} (x, y) \qquad (x \in G),$$

the Schwarz inequality shows that

$$\|K\|_1 \leqslant \frac{1}{m(S)} \|f_1\|_2 \|f_2\|_2 = \frac{1}{m(S)} \{m(S)\, m(T)\}^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}} < 2.$$

To complete the lemma, we rewrite $K(x)$ in the form

$$K(x) = \frac{1}{m(S)} \sum (x, y'' - y') \qquad (y' \in S,\ y'' \in T). \tag{1.2.2}$$

For any fixed $y \in E$ and any $y' \in S$, we have $y + y' \in T$, so that $y = y'' - y'$, for some $y'' \in T$. It follows that the term (x, y) occurs precisely $m(S)$ times in the sum (1.2.2) if $y \in E$, and it occurs no more than $m(S)$ times for any $y \in \Gamma$.

Note. If Γ is the group of all integers, the familiar trapezoidal function for $\hat{K}$ is obtained by the above construction: if

$$K(e^{i\theta}) = \frac{1}{2N+1} \sum_{p=-N}^{N} e^{-ip\theta} \sum_{q=-2N}^{2N} e^{iq\theta},$$

then

$$\hat{K}(n) = \begin{cases} 1 & \text{if } |n| \leq N, \\ 2 - \dfrac{|n|}{N} & \text{if } N < |n| \leq 2N, \\ 0 & \text{if } 2N < |n|. \end{cases} \tag{1.2.3}$$

Formula (1.2.3) also applies if Γ is a finite cyclic group, provided its order exceeds $4N$; this remark will be used in 5.5.

1.3. LEMMA. *Suppose Γ is discrete and $\varepsilon > 0$. Suppose that $F(f) \in B(\Gamma)$ for all $f \in B_I(\Gamma)$ such that $\|f\| < \varepsilon$. If $\varepsilon \leq 1$, F is continuous in the segment $(-\varepsilon, \varepsilon)$; if $\varepsilon > 1$, F is continuous on I.*

Proof. Fix $t \in I$ such that $|t| < \varepsilon$, and assume that F is discontinuous at t. Then there exist real numbers a_n such that $t + a_n \in I$,

$$\sum_{n=1}^{\infty} a_n^2 < \{\varepsilon - |t|\}^2 \tag{1.3.1}$$

and

$$|F(t + a_n) - F(t)| > \eta \qquad (n = 1, 2, 3, \ldots) \tag{1.3.2}$$

for some $\eta > 0$.

Since Γ is infinite (our standing assumption), there is a sequence $\{y_n\}$ $(n = 1, 2, 3, \ldots)$ in Γ, with the following properties:

(a) $y_n \neq y_{i_1} + y_{i_2} - y_{i_3}$ if i_1, i_2, i_3 are all less than n;

(b) if E_j consists of $y_1, \ldots, y_j$ and if S_j and T_j are associated with E_j as in Lemma 1.2, then $y_n \notin T_j - S_j$ if $j < n$.

Define

$$f(y) = \begin{cases} t + a_n & \text{if } y = y_n \quad (n = 1, 2, 3, \ldots), \\ t & \text{for all other } y \in \Gamma. \end{cases} \tag{1.3.3}$$

Then $f \in B_I(\Gamma)$, since f is the Fourier-Stieltjes transform of the measure

$$t\,\delta_0 + \sum_{n=1}^{\infty} a_n\,(x, y_n) \tag{1.3.4}$$

on G, where δ_0 is the unit mass concentrated at the identity element of G; the series in (1.3.4) belongs to $L^2(G)$, and (1.3.1) shows furthermore that $\|f\| < \varepsilon$.

The hypotheses of the lemma thus imply that the function g defined by

$$g(y) = \begin{cases} F(t + a_n) & \text{if } y = y_n \\ F(t) & \text{for all other } y \in \Gamma \end{cases}$$

belongs to $B(\Gamma)$, and we conclude that there is a measure μ on G such that

$$\hat{\mu}(y) = \begin{cases} F(t + a_n) - F(t) & \text{if } y = y_n, \\ 0 & \text{for all other } y \in \Gamma. \end{cases} \tag{1.3.5}$$

Put

$$P_j(x) = \sum_{n=1}^{j} \hat{\mu}(y_n)\,(x, y_n) \qquad (j = 1, 2, 3, \ldots;\ x \in G). \tag{1.3.6}$$

Property (a) of $\{y_n\}$ implies, by an argument familiar from the study of lacunary trigonometric series [15; p. 217], that $\|P_j\|_4 \leqslant 2^{\frac{1}{4}}\,\|P_j\|_2$, and Hölder's inequality then shows that

$$\|P_j\|_2 \leqslant 2\,\|P_j\|_1. \tag{1.3.7}$$

But if K_j is the polynomial associated with E_j as in Lemma 1.2, property (b) of $\{y_n\}$ implies that $\hat{P}_j = \hat{K}_j\,\hat{\mu}$, so that $\|P_j\|_1 \leqslant 2\,\|\mu\|$. It follows that

$$\left\{\sum_{n=1}^{j} |\hat{\mu}(y_n)|^2\right\}^{\frac{1}{2}} = \|P_j\|_2 \leqslant 4\,\|\mu\| \qquad (j = 1, 2, 3, \ldots),$$

which is impossible, since $|\hat{\mu}(y_n)| > \eta > 0$, by (1.3.2).

This contradiction proves that F is continuous at t.

II. The principal lemmas

2.1. LEMMA. *If Γ is any infinite locally compact abelian group, then*

$$\sup \|e^{if}\| = e^r, \tag{2.1.1}$$

where f ranges over all functions in $B_R(\Gamma)$ with $\|f\| \leqslant r$.

Proof: Since $e^{if} = \sum_{0}^{\infty} (if)^n/n!$, it is clear that $\|e^{if}\| \leqslant e^{\|f\|}$, and the left member of (2.1.1) does not exceed e^r.

Given $\varepsilon > 0$, we choose a positive integer $n > 2r$, so large that

$$e^r \left\{ \left(1 + \frac{r^2}{n^2}\right)^n - 1 \right\} < \varepsilon/2 \tag{2.1.2}$$

and

$$\left(1 + \frac{r}{n}\right)^n > e^r - \varepsilon/2. \tag{2.1.3}$$

We choose points $x_1, \ldots, x_n$ in G (not 0), such that

$$x_{k+1} \neq \pm x_1 \pm x_2 \pm \cdots \pm x_k \qquad (k = 1, \ldots, n-1), \tag{2.1.4}$$

no matter how the signs are chosen.

