SOME CHARACTERIZATIONS OF CONVEX POLYHEDRA

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1. Introduction

This paper deals with subsets of a finite-dimensional euclidean space E, a set being called *polyhedral* or a *polyhedron* provided it is the intersection of a finite number of closed halfspaces. Thus as the term is used here, a polyhedron is closed and convex but need not be bounded. A set will be called *boundedly polyhedral* provided its intersection with each bounded polyhedron is polyhedral. Our principal goal is to characterize polyhedra as convex sets, certain of whose projections or sections are polyhedral. In connection with this task, we are led to develop various properties of polyhedra and of boundedly polyhedral sets which seem to be available in the literature only for bounded polyhedra or polyhedral cones. Some of our proofs could be simplified a bit by working in projective space. However, since many of the results on unbounded convex subsets of the affine space seem to be not so simply obtainable in this way, we have chosen to work entirely in the affine space E.

Section 2 begins with a simple but useful theorem on the facial structure of an arbitrary convex set, generalizing from the fact that a bounded closed convex set is the convex hull of its set of extreme points. This theorem supplies one step in proving equivalence of the following five conditions on a subset K of E: K is the intersection of a finite system of closed halfspaces; K is a closed convex set with only finitely many faces; K is closed and is the convex hull of a finite system of points and rays; K is the closed convex hull of the union of a bounded polyhedron and a polyhedral cone; K is the linear sum of a bounded polyhedron and a polyhedral cone. Surely the equivalence is generally "known", but it seems not to be available elsewhere in precisely this form. In the present paper, we

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have need of the equivalence and have included the proof in an effort to smooth the reader's path.

In § 3 we exploit some connections between polyhedra and convex cones, in preparation for the main results in § 4. These are as follows (rendering "if and only if" by "iff"). Suppose K is a convex subset of E^n and $2 \leqslant j \leqslant n-1$. Then (1) K is polyhedral iff all its j-sections are polyhedral; (2) if K is bounded, K is polyhedral iff all its j-projections are polyhedral; (3) if $j \geqslant 3$, K has polyhedral closure iff all its j-projections have polyhedral closure; (4) if K is a cone, K is polyhedral iff all its j-projections are closed. These theorems are believed to be new, except that for $j=2$ and K closed; the result (4) was recently given by Mirkil [7]. Our general procedure has much in common with his, and some of our propositions amount to formalizations of steps in his proof. The results (1) and (2) reduce trivially to the case $j=2$, but this does not appear to be true of (4).

Section 5 is devoted to some types of "nearly polyhedral" sets, characterizing them in various ways, identifying their polars, etc. The results are applied in discussion of two interesting examples in E^3, one a nonpolyhedral closed convex set all of whose 2-dimensional projections are polyhedral, and the other a set which is nonpolyhedral even though it and its polar are both boundedly polyhedral. In § 6 it is proved that every convex F_σ set is a projection of some closed convex set, and the projections of boundedly polyhedral sets are also characterized. Some approximation theorems are given which are valid for unbounded convex sets and reduce in the bounded case to the classical result on approximation by polyhedra.

In the concluding § 7, the term *polyhedral* is employed in its more customary sense to describe a set which is the union of a finite number of geometric simplexes. There is constructed in E^3 a nonpolyhedral 3-cell all of whose 2-sections and 2-projections are polyhedral. Thus convexity seems essential for the results of § 4.

For basic material on convex sets, including proofs of results used here without specific reference, the reader is referred to [1, 2, 3, 4], especially [4]. I am indebted to Professor Fenchel for some helpful suggestions, especially in connection with § 2.

Notation and terminology. A *j-flat* is a j-dimensional affine subspace of E, and the term *subspace* will be reserved hereafter for linear subspaces. A *j-section* of a set $X \subset E$ is the intersection of X with a j-flat, and a j-projection of X is the image of X under an affine projection of E onto a j-flat. (Since such an affine projection can always be obtained as the composition of a translation with a linear projection onto a j-subspace of E, conditions on the j-projections of a set may be regarded equivalently in terms of linear projections.)

We denote the empty set by Λ, and the origin in E by ϕ. Set-theoretic union, intersection, and difference will be denoted by $\cup$, $\cap$, and $\sim$, the closure, interior, and convex hull of a set X by cl X, int X, and conv X. The smallest flat containing X will be denoted by fl X and the interior of X relative to fl X by relint X (called the *relative interior of* X). The *relative boundary* of X is the set $X \sim$ relint X. For x and y in E, $[x, y] = \{rx + (1-r)y: 0 \leqslant r \leqslant 1\}$, $]x, y[= \{rx + (1-r)y : 0 < r < 1\}$, etc. For $X \subset E$, $Y \subset E$, $\varepsilon > 0$ and $A \subset R$ (the real number field), $X \pm Y = \{x \pm y : x \in X, y \in Y\}$, $AX = \{ax : a \in A, x \in X\}$, and $S(X, \varepsilon)$ is the union of the open ε-neighborhoods of the points of X.

2. Facial structure, polarity, and polyhedra

We prove here a useful result on the facial structure of convex sets, review the notion of polarity, and establish some basic properties of polyhedra which will be used later in the paper. Fenchel's book [4] may be mentioned as a basic reference for the methods employed and for Proposition 2.2, Goldman's paper [5] for the result 2.12 (v), and my paper [6] for 2.3. The remarks on polarity and at least special cases of the results on polyhedra have appeared in print many times (see, for example, [2, 4, 5, 9]).

A convex set will be called *reducible* provided it is the convex hull of its relative boundary; otherwise it is *irreducible*. By a *face* of a convex set K we shall mean a convex subset F of K such that whenever x and y are points of K for which F is intersected by the open segment $]x, y[$, then $x \in F$ and $y \in F$. The convex set K is said to be *generated* by a family $\mathfrak{F}$ of sets provided K is the convex hull of the union of the members of $\mathfrak{F}$. (In using this term, we shall not always distinguish carefully between a point $p \in E$ and the corresponding set $\{p\} \subset E$.)

2.1. Theorem. *Every convex set is generated by its irreducible faces.*

Proof. As always in this paper, we are concerned only with finite-dimensional sets; 2.1 is trivial for those of dimension zero. Suppose it is known for all convex sets of dimension $< n$, and consider an n-dimensional convex set K. Let D denote the relative boundary of K. If $K \neq$ conv D, then K itself is an irreducible face of K. If $K =$ conv D, then by the support theorem K must be generated by the sets $H \cap D$, where H is a hyperplane which meets D but not relint K. But each face of a set $H \cap D$ is easily seen to be a face of K, and by the inductive hypothesis each set $H \cap D$ is generated by its irreducible faces. This completes the proof.

Theorem 2.1 is especially useful for closed convex sets, in view of the following.

2.2. Proposition. *The only irreducible closed convex sets are the flats and the half-flats.*

Proof. Obviously all flats and half-flats are irreducible. Now consider an irreducible closed convex set K and let D denote its relative boundary. Since conv $D \neq K$, we have conv $D \not\supset$ relint K, and by the separation theorem there must be an open halfspace Q in

$\mathrm{fl}\, K$ which misses $\mathrm{conv}\, D$ but meets $\mathrm{relint}\, K$. Since K is closed, it can be seen that $Q \subset \mathrm{relint}\, K$, and is then easy to verify that $K = \mathrm{fl}\, K$ or K is a closed halfspace in $\mathrm{fl}\, K$.

2.3. COROLLARY. *A closed convex set which contains no line is generated by its extreme points and extreme rays.*

Let us include a few more remarks on facial structure, even though they are not needed for the sequel. A face of a convex set K will be called *essential* provided it lies in some member of each generating family of faces of K, and *strictly essential* provided, in addition, it is not contained in another essential face. We shall prove together the following two results.

2.4. PROPOSITION. *Every irreducible face is essential, and every strictly essential face is irreducible. Thus the strictly essential faces are exactly the maximal irreducible faces.*

2.5. PROPOSITION. *Every convex set is generated by its family of strictly essential faces, but not by any proper subfamily of that.*

Proofs. We note: 1° a convex set is irreducible iff it is an essential face of itself; 2° if $X \subset K$ and F is a face of K, then $\mathrm{conv}\,(X \cap F) = (\mathrm{conv}\, X) \cap F$; 3° if G is an essential face of F and F is a face of K, then G is an essential face of K. The assertion 1° is trivial. As for 2°, it is obvious that $\mathrm{conv}\,(X \cap F) \subset (\mathrm{conv}\, X) \cap F$. Now consider points x_i of X and positive numbers t_i whose sum is 1, such that $\sum t_i x_i = p \in F$. Then for each j we have $p = t_j x_j + (1 - t_j) y_j$, where $y_j = \sum_{i \neq j} (1 - t_j)^{-1}\, t_i x_i \in K$. Since F is a face, it follows that $x_j \in F$. Thus $p \in \mathrm{conv}\,(X \cap F)$ and 2° has been proved. Finally, suppose G and F are as in 3° and $\mathfrak{S}$ is a generating family of faces of K. From 2° it follows that F is generated by its family of faces, $\{J \cap F : J \in \mathfrak{S}\}$, and thus G, being an essential face of F, must be in some set $J \cap F$. This establishes 3°, and we are ready to prove 2.4 and 2.5.

That every irreducible face is essential follows from 1° and 3°. Now let $\mathcal{S}$ be the family of all strictly essential faces of K. From finite-dimensionality it follows that every essential face (and hence every irreducible face) lies in some strictly essential face, so from 2.1 it follows that $\mathcal{S}$ generates K. Consider an arbitrary $S \in \mathcal{S}$ and let $\mathcal{J}$ be the family of all proper faces of S. If K is generated by $(\mathcal{S} \sim \{S\}) \cup \mathcal{J}$, then S, being essential, must lie in some member of $\mathcal{S} \sim \{S\}$ or in some member of $\mathcal{J}$. Since this is impossible, K is not generated by $\mathcal{S} \sim \{S\}$ (completing the proof of 2.5) and S is not generated by $\mathcal{J}$, whence S is irreducible and the proof of 2.4 is complete.

In the case of closed convex sets, a more complete picture of the facial structure can be obtained from the following two remarks, whose proof will be left for the reader.

2.6. PROPOSITION. *Suppose L and M are supplementary linear subspaces of E, J is a convex subset of M, and π is the projection of E onto M whose kernel is L. Then if G is* *a face of the set* $J + L$, πG *is a face of* J, *and if* F *is a face of* J, $\pi^{-1}F$ *is a face of* $J + L$. *The same assertion is valid for essential faces and for strictly essential faces.*

2.7. PROPOSITION. *Suppose* K *is a closed convex subset of* E *with* $\phi \in K$, *and* $L = \{x : Rx \subset K\}$, *so that* L *is a linear subspace of* E. *Let* M *be a subspace supplementary to* L. *Then* $K = L + (M \cap K)$ *and the closed convex set* $M \cap K$ *contains no lines.*

It follows that the essential faces of K are exactly the sets $\varrho + L$ where ϱ is an extreme ray of $M \cap K$ and the sets $p + L$ where p is an extreme point of $M \cap K$. These are all strictly essential except for the sets $p + L$ where p is an endpoint of an extreme ray.

For a subset X of E, the *polar* X^0 of X is defined as the set of all linear functionals f on E such that $fx \leqslant 1$ for all $x \in X$. Thus X^0 is a subset of the space E' conjugate to E, and the *bipolar* X^{00} is a subset of E under the usual identification of E with the conjugate space of E'.

The propositions 2.8–2.10 below summarize some well-known facts which will be used freely in the sequel.

2.8. PROPOSITION. *For* $X \subset E$, *the polar* X^0 *is a closed convex set which includes the origin* ϕ' *of* E', $X^0 = [\mathrm{cl\,conv}\,(X \cup \{\phi\}]^0$ *and* $X^{00} = \mathrm{cl\,conv}\,(X \cup \{\phi\})$. *Thus always* $X^{000} = X^0$, *while* $X^{00} = X$ *iff* X *is closed and convex and* $\phi \in X$.

