TRANSLATION INVARIANT SPACES

To Karl Loewner

BY

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1. Let h denote the space of complex-valued square integrable functions $u(x)$ defined for x real which are zero for x negative. Let H denote the space of functions $U(z)$ which are Fourier transforms of functions in h. The space H is characterized by the one-sided

PALEY–WIENER THEOREM. (2) *Every function U in H can be extended as regular analytic to the upper half-plane, so that*

$$\int_{-\infty}^{\infty} U^*(i\tau+\sigma)\, U(i\tau+\sigma)\, d\sigma \leqslant \text{const},$$

for all τ positive. Conversely, the restriction to the real axis of any such function belongs to H.

For fixed τ, $U(i\tau+\sigma)$ is the Fourier transform of $e^{-x\tau}u(x)$; since $u(x)$ vanishes for negative x, the L_2 norm of $e^{-x\tau}\,u(x)$ decreases with increasing τ. So by Parseval's formula we have this

COROLLARY. *If U lies in H, its L_2 norm along the line* $\operatorname{Im} z=\tau$, $\tau \geqslant 0$, *decreases with increasing τ.*

The orthogonal complement of h with respect to the space of square integrable functions on the entire real axis is the space of square integrable functions which vanish for x positive. The Fourier transforms of these functions form the orthogonal complement $H^{\perp}$ of H. Functions in $H^{\perp}$ can be continued analytically into the lower halfplane. Also, it is easy to show that $H^{\perp}$ is the conjugate of H:

$$H^{\perp}=H^*$$

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(2) We denote the conjugate of a complex number by *; in section 4 where we deal with matrix valued functions the * denotes the adjoint.

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We denote by r any subspace of h which is invariant under right translation. I.e., whenever $g(x)$ belongs to r, we require that $g(x-s)$ should belong to r for all positive s. A subspace will be called left translation invariant and denoted by l if, whenever $g(x)$ belongs to it, the projection of $g(x+s)$ into h (i.e., its restriction to the positive axis) also belongs to l for all positive values of s.

The closure of translation invariant spaces is translation invariant.

The orthogonal complement with respect to h of an r-space is an l-space, and vice versa.

The Fourier transform of an r-space will be denoted by R. Such an R-space can be characterized intrinsically as a subspace of H such that $e^{isz}R$ is contained in R for all positive s.

In this paper we study R-spaces of *vector-valued* functions, i.e., functions whose values lie in a finite-dimensional Hilbert space S over the complex numbers. When we wish to make a distinction, we shall denote the H-space of functions with values lying in S by H_S. Our main result is a unique representation for such spaces:

REPRESENTATION THEOREM. *Every closed R-space is of the form FH_T, where $F(z)$ is an operator-valued function of z mapping a Hilbert space T of possibly lower dimension than S into S. $F(z)$ is regular in the upper half-plane, $\|F(z)\| \leq 1$ there, and for z real F is an isometry. This representation of R is unique, save for a multiplication of F on the right by a constant unitary matrix.*

In the scalar case, such a representation of the Fourier transform of an r-space spanned by the translates of a single function has been given by Karhunen in [5]. A similar representation theorem for the Fourier transform of an r-space spanned by the translates of a finite number of functions defined on the positive integers has been given by Beurling in [1]. So in the scalar case my representation theorem is a slight extension of their results.

Beurling and Karhunen use a function-theoretic method, relying on the factorization due to Riesz, Herglotz and Nevanlinna of functions, analytic in the upper half-plane and bounded in a certain integral sense, into an inner and outer factor. The outer factor is the exponential of a Poisson integral of an absolutely continuous measure, the inner factor is the exponential of a Poisson integral with respect to a singular measure times a Blaschke product. My proof employs only Hilbert space methods, specifically the projection of the exponential function into r. The significance of this projection has already been pointed out by Beurling in [1].

In Section 3 we use the representation theorem to reduce problems of division in the ring of bounded analytic functions to problems in the Boolean algebra of r-spaces. In particular we are able to factor functions into inner and outer factors. This decomposition is used to give a new proof of the Titchmarsh convolution theorem.

The proof of the representation theorem in the scalar case is given in Section 2, for the vector-valued case in Section 4.

Many problems of analysis are about translation invariant spaces, such as occur in the theory of approximation by exponentials, in Wiener's theory of Tauberian theorems, and in many others. A representation theorem such as the one given here is often useful in such problems, see e.g. [9]. My own interest in the subject came from the study of solutions of partial differential equations in a half-cylinder which, as explained in [8], can be regarded as an l-space of functions whose values lie in an *infinite-dimensional* space. Whether the theory given in the following pages applies to that situation and just how useful it might be is still to be seen.

2.1. In this section we treat the scalar case for which the representation theorem asserts:

SCALAR REPRESENTATION THEOREM. *Every nonempty closed R-space is of the form FH, where $F(z)$ is a regular analytic function in the upper half-plane, $|F(z)| \leqslant 1$ there. For z real, $|F(x)| = 1$. F is uniquely determined by R, save for multiplication by a complex constant of modulus* 1.