Let δ_x denote the measure of total mass 1, concentrated at the point $x \in G$, put

$$\sigma_k = \tfrac{1}{2}(\delta_{x_k} + \delta_{-x_k}) \qquad (k = 1, \ldots, n) \tag{2.1.5}$$

and

$$\mu = \frac{r}{n}(\sigma_1 + \cdots + \sigma_n). \tag{2.1.6}$$

It is clear that $\|\mu\| = r$ and that $\hat{\mu}$ is real, since $\hat{\sigma}_k(y)$ is the real part of (x_k, y). We shall prove that

$$\|e^{i\mu}\| > e^r - \varepsilon. \tag{2.1.7}$$

Using $*$ to denote convolution, and writing σ^p for the convolution of the measure σ with itself, p times, we have

$$e^{i\mu} = \left(\delta_0 + \frac{ir}{n}\sigma_1 + \tau_1\right) * \cdots * \left(\delta_0 + \frac{ir}{n}\sigma_n + \tau_n\right), \tag{2.1.8}$$

where

$$\tau_k = \sum_{p=2}^{\infty} \frac{1}{p!}\left(\frac{ir}{n}\right)^p \sigma_k^p \qquad (k = 1, \ldots, n).$$

To estimate the norm of τ_k, we use the assumption $n > 2r$:

$$\|\tau_k\| \leqslant \sum_{p=2}^{\infty} \frac{1}{p!}\left(\frac{r}{n}\right)^p < \frac{r^2}{n^2} \qquad (k = 1, \ldots, n).$$

We also note that

$$\left\| \delta_0 + \frac{ir}{n}\sigma_k \right\| = 1 + \frac{r}{n} \qquad (k = 1, \ldots, n).$$

Hence (2.1.8) may be rewritten in the form

$$e^{i\mu} = \left(\delta_0 + \frac{ir}{n}\sigma_1\right) * \cdots * \left(\delta_0 + \frac{ir}{n}\sigma_n\right) + \lambda, \tag{2.1.9}$$

where the norm of remainder λ satisfies the inequality

$$\|\lambda\| \leqslant \sum_{k=1}^{n} \binom{n}{k} \left(1+\frac{r}{n}\right)^{n-k} \left(\frac{r}{n}\right)^{2k} \leqslant e^r \sum_{k=1}^{n} \binom{n}{k} \left(\frac{r}{n}\right)^{2k} = e^r \left\{\left(1+\frac{r^2}{n^2}\right)^n - 1\right\} < \varepsilon/2, \quad (2.1.10)$$

by (2.1.2). Finally, (2.1.3) and (2.1.4) imply that

$$\left\| \left(\delta_0 + \frac{ir}{n}\sigma_1\right) * \cdots * \left(\delta_0 + \frac{ir}{n}\sigma_n\right) \right\| = \prod_{k=1}^{n} \left\| \delta_0 + \frac{ir}{n}\sigma_k \right\| = \left(1+\frac{r}{n}\right)^n > e^r - \varepsilon/2,$$

and (2.1.7) follows from this, together with (2.1.9) and (2.1.10).

2.2. THE SCHOENBERG CRITERION. This theorem, for whose rather simple proof we refer to [9] or [1], asserts that each of the following two statements about a function f, defined on Γ, implies the other:

(a) *$f \in B(\Gamma)$ and $\|f\| \leqslant M$.*

(b) *f is continuous, and for every $g \in L^1(\Gamma)$*

$$\left|\int_\Gamma f(y)\, g(y)\, dy\right| \leqslant M \cdot \sup_{x \in G} \left|\int_\Gamma g(y)\, (x, y)\, dy\right|.$$

We shall use the following corollary: *If $f_n \in B(\Gamma)$, $\|f_n\| \leqslant M$,*

$$f(y) = \lim_{n\to\infty} f_n(y) \qquad (y \in \Gamma),$$

and f is continuous, then $f \in B(\Gamma)$ and $\|f\| \leqslant M$.

2.3. LEMMA. *Suppose $r > 0$, $M < \infty$, and Φ is a periodic function on the real line, with period 2π. Suppose $\Phi(f+c) \in B(\Gamma)$ and $\|\Phi(f+c)\| \leqslant M$ for every real number c, provided that $f \in B_R(\Gamma)$ and $\|f\| \leqslant r$.*

Then Φ can be extended to a function which is analytic in the strip $|y| < r$ $(z = x + iy)$.

Proof: By 1.1 and Lemma 1.3, Φ is continuous in some segment; since the hypothesis involves arbitrary c, Φ is continuous on the whole line, and we can therefore expand it in a Fourier series:

$$\Phi(t) \sim \sum_{-\infty}^{\infty} a_n e^{int}. \quad (2.3.1)$$

Fix n, choose $f \in B_R(\Gamma)$ with $\|f\| \leqslant r$, and define

$$f_p(y) = \frac{1}{p} \sum_{k=1}^{p} \Phi\left(f(y) + \frac{2k\pi}{p}\right) e^{-2\pi i k n/p} \qquad (y \in \Gamma,\ p = 1, 2, 3, \ldots). \quad (2.3.2)$$

Since Φ is continuous, we obtain

$$\lim_{p\to\infty} f_p(y) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(f(y)+t)\, e^{-int}\, dt = a_n\, e^{inf(y)} \qquad (y \in \Gamma). \tag{2.3.3}$$

On the other hand, the hypotheses of the lemma imply that $f_p \in B(\Gamma)$ and that $\|f_p\| \leqslant M$. The corollary to Schoenberg's criterion therefore shows that

$$|a_n| \cdot \|e^{inf}\| \leqslant M \tag{2.3.4}$$

for every $f \in B_R(\Gamma)$ with $\|f\| \leqslant r$, and we can conclude from Lemma 2.1 that

$$|a_n| \leqslant M \cdot e^{-|n|r} \qquad (n = 0, \ \pm 1, \ \pm 2, \ \dots). \tag{2.3.5}$$

These bounds on the coefficients of the series (2.3.1) show that the series

$$\sum_{-\infty}^{\infty} a_n\, e^{in(x+iy)} \qquad (x, y \text{ real}) \tag{2.3.6}$$

converges uniformly in every compact subset of the strip $|y| < r$, and the sum of (2.3.6) is the desired analytic extension of Φ.