2.9. PROPOSITION. *Suppose* X *is closed and convex with* $\phi \in X$. *Then* X *is a* j*-subspace iff* X^0 *is a* $(\dim E - j)$*-subspace*, X *is a cone with vertex* ϕ *iff* X^0 *is a cone with vertex* ϕ', *and* X *is bounded iff* $\phi' \in \mathrm{int}\, X^0$.

2.10. PROPOSITION. *For a family* $\{X_a : a \in A\}$ *of subsets of* E, $(\bigcup_{a\in A} X_a)^0 = \bigcap_{a\in A} X_a^0$ *and* $(\bigcap_{a\in A} X_a)^0 = \mathrm{cl\,conv} \bigcup_{a\in A} X_a^0$.

We shall prove together 2.11 and 2.12 below, supplying five useful characterizations of polyhedra. Essential tools in the proof are 2.1, 2.2, and parts of 2.8 and 2.10.

2.11. THEOREM. *If* K *is a closed convex set with* $\phi \in K$, *then* K *is polyhedral iff* K^0 *is polyhedral.*

2.12. THEOREM. *For a subset* K *of* E, *the following five assertions are equivalent*:

(i) K *is the intersection of a finite system of closed halfspaces*;

(ii) K *is closed, convex, and has only finitely many faces*;

(iii) K *is closed and is the convex hull of a finite system of points and rays*;

(iv) K *is the closed convex hull of the union of a bounded polyhedron and a polyhedral cone*;

(v) K *is the linear sum of a bounded polyhedron and a polyhedral cone.*

Proofs. We show first that (i) implies (ii). Let $\mathcal{G}$ be a finite family of closed halfspaces whose intersection is K, and for each $\mathfrak{S} \subset \mathcal{G}$ let $V\mathfrak{S}$ be the intersection of the bounding hyperplanes of the members of $\mathfrak{S}$. There are only finitely many sets of the form $K \cap V\mathfrak{S}$, and we shall show that each face of K has that form. Consider an arbitrary face F of K, and let $\mathfrak{S}$ be the set of all members of $\mathcal{G}$ whose bounding hyperplanes contain F; obviously $F \subset K \cap V\mathfrak{S}$. Now for each $G \in \mathcal{G} \sim \mathfrak{S}$ we may choose a point p_G of $F \cap \operatorname{int} G$. With p denoting the centroid of the chosen points p_G, it is evident that $p \in U = \bigcap_{G \in \mathcal{G} \sim \mathfrak{S}} \operatorname{int} G$. Consider an arbitrary point $q \in K \cap V\mathfrak{S}$. Clearly $V\mathfrak{S}$ contains the entire line of points $y_r = p + r(p - q)$ (for $r \in R$), and since U is open we have $y_t \in U$ for some $t > 0$. Then $y_t \in K$; since F is a face and $p \in]q, y_t[$, it follows that $q \in F$. Thus $K \cap V\mathfrak{S} \subset F$ and we know that (i) implies (ii).

That (ii) implies (iii) is a consequence of 2.1 and 2.2, for obviously condition (iii) is satisfied by every flat and every half-flat.

We next establish

(1) Suppose $\phi \in K$ and K is the closed convex hull of a finite system of points and rays. Then K is the closed convex hull of a finite system of points and of rays emanating from ϕ; further, the polar K^0 is polyhedral.

Let $\varrho_1, \ldots, \varrho_m$ and $x_{m+1}, \ldots, x_n$ be the given rays and points. For each ϱ_i, let x_i be the endpoint of ϱ_i and let ϱ_i^1 be the ray $\varrho_i - x_i$, emanating from ϕ. Since always $\varrho_i \subset \operatorname{cl\,conv}(\{x_i\} \cup \varrho_i^1)$ and $\varrho_i^1 \subset \operatorname{cl\,conv}(\{\phi\} \cup \varrho_i)$, it is evident that K is the closed convex hull of the system of points $x_i (1 \leqslant i \leqslant n)$ and rays $\varrho_i^1 (1 \leqslant i \leqslant m)$. With $y_i \in \varrho_i^1 \sim \{\phi\}$, it follows from 2.10 that K^0 is the intersection of n halfspaces of the form $\{f : fx_i \leqslant 1\}$ and m of the form $\{f : fy_i \leqslant 0\}$, so K^0 is polyhedral.

Now consider a closed convex set $K \ni \phi$. From (1), in conjunction with the fact that 2.12 (i) implies 2.12 (iii), it follows that if K is polyhedral, so is K^0. But this implies that if K^0 is polyhedral, so is $K^{00} = K$, and thus 2.11 has been proved. Now if K satisfies 2.12 (iii), then K^0 is polyhedral by (1) and K is polyhedral by 2.11; thus 2.12 (iii) implies 2.12 (i).

We are now in a position to prove

2.13. COROLLARY. *A set is a bounded polyhedron iff it is the convex hull of a finite set.*

2.14. COROLLARY. *A set is a polyhedral cone with vertex z iff it is the convex hull of a finite system of rays emanating from z.*

For closed sets, these characterizations come immediately from equivalence of conditions (i) and (iii) in 2.12. From compactness of $[0, 1]$ it follows readily that the convex hull of a finite set is compact, hence closed, and this completes the proof of 2.13. For 2.14, it suffices to show that if $x_1, \ldots, x_k$ are points of E and C is the set of all linear combinations of the x_i's with non-negative coefficients, then C is closed. For $k = 1$, this is obvious. Suppose it is known for $k = n - 1$ and consider the case $k = n$. Let $J = \{\sum_1^{n-1} t_i x_i : t_i \geqslant 0\}$. Then $C = J + [0, \infty[x_n$, and J is closed by the inductive hypothesis. If $x_n \in J$, then $C = J$, so we may assume $x_n \notin J$. Now consider an arbitrary point $p \in \mathrm{cl}\, C$. There are sequences t_α in $[0, \infty[$ and y_α in J such that $t_\alpha \to t \in [0, \infty]$ and $y_\alpha + t_\alpha x_n \to p$. If $t = \infty$, then $t_\alpha^{-1} y_\alpha + x_n \to \phi$ and thus $x_n \in J$, a contradiction. If $t \in [0, \infty[$, then $y_\alpha \to p - t x_n$, whence $p - t x_n = y \in J$ and $p \in C$. The proof of 2.14 is complete, and we may continue with the proof of 2.12.

Note that in the proof of (1) above we have

$$(^*)\ K = \mathrm{cl\ conv}\,(\mathrm{conv}\{x_i : 1 \leqslant i \leqslant n\} \cup \mathrm{conv} \bigcup_{i=1}^{m} \varrho_i^1),$$

so by using the characterizations 2.13 and 2.14 we see that (iii) implies (iv) in 2.12. On the other hand, if K is the closed convex hull of the union of a bounded polyhedron and a polyhedral cone with vertex ϕ, it follows by 2.13 and 2.14 that K has the form (*), by (1) that K^0 is polyhedral, and then by 2.11 that K is polyhedral. Thus (iv) implies (i) in 2.12, and it remains only to prove that (iv) and (v) are equivalent. To this end, we establish

(2) Suppose X is a compact convex set with $\phi \in X$ and Y is a closed convex cone with vertex ϕ. Then $X + Y = \mathrm{cl\ conv}\,(X \cup Y)$.

With $\phi \in X \cap Y$, it is clear that $X + Y \supset X \cup Y$, and then since $X + Y$ is closed and convex it follows that $X + Y \supset \mathrm{cl\ conv}\,(X \cup Y)$. The reverse inclusion stems from the fact that if $x \in E$ and ϱ is a ray emanating from ϕ, then $x + \varrho \subset \mathrm{cl\ conv}\,(\{x\} \cup \varrho)$.

Now suppose B is a bounded polyhedron and C is a polyhedral cone with vertex ϕ. Let $B' = \mathrm{conv}\,(B \cup \{\phi\})$ and $B'' = B - b$ for some $b \in B$. From (2) we see that if $K = B + C$, then $K - b = B'' + C = \mathrm{cl\ conv}\,(B'' \cup C)$. Thus (iv) is equivalent to (v) and the proof of 2.12 is complete.

2.15. Corollary. *If K is a polyhedron (a bounded polyhedron, a polyhedral cone), then so is every affine image of K.*

Proof. For bounded polyhedra and polyhedral cones, this is evident by 2.13 and 2.14. Then use 2.12 (v) for the general case.

2.16. Corollary. *If K_1 and K_2 are polyhedra (bounded polyhedra, polyhedral cones with vertex ϕ), then so are the sets $K_1 + K_2$ and* cl conv $(K_1 \cup K_2)$.

Proof. From 2.13 it follows that if B_1 and B_2 are bounded polyhedra, then so are $B_1 + B_2$ and $\mathrm{conv}\,(B_1 \cup B_2)$. From 2.14 it follows that if C_1 and C_2 are polyhedral cones, then $C_1 + C_2$ is a polyhedral cone, as is $\mathrm{conv}\,(C_1 \cup C_2)$ under the additional assumption that C_1 and C_2 have the same vertex. Thus if K_1 and K_2 are polyhedral (say $K_i = B_i + C_i$, so that $K_1 + K_2 = (B_1 + B_2) + (C_1 + C_2)$), we deduce from 2.12 (v) that $K_1 + K_2$ is polyhedral. Observe, finally, that when K_1 and K_2 are polyhedral it is evident from 2.12 (iii) that the set $K = \text{cl conv}(K_1 \cup K_2)$ is the closed convex hull of a finite system of points and rays; that K is polyhedral is then a consequence of (1) above in conjunction with 2.11. The proof of 2.16 is complete.

In concluding this section, we mention two more notions which play an important role in the sequel. A subset K of E is said to be *boundedly polyhedral* provided its intersection with each bounded polyhedron in E is polyhedral, and to be *polyhedral at* a point $p \in K$ provided some neighborhood of p relative to K is polyhedral.

2.17. PROPOSITION. *A set is boundedly polyhedral iff it is closed, and convex, polyhedral at all its points.*

Proof. Suppose K is closed, convex, and polyhedral at all its points, and consider a bounded polyhedron B. Each point $x \in K$ admits a bounded polyhedral neighborhood N_x relative to K, and by compactness of $B \cap K$ there must be a finite set $X \subset K$ with $B \cap K \subset \bigcup_{x \in X} N_x$. Let $Z = \text{conv} \bigcup_{x \in X} N_x$. Then Z is polyhedral, whence so is $B \cap Z$, and we have $B \cap K \subset Z \subset K$. It follows that $B \cap K = B \cap Z$ and K must be boundedly polyhedral.

3. Cones and polyhedra

For a point $p \in E$ and a set $X \subset E$, cone (p, X) will denote the set $p +]0, \infty[(X - p)$, the smallest cone which contains X and has vertex p. (Note that $p \in \text{cone}(p, X)$ iff $p \in X$.) The present section consists largely of exploiting the connection between this notion and polyhedra. We begin with a collection of elementary but useful facts about convex cones, supplying some of the machinery to be employed in proving the main theorems.