Let l and r be a pair of closed translation invariant subspaces of h which are orthogonal complements of each other with respect to h. Let λ be any complex number in the upper half-plane; the function defined as $e^{i\lambda x}$ for x positive, zero for x negative, belongs then to h. Decompose this function into components by orthogonal projection into l and r:

$$e^{i\lambda x} = a_\lambda(x) + b_\lambda(x), \quad 0 < x, \tag{2.1}$$

a_λ in l, b_λ in r. Take the complex conjugate of (2.1), multiply both sides by $b_\mu(x-s)$, where μ is any complex number in the upper half-plane and s is non-negative, and integrate with respect to x from 0 to ∞:

$$\int_0^\infty b_\mu(x-s)\, e^{-i\lambda^* x}\, dx = \int_0^\infty b_\mu(x-s) a_\lambda^*(x)\, ds + \int_0^\infty b_\mu(x-s)\, b_\lambda^*(x)\, dx. \tag{2.2}$$

Denote by $B_\mu(z)$ the Fourier transform of $b_\mu(x)$. The left side of (2.2) is equal to $e^{-i\lambda^* s} B_\mu(-\lambda^*)$. On the right, the first term vanishes since $a_\lambda(x)$ belongs to l while $b_\mu(x-s)$ belongs to r. We transform the second term by Parseval's theorem, [1] using the fact that the Fourier transform of $b_\mu(x-s)$ is $B_\mu(z) e^{izs}$. So we get from (2.2)

[1] dz denotes $dz/2\pi$.

$$e^{-i\lambda^* s} B_\mu(-\lambda^*) = \int_{-\infty}^{\infty} B_\lambda^*(z) B_\mu(z) e^{izs} dz, \quad 0 \leqslant s. \tag{2.3}$$

Take the complex conjugate of both sides, interchange the role of λ and μ and replace s by $-s$. We get

$$e^{-i\mu s} B_\lambda^*(-\mu^*) = \int_{-\infty}^{\infty} B_\lambda^*(z) B_\mu(z) e^{izs} dz, \quad s \leqslant 0. \tag{2.3'}$$

Putting $s=0$ in (2.3), (2.3′) shows that $B_\mu(-\lambda^*)$ and $B_\lambda^*(-\mu^*)$ are equal; we denote their common value by $B_{\lambda\mu}$:

$$\int_{-\infty}^{\infty} B_\lambda^*(z) B_\mu(z) dz = B_{\lambda\mu} = B_\lambda^*(-\mu^*) = B_\mu(-\lambda^*). \tag{2.4}$$

Equations (2.3) and (2.3′) give the Fourier transform of $B_\lambda^*(z) B_\mu(z)$ in the ranges $s \geqslant 0$ and $s \leqslant 0$ respectively. Therefore, the value of $B_\lambda^*(z) B_\mu(z)$ for z real can be found by Fourier inversion:

$$\begin{aligned} B_\lambda^*(z) B_\mu(z) &= B_{\lambda\mu} \int_{-\infty}^{0} e^{-i\mu s - izs} ds + B_{\lambda\mu} \int_{0}^{\infty} e^{-i\lambda^* s - izs} ds \\ &= B_{\lambda\mu} \left\{ \frac{i}{\mu + z} - \frac{i}{\lambda^* + z} \right\} = \frac{i B_{\lambda\mu}(\lambda^* - \mu)}{(\mu+z)(\lambda^*+z)}. \end{aligned} \tag{2.5}$$

Set $\mu = \lambda$ in (2.4); we get

$$B_{\lambda\lambda} = \int_{-\infty}^{\infty} |B_\lambda(z)|^2 dz$$

which, by Parseval's formula, is equal to

$$\int_0^\infty |b_\lambda(x)|^2 dx.$$

In particular if $B_{\lambda\lambda} = 0$ it follows that $b_\lambda(x) \equiv 0$ and therefore, in view of (2.1), that $e^{i\lambda x}$ is orthogonal to r. We have assumed that r contains non-zero elements; the Fourier transform of one of these cannot vanish for all z in the upper half-plane; say it does not vanish at $z = \lambda$. Then $B_{\lambda\lambda} \neq 0$. Set μ equal to this λ in (2.5); we get, for real z,

$$B_\lambda^*(z) B_\lambda(z) = \frac{2 \operatorname{Im} \lambda \, B_{\lambda\lambda}}{|\lambda + z|^2}$$

from which we deduce that

$$B_\lambda(z) = \frac{F(z)}{\lambda + z} G, \tag{2.6}$$

where $G = (2 \operatorname{Im} \lambda \, B_{\lambda\lambda})^{\frac{1}{2}}$ and $F(z)$ *has modulus* 1.