2.4. LEMMA. *Suppose $F(f) \in B(\Gamma)$ and $\|F(f)\| \leqslant M$ for every $f \in B_I(\Gamma)$ with $\|f\| < \varepsilon$, for some $\varepsilon > 0$. Then F is analytic in some neighborhood of the origin.*

Proof: Put

$$\Phi(x) = F(R \sin x),$$

where $0 < R < 1$ and $eR < \varepsilon$. Observe that if $\|f\| \leqslant 1$, then $\|\sin(f+c)\| = \|\sin c \cdot \cos f + \cos c \cdot \sin f\| \leqslant \|\cos f\| + \|\sin f\| \leqslant e^{\|f\|} \leqslant e$, for every real constant c, so that

$$\|R \sin(f+c)\| \leqslant Re < \varepsilon.$$

Thus Φ satisfies the hypotheses of Lemma 2.3, with $r = 1$. The formula

$$F(x) = \Phi\left(\arcsin \frac{x}{R}\right),$$

valid for $-R < x < R$, then shows that F is analytic in a neighborhood of the origin.

In order not to interrupt the argument later, we include one more lemma in this section.

2.5. LEMMA. *If G is a compact abelian group which is not of bounded order, then G contains an element of infinite order.*

(We recall that a group G is said to be of bounded order if there is a positive integer q such that $qx = 0$ for every $x \in G$.)

Proof: For $n = 1, 2, 3, \ldots$, let E_n be the set of all $x \in G$ such that $nx = 0$, and assume that E_k contains a non-empty open set V, for some k. If $W = V - V$, then W is a neighborhood of 0 and $kx = 0$ for every $x \in W$. The group H generated by W is compact and open, and G/H is finite (being compact and discrete). If G/H has p elements, it follows that $px \in H$ and $kpx = 0$ for every $x \in G$; thus G is of bounded order.

This contradiction shows that none of the compact sets E_n contains a non-empty open subset of G. It follows that the set of all elements of infinite order, which is the complement of $\bigcup_1^\infty E_n$, is actually a dense subset of G.

III. Proof of Theorem 1

3.1. If Γ is discrete, so that G is compact, and if P is a polynomial on G, of the form

$$P(x) = \sum_y a_y (x, y) \qquad (x \in G,\ y \in \Gamma), \tag{3.1.1}$$

with only finitely many $a_y \neq 0$, we write $F \circ P$ for the polynomial

$$(F \circ P)(x) = \sum_y F(a_y)(x, y) \qquad (x \in G,\ y \in \Gamma). \tag{3.1.2}$$

3.2. Lemma. *Suppose Γ is countable and discrete, F is defined on I, and there exist constants $\varepsilon > 0$, $M < \infty$, with the following property:*

$$\|F \circ P\|_1 \leqslant M \tag{3.2.1}$$

for all P of the form (3.1.1), *with $a_y \in I$, whose norm satisfies*

$$\|P\|_1 \leqslant 2\varepsilon. \tag{3.2.2}$$

Then if $f \in B_I(\Gamma)$ and if $\|f\| \leqslant \varepsilon$, we also have $F(f) \in B(\Gamma)$, and $\|F(f)\| \leqslant M$.

Proof: Choose $f \in B_I(\Gamma)$, with $\|f\| \leqslant \varepsilon$, let $\{E_n\}$ be an expanding sequence of finite sets whose union is Γ, and associate with each set E_n a polynomial K_n, as in Lemma 1.2.

Since $\|f\| \leqslant \varepsilon$, we have $\|\hat{K}_n f\| \leqslant 2\varepsilon$; also, $\hat{K}_n f$ is the transform of a polynomial and the range of $\hat{K}_n f$ lies in I; thus (3.2.1) applies:

$$\|F(\hat{K}_n f)\| \leqslant M \qquad (n = 1, 2, 3, \ldots). \tag{3.2.3}$$

But for every $y \in E_n$ we have

$$F(\hat{K}_n(y) f(y)) = F(f(y)),$$

so that

$$F(f(y)) = \lim_{n\to\infty} F(\hat{K}_n(y)\, f(y)) \qquad (y \in \Gamma).$$

If we apply the corollary to Schoenberg's criterion to the functions $F(\hat{K}_n f)$, (3.2.3) shows that $F(f) \in B(\Gamma)$ and $\|F(f)\| \leq M$.

Note: The assumption that Γ is countable may be removed from this lemma, but we only require the countable case.

3.3. We now assume that F, defined on I, operates in $A(\Gamma)$, for some discrete Γ. If Γ_1 is any countable subgroup of Γ, it is clear that F also operates in $A(\Gamma_1)$; we may accordingly assume, without loss of generality, that Γ is countable, and we shall prove that the hypotheses of Lemma 3.2 are satisfied. Once this is done, Lemmas 3.2 and 2.4 give the conclusion of Theorem 1.

If the hypotheses of Lemma 3.2 are not satisfied, then there exist polynomials P_n on G, with coefficients in I, such that

$$\|P_n\|_1 < n^{-2} \tag{3.3.1}$$

but

$$\|F \circ P_n\|_1 \to \infty \qquad (n \to \infty). \tag{3.3.2}$$

Let E_n be the set of all $y \in \Gamma$ such that $\hat{P}_n(y) \neq 0$, associate sets S_n, T_n, and polynomials K_n with E_n, as in Lemma 1.2, and let $\{y_n\}$ be a sequence in Γ such that the sets $y_n + T_n - S_n$ are mutually disjoint. The series

$$\sum_{n=1}^{\infty} (x, y_n)\, P_n(x) \tag{3.3.3}$$

converges, in the L^1-norm, to a function $g \in L^1(G)$, by (3.3.1); since $\hat{g} \in A_I(\Gamma)$, and since F operates in $A(\Gamma)$, $F(\hat{g}) \in A(\Gamma)$. Let h be the function in $L^1(G)$ such that $\hat{h} = F(\hat{g})$. Our choice of y_n implies

$$\hat{h}(y) = F(\hat{P}_n(y - y_n)) \qquad (y \in E_n + y_n). \tag{3.3.4}$$

and hence

$$(\hat{h} \cdot \hat{Q}_n)(y) = F(\hat{P}_n(y - y_n)) \qquad (y \in \Gamma), \tag{3.3.5}$$

where

$$Q_n(x) = (x, y_n)\, K_n(x) \qquad (x \in G,\ n = 1, 2, 3, \ldots). \tag{3.3.6}$$

By (3.3.5), we have

$$\|F \circ P_n\|_1 \leq \|Q_n\|_1 \cdot \|h\|_1 < 2\,\|h\|_1, \tag{3.3.7}$$

which contradicts (3.3.2). Theorem 1 is thus proved.