3.1. PROPOSITION. *Suppose C is a convex cone in E with vertex ϕ, and let L denote the lineality space of* cl C ($L = \text{cl}\, C \cap -\text{cl}\, C$). *Then C is linear iff $C \subset L$. Suppose now that C is not linear, so $C \not\subset L$. Then there is a linear functional f on E such that $f = 0$ on L and $f > 0$ on* cl $C \sim L$. *With $t > 0$, let $H_0 = f^{-1}0$ and $H_t = f^{-1}t$. Then the following statements are true:*

(i) $C = (H_0 \cap C) \cup]0, \infty[(H_t \cap C)$;

(ii) $H_t \cap C \supset (H_0 \cap C) + (H_t \cap C)$ *and* $C \supset (H_0 \cap C) + [0, \infty[(H_t \cap C)$, *with equality when $\phi \in C$ (and also under certain other conditions)*;

(iii) *if $p \in H_t \cap C$, then* cone $(p, H_t \cap C) = \pi C + p$, *where π is the projection of E onto H_0 which is the identity on H_0 and maps p onto ϕ*;

(iv) *if S is a subspace supplementary to L in H_0 and S_t is a translate of S to H_t then $S_t \cap C$ is bounded;*

(v) *if* $L \subset C$, *then* $H_t \cap C = L + (S_t \cap C)$ *and* $C = L + [0, \infty[(S_t \cap C)$;

(vi) C *is closed iff* $H_t \cap C$ *is closed and* $L \subset C$;

(vii) C *is polyhedral iff* $H_t \cap C$ *is boundedly polyhedral and* $L \subset C$.

Proof. The existence of f as described is well-known. Since $f > 0$ on $C \sim L$, it is clear that each point of $C \sim L$ has a positive multiple in $H_t \cap C$, whence condition (i) holds. If $H_0 \cap C = \Lambda$, the inclusions of (ii) hold trivially, for always $\Lambda + X = \Lambda$. Suppose on the other hand that $x \in H_0 \cap C$ and $y \in H_t \cap C$. Then $x + y \in C$ (since $C + C \subset C$) and $f(x + y) = 0 + t$, so $x + y \in H_t \cap C$. This justifies the first inclusion in (ii). For the second, we see from (i) that $C \supset]0, \infty[(H_t \cap C)$, whence $C \supset]0, \infty[\{(H_0 \cap C) + (H_t \cap C)\} = (H_0 \cap C) +]0, \infty[(H_t \cap C)$. And, by (i), $C \supset H_0 \cap C = (H_0 \cap C) + \{0\}(H_t \cap C)$, so the second inclusion of (ii) is established. The assertions about equality in (ii) are easily verified.

Now to establish (iii), let $K = (H_t \cap C) - p$, so that $K \subset H_0$ and $H_t \cap C = K + p$. Then

$$\text{cone}\,(p, H_t \cap C) \subset \text{cone}\,(p, K + p) = p +]0, \infty[K$$

and

$$\pi C = \pi(H_0 \cap C) \cup \pi]0, \infty[(K + p) = (H_0 \cap C) \subset]0, \infty[K.$$

From (ii) we see that $K \supset (H_0 \cap C) + K$, whence $K \supset H_0 \cap C$ (since $\phi \in K$) and $]0, \infty[K \supset]0, \infty[(H_0 \cap C) = H_0 \cap C$. It follows that $\pi C =]0, \infty[K$ and (iii) has been proved.

For (iv), observe that if $S_t \cap C$ is unbounded it must contain a ray and then the parallel ray emanating from ϕ must lie in $S \cap \text{cl}\, C$, contradicting the fact that $H_0 \cap \text{cl}\, C = L$.

Now if $L \subset C$, then $L + C \subset C$ and hence $L + (S_t \cap C) \subset H_t \cap C$. The reverse inclusion follows from the fact that $H_t = L + S_t$, and then using (ii) we obtain

$$C = L + [0, \infty[\{L + (S_t \cap C)\} = L + [0, \infty[(S_t \cap C),$$

so (v) is proved.

It is easy to verify that if $H_t \cap C$ is closed, so is $[s, \infty[(H_t \cap C)$ for each $s > 0$. And since $H_0 \cap \text{cl}\, C = L$, it follows that C is closed when $H_t \cap C$ is closed and $L \subset C$. This establishes (vi).

Now suppose $H_t \cap C$ is boundedly polyhedral and $L \subset C$. Since $S_t \cap C$ is bounded, it is the intersection with $H_t \cap C$ of a bounded polyhedron in S_t, and thus $S_t \cap C$ is polyhedral. With V denoting the set of all extreme points of $S_t \cap C$ and B a basis for L, it is clear that C consists of all non-negative combinations of elements of $B \cup -B \cup V$, for $C = L + [0, \infty[(S_t \cap C)$ by (v). Thus C is polyhedral and the proof of 3.1 is complete.

3.2. Proposition. *Suppose X and Y are convex subsets of E, $p \in X \cap Y$, and $Y \subset$* cone *(p, X). Then if Y is polyhedral, the set $X \cap Y$ is a neighborhood of p relative to Y.*

Proof. Let L and S_t be as in 3.1, so that $C = L + [0, \infty[(S_t \cap C)$. We assume without loss of generality that $p = \phi$. Now Y can be expressed as the intersection of closed half-spaces $Q_1, \ldots, Q_k, Q_{k+1}, \ldots, Q_m$ such that $\phi \in \operatorname{int} Q_i$ iff $k+1 \leqslant i \leqslant m$. Let $C = \bigcap_1^k Q_i$. Then $Y \subset C \subset \operatorname{cone}(p, X)$ and C is a polyhedral cone with vertex ϕ. To prove 3.2 it suffices to prove that $X \cap C$ is a neighborhood of ϕ relative to C. Since $S_t \cap C$ is a bounded polyhedron, there is a finite set V such that $\operatorname{conv} V = S_t \cap C$, and then for each $v \in V$ there exists $r_v > 0$ such that $[\phi, r_v v] \subset X$. The number $r = \inf\{r_v : v \in V\}$ is positive and $[0, r]\, V \subset X$, whence $[0, r](S_t \cap C) \subset X$. Observe also that ϕ is an inner point of $T \cap X$ for each line T' through ϕ in L, and hence $X \cap L$ is a neighborhood of ϕ relative to L. (We are using here the well-known fact that 3.2 is valid when $Y = L$.) Now each point u of C has a unique expression in the form $u = w_u + a_u z_u$ with $w_u \in L$, $a_u \in [0, \infty[$, and $z_u \in S_t \cap C$. Clearly there is a neighborhood N of ϕ such that $w_u \in \frac{1}{2}(X \cap L)$ and $a_u \in [0, \frac{1}{2} r]$ for all $u \in N \cap Y$; we then have

$$N \cap Y \subset \tfrac{1}{2}(X \cap L) + \tfrac{1}{2}[0, r](S_t \cap C) \subset \operatorname{conv} X = X,$$

and the proof of 3.2 is complete.

3.3. Corollary. *A convex set K is polyhedral at a point $p \in K$ iff cone (p, K) is polyhedral.*

Proof. If cone (p, K) is polyhedral, it follows from 3.2 that K is a neighborhood of p relative to cone (p, K). But then p has a polyhedral neighborhood N in E such that $N \cap \operatorname{cone}(p, K) = P \subset K$, and P is a polyhedral neighborhood of p relative to K.

Conversely, if K is polyhedral at p there is a bounded polyhedral neighborhood J of p relative to K. From convexity of K it is clear that cone $(p, K) =$ cone (p, J), which is obviously polyhedral.

3.4. Corollary. *The result 3.2 is valid under the assumption that X is polyhedral rather than Y.*

Proof. If X is polyhedral, then cone (p, X) is polyhedral by 3.3, and it follows from 3.2 that X is a neighborhood of p relative to cone (p, X). With $Y \subset \operatorname{cone}(p, X)$, this is sufficient.

3.5. Corollary. *If p is a point of a convex set K, then cone (p, K) is closed iff $J \cap K$ is polyhedral at p for each 2-flat J through p.*

Proof. Use 3.3 in conjunction with the following two facts: a two-dimensional convex cone is polyhedral iff it is closed; a convex set is closed iff all its 2-sections through a given point are closed.

We conclude this section with two basic lemmas which will be subsumed by the theorems of § 4.

3.6. LEMMA. *Suppose $2 \leqslant j \leqslant n$ and K is a convex subset of E^n all of whose j-projections are closed. Then if all 2-sections of* $\mathrm{cl}\, K$ *are boundedly polyhedral, K must be closed.*

Proof. We may assume that $\phi \in K$, $j < n$, and let L denote the union of all lines through ϕ which lie in $\mathrm{cl}\, K$, M a subspace supplementary to L in E. Then (by 2.7) $\mathrm{cl}\, K = L + M \cap (\mathrm{cl}\, K)$, where the set $A = M \cap \mathrm{cl}\, K$ contains no lines and thus by 2.3 $A = \mathrm{conv}\, (\mathrm{ex}\, A \cup \mathrm{rex}\, A)$, where $\mathrm{ex}\, A$ is the set of all extreme points of A and $\mathrm{rex}\, A$ the union of its extreme rays. Clearly A inherits from $\mathrm{cl}\, K$ the property that all its 2-sections are boundedly polyhedral, and we can show that K is closed by showing that $L + \mathrm{ex}\, A \subset K$ and $L + \mathrm{rex}\, A \subset K$.

We use the following two facts: (a) $L + \mathrm{relint}\, A \subset K$; (b) if $J = \{p\}$ for $p \in \mathrm{ex}\, A$ or $J = \mathrm{cl}\, \varrho$ for some extreme ray ϱ of A, then A is supported by a hyperplane H in M such that $H \cap A = J$. The assertion (a) follows from the readily verified facts that each finite-dimensional convex set contains the interior of its closure, and $L + \mathrm{relint}\, A \subset \mathrm{relint}\, \mathrm{cl}\, K$. The assertion (b) is an easy consequence of 2.17, 3.5, and the existence of f as described in 3.1.

Consider first an extreme point p of A. The assertion (b) guarantees the existence of a supporting hyperplane U of K such that $U \cap \mathrm{cl}\, K = p + L$. And if $p + L \not\subset K$, there must be a hyperplane V in U bounding an open halfspace W in U such that W misses $(p + L) \cap K$ but includes a point w of $p + L$. Let X be a $(j-2)$-dimensional flat in V, x a point of X, $q \in \mathrm{relint}\, A$ and Y the j-dimensional flat containing $X \cup \{q, w\}$. Let ξ be an affine projection of V onto X. Each point $z \in E$ has a unique expression in the form

$$z = x + (z_1 - x) + z_2(w - x) + z_3(q - x) \text{ with } z_1 \in V,\ z_2 \in R,\ z_3 \in R,$$

and for each z we define

$$\eta z = x + (\xi z_1 - x) + z_2(w - x) + z_3(q - x),$$

so that η is an affine projection of E onto Y. By hypothesis, ηK is closed. Now if $z \in K \sim U$, then $z_3 > 0$ and $\eta z \neq w$; if $z \in U \cap K$, then $z_2 \leqslant 0$ and $\eta z \neq w$. Thus $w \notin \eta K$. But $\eta]w, q] =]w, q] \subset K$, so it follows that $w \in \mathrm{cl}\, \eta K \sim \eta K$, an impossibility. We conclude that $p + L \subset K$ for each $p \in \mathrm{ex} A$.

Now consider an extremal ray ϱ of A—say $\varrho = p +]0, \infty[(u - p)$ where $u \in \varrho$ and p is the endpoint of ϱ. By (b), there is a supporting hyperplane U of K such that $U \cap \mathrm{cl} K = \mathrm{cl}\, \varrho + L$. Since $p \in \mathrm{ex}\, A$, the result of the last paragraph shows that $p + L \subset K$, and it follows easily that $(\varrho + L) \cap K = \sigma + L$ for some "initial segment" σ of ϱ. We wish to show that $\sigma = \varrho$. Suppose not, and let $w \in \varrho \sim \mathrm{cl}\,\sigma$. It can be verified that U must contain a hyperplane V bounding an open halfspace W in U such that $w \in W$ but W misses $(\varrho + L) \cap K$. A contradiction is reached as in the preceding paragraph, and the proof of 3.6 is complete.

3.7. Lemma. *Suppose $2 \leqslant j \leqslant n$, and C is a convex cone in E^n all of whose j-projections are closed. Then all $(n - j + 1)$-sections of C are polyhedral.*

Proof. Application of 3.1 (vii) shows that all r-sections of a convex cone C are polyhedral iff $M \cap C$ is polyhedral for every $(r + 1)$-flat M through the vertex of C. This equivalence will be used in the present proof, and we refer to it as (**).