IV. Proof of Theorem 2

4.1. To begin with, we shall assume that Γ is compact, and our first aim will be the proof that F is analytic in a neighborhood of the origin if F operates in $A(\Gamma)$.

If $f \in A_I(\Gamma)$, then $F(f) \in A(\Gamma)$, and

$$F(f(y)) = \sum_{x \in G} a_x(f)(x, y) \qquad (y \in \Gamma), \tag{4.1.1}$$

where

$$a_x(f) = \int_\Gamma F(f(y))(-x, y)\, dy \qquad (x \in G). \tag{4.1.2}$$

Since F is continuous on I (see 1.1), (4.1.2) shows that the mapping

$$f \to a_x(f) \tag{4.1.3}$$

is, for each $x \in G$, a real valued function on $A_I(\Gamma)$, which is continuous in the norm topology of $A(\Gamma)$.

Let J be the set of all $f \in A_I(\Gamma)$ which vanish in some neighborhood of 0 (the identity element of Γ), and let C be the closure of J, in the norm topology; C is a complete metric space, and the mapping

$$f \to \|F(f)\| = \sum_{x \in G} |a_x(f)| \tag{4.1.4}$$

is a real-valued (finite) lower semi-continuous function on C. The Baire category theorem implies that $\|F(f)\|$ is bounded in some open set of C. Since J is dense in C, there is an $f_0 \in J$ with the following property: if $f \in C$ and if $\|f - f_0\| \leqslant \varepsilon$, then $\|F(f)\| \leqslant M$, where ε, M are suitable positive numbers.

Let U be a neighborhood of 0 on which f_0 vanishes, and choose a non-empty open set V, whose closure lies in U and does not contain 0.

We now consider any $g \in A_I(\Gamma)$ which vanishes outside V, such that $\|g\| \leqslant \varepsilon$. Putting $f = f_0 + g$, the above remarks imply that $\|F(f_0 + g)\| \leqslant M$ for some $M < \infty$. But f_0 and g have disjoint supports, so that

$$F(f_0 + g) = F(f_0) + F(g). \tag{4.1.4}$$

It follows that

$$\|F(g)\| \leqslant \|F(f_0 + g)\| + \|F(f_0)\| \leqslant 2M. \tag{4.1.5}$$

Let us summarize what we have proved so far: *There exists an open set* $V \subset \Gamma$ *and there exists* $\varepsilon > 0$, $M < \infty$, *such that* $\|F(g)\| \leqslant M$ *whenever* $g \in A_I(\Gamma)$, $g = 0$ *outside* V, *and* $\|g\| \leqslant \varepsilon$. By translation, we may assume that V is a neighborhood of 0.

4.2. We now consider three cases.

Case A. Suppose Γ *is totally disconnected.* Then the open set V which we have just described contains an open-closed subgroup Γ_1 of Γ [11; p. 19], and the result of 4.1 implies that $\|F(g)\| \leqslant M$ for every $g \in A_I(\Gamma_1)$, provided $\|g\| \leqslant \varepsilon$.

Since Γ_1 is compact, $A(\Gamma_1) = B(\Gamma_1)$, so that Lemma 2.4 applies, and we conclude that F is analytic near the origin.

Case B. Suppose Γ *is the unit circle* (this is the case to which the original theorem of Wiener and Lévy applied). Our set V is now a segment $(-\delta, \delta)$, where $0 < \delta < \pi$. If $0 < 2b < \delta$, define

$$Q_b(e^{i\theta}) = \begin{cases} 1 & \text{if } |\theta| < b, \\ 2 - \dfrac{|\theta|}{b} & \text{if } b < |\theta| \leqslant 2b, \\ 0 & \text{if } 2b < |\theta| < \pi. \end{cases}$$

It is then easy to see, as in Lemma 1.2, that $Q_b \in A(\Gamma)$, and

$$1 < \|Q_b\| < 2. \tag{4.2.1}$$

We now consider any $f \in A_I(\Gamma)$ for which $\|f\| \leqslant \varepsilon/2$. Set $a = \pi/6N$, where N is a positive integer and $3N\delta > 2\pi$. Then

$$\|Q_{2a} \cdot f\| \leqslant 2\|f\| \leqslant \varepsilon, \tag{4.2.2}$$

and $Q_{2a} \cdot f$ vanishes outside V. Hence

$$\|F(Q_{2a} \cdot f)\| \leqslant M, \tag{4.2.3}$$

and the identity

$$Q_a \cdot F(f) = Q_a \cdot F(Q_{2a} \cdot f), \tag{4.2.4}$$

together with (4.2.1) and (4.2.3), implies that

$$\|Q_a \cdot F(f)\| \leqslant 2M. \tag{4.2.5}$$

It is now clear that (4.2.5) also holds if Q_a is replaced by any of its translates. Since there are N of these translates, say $Q_{a,1}, \ldots, Q_{a,N}$, whose sum is 1, we have $F(f) = \sum_1^N Q_{a,k} \cdot F(f)$, and we conclude that $\|F(f)\| \leqq 2MN$.

Since $A(\Gamma) = B(\Gamma)$, Lemma 2.4 applies, so that F is again analytic near the origin.

Case C. Suppose Γ *is not totally disconnected.*

Since Γ is not zero-dimensional, G contains an element of infinite order [11; p. 111], so that G contains an infinite cyclic group Z. Let Γ_1 be the group of all $y \in \Gamma$ such that

$(x, y) = 1$ whenever $x \in Z$. Then Γ/Γ_1 is isomorphic to the circle group, and if F operates in $A(\Gamma)$, then F also operates on those functions in $A(\Gamma)$ which are constant on the cosets of Γ_1. This takes us back to Case B.

4.3. We can now prove Theorem 2 under the assumption that Γ is compact.

We have already proved that F is analytic at the origin. Suppose $-1 < a < 1$, and put

$$F_a(t) = F(a + (1-|a|)t) - F(a) \qquad (-1 \leqslant t \leqslant 1). \tag{4.3.1}$$

Since F_a evidently operates in $A(\Gamma)$, F_a is analytic at the origin, which implies that F is analytic at a. To prove analyticity of F at the end-points of I, put

$$F_1(t) = F(1-t^2) \qquad (-1 \leqslant t \leqslant 1). \tag{4.3.2}$$

Again, F_1 operates in $A(\Gamma)$; since F_1 is an even function, we have

$$F_1(t) = \sum_0^\infty c_n t^{2n} \qquad (-\delta < t < \delta) \tag{4.3.3}$$

for some $\delta > 0$. Hence

$$F(1-x) = \sum_0^\infty c_n x^n \qquad (0 \leqslant x < \delta^2), \tag{4.3.4}$$

and this proves that F is analytic at the right end-point of I. The other end-point can be treated similarly, and the proof is complete for compact Γ.

4.4. We shall prove Theorem 2 for non-compact groups with the aid of the following structure theorem [11; p. 110].

Every locally compact abelian group Γ contains an open subgroup Γ_0 which is the direct sum of a compact group H and a p-dimensional euclidean space R^p.

(Note that open subgroups are also closed [11; p. 13].)

4.5. Suppose now that Γ is not discrete. If F operates in $A(\Gamma)$, then F also operates in $A(\Gamma_0)$, and we consider two cases (in the notation of 4.4): $p = 0$ or $p > 0$.

If $p = 0$ then Γ_0 is compact, and since Γ_0 is not discrete, Γ_0 is infinite. The conclusion of Theorem 2 then follows from 4.3.

If $p > 0$, we observe that F also operates in the subalgebra of $A(\Gamma_0)$ consisting of those $f \in A(\Gamma_0)$ which are constant on the cosets of H; that is to say, F operates in $A(R^p)$. We shall prove that this implies that F operates in $A(T^p)$, where T^p is the p-dimensional torus; the analyticity of F on I will then again follow from 4.3.

With every $f \in A_I(T^p)$ we associate a function f^*, defined on R^p by

$$f^*(x_1, \ldots, x_p) = f(e^{ix_1}, \ldots, e^{ix_p}),$$

which has period 2π in each of the variables $x_1, \ldots, x_p$. Choose a function $g \in A(R^p)$ whose range lies in I, such that

$$g(x_1, \ldots, x_p) = 1 \quad (0 \leqslant x_j \leqslant 2\pi,\ 1 \leqslant j \leqslant p),$$

and such that g vanishes outside some compact set. Then $f^* g \in A_I(R^p)$, so that $F(f^* g) \in A(R^p)$. Since $(f^* g)(x) = f^*(x)$ over a full period of f^*, we conclude that F operates in $A(T^p)$.

This completes the proof of Theorem 2.

V. Proof of Theorem 3

5.1. In our proof of Theorem 3, the fact that F is defined only on the interval I, and not on the whole real axis, causes some inconvenience. To avoid this, let us consider the functions

$$\Phi_1(x) = F(\sin x),\ \Phi_2(x) = F(b \sin x) \qquad (-\infty < x < \infty), \tag{5.1.1}$$

where $0 < b < 1$. Since every entire function operates in $B(\Gamma)$, so do Φ_1 and Φ_2 if F does.

Suppose we can prove that Φ_1 and Φ_2 can be extended to entire functions in the complex plane, and let us solve for F: we obtain

$$F(x) = \begin{cases} \Phi_1(\operatorname{arc\,sin} x) & (-1 \leqslant x \leqslant 1), \\ \Phi_2\left(\operatorname{arc\,sin} \dfrac{x}{b}\right) & (-b \leqslant x \leqslant b). \end{cases}$$

The first of these formulas shows that F can be expanded in a power series about the origin, and that this power series can be analytically continued to a (possibly multi-valued) function in the whole plane, except for possible branch points at $x = \pm 1$. The second formula shows, in the same way, that $x = \pm b$ are the only possible singular points of F in the finite plane. Since $b \neq 1$, we conclude that the analytic extension of F is an entire function.

These remarks show: *It suffices to prove Theorem* 3 *under the stronger assumption that* F *is defined on the whole real axis, and has period* 2π.

5.2. We next show that *it suffices to prove Theorem* 3 *for countable discrete groups* Γ.

Let Γ_0 be the open subgroup of Γ which is mentioned in the structure theorem 4.4. If Γ_0 has infinitely many cosets in Γ, and if F operates in $B(\Gamma)$, then F operates in the algebra of all $f \in B(\Gamma)$ which are constant on these cosets, which means that F operates in $B(\Gamma_1)$, where Γ_1 is the discrete quotient group Γ/Γ_0. It is clear that F then also operates in $B(\Gamma_2)$, where Γ_2 is any countable subgroup of Γ_1.

If Γ_0 has only finitely many cosets in Γ, then $p > 0$ (in the notation of 4.4), since Γ is not compact, and we observe that F operates in the subalgebra of all $f \in B(\Gamma)$ which are constant on the cosets of H in Γ_0 and which vanish outside Γ_0. This means that F operates in $B(R^p)$. Let Λ^p be the p-dimensional lattice group in R^p, i.e., the set of all points $x = (x_1, \ldots, x_p)$ in R^p all of whose coordinates are integers. It is quite easy to see that the restriction of every $f \in B(R^p)$ to Λ^p belongs to $B(\Lambda^p)$, and conversely that every $f \in B_R(\Lambda^p)$ can be extended to R^p so as to belong to $B_R(R^p)$. Hence F operates in $B(\Lambda^p)$.

Hence Theorem 3 will follow if we can show that the entire functions are the only ones which operate in $B(\Gamma)$, for any countable discrete group Γ.

5.3. Consider now the following two conditions on a function F, defined on the real axis, with period 2π:

(α) *F operates in $B(\Gamma)$ for some countable discrete group Γ.*

(β) *For some countable discrete group Γ, there is associated with every $r > 0$ a number $M(r) < \infty$, such that*

$$\|F \circ P\|_1 \leqslant M(r),$$

whenever P is a polynomial on G, with real coefficients, whose norm satisfies

$$\|P\|_1 \leqslant r.$$

(We refer to 3.1 for the notation $F \circ P$; the norms here are the L^1-norms over the compact group G.)

Suppose (β) holds. Choose $\eta > 0$, and suppose $f \in B_R(\Gamma)$, with $\|f\| \leqslant \eta$. Then $\|f + c\| \leqslant \eta + \pi$ if c is a constant and $-\pi \leqslant c \leqslant \pi$. We apply ($\beta$) with $r = 2(\eta + \pi)$, and we apply a slight modification of Lemma 3.2 (R in place of I) with $\varepsilon = \eta + \pi$, $M = M(2\eta + 2\pi)$; the periodicity of F then implies that

$$\|F(f + c)\| \leqslant M(2\eta + 2\pi) \quad (-\infty < c < \infty).$$

By Lemma 2.3, F can therefore be extended to a function which is analytic in the strip $|y| < \eta$.

Since this is true for every η, we see that the entire functions are the only ones which satisfy (β). On the other hand, our discussion in 5.1 and 5.2 shows that Theorem 3 will be proved if we can show that the entire functions are the only ones which satisfy (α).