Now consider a fixed $j \geqslant 2$, and let N_j denote the set of all integers $n \geqslant j$ such that each convex cone in E^n which has all its j-projections closed must also have all its $(n - j + 1)$-sections polyhedral. Clearly $j \in N_j$. Now suppose $k - 1 \in N_j$, and consider a convex cone C with vertex ϕ in E^k, all j-projections of C assumed to be closed. We wish to show that all $(k - j + 1)$-sections of C are polyhedral, and observe that it suffices to do this for $\mathrm{cl}\, C$, for then it follows by 3.6 that C is closed.

Let L, H_0, and H_t be as in 3.1. Consider an arbitrary point $p \in H_t \cap \mathrm{cl}\, C$, and let π be as in 3.1 (iii). Then $\pi\,\mathrm{cl}\, C$ is a convex cone in the $(k - 1)$-dimensional space H_0 and all its j-projections are closed (being j-projections of C), so from the inductive hypothesis it follows that all $(k - j)$-sections of $\pi\,\mathrm{cl}\, C$ are polyhedral. Now 3.1 (iii) shows that $\pi\,\mathrm{cl}\, C$ is merely a translate of cone $(p, H_t \cap \mathrm{cl}\, C)$, so all $(k - j)$-sections of the latter are polyhedral. Using (**) above we see that cone $(p, J \cap \mathrm{cl}\, C)$ is polyhedral for each $(k - j + 1)$-flat J through p in H_t. An application of 3.3 shows that for each $(k - j + 1)$-flat G_t in H_t, the set $G_t \cap \mathrm{cl}\, C$ is polyhedral at each of its points, and hence by 2.17 must be boundedly polyhedral. From 3.1 (vii) we conclude that the cone $\mathrm{cl}\,[0, \infty[(G_t \cap C)$ is polyhedral and hence that $G_t \cap \mathrm{cl}\, C$ is polyhedral. Since G_t is an arbitrary $(k - j + 1)$-flat in H_t, it follows from 3.1 (vii) and (**) that all $(k - j + 1)$-sections of $\mathrm{cl}\, C$ are polyhedral. Thus C is closed by 3.6 and all $(k - j)$-sections of C are polyhedral.

In the preceding two paragraphs we showed that if $k - 1 \in N_j$, then $k \in N_j$. It follows by mathematical induction that N_j includes all integers $\geqslant j$, and 3.7 has been proved.

4. Projections and sections

We start with the fundamental

4.1. Theorem. *Suppose K is a convex subset of E^n, $p \in K$, and $2 \leqslant j \leqslant n$. Then K is polyhedral at p iff πK is polyhedral at p whenever π is an affine projection of K onto a j-flat through p.*

Proof. For each π as described, it is evident that cone $(p, \pi K) = \pi$ cone (p, K). Then if K is polyhedral at p, cone (p, K) is polyhedral by 3.3, whence of course π cone (p, K) is polyhedral and a second application of 3.3 shows that πK is polyhedral at p. On the other hand, if πK is polyhedral at p for each π as described, then all j-projections of cone (p, K) are polyhedral, whence all 2-projections are polyhedral. It follows by 3.7 that all $(n-1)$-sections of cone (p, K) are polyhedral, whence cone (p, K) is itself polyhedral by 3.1 (vii). We conclude from 3.3 that K is polyhedral at p, and the proof of 4.1 is complete.

4.2. COROLLARY. *With $2 \leqslant j \leqslant n$, a convex subset of E^n is polyhedral at all its points iff all its j-projections have this property.*

4.3. COROLLARY. *With $2 \leqslant j \leqslant n$, a closed convex subset of E^n is boundedly polyhedral iff all its j-projections are polyhedral at each point.*

4.4. COROLLARY. *With $2 \leqslant j \leqslant n$, if all j-projections of a convex subset of E^n are boundedly polyhedral, the set itself must be boundedly polyhedral. Conversely, each closed j-projection of a boundedly polyhedral set in E^n is boundedly polyhedral (but of course there may be j-projections which are not closed).*

4.5. COROLLARY. *With $2 \leqslant j \leqslant n$, a bounded convex subset of E^n is polyhedral iff all its j-projections are polyhedral.*

In § 5 we construct a nonpolyhedral convex set in E^3 all of whose 2-projections are polyhedral. (The set is necessarily unbounded and boundedly polyhedral.) Thus for $j = 2$, the restriction to bounded sets is essential in 4.5, but we shall see below that for $j \geqslant 3$ the restriction can be removed.

Since we know already that all j-projections and j-sections of a polyhedron are polyhedral, and all j-sections of a boundedly polyhedral set are boundedly polyhedral, the remaining results of this section are most simply stated not in terms of a range of values for j, but instead in terms of the "best" value for j.

We need the following

4.6. REMARK. *Suppose K is a closed convex set in E with $\phi \in K$, L is a subspace of E, and π is a linear projection of E' whose kernel is L^0. Then $(L \cap K)^0 = L^0 + \operatorname{cl} \pi K^0 = \pi^{-1}$ $(\operatorname{cl} \pi K^0)$.*

Proof. Since L^0 is a subspace of E' and πK^0 lies in the supplementary subspace $\pi E'$, it is easy to check that

$$\begin{aligned}(L \cap K)^0 &= \operatorname{cl} \operatorname{conv} (L^0 \cup K^0) = \operatorname{cl} (L^0 + K^0) \\ &= \operatorname{cl} (L^0 + \pi K^0) = L^0 + \operatorname{cl} \pi K^0 = \pi^{-1} (\operatorname{cl} \pi K^0).\end{aligned}$$

4.7. THEOREM. *If K is convex and $p \in \operatorname{int} K$, then K is polyhedral [resp. boundedly polyhedral] iff all its 2-sections through p are polyhedral [resp. boundedly polyhedral].*

Proof. Obviously K is closed and we may assume that $p = \phi$, whence $K = K^{00}$ and K^0 is bounded. Now consider a linear projection π of E' onto one of its 2-subspaces and let M be the kernel of π, $L = M^0$. Then $L^0 = M$ and we see from 4.6 that $(L \cap K)^0 = M + \operatorname{cl} \pi K^0$. But K^0 is compact, so πK^0 is closed, and thus if $L \cap K$ is polyhedral it follows that $M + \pi K^0$ is polyhedral, whence πK^0 is itself polyhedral. Thus if all of K's 2-sections through $\phi \in \operatorname{int} K$ are polyhedral, all 2-projections of K^0 are polyhedral. From 4.5 it follows that K^0 is a polyhedron, and then by 2.11 that K is one. Thus 4.7 has been proved for polyhedra, and it remains only to consider the case of boundedly polyhedral sets.

Suppose all 2-sections of K through $p \in \operatorname{int} K$ are boundedly polyhedral, and consider an arbitrary bounded polyhedron B. There is a bounded polyhedron $Q \supset B$ such that $p \in \operatorname{int} Q$; of course all 2-sections of $Q \cap K$ through p are polyhedral and hence $Q \cap K$ is polyhedral by the result just established. But then $B \cap K$ must be polyhedral, so K is boundedly polyhedral and the proof of 4.7 is complete.

4.8. COROLLARY. *If K is convex and $q \in E$, then K is polyhedral [resp. boundedly polyhedral] iff all its sections by 3-flats through q are polyhedral [resp. boundedly polyhedral].*

4.9. THEOREM. *A convex set $K \subset E^n$ (with $n \geqslant 3$) has polyhedral closure iff all its 3-projections have polyhedral closure.*

Proof. Assume $\phi \in K$. Then if all 3-projections of K have polyhedral closure, it follows by 4.6 that all 3-sections of K^0 through ϕ' are polyhedral, whence K^0 is polyhedral by 4.8 and the set $\operatorname{cl} K = K^{00}$ is polyhedral by 2.11.

4.10. COROLLARY. *A convex set $K \subset E^n$ (with $n \geqslant 3$) is polyhedral iff all its 3-projections are polyhedral.*

Proof. Use 4.9 and 3.6.

4.11. THEOREM. With $2 \leqslant j \leqslant n-1$, *a convex cone in E^n is polyhedral iff all its j-projections are closed.*

Proof. Let C be the cone in question, q a vertex of C. If all j-projections of C are closed, then all $(n-j+1)$-sections of C are polyhedral by 3.7, whence all 2-sections of C are polyhedral. Each 3-section of C through q must then be polyhedral by 3.1 (vii), and from 4.8 we conclude that C is polyhedral.

We wish to deduce from 4.11 a result on the extension of positive linear functionals. The connection between projections and extensions may be stated as follows:

4.12. Remark. *Suppose C is a closed convex cone in E with vertex ϕ, L is a subspace of E, and π is a linear projection of E' whose kernel is L^0. Then πC^0 is closed iff every linear functional on L which is $\geqslant 0$ on $L \cap C$ can be extended to a linear functional on E which is $\geqslant 0$ on C.*

Proof. For each $f \in E'$, let ξf be the restriction of f to L, so that ξ is a linear map of E' onto L'. The extendability condition above is equivalent to the condition that $\xi C^0 = \xi(L \cap C)^0$, or, since L^0 is the kernel of ξ, that $L^0 + C^0 = L^0 + (L \cap C)^0$. But $L^0 + C^0 = L^0 + \pi C^0$, and $(L \cap C)^0 = L^0 + \text{cl}\, \pi C^0$ by 4.6. Thus the desired conclusion follows.

The following consequence of 4.11 and 4.12 was proved by Mirkil [7] for $j = 2$:

4.13. Theorem. *Suppose C is a closed convex cone with vertex ϕ in E^n, and let us say that C has the property P_k (for $0 \leqslant k \leqslant n$) iff every linear functional on a k-subspace L of E^n which is $\geqslant 0$ on $L \cap C$ can be extended to a linear functional on E^n which is $\geqslant 0$ on C. Then C must have the properties P_0, P_1, and P_n; but for $2 \leqslant j \leqslant n-1$, C has property P_j iff C is polyhedral.*

5. Some examples and further results

This section contains a rather discursive treatment of material which was suggested by the results and methods of earlier sections, but was not essential in dealing with the principal theorems in § 4. We discuss primarily the sets which are boundedly polyhedral and those which are polyhedral away from ϕ (a notion defined below), characterizing these in various ways and describing their polar sets. We construct in E^3 a nonpolyhedral set $K \ni \phi$ such that both K and K^0 are boundedly polyhedral, and also a nonpolyhedral convex set all of whose 2-projections are polyhedral.

In analogy with earlier definitions, a convex set K is said to be *polyhedral at* ∞ iff each polyhedron (or, equivalently, each closed halfspace) in E has a translate whose intersection with K is polyhedral, and K is said to be *polyhedral away from* the point $p \in \text{cl}\, K$ iff K has polyhedral intersection with each polyhedron in $E \sim \{p\}$. The following two results should be compared with 2.17.

5.1. Proposition. *A convex set K is polyhedral iff it is closed and is polyhedral at each point of $K \cup \{\infty\}$.*

Proof. Let F be a basis for the conjugate space E'. Then if K is polyhedral at ∞, there exists $t > 0$ such that each of the sets $P_f = f^{-1}]-\infty, -t] \cap K$ and $Q_f = f^{-1}[t, \infty[\cap K$ is polyhedral. And if K is closed and is polyhedral at each point of K, then K is boundedly polyhedral by 2.17 and hence its intersection with the set $\bigcap_{f\in F} f^{-1}[-t, t]$ is a polyhedron B. But of course $K = \mathrm{cl\ conv}\,(B \cup (\bigcup_{f\in F} P_f) \cup (\bigcup_{f\in F} Q_f))$, and it follows by 2.16 that K is polyhedral.