Hence the proof will be complete if we can show that (α) implies (β). We shall do this in two steps:

Step 1: If (α) holds for some Γ of bounded order, then (β) holds for the same Γ.

Step 2: If (α) holds for some Γ which is not of bounded order, then (β) holds for the group of all integers.

(We refer to the definition which follows Lemma 2.5.)

5.4. *Proof of Step* 1. We may assume, without loss of generality, that Γ is the direct sum of infinitely many cyclic groups, all of which have the same order. To justify this simplification, we remark (1) that if 5.3(α) holds for some Γ, then it holds for every infinite subgroup of Γ, (2) that every group of bounded order is a direct sum of cyclic groups [5; p. 17] and (3) that among these direct summands infinitely many have the same order.

If 5.3(β) is false, then for some $r > 0$ there are polynomials P_n on G, with real coefficients, such that $\|P_n\|_1 \leq r$, but

$$\|F \circ P_n\|_1 \to \infty \quad (n \to \infty). \tag{5.4.1}$$

Let E_n be the smallest subgroup of Γ which contains the support of $\hat{P}_n$. It is clear that each E_n is a finite group; hence we can choose y_n in the complement of E_n, and we let Γ_n be the group generated by E_n and y_n. The simplifying assumption made in the first paragraph of 5.4 shows that we can transform the polynomials P_n by automorphisms t_n of Γ, replacing $\Sigma \hat{P}_n(y)(x,y)$ by $\Sigma \hat{P}_n(y)(x,t_n y)$, so that no two of the groups Γ_n have a non-zero element in common; these transformations do not change any of the norms with which we are concerned, and we assume that they are carried out.

We shall show that the assumption $\|P_n\|_1 \leq r$ for $n = 1, 2, 3, \ldots$ implies that there is a measure μ on G, with $\hat{\mu}$ real, such that

$$\hat{\mu}(y + y_n) = \hat{P}_n(y) \quad (y \in E_n;\ n = 1, 2, 3, \ldots). \tag{5.4.2}$$

Once this is done, we let $\hat{K}_n$ be the characteristic function of the set $E_n + y_n$; obviously $\hat{K}_n \in B(\Gamma)$, and since E_n is a group, $\|\hat{K}_n\| = 1$. Also, $F(\hat{\mu}) \in B(\Gamma)$, by 5.3($\alpha$), and

$$F(\hat{P}_n(y)) = F(\hat{\mu}(y + y_n))\hat{K}(y + y_n)$$

for all $y \in \Gamma$. Hence $\|F \circ P_n\|_1 \leq \|F(\hat{\mu})\|$, contradicting (5.4.1).

The problem is thus reduced to exhibiting a measure on G which satisfies (5.4.2).

Let S be the linear space of all finite sums f of the form

$$f(x) = \Sigma (x, y_n) Q_n(x) \quad (x \in G), \tag{5.4.3}$$

where the Q_n are polynomials, such that $\hat{Q}_n$ is real and has its support in E_n. Our assumptions about the groups Γ_n show that each $f \in S$ has a unique representation of the form (5.4.3). Since $y_n \notin E_n$, (x, y_n) takes each of its values on each of the subsets of G on which Q_n is constant; hence

$$\max_{x \in G} \operatorname{Re}\,[(x, y_n)\, Q_n(x)] \geq \tfrac{1}{2} \| Q_n \|_\infty, \tag{5.4.4}$$

where $\| \;\; \|_\infty$ denotes the supremum norm.

Moreover, our assumptions about Γ_n imply that if $n \neq m$, then each of the sets on which $(x, y_n)\, Q_n(x)$ is constant intersects each of the sets on which $(x, y_m)\, Q_m(x)$ is constant, and we conclude from (5.4.4) that

$$\|f\|_\infty \geqslant \tfrac{1}{2} \sum \|Q_n\|_\infty \tag{5.4.5}$$

if f is given by (5.4.3).

We consider S as a subspace of $C(G)$, the space of all continuous functions on G; by (5.4.5), the mapping

$$f \to Tf = \sum \int_G Q_n(-x)\, P_n(x)\, dx \tag{5.4.6}$$

is a linear functional on S, of norm at most $2r$, which can be extended, by the Hahn-Banach Theorem, to a bounded linear functional on $C(G)$. Thus there is a measure μ on G, such that

$$\int_G f(-x)\, d\mu(x) = \sum \int_G Q_n(-x)\, P_n(x)\, dx \tag{5.4.7}$$

for all f of the form (5.4.3).

In particular, if we fix $y \in E_n$ and take $f(x) = (x, y + y_n)$, then (5.4.7) applies, and gives precisely (5.4.2). Finally, if $\hat{\mu}(y)$ is not real for all $y \in \Gamma$, we replace $\hat{\mu}$ by its real part; since $\hat{P}_n(y)$ is real, this change does not affect (5.4.2), and since $\mathrm{Re}[\hat{\mu}] \in B(\Gamma)$, the proof is complete.

Remark: If F had been defined only on I, and $\hat{P}_n(y)$ had been in I, it is not clear that we can find μ such that (5.4.2) holds, with $\hat{\mu}(y) \in I$ for all $y \in \Gamma$, and hence $F(\hat{\mu})$ might not be defined. This is one of our reasons for making the simplifying assumptions 5.1 at the very beginning of the proof of Theorem 3.

5.5. *Proof of Step* 2. This is similar in outline to the proof of Step 1, but the details are a little more complicated.

We assume that F, defined on the real axis, operates in $B(\Gamma)$, where Γ is a discrete group which is not of bounded order, and, to obtain a contradiction, we assume that 5.3 (β) does not hold for the group of all integers. That is to say, we assume that there are polynomials with real coefficients,

$$P_j(e^{i\theta}) = \sum_{-N_j}^{N_j} a_{n,j}\, e^{in\theta} \tag{5.5.1}$$

such that

$$\|P_j\|_1 \leqslant r \qquad (j = 1, 2, 3, \ldots) \tag{5.5.2}$$

but

$$\|F \circ P_j\|_1 \to \infty \qquad (j \to \infty). \tag{5.5.3}$$

The norms here are the L^1-norms with respect to the Haar measure of the unit circle.

Consider polynomials of the form

$$Q_j(e^{i\theta}) = \sum_{-2N_j}^{2N_j} b_{n,j} e^{in\theta}. \tag{5.5.4}$$

Since

$$\|Q_j'\|_\infty \leqslant \sum_{-2N_j}^{2N_j} |n|\,|b_{n,j}| \leqslant \|Q_j\|_\infty \cdot \sum_{-2N_j}^{2N_j} |n| \leqslant 5N_j^2 \|Q_j\|_\infty,$$

there exists an integer m_j and a $\delta_j > 0$, depending only on N_j, such that

$$\operatorname{Re}\,[e^{im_j\theta} Q_j(e^{i\theta})] \geqslant \tfrac{1}{2} \|Q_j\|_\infty \tag{5.5.5}$$

on some arc of length δ_j.

By Lemma 2.5, G contains an element x_0 of infinite order; hence there are real numbers α_j and elements $y_j \in \Gamma$ $(j = 1, 2, 3, \ldots)$ such that

(a) $(x_0, y_j) = e^{i\alpha_j}, \quad 0 < 4\pi\alpha_{j+1} < \alpha_j \delta_{j+1}$;

(b) the order of y_j exceeds $2m_j + 6N_j$, and the sets

$$E_j = \{k y_j \mid m_j - 2N_j \leqslant k \leqslant m_j + 2N_j\}$$

are disjoint.

Having done this, we associate polynomials P_j^*, Q_j^* on G with P_j and Q_j:

$$P_j^*(x) = (x, m_j y_j) P_j((x, y_j)) = \sum_{-N_j}^{N_j} a_{n,j} (x, y_j)^{m_j+n},$$

$$Q_j^*(x) = (x, m_j y_j) Q_j((x, y_j)) = \sum_{-2N_j}^{2N_j} b_{n,j} (x, y_j)^{m_j+n},$$

and we let S be the linear subspace of $C(G)$ which consists of all finite sums of the form

$$f(x) = \sum Q_j^*(x) \qquad (x \in G); \tag{5.5.6}$$

S depends on $\{N_j\}$, $\{m_j\}$, $\{\delta_j\}$, and $\{y_j\}$, and property (b) of $\{y_j\}$ implies that each $f \in S$ has a unique representation of the form (5.5.6). Since

$$Q_j^*(n x_0) = e^{i m_j n \alpha_j} Q_j(e^{in\alpha_j}) \qquad (n = 0, \pm 1, \pm 2, \ldots), \tag{5.5.7}$$

(5.5.5) shows that

$$\operatorname{Re}\,[Q_j^*(n x_0)] \geqslant \tfrac{1}{2} \|Q_j\|_\infty$$

for repeated stretches of $[\delta_j/\alpha_j]$ consecutive integers n. If n were a continuous variable, $2\pi/\alpha_j$ would be a period of $Q_j^*(nx_0)$, by (5.5.7), and property (a) of $\{\alpha_j\}$ implies that this period is less than one half of $[\delta_{j+1}/\alpha_{j+1}]$. From this it follows that

$$\sup_n \operatorname{Re}[f(nx_0)] \geq \tfrac{1}{2}\sum \|Q_j\|_\infty$$

for every f of the form (5.5.6), and hence

$$\|f\|_\infty \geq \tfrac{1}{2}\sum \|Q_j\|_\infty; \tag{5.5.8}$$

note that $\|f\|_\infty$ is the supremum of $|f|$ over G, whereas $\|Q_j\|_\infty$ refers to the unit circle.

We insert here a remark concerning the relation between $\|Q_j\|_1$ and $\|Q_j^*\|_1$, the L^1-norms over the unit circle and over G, respectively. If y_j is of infinite order, then $\|Q_j\|_1 = \|Q_j^*\|_1$. But if y_j has order q, say, then

$$\|Q_j^*\|_1 = \int_G |Q_j((x, y_j))|\,dx = \frac{1}{q}\sum_{n=1}^{q}\left|Q_j\left(\exp\frac{2\pi i n}{q}\right)\right|,$$

and this differs from

$$\|Q_j\|_1 = \sum_{n=1}^{q}\frac{1}{2\pi}\int_{(2n-1)\pi/q}^{(2n+1)\pi/q}|Q_j(e^{i\theta})|\,d\theta$$

by not more than

$$\frac{\pi}{q^2}\|Q_j'\|_\infty \leq \frac{5\pi}{q^2}N_j^2\|Q_j\|_1 < \tfrac{1}{2}\|Q_j\|_1,$$

by property (b) of $\{y_j\}$. Hence, in any case, we have

$$\|Q_j^*\|_1 > \tfrac{1}{2}\|Q_j\|_1. \tag{5.5.9}$$

Returning to our proof, we note that (5.5.2) and (5.5.8) imply that the mapping

$$f \to Tf = \sum \frac{1}{2\pi}\int_{-\pi}^{\pi} Q_j(e^{-i\theta})\,P_j(e^{i\theta})\,d\theta$$

is a bounded linear functional on S; hence there is a measure μ on G (with real $\hat{\mu}$; see 5.4), such that

$$\int_G f(-x)\,d\mu(x) = \sum\frac{1}{2\pi}\int_{-\pi}^{\pi} Q_j(e^{-i\theta})\,P_j(e^{i\theta})\,d\theta \tag{5.5.10}$$

for every f of the form (5.5.6).

In particular, if $y \in E_j$ and $f(x)=(x, y)$, then (5.5.10) applies, since $y=(m_j+t)\,y_j$, with $|t| \leqslant 2N_j$, and we obtain

$$\hat{\mu}(y)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-it\theta} P_j(e^{i\theta})\,d\theta = a_{t,j} = \hat{P}_j^*(m_j y_j + t y_j) = \hat{P}_j^*(y).$$

If we now define

$$\hat{K}_j(y)=\begin{cases} 1 & \text{if } y=(m_j+t)\,y_j \text{ and } |t| \leqslant N_j, \\ 2-\dfrac{|t|}{N_j} & \text{if } y=(m_j+t)\,y_j \text{ and } N_j<|t| \leqslant 2N_j, \\ 0 & \text{for all other } y \in \Gamma, \end{cases}$$

then

$$\hat{K}_j(y)\,F(\hat{\mu}(y)) = F(\hat{P}_j^*(y)) \qquad (y \in \Gamma),$$

so that $\|F \circ \hat{P}_j^*\|_1 \leqslant 2\,\|F(\hat{\mu})\|$. Since (5.5.9) applies to $F \circ P_j$ in place of Q_j, we have

$$\|F \circ P_j\|_1 \leqslant 2\,\|F \circ P_j^*\|_1 \leqslant 4\,\|F(\hat{\mu})\| \qquad (j=1, 2, 3, \dots), \tag{6.1.1}$$

contradicting (5.5.3).

This completes the proof of Theorem 3.