5.2. COROLLARY. *If K is convex and $p \in \mathrm{cl}\, K$, then K is polyhedral away from p iff* $\mathrm{cl}\, K \subset K \cup \{p\}$ *and K is polyhedral at each point of* $(K \cup \{\infty\}) \sim \{p\}$.

Proof. The "only if" part follows at once from the fact that each point of $E \sim \{p\}$ has a polyhedral neighborhood in E which misses p. For the "if" part, consider a polyhedron J in $E \sim \{p\}$. The hypotheses imply that $J \cap K$ is closed and is polyhedral at each point of $(J \cap K) \cup \{\infty\}$, whence application of 5.1 completes the proof.

We see from 2.16 that the convex hull of the union of two polyhedra is polyhedral if it is closed. In addition to the obvious result for bounded polyhedra, we note

5.3. PROPOSITION. *If X and Y are polyhedral cones each of which contains a vertex of the other, then* $\mathrm{conv}\,(X \cup Y)$ *is polyhedral.*

Proof. Let p be a vertex of X in Y, q a vertex of Y in X, and let the origin in E be so chosen that $q = -p$. Then there are finite subsets U and V of E such that $-p \in U$, $p \in V$, $X - p$ consists of all non-negative combinations of U, and $Y - q$ consists of all non-negative combinations of V. But then it is easy to verify that $\mathrm{conv}\,(X \cup Y)$ consists of all non-negative combinations of $U \cup V$.

The following result has some useful corollaries.

5.4. PROPOSITION. *If x and z are points of a convex set K and $y \in]x,z[$, then* $\mathrm{cone}\,(y, K) = \mathrm{conv}\,[\mathrm{cone}\,(x, K) \cup \mathrm{cone}\,(z, K)]$.

Proof. We may assume that $x \neq z$ and let $J = \mathrm{conv}\,[\mathrm{cone}\,(x, K) \cup \mathrm{cone}\,(z, K)]$. Consider a point $u \in \mathrm{cone}\,(y, K)$, not collinear with $[x, z]$. There exists $v \in]u, y[\cap K$. It is easy to verify that u lies on a segment joining the ray from x through v to the ray from z through v, and we conclude that $\mathrm{cone}\,(y, K) \subset J$.

Now consider a point $w \in]x, y[$, and observe that $[x, v] \subset K$ and $]u, w[$ intersects $[x, v]$, so $u \in \mathrm{cone}\,(w, K)$. Thus $\mathrm{cone}\,(y, K) \subset \mathrm{cone}\,(w, K)$. A similar argument establishes the reverse inclusion, and we conclude that $\mathrm{cone}\,(s, K)$ is constant for $s \in]x, z[$. Thus to prove that $J \subset \mathrm{cone}\,(y, K)$ it suffices to show that if $p \in \mathrm{cone}\,(x, K)$, $q \in \mathrm{cone}\,(z, K)$, and $c = rp + (1-r)q$ for $r \in]0, 1[$, then $c \in \mathrm{cone}\,(s, K)$ for some $s \in]x, z[$. Under these conditions there are positive numbers α and β such that $x + [0, \alpha](p - x) \subset K$ and $z + [0, \beta](q - z) \subset K$. With $m = \min\,(\alpha, \beta)$, $p' = x + m(p - x)$, $q' = z + m(q - z)$, and $s = rx + (1-r)y$, it can be verified that $c = s + m^{-1}[rp' + (1-r)q' - s]$, whence $c \in \mathrm{cone}\,(s, K)$ and the proof of 5.4 is complete.

5.5. Corollary. *If K is convex and Y is the set of all points of K at which K is locally polyhedral, then Y is convex.*

Proof. Use 3.3, 5.4, and 5.3.

5.6. Corollary. *If a nonpolyhedral closed convex set K is polyhedral away from $p \in K$, then p is an extreme point of K.*

Proof. Use 5.2 and 5.5.

5.7. Corollary. *If K is a closed convex set and $K = \text{conv}\, S$, then K is boundedly polyhedral iff K is polyhedral at each point of S.*

Proof. Use 2.17 and 5.5.

In connection with 5.7, we note

5.8. Proposition. *A closed convex set K is boundedly polyhedral iff cone (p, K) is closed for each $p \in K$.*

Proof. If cone (p, K) is closed for each $p \in K$, it follows from 3.5 and 5.7 that each 2-section of K is boundedly polyhedral, whence K must be boundedly polyhedral by 4.7.

Comparing 5.8 with 5.7, it is natural to ask whether a closed convex set K must be boundedly polyhedral if cone (p, K) is closed for each point p of a set S such that conv $S = K$. The answer is negative, as the following example shows. In E^3, let J be a non-polyhedral two-dimensional compact convex set which is polyhedral away from ϕ and let S be a segment which has ϕ as an inner point and is not coplanar with J. Let $K = \text{conv}\, (S \cup J)$. Then cone (p, K) is closed except when p is an inner point of S.

The following is an analogue of 5.5.

5.9. Proposition. *Suppose K is a closed convex subset of E and F is the set of all $f \in E'$ such that $f^{-1}[t, \infty[\cap K$ is polyhedral for some $t < \infty$. Then F is a convex cone with vertex ϕ'.*

Proof. Clearly $[0, \infty[F \subset F$, and it remains to prove that $F + F \subset F$. Consider f, $g \in F$, with $h = f + g$ and $r, s \in R$ such that the sets $P = f^{-1}[r, \infty[\cap K$ and $Q = f^{-1}[s, \infty[\cap K$ are both polyhedral. Then $h^{-1}[r + s, \infty[\subset f^{-1}[r, \infty[\cup g^{-1}[s, \infty[$, and it follows that

$$h^{-1}[r + s, \infty[\cap K = h^{-1}[r + s, \infty[\cap \text{cl conv}\, (P \cup Q).$$

But cl conv $(P \cup Q)$ is polyhedral by 2.16 and the desired conclusion follows.

We wish next to describe the polars of boundedly polyhedral sets. In doing this we employ the following proposition, which goes a bit beyond our immediate use for it.

5.10. Proposition. *Suppose K is a closed convex set in E, $p \in K$, K contains no line, $C = \{x : [0, \infty[x \subset K - p\}$, and F is the set of all linear functionals f on E such that $f > 0$*

on $C \sim \{\phi\}$. *Then* F *is a nonempty open convex cone in* E' *with vertex* ϕ'. *A closed halfspace in* E *has the form* $f^{-1}]-\infty, r]$ *for some* $f \in F$ *and* $r \in R$ *iff each of its translates has bounded intersection with* K. *If* $C \neq K - p$ *there exists* $f \in F$ *such that* $f^{-1}]-\infty, fp] \cap K$ *has nonempty interior relative to the smallest flat containing* K.

Proof. Clearly C^0 is a closed convex cone with vertex ϕ', and since C contains no line C^0 is not contained in any hyperplane in E. But then C^0 must have interior points, and it is not hard to verify that $F = -\operatorname{int} C^0$, proving the first assertion of 5.10. Now if a halfspace in E has the form $f^{-1}]-\infty, r]$, then each of its translates has the form $f^{-1}]-\infty, s]$, and for $f \in F$ it is evident that the convex set $f^{-1}]-\infty, s] \cap K$ contains no ray and hence is bounded. On the other hand, if $f \in E' \sim F$ then $f^{-1}]-\infty, fp] \cap K$ is easily seen to be unbounded. It remains to prove the last assertion of 5.10.

If $C \neq K - p$ there must be a point $y \in \operatorname{relint} K \sim \{p\}$ such that K does not contain the entire ray from p through y. We may assume without loss of generality that $y = \phi$, whence $C = \{x\colon [0, \infty[x \subset K\}$. The choice of y assures that $p \notin -C$, so either $p \in C$ or $Rp \cap C = \{\phi\}$. By a separation theorem for convex cones there exists $f \in F$ such that $p \in C$ or $fp = 0$, and then from the fact that $y \in \operatorname{relint} K$ it follows readily that the interior of $f^{-1}]-\infty, fp] \cap K$ relative to $\operatorname{fl} K$ is nonempty.

5.11. PROPOSITION. *For a closed convex set* $K \subset E$ *with* $\phi \in K$, *the following assertions are equivalent:*

(i) K *is boundedly polyhedral;*

(ii) *if* f *is a linear functional on* E *such that* $fx > 0$ *whenever* $[0, \infty[x \subset K$ *but* $]-\infty, 0]x \not\subset K$ *then* $f^{-1}]-\infty, s] \cap K$ *is polyhedral for each* $s \in R$;

(iii) *there exists a linear functional* f *on* E *such that* $f^{-1}]-\infty, s] \cap K$ *is polyhedral for each* $s \in R$;

(iv) $\operatorname{cl}\operatorname{conv}(K^0 \cup N)$ *is polyhedral for each polyhedral neighborhood* N *of* ϕ' *in* E';

(v) $\operatorname{cl}\operatorname{conv}(K^0 \cup [\phi', -g])$ *is polyhedral for each* $g \in \operatorname{relint} K^0$;

(vi) *there exists* $f \in E'$ *such that* $\operatorname{cl}\operatorname{conv}(K^0 \cup [\phi', tf])$ *is polyhedral for each* $t > 0$.

Proof. It is evident that conditions (i) and (iv) are dual under polarity, and hence equivalent by 2.8 and 2.11, as are (iii) and (vi). Now let L and M be as in 2.7, so that $K = L + (M \cap K)$ and $M \cap K$ contains no line, and let $C = \{x\colon [0, \infty[x \subset M \cap K\}$. It can be verified that f is as described in condition (ii) above iff $f > 0$ on $C \sim \{\phi\}$, so it follows from 5.10 that (i) implies (ii). By further use of 5.10 we see that (ii) implies (iii) and that (ii) and (v) are dual under polarity. Since obviously (iii) implies (i), the proof of 5.11 is complete.

The condition (vi) of 5.11 is much less restrictive than might at first be imagined. Indeed, from 4.6 and our later result 6.2 on the projections of boundedly polyhedral sets, it follows that if $k < n$ and X is an arbitrary k-dimensional closed convex subset of E^n with $\phi \in X$, then there exists in E^n a closed convex set $K \ni \phi$ such that K^0 is boundedly polyhedral and X is a k-section of K. The following example is also of interest in this connection.

5.12. Example. *There is in E^3 a nonpolyhedral set $K \ni \phi$ such that both K and K^0 are boundedly polyhedral.*

Proof. We shall construct a nonpolyhedral set $W \ni (-1, 0, 0)$ such that W is boundedly polyhedral and cl conv $(\{(-1, 0, -t)\} \cup W)$ is polyhedral for each $t > 0$. From 5.11 it follows that the set $K = W + (1, 0, 0)$ has the stated properties.

Let f, g and h be sectionally linear positive convex functions on $[-1, 0[$, $]0, 1]$, and $]0, 1]$ respectively such that $f(-1) = 0 = g1$, $\lim_{s\to 0} fs = \infty = \lim_{t\to 0} gt$, and $\lim_{t\to 0} ht = 0 = \lim_{t\to 0} h^+ t$, where h^+ is the right-hand derivative of h. (Thus f, g, and h are all nonpolygonal but are polygonal away from 0.) Let $S' = \{(s, fs, 0) : s \in [-1, 0[\}$ and $T' = \{(t, gt, ht) : t \in]0, 1]\}$. Let Q be the plane quadrant consisting of all $x = (x^1, x^2, x^3) \in E^3$ for which $x^1 = 0$ and $x^2 \geq 0 \leq x^3$, and let $S = S' + Q$, $T = T' + Q$. It can be verified that both S and T are boundedly polyhedral. For example, if $m < \infty$, then the intersection of T with the half-space $\{x : x^2 \leq m\}$ is the closed convex hull of the union of the (finitely many) sets of the form $\{x : x^2 \leq m\} \cap ((t, gt, hv) + Q)$ for t such that $gt \leq m$ and $t = 1$, $gt = m$, or g or h has a corner at t.