VI. Consequences of Theorem 3

6.1. Let $M(G)$ be the convolution algebra of all bounded complex Borel measures on G; $M(G)$ is isomorphic to $B(\Gamma)$, it is a commutative Banach algebra with unit, and it therefore has a compact maximal ideal space Δ. We may think of Δ as the set of all homomorphisms h of $M(G)$ onto the complex field, and we define the Gelfand transform $\hat{\mu}$, as a function on Δ, by

$$\hat{\mu}(h) = h(\mu) \qquad (\mu \in M(G)). \tag{6.1.1}$$

With every $y \in \Gamma$ these is associated a homomorphism

$$\mu \to h_y(\mu) = \int_G (-x, y)\,d\mu(x) \qquad (\mu \in M(G)) \tag{6.1.2}$$

and we may thus consider Γ as a subset of Δ. Comparison with our earlier definition of $\hat{\mu}$ shows that every $f \in B(\Gamma)$ may be extended to a Gelfand transform on Δ (the uniqueness theorem for Fourier-Steltjes transforms shows that there is only one such extension) and, conversely, that the restrictions of the Gelfand transforms to Γ belong to $B(\Gamma)$. We also note that our embedding of Γ into Δ is a homeomorphism.

We now show that the range of a Gelfand transform can extend into the imaginary part of the complex plane, although the corresponding Fourier-Stieltjes transform is real:

6.2. LEMMA. *Suppose* Γ *is not compact, and let* z_0 *be a complex number. Then there exists a measure* μ *on* G *whose Fourier-Stieltjes transform has its range in the interval* $[-1,1]$ *on the real axis, such that* $h(\mu) = z_0$ *for some* $h \in \Delta$.

Proof: If this were not so, then the function

$$F(x) = (x - z_0)^{-1} \qquad (-1 \leqslant x \leqslant 1)$$

would operate in $B(\Gamma)$, by one of the basic theorems of the Gelfand theory of normed rings, and this would contradict Theorem 3.

6.3. THEOREM. *Suppose* F *is defined in the whole complex z-plane, and suppose* F *operates in the algebra of all Gelfand transforms of measures on* G, *where* G *is not discrete* (i.e., Γ *is not compact*). *Then* F *is an entire function of* z.

(The hypothesis may be restated by saying that F associates with each $\mu \in M(G)$ a measure $\sigma \in M(G)$ such that $h(\sigma) = F(h(\mu))$ for every $h \in \Delta$.)

Proof. The restriction of F to the real axis operates in $B(\Gamma)$, and by Theorem 3 there is an entire function F_1 such that $F_1(x) = F(x)$ for all real x. Being entire, F_1 operates in the algebra of all Gelfand transforms, and so does $F - F_1$. But $(F - F_1)(f) = 0$ for every $f \in B_R(\Gamma)$. Hence $F - F_1$ associates the null-measure to each $\mu \in M(G)$ whose Fourier-Stieltjes transform is real, and Lemma 6.2 implies that $F(z_0) - F_1(z_0) = 0$ for every complex z_0. The theorem follows.

Remark A. The proof shows that the hypothesis of the Theorem can be weakened: it suffices to assume that F operates on the Gelfand transforms of those $\mu \in M(G)$ whose Fourier-Stieltjes transform has its range in I.

Remark B. The algebra $M(G)$ is said to be *asymmetric* if the set of all Gelfand transforms is not closed under conjugation, i.e., if the function $F(z) = \bar{z}$ does not operate in the algebra of all Gelfand transforms.

Theorem 6.3 shows immediately that $M(G)$ is asymmetric for every non-discrete G; for the real line this was proved by Šreider [10]; Williamson [14] recently obtained this result for the general case. (For discrete G, Γ is compact, $\Delta = \Gamma$, and $M(G) = L^1(G)$ is symmetric.)

Also, Lemma 6.2 implies that the closure of Γ in Δ is not the Šilov boundary of $M(G)$, and that there are functions $f \in B(\Gamma)$ such that $f^{-1} \notin B(\Gamma)$, although f^{-1} is bounded on Γ; on the real line, this phenomenon was noted by Wiener and Pitt [13].

Each of these facts leads to the conclusion that Γ is not dense in Δ, if Γ is not compact.

VIII. Operating functions defined in plane regions

7.1. A function F, defined on a set E in the plane, is said to be *real-analytic* if to every point $(x_0, y_0) \in E$ there is an expansion with complex coefficients

$$F(x, y) = \sum_{n, m=0}^{\infty} a_{n, m} (x - x_0)^n (y - y_0)^m$$

which converges absolutely for all $(x, y) \in E$ such that $|x - x_0| < \delta$, $|y - y_0| < \delta$, for some $\delta > 0$.

If F is defined in the whole plane by a series

$$F(x, y) = \sum_{n, m=0}^{\infty} a_{n, m} x^n y^m$$

which converges absolutely for every (x, y), we call F *real-entire.*

Note that a function may be real-analytic at every point of the plane without being real-entire: consider

$$F(x, y) = \{(1 + x^2)(1 + y^2)\}^{-1}$$

7.2. Suppose now that F is defined in a plane open set E which contains the origin. The analogues of Theorems 1, 2, 3 are as follows:

If F operates in $A(\Gamma)$, and Γ is discrete, then F is real-analytic in some neighborhood of the origin; if Γ is not discrete, then F is real-analytic in E. If Γ is not compact and if F operates in $B(\Gamma)$, then F can be extended to a real-entire function in the plane.

The proofs are almost identical with those of Theorems 1, 2, 3; the only significant difference is that in place of the functions

$$\Phi(x) = F(r \sin x)$$

we now introduce doubly periodic functions

$$\Phi(x, y) = F(r \sin x, r \sin y)$$

which we expand in double Fourier series $\sum_{-\infty}^{\infty} a_{n, m} e^{i(nx + my)}$; the coefficients $a_{n, m}$ can be estimated as in 2.3.

7.3. If E is a closed convex set in the plane, if F, defined in E, operates in $A(\Gamma)$, where Γ is not discrete, then one can prove the full analogue of Theorem 2: E is real-analytic on E (not just in the interior). One uses the result stated in 7.2, and an argument similar to the one which was used in 4.3 to establish the analyticity of F at the end-points of I.

For closed sets in general, the problem seems to be open.

7.4. As a final remark, we point out that the converses of Theorems 1, 2, 3 are of course true; hence we have obtained complete characterizations of the functions which operate in $A_R(\Gamma)$ and $B_R(\Gamma)$.

Since $A(\Gamma)$ and $B(\Gamma)$ are closed under conjugation, it is not hard to see that the converses of the results stated in 7.2 are also valid.

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