Now let $W = \text{conv}(S \cup T) = S \cup \text{conv}(S' \cup T)$. Then $W \cap \{x : x^3 = 0\}$ is the non-polyhedral set of all points $(s, a, 0)$ for which $s \in [-1, 0[$ and $a \geq fs$, so it follows that W is nonpolyhedral. To show that W is closed, we consider an arbitrary point $w \in \text{cl}\, W$. There exist sequences λ_α in $[0, 1]$, s_α in $[-1, 0[$, t_α in $]0, 1]$, and p_α and q_α in Q such that $(1-\lambda_\alpha)((s_\alpha, fs_\alpha, 0) + p_\alpha) + \lambda_\alpha((t_\alpha, gt_\alpha, ht_\alpha) + q_\alpha) \to w$, $\lambda_\alpha \to \lambda \in [0, 1]$, $s_\alpha \to s \in [0, 1]$, and $t_\alpha \to t \in [0, 1]$. Let $z_i = (1-\lambda_i)p_i + \lambda_i q_i \in Q$. Now if $s < 0 < t$, then $fs_\alpha \to fs$, $gt_\alpha \to gt$, $ht_\alpha \to ht$, and it follows that $z_\alpha \to w - (1-\lambda)(s, fs, 0) - \lambda(t, gt, ht)$. Denoting this limit by z, we have $z \in Q$ and

$$w = (1-\lambda)((s, fs, 0) + z) + \lambda((t, gt, ht) + z) \in W.$$

If $t = 0$, then $gt_\alpha \to \infty$; since always $fs_i > 0$, and $fs_\alpha \to \infty$ if $s = 0$, we conclude that $\lambda = 0$ and $s < 0$. Now with $q_i^1 = (0, gt_i, ht_i) \in Q$, the distance from q_i^1 to the point (t_i, gt_i, ht_i) tends to 0 as $i \to \infty$, and it is easily verified that $z_\alpha + \lambda_\alpha q_\alpha^1 \to w - (s, fs, 0)$, whence $w \in (s, fs, 0) + Q \subset S \subset W$. A similar argument applies when $s = 0$, and we conclude that W is closed.

Now to complete the proof we must show W is boundedly polyhedral and cl conv $(\{(-1, 0, -t)\} \cup W)$ is polyhedral for each $t > 0$. In view of 2.17, 5.5, and the fact that W is closed, it suffices for the former to show that W is polyhedral at each point of $S \cup T$, or equivalently that cone (p, W) is polyhedral for each $p \in S \cup T$. Consider an arbitrary $p \in S$. It can be verified that $\sup \{x^2 : x \in T \sim \text{cone}(p, S)\} < \infty$, whence follows the existence of a bounded polyhedron B such that $T \sim \text{cone}(p, S) \subset B + [0, \infty[z \subset T$, with $z = (0, 0, 1)$. Let $U = \text{cone}(p, S)$ and $V = \{p\} \cup \text{cone}(p, B)$. We have

$$\begin{aligned}\text{cone}(p, W) &= \text{cone}(p, \text{conv}(S \cup T)) = \text{conv}[\text{cone}(p, S) \cup \text{cone}(p, T)]\\ &= \text{conv}[U \cup \text{cone}(p, B + [0, \infty[z)] = \text{conv}[U \cup (V + [0, \infty[z)].\end{aligned}$$

It can be verified that for any convex sets X and Y, and convex cone Z with vertex ϕ,

$$\text{conv}[(X + Z) \cup Y] \cup (Y + Z) = \text{conv}[X \cup (Y + Z)] \cup (X + Z).$$

Applying this with $X = U$, $Y = V$, and $Z = [0, \infty[z$, we conclude that

$$\text{cone}(p, W) \cup (U + [0, \infty[z) = \text{conv}[(U + [0, \infty[z) \cup V] \cup (V + [0, \infty[z).$$

Since evidently $U + [0, \infty[z = U \subset \text{cone}(p, W)$,
this reduces to

$$\text{cone}(p, W) = \text{conv}(U \cup V) \cup (V + [0, \infty[z).$$

Now V and $V + [0, \infty[z$ are polyhedral by 2.13, 2.14, and 2.16, and U is polyhedral because $p \in S$ and S is boundedly polyhedral. But then conv $(U \cup V)$ is polyhedral, for U and V are polyhedral cones with common vertex p. Thus cone (p, W) is the union of two polyhedra and, being convex, must be polyhedral. It follows that W is polyhedral at each point of S. The same argument with S and T interchanged shows that W is polyhedral at each point of T, and we conclude that W is boundedly polyhedral.

Now consider an arbitrary $t > 0$ and let $u_t = (-1, 0, -t)$. It can be verified that if r is a sufficiently small positive number (depending on t) and $J_r = \text{cl conv}(\{u_t\} \cup \{x : x \in T, x^1 = r\})$, then $\inf \{|x^1| : x \in (S \cup T) \sim J_r\} > 0$, whence there are polyhedra M and N such that $S \sim J_r \subset M$ and $T \sim J_r \subset N$. We then have

$$\text{cl conv}(\{u_t\} \cup W) = \text{cl conv}(\{u_t\} \cup S \cup T) = \text{cl conv}(\{u_t\} \cup J_r \cup A \cup B),$$

which is of course polyhedral and the proof of 5.12 is complete.

5.13. PROPOSITION. *For a closed convex set $K \subset E$ with $\phi \in K$, the assertions* (ii)–(iv) *below are equivalent, are implied by* (i), *and imply* (i) *when K is bounded.*

(i) *K is polyhedral away from ϕ;*

(ii) cl conv $(K \cup [\phi, -x])$ *is polyhedral for each $x \in K \sim \{\phi\}$;*

(iii) *for each $y \in E \sim \{\phi\}$,* $\sup yK^0 = \infty$ *or $y^{-1}[t, \infty[\cap K^0$ is polyhedral for all $t \in R$;*

(iv) *if A and B are "opposite" closed halfspaces in E having the same bounding hyperplane and K^0 lies in some translate of A, then $B \cap K^0$ is polyhedral.*

Proof. It is evident that (iii) and (iv) are equivalent and (ii) is dual to (iii). Now if K is bounded and (ii) holds, then for each $x \in K \sim \{\phi\}$ we have $\operatorname{conv}(K \cup [\phi, -x]) = \operatorname{cl}\operatorname{conv}(K \cup [\phi, -x])$, which is polyhedral by hypothesis, and

$$\operatorname{cone}(x, K) = \operatorname{cone}(x, \operatorname{conv}(K \cup [\phi, -x]),$$

whence $\operatorname{cone}(x, K)$ is polyhedral. But then K is polyhedral at each point of $K \sim \{\phi\}$, and with K bounded it follows by 5.2 that K is polyhedral away from ϕ.

Now suppose (i) holds and $x \in K \sim \{\phi\}$. We wish to show that $\operatorname{cl}\operatorname{conv}(K \cup [\phi, -x])$ is polyhedral, and for this it suffices in view of 2.16 to show that $\operatorname{cl}\operatorname{conv}(K \cup [\phi, -y])$ is polyhedral with $y = \frac{1}{2}x$. Let N be a bounded polyhedral neighborhood of y relative to K and let $M = \operatorname{conv}(\{-y\} \cup N)$. Then of course M is polyhedral and $\phi \in M$. Consider an arbitrary point $p \in K$. Since $[y, p] \subset K$ and N is a neighborhood of y relative to K, there exists $q \in]y, p[$ such that $[y, q] \subset N$. But then M must contain the segment $[\phi, z]$ for some $z \in]\phi, p[$. Since M is polyhedral we conclude from 3.2 that $M \cap K$ is a neighborhood of ϕ relative to K.

Now let $W = \operatorname{cl}\operatorname{conv}(K \sim M)$. Then if $\phi \in W$ we have $\phi \in \operatorname{ex} W$ (for $\phi \in \operatorname{ex} K$ by 5.6) and hence $\phi \in \operatorname{cl}(K \sim M)$ by the result (3.5) of [6]. Since this is impossible, it follows that $\phi \notin W$ and there must be a closed halfspace J with $W \subset J \subset E \sim \{\phi\}$. The set $J \cap K$ is polyhedral since K is polyhedral away from ϕ, and we have

$$\operatorname{cl}\operatorname{conv}(K \cup [\phi, -y]) = \operatorname{cl}\operatorname{conv}((J \cap K) \cup M \cup [\phi, -y]),$$

which is polyhedral by 2.16. Thus (i) implies (ii).

I do not know whether conditions 5.13 (i) and 5.13 (ii) are equivalent even when K is unbounded.

5.14. COROLLARY. *If K is polyhedral away from ϕ, K^0 is boundedly polyhedral.*

Proof. Compare conditions 5.11 (v) and 5.13 (ii).

5.15. COROLLARY. *The set $K \ni \phi$ is polyhedral iff K and K^0 are both polyhedral away from ϕ.*

We turn now to another example, promised earlier in connection with 4.5 and 5.10.

5.16. EXAMPLE. *There is in E^3 a nonpolyhedral convex set all of whose 2-projections are polyhedral.* (*Such a set must be unbounded and boundedly polyhedral.*)

Proof. Let $E^2 = \{x \in E^3 : x^3 = 0\}$, and let P be a nonpolyhedral convex subset of E^2 such that ϕ is interior to P in the relative topology of E^2 and P satisfies the condition 5.13 (iv) relative to E^2. (For example, P may be taken as a boundedly polyhedral set inscribed in a parabola.) By 5.13, the polar Q of P in $E^{2\prime}$ is a bounded nonpolyhedral set which is polyhedral away from ϕ'. Let $z = (0, 0, 1) \in E^3$. For notational convenience we shall employ the usual identification of E^3 with its conjugate space $E^{3\prime}$, implying $E^{2\prime} = E^2$, $(E^2)^0 = Rz$, $P^0 = Q + Rz$, etc., where the symbol ^{0} will denote the polarity between E^3 and $E^{3\prime}$.

Now let $K = \text{cone}\,(z, P) \cap \{x : x^3 \leqslant 0\}$, or equivalently, $K = \{(\lambda p^1, \lambda p^2, 1 - \lambda) : \lambda \geqslant 1, p \in P\}$. Then of course K is convex and it is easy to check that K is closed and non-polyhedral. A function $f = (f^1, f^2, f^3) \in E^3$ is a member of K^0 iff $\lambda(f^1p^1 + f^2p^2) \leqslant 1 + (\lambda - 1)f^3$ for all $\lambda \geqslant 1$, $p \in P$, or equivalently iff $f^1p^1 + f^2p^2 \leqslant \inf\,(1, f^3)$ for all $p \in P$. It follows that $K^0 = [0, 1](Q + z) \cup (Q + z) \cup (Q + [1, \infty[z)$. (Compare this with the example immediately following 5.8.) But then from the fact that Q is polyhedral away from ϕ' we conclude by an easy argument that all 2-sections of K^0 through ϕ' are polyhedral, whence from 4.6 it follows that all 2-projections of K have polyhedral closure. To complete the proof we shall show that all 2-projections of K are closed, or equivalently that $K + L$ is closed for each line L in E.

If L is a line in E^2 through ϕ, then $K + L = \bigcup_{\lambda \geqslant 1} \lambda(P + L) + (1 - \lambda)z$. That $P + L$ is closed follows from the fact that P satisfies condition 5.13 (iv), and thus $K + L$ must be closed also. This handles the case of all lines in E^2 or parallel to E^2. To deal with the remaining lines, it suffices to prove the following: *If X is a boundedly polyhedral subset of a hyperplane H in $E \sim \{\phi\}$ and $u \in H$, then* $[1, \infty[X + Ru$ *is closed.* To prove this, let $f \in E'$ be such that $H = f^{-1}1$ and consider a sequence $t_\alpha x_\alpha + r_\alpha u$ converging to a point $q \in E$, where t_α is a sequence in $[1, \infty[$ with $t_\alpha \to t \in [1, \infty]$, x_α is a sequence in X, and r_α is a sequence in R with $r_\alpha \to r \in [-\infty, \infty]$. Note that $t = fq - r$. Now if $t < \infty$, then $r \in R$ and $x_\alpha \to t^{-1}(q - ru) = x \in X$, whence $q = tx + ru$. If $t = \infty$, then $x_\alpha + r_\alpha t_\alpha^{-1} u \to \phi$, and since $f(x_i + r_i t_i^{-1} u) = 1 + r_i t_i^{-1}$ it follows that $r_\alpha t_\alpha^{-1} \to -1$ whence $x_\alpha \to u$ and $u \in X$. But X is boundedly polyhedral, hence polyhedral at u, so u admits a bounded polyhedral neighborhood N relative to X. For all sufficiently large i we have

$$t_i x_i + r_i u \in [1, \infty[N + Ru \subset [1, \infty[X + Ru;$$

since $[1, \infty[N + Ru$ is polyhedral by 2.16, and hence closed, the desired conclusion follows and the proof of 5.16 is complete.

6. Projections and approximations

Before characterizing the projections of boundedly polyhedral sets, we perform the same task for closed convex sets.

6.1. THEOREM. *For a subset X of a proper subspace of E, the following assertions are equivalent:*

(i) *X is a convex F_σ set;*

(ii) *X is a projection of some closed convex set;*

(iii) *X is a projection of some convex F_σ set.*

Proof. Obviously (ii) implies (iii). Now suppose (iii) holds, so there are an increasing sequence U_α of closed subsets of E and a linear projection π of E into E such that $\bigcup_1^\infty U_i$ is convex and $\pi(\bigcup_1^\infty U_i) = X$. For each i, let $W_i = \operatorname{conv}\{x: x \in U_i, \|x\| \leqslant i\}$. It can be verified that each W_i is compact (being the convex hull of a compact set) and $X = \pi \bigcup_1^\infty W_i = \bigcup_1^\infty \pi W_i$. But π is linear and continuous, so πW_α is an increasing sequence of compact convex sets and it follows that X is a convex F_σ set. Thus (iii) implies (i).

Now suppose (i) holds and let Y_α be an increasing sequence of compact convex sets such that $X = \bigcup_1^\infty Y_i$ and $\|y\| \leqslant i$ for all $y \in Y_i$. Let L be a hyperplane through ϕ which contains X and z a unit vector orthogonal to L, so each point $w \in E$ has a unique expression in the form $w = w' + w''z$ with $w' \in L$ and $w'' \in R$. Let $K = \operatorname{conv} \bigcup_1^\infty (Y_i + i^2 z)$, so X is the image of K under the projection of E onto L whose kernel is Rz (i.e., which sends w to w'). Note that since Y_α is an increasing sequence, $Y_i + [i^2, \infty[z \subset K$ for each i, whence $K + [0, \infty[z \subset K$. To complete the proof of 6.1, we shall show that K is closed.

Consider a sequence p_α in K converging to a point $p \in E$, and let d be the dimension of E. For each $j \in I$ (the set of positive integers) there are $(d+1)$-tuples $\{t_j^i\}_{i=1}^{d+1}$, $\{n_j^i\}_{i=1}^{d+1}$, and $\{y_j^i\}_{i=1}^{d+1}$ such that the t_j^i's are in $]0, 1]$ with $\sum_{i=1}^{d+1} t_j^i = 1$, the n_j^i's are in I, always $y_j^i \in Y_{n_j^i}$ and $p_j = \sum_{i=1}^{d+1} t_j^i(x_j^i + (n_j^i)^2 z)$. We may assume that each sequence $(n_j^i)_{j\in I}$ (for $1 \leqslant i \leqslant d+1$) is either constant or strictly increasing. (If necessary, select the appropriate subsequences and change the notation to achieve this condition.) Note that $(p_j'')_{j\in I}$ is bounded and always $p_j'' \geqslant \min\{n_j^i\}_{i=1}^{d+1}$, so at least one of the sequences $(n_j^i)_{j\in I}$ must be constant. Let m be the largest constant value attained and $J = \operatorname{conv} \bigcup_1^m (Y_i + i^2 z)$. We shall prove that $p \in J + [0, \infty[z$, whence $p \in K$ and K must be closed.

With the situation as described, it is easy to obtain a subsequence q_α of p_α, a convergent sequence u_α of points of J (say $u_\alpha \to u$), a sequence λ_α in $[0, 1]$ with $\lambda_\alpha \to 0$, and a sequence v_α such that $v_j \in K_j = \operatorname{conv} \bigcup_j^\infty (Y_i + i^2 z)$ and $q_j = (1-\lambda_j)u_j + \lambda_j v_j$ for each $j \in I$. Then of course $\lambda_\alpha v \to p - u$. Now $(j\|w'\| - w'')\,|\, w \in E$ is a convex function on E which is $\leqslant 0$ on $Y_i + i^2 z$ for all $i \geqslant j$, and hence $\leqslant 0$ on K_j. Thus for all $w \in K_j$ we have

$$w''/\|w\| = [(\|w'\|/w'')^2 + 1]^{-\frac{1}{2}} \geqslant (j^2+1)^{-\frac{1}{2}}$$

and hence

$$\|(v_j/\|v_j\|) - z\|^2 = 2 - 2v_j''/\|v_j\| \leqslant 2[1 - (j^2+1)^{-\frac{1}{2}}].$$

It follows that $v_\alpha/\|v_\alpha\| \to z$ and hence that $p - u \in [0, \infty[z$, completing the proof of 6.1.

6.2. Theorem. *For a subset X of a proper subspace of E, the following assertions are equivalent:*

(i) *X is a convex set which is polyhedral at each of its points;*

(ii) *X is a projection of some boundedly polyhedral set;*

(iii) *X is a projection of some convex set which is polyhedral at each of its points;*

Proof. Obviously (ii) implies (iii), and that (iii) implies (i) follows from 4.2. Now suppose (i) holds, so for each $p \in X$ there is a bounded polyhedron N_p which is a neighborhood of p relative to X. Since X has the Lindelöf property, there is a sequence q_α in X with $X = \bigcup_1^\infty N_{q_i}$. We wish to produce a sequence Y_α of bounded polyhedra in X such that for each $j \in I$, $Y_j \supset N_{q_j}$ and Y_j is interior to Y_{j+1} in the relative topology of X. Start with $Y_1 = N_{q_1}$. And having obtained Y_i as required for all $i < k$, observe that by compactness of Y_{k-1} there must be a finite set $G \subset Y_{k-1}$ such that Y_{k-1} is interior to $\bigcup_{p \in G} N_p$ relative to X; then let $Y_k = \operatorname{conv}(N_{q_k} \cup \bigcup_{p \in G} N_p)$. Proceeding by mathematical induction, we obtain a sequence Y_α of polyhedra in X such that $X = \bigcup_1^\infty Y_i$; there exist sequences B_α and ε_α in $]0, \infty[$ such that for all $j \in I$, $Y_j \subset S(\{\phi\}, B_j)$ and $S(Y_j, \varepsilon_j) \cap X \subset Y_{j+1}$. It is easy to produce an increasing sequence r_α in $]0, \infty[$ such that $r_\alpha \to \infty$ and $(B_k + B_i)(r_j - r_i) < \varepsilon_i(r_k - r_i)$ whenever $i < j < k$. Let L and z be as in the proof of 6.1 and $K = \operatorname{conv} \bigcup_1^\infty V_i$ with $V_i = Y_i + r_i z$. It is evident that X is a projection of K and to complete the proof of 6.2 it suffices to show that K is boundedly polyhedral.

Suppose $i < j < k$, $p \in V_i$, and $q \in V_k$. Then of course $p' \in Y_i$ and $\|p' - q'\| < B_k + B_i$. With

$$u = \frac{r_k - r_j}{r_k - r_i} p + \frac{r_j - r_i}{r_k - r_i} q,$$

we have $u' \in X$, $u'' = r_j$, and

$$\|u' - p'\| = (r_j - r_i)(r_k - r_i)^{-1}\|p' - q'\| < \varepsilon_i.$$

Since $S(Y_i, \varepsilon_i) \cap X \subset Y_{i+1} \subset Y_j$, it follows that $u' \in Y_j$ and $u \in V_j$. Thus each segment from V_i to V_k intersects V_j. Now $\operatorname{conv}(V_i \cup V_j \cup V_k)$ is the union of all segments $[p, q]$ such that $q \in V_k$ and $p \in [v, w]$ for some $v \in V_i$ and $w \in V_j$. For such p, q, v, and w, the segment

$[q, v]$ must intersect V_j at some point s and it is evident that $[p, q] \subset \operatorname{conv}\{w, s, q\} \cup \operatorname{conv}\{w, s, v\}$. Thus

$$\operatorname{conv}(V_i \cup V_j \cup V_k) = \operatorname{conv}(V_i \cup V_j) \cup \operatorname{conv}(V_j \cup V_k).$$

A straightforward application of this fact shows that $K = \bigcup_1^\infty \operatorname{conv}(V_i \cup V_{i+1})$, whence it is clear that K is boundedly polyhedral and the proof of 6.2 is complete.

Note that if N_p is a polyhedral neighborhood of $p \in X$ relative to the convex set X, then $\operatorname{cone}(p, X) = \operatorname{cone}(p, N_p) = \operatorname{cone}(p, \operatorname{cl} X)$, where the last equality follows from the fact that $\operatorname{cone}(p, N_p)$ is polyhedral and hence closed. It then follows from 3.2 that N_p is a neighborhood of p relative to $\operatorname{cl} X$. Thus if a convex set is polyhedral at all its points, it must be relatively open in its closure.

6.3. Theorem. *Suppose K is a closed convex subset of E, K contains no line, and μ is a continuous function on K to $]0, \infty[$. Then there is a boundedly polyhedral set P such that $P \subset K \subset \bigcup_{x \in P} S(x, \mu x)$.*

Proof. We know by 5.10 that E admits a linear functional f such that $f^{-1}]-\infty, r] \cap K$ is bounded for each $r \in R$. Let $m = \inf fK > 0$ and for each $r \geqslant m$ let $K_r = f^{-1}r \cap K$ and $d_r = \inf \mu K_r > 0$. From the fact that each set $f^{-1}[m, r] \cap K$ is compact it is easy to establish the existence of an increasing sequence $r\alpha$ in $[m, \infty[$ such that $K \subset \bigcup_1^\infty S(K_{ri}, \frac{1}{2} d_{ri})$. Let F_1 be a finite set in relint K_{r1} such that $K_{r1} \subset S(F_1, \frac{1}{2} d_{r1})$ and let $B_1 = \operatorname{conv} F_1$, so B_1 is a polyhedron, $B_1 \subset \operatorname{relint} K_{r1}$, and $K_{r1} \subset S(B_1, \frac{1}{2} d_{r1})$. With $J_{13} = \operatorname{conv}(B_1 \cup K_{r3})$, observe that J_{13} is compact and $J_{13} \sim K_{r3} \subset \operatorname{relint} K$, whence $J_{13} \cap K_{r2} \subset \operatorname{relint} K_{r2}$. It is thus possible to produce a polyhedron B_2 such that

$$J_{13} \cap K_{r2} \subset B_2 \subset \operatorname{relint} K_{r2} \subset K_{r2} \subset S(B_2, \tfrac{1}{2} d_{r2}).$$

Proceeding in this manner, we obtain a sequence B_α of polyhedra such that for each $n \geqslant 2$,

$$(^*)\ \operatorname{conv}(B_{n-1} \cup K_{r(n+1)}) \cap K_{rn} \subset B_n \subset \operatorname{relint} K_{rn} \subset S(B_n, \tfrac{1}{2} d_{rn}).$$

Let $P = \operatorname{conv} \bigcup_1^\infty B_i$. Then certainly $P \subset K$, and for each n it is true that

$$S(K_n, \tfrac{1}{2} d_{rn}) \subset S(B_n, d_{rn}) \subset \bigcup_{x \in B_n} S(x, \mu x),$$

whence $K \subset \bigcup_{x \in K} S(x, \mu x)$. Now for $k \geqslant i + 2$ it is evident from convexity of K that each segment from B_k to B_i intersects K_{i+2}, and hence by (*) above intersects B_{i+1}. From this it follows that whenever $i < j < k$, every segment from B_k to B_i intersects B_j, and we conclude P is boundedly polyhedral. This completes the proof of 6.3.

6.4. COROLLARY. *If K is a closed convex subset of E and $\varepsilon > 0$, there are boundedly polyhedral sets P and Q such that $P \subset K \subset S(P, \varepsilon)$ and $K \subset Q \subset S(K, \varepsilon)$.*

Proof. In view of 2.7, it suffices to consider the case in which K contains no line, and here the existence of P as stated in 6.4 is an immediate consequence of 6.3. Application of this result to $\text{cl}\, S(K, \varepsilon)$ produces a boundedly polyhedral set Q such that $Q \subset \text{cl}\, S(K, \varepsilon) \subset S(Q, \frac{1}{2}\varepsilon)$, and it is easy to verify that $K \subset Q$, so 6.4 is proved.

If $\varepsilon > 0$ and K is a bounded convex set in E with boundary J, then K can be ε-approximated in the sense of 6.4 by polyhedra of the form conv Y for $Y \subset J$, and by polyhedra which are intersections of supporting halfspaces of J. But if K is unbounded, boundedly polyhedral approximations of these special types may not exist, even when K contains no line. For example, neither type of boundedly polyhedral approximation is available for a circular cone. Though we have not done so, it might be of interest to study this situation, seeking to characterize those convex sets which admit boundedly polyhedral approximations of these special types, and searching for a weaker type of "uniform" approximation (for example, in terms of uniform structures for E other than the usual metric uniformity) under which all convex sets admit such approximations.

The result 6.4 extends the classical theorem on approximation of compact convex sets by means of polyhedra. In preparation for another such extension, we state the following

6.5. PROPOSITION. *Suppose X and Y are closed convex subsets of E, and Y is bounded. Then for each extreme point z of $X + Y$ there are unique points x_z of X and y_z of Y such that $z = x_z + y_z$; further, $\{x_z : z \in \text{ex}\,(X + Y)\} = \text{ex}\, X$ and $\{y_z : z \in \text{ex}\,(X + Y)\} \subset \text{ex}\, Y$. A similar relationship holds among the extreme rays of $X + Y$, the extreme rays of X, and the extreme points of Y.*

Proof. We consider only the case of extreme points, for the other is similar. The only assertions which may not be quite obvious are the uniqueness of x_z and y_z, and the fact that each point of ex X appears as x_z for some $z \in \text{ex}\,(X + Y)$. For the uniqueness, suppose we have $x + y = u + v = z$ with $x, u \in X$ and $y, v \in Y$. Then $z = \frac{1}{2}(x + v) + \frac{1}{2}(u + y)$, so if $z \in \text{ex}\,(X + Y)$ we have $x + v = u + y$, whence $x = u$ and $y = v$.

The assertion about ex X will be proved by induction on the dimension d of E, being obvious when $d = 1$. Suppose it is known for $d = k - 1$, and consider the case of a k-dimensional E. For $x \in \text{ex}\, X$, let Q be a closed halfspace with bounding hyperplane H such that

$\phi \in H$ and $X \subset Q + x$. Since Y is compact, there exists $y \in Y$ for which $Y \subset Q + y$. It can be verified that

$$(H + x + y) \cap (X + Y) = (H + x) \cap X + (H + y) \cap Y;$$

in conjunction with the inductive hypothesis and the fact that x is an extreme point of the set $(H + x) \cap X$, this yields the desired conclusion.

6.6. Corollary. *Suppose X is a closed convex subset of E and Y is a bounded polyhedron in E. Then if $X + Y$ is polyhedral, so is X.*

Proof. If X contains no line, then neither does $X + Y$, and each set is the convex hull of its extreme points together with its extreme rays. If $X + Y$ is polyhedral, it has only finitely many of these and from 6.5 it follows that the same is true of X, whence X is polyhedral. If X contains a line, then use 2.7 in conjunction with the case just discussed.

Now the *Hausdorff distance* $h(X, Y)$ between two sets X and Y in E is defined as the greatest lower bound of numbers d such that $X \subset S(Y, d)$ and $Y \subset S(X, d)$. This may of course be infinite when the sets are unbounded. Our approximation theorem is as follows:

6.7. Theorem. *If a closed convex subset Q of E is a finite Hausdorff distance d from some polyhedron P, then it is uniformly approximable by means of polyhedra. (I.e., for each $\varepsilon > 0$ there is a polyhedron P_ε with $h(P_\varepsilon, Q) \leqslant \varepsilon$.)*

Proof. We assume without loss of generality that $d < 1$, and deal at first with the case in which P contains the unit cell U of E. For each $f \in E'$, let $\mu f = \sup fP$ and $\nu f = \sup fQ$. Since $U \subset P$, $\mu f \geqslant \|f\|$. From the fact that $h(P, Q) = d$ it follows by an easy application of the separation theorem that whenever $\|f\| = 1$, then $|\nu f - \mu f| \leqslant d$ and thus (†): $\nu f \in [1 - d, 1 + d]\mu f$ whenever $\|f\| = 1$. Let F be the set of all $f \in E'$ such that $\|f\| = 1$ and $\mu f < \infty$ (or, equivalently, $\nu f < \infty$), and for each convex $K \subset E'$ with $\phi' \in K$, let βK denote the set of all $f \in E'$ such that $[0, 1[f \subset K$ and $]1, \infty[f \subset E' \sim K$. Then we have $\beta P^0 = \{(1/\mu f)f : f \in F\}$, $P^0 = [0, 1]\beta P^0$, $\beta Q^0 = \{(1/\nu f) f : f \in F\}$, $Q^0 = [0, 1]\beta Q^0$. Since P^0 is a compact polyhedron, βP^0 must be compact, and with the aid of (†) we see that βQ^0 is also compact. It follows that $\sup \nu F = s < \infty$. Note further that $[0, \infty[\beta Q^0 = [0, \infty[P^0$, and hence is a polyhedral cone. Let $Y = \beta Q^0 \cap \operatorname{rex} [0, \infty[P^0$.

Now let ε be an arbitrary positive number. Since βQ^0 is compact, it must have a finite subset $Z \supset Y$ such that with $M = \operatorname{conv}(Z \cup \{\phi'\})$, then M is a polyhedron in E' with $\beta M \subset [1 - \varepsilon/s, 1]\beta Q^0$. Now let $P = M^0 \subset E$, and for each $f \in E'$ let $\zeta f = \sup fP_\varepsilon \, \varepsilon [1 - \varepsilon/s, 1]\nu f$. Since $s = \sup \nu F$, we see that $|\nu f - \zeta f| \leqslant \varepsilon$ whenever $\|f\| = 1$, and it follows that $h(P_\varepsilon, Q) \leqslant \varepsilon$. Thus the theorem is proved for the special case $U \subset P$.

In treating the general case, we employ the following result due to Rådström [8]: if A and B are closed convex subsets of E and X is a bounded subset of E, then

$h(A, B) = h(A + X, B + X)$. Now consider a closed convex set Q and a polyhedron P with $h(P, Q) = d < 1$. Let X be a bounded polyhedron in E such that $P + X \supset U$. Then $h(P + X, Q + X) = d$, and the result above implies the existence of a polyhedron Y with $h(Y, Q + X) < \varepsilon$. Let $Z = \{y \in Y: y + X \subset Y\}$. Then Z is closed and convex and $Z + X = Y$, so it follows by Rådström's theorem that $h(Z, Q) < \varepsilon$ and by 6.5 that Z is polyhedral. The proof of 6.7 is complete.

(Before I found the above proof, John Isbell supplied me with one which used the projective space in an interesting way.)

7. Failure of the main theorems for nonconvex sets

In this section only, we employ the term *polyhedral* in its more general and customary sense to describe a set in E which is the union of a finite number of geometric simplexes. To show that the theorems of § 4 do not apply to nonconvex sets, we construct in E^3 a nonpolyhedral 3-cell of which all 2-sections and all 2-projections are polyhedral.

Let f be a continuous convex function on $[0, \infty[$ and a_α and b_α sequences in $]0, \infty[$ such that the following conditions are satisfied: $f0 = 0$ and $ft > 0$ for all $t > 0$; for each n, $b_n = fa_n$; $a_\alpha \to 0$, $b_\alpha / a_\alpha \to 0$, $a_\alpha / a_{\alpha+1} \to 1$, and $b_\alpha / b_{\alpha+1} \to 1$. (For example, take $ft = t^2$, $a_n = 1/n$, $b_n = 1/n^2$.) Let c_α and d_α be sequences in $]0, \infty[$ such that $b_\alpha / c_\alpha \to 0$ and always $d_n \in]0, c_n[$. For each n, let u_n denote the point $(a_n, b_n, c_n) \in E^3$ and v_n the point (a_n, b_n, d_n). Let P_n denote the "pyramid" conv $\{\phi, u_n, v_n, u_{n+1}, v_{n+1}\}$ and $T = \bigcup_1^\infty P_n$. Clearly T is not polyhedral, for its projection onto the xy-plane is not. It can be verified that T is a 3-cell (i.e., T is homeomorphic with the unit cell in E^3). We shall prove that all 2-sections of T are polyhedral, and to do this it suffices to show that each plane intersects $P_n \sim \{\phi\}$ for only finitely many values of n. This is evident for planes which miss ϕ, for $u_\alpha \to \phi$ and $v_\alpha \to \phi$. And it is evident for planes containing the z-axis, for the xy-projection of such a plane intersects the xy-projection of at most one set $P_n \sim \{\phi\}$. It remains to consider a plane Π whose equation has the form $z = rx + sy$.

If the plane $\Pi (z = rx + sy)$ intersects $P_n \sim \{\phi\}$, then it intersects at least two of the segments $[u_n, v_n]$, $[u_n, u_{n+1}]$, $[v_n, v_{n+1}]$, and $[u_{n+1}, v_{n+1}]$. If Π intersects $[u_n, v_n]$, there exists $\lambda_n \in [0, 1]$ such that

$$\lambda_n c_n + (1 - \lambda_n) d_n = ra_n + sb_n.$$

Since c_α, d_α, and b_α are all $o(a_\alpha)$, this cannot occur for infinitely many values of n unless $r = 0$. But if $r = 0$, note that c_α and d_α are both $o(b_\alpha)$, so the equality cannot hold for infinitely many values of n unless $s = 0$. Since the xy-plane misses $T \sim \{\phi\}$ entirely, we conclude that $\Pi \cap [u_n, v_n]$ is empty for all but finitely many values of n. The cases of $\Pi \cap [u_n, u_{n+1}]$ and $\Pi \cap (v_n, v_{n+1}]$ can be handled by a similar argument, using the facts about orders of convergence employed above and also the information that $a_{\alpha+1}$ is $O(a_\alpha)$ and and $b_{\alpha+1}$ is $O(b_\alpha)$. It follows that all 2-sections of T are polyhedral.

Now in E^3, let Q be a cube such that the "face" conv $\{\phi, u_1, v_1\}$ of T lies (relatively) interior to some face of Q and the rest of T lies properly interior to Q. Then the set $Q \sim \operatorname{int} T$ is a nonpolyhedral 3-cell of which all 2-sections and all 2-projections are polyhedral.

Though we have not done so, it might be of interest to investigate in detail the interrelations of the following three conditions, as applied to a 3-cell J in E^3: (i) J is polyhedral; (ii) each 2-section of J is polyhedral; (iii) each maximal convex subset of J is polyhedral. We know merely that (i) implies both (ii) and (iii) but is not implied by either of them.

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