Equation (2.6) provides an extension of $F(z)$ into the upper half-plane, as a regular analytic function. *We claim that* $F(z)$, *thus extended, is a bounded function.* For let w be any point in the upper half plane. Then we have (omitting the subscript λ) a Poisson formula for $wB(w)$:

$$B(w)=\int_{-\infty}^{\infty} b(x)\,e^{ixw}\,dx=\frac{i}{w}\int_{-\infty}^{\infty} b'(x)\,e^{ixw}\,dx$$

$$=\frac{i}{w}\int_{-\infty}^{\infty} b'(x)\,e^{ix\operatorname{Re}w-|x|\operatorname{Im}w}\,dx$$

$$=\frac{i}{w}\int_{-\infty}^{\infty} z\,B(z)\,P(z-\operatorname{Re}w)\,dz.$$

The first equality is obtained by integrating by parts, the second by noting that the support of b' is contained in $(0,\infty)$, the third by Parseval's formula, noting that the Fourier transform of b' is $-iz\,B(z)$, and denoting the Fourier transform of $e^{-|x|\operatorname{Im}w}$ by $P(z)$. Now $P(z)=2\operatorname{Im}w/(z^2+(\operatorname{Im}w)^2)$ is positive and its integral is equal to 1. On the other hand, according to (2.6), $zB(z)$ is bounded on the real axis. Therefore, from the last integral formula it follows that $wB(w)$—and thereby $F(w)$—is bounded in the upper halfplane. Hence by a standard extension of the maximum principle, $|F(z)|$ assumes its maximum on the real axis where it has modulus 1.

Multiply both sides of (2.5) by $B_\lambda(z)$; using formula (2.6) for B_λ, the above formula for $B_{\lambda\mu}$ and the relation $F_\lambda(z)\,F_\lambda^*(z)=1$, we obtain the following expression for B_μ:

$$B_\mu(z)=i\,\frac{F(z)\,F^*(-\mu^*)}{\mu+z} \tag{2.7}$$

for z real. Since both sides are regular analytic in the upper halp-plane, (2.7) holds for all z in the upper half-plane as well.

This completes the construction of the function F. Formula (2.7) shows as well that $F(z)$ is uniquely determined up to a constant factor of modulus 1. We turn now to showing that the space R is equal to FH.

Denote the space FH by R'. Since $|F(z)|\leqslant 1$ in the upper half-plane, we conclude by the Paley-Wiener theorem that R' is a subspace of H. R' is the Fourier transform of a right translation invariant subspace of h, since $e^{isz}R'=e^{isz}FH=Fe^{isz}H\subset FH=R'$ for s positive. Finally, since $|F(z)|=1$ on the real axis, R' is closed.

As before, we project $e^{i\mu x}$ into r'.

$$e^{i\mu x} = a'_\mu(x) + b'_\mu(x).$$

Take the Fourier transform of both sides:

$$\frac{i}{\mu+z} = A'_\mu(z) + B'_\mu(z), \tag{2.8}$$

where B'_μ lies in R', A'_μ in L', the orthogonal complement of R' with respect to H.

The spaces H and H^* are orthogonal complements of each other in L_2. Multiplication by F is a unitary mapping of L_2 into itself and therefore complements are preserved. Hence the orthogonal complement in L_2 of $R' = FH$ is FH^*. So A'_μ and B'_μ are of the form

$$A'_\mu = F\,D, \qquad B'_\mu = F\,E, \qquad D \in H^*, \qquad E \in H. \tag{2.9}$$

Substitute (2.9) into (2.8) and multiply it by $F^*(z)$. Using the facts that, for z real, $F^*(z)F(z) = 1$ and $z = z^*$ we get

$$\frac{i\,F^*(z^*)}{\mu+z} = D(z) + E(z). \tag{2.10}$$

We are now in a position to determine D and E explicitly:

$$\left.\begin{aligned} D &= i\frac{F^*(z^*) - F^*(-\mu^*)}{\mu+z}, \\ E &= i\frac{F^*(-\mu^*)}{\mu+z}. \end{aligned}\right\} \tag{2.11}$$

To verify these expressions for D and E, we have to show that they belong to H^* and H respectively. Clearly, $D(z)$ is regular in the lower half-plane and its square integral along any line parallel to the real axis is uniformly bounded. Therefore, by the Paley–Wiener theorem, D belongs to H^*. E, on the other hand, clearly belongs to H.

From (2.10) we get

$$B'_\mu(z) = FE = i\frac{F(z)\,F^*(-\mu^*)}{\mu+z}.$$

Comparing this with (2.7) we conclude that $B'_\mu(z)$ and $B_\mu(z)$ are identical, i.e., that the projections of $e^{i\mu x}$ into r and r' are identical. Since projections are linear and bounded, it follows that also all linear combinations of exponential functions and their closures have identical projections. Since the set of all functions $e^{i\mu x}$ spans h, it follows that r and r' coincide.

Observe the curious skew symmetry in the dependence of B on μ and z displayed by formula (2.7).

2.2. Denote by d_λ the distance of the normalized exponential function $(2\operatorname{Im}\lambda)^{\frac{1}{2}} e^{i\lambda x}$ from the space l. From (2.4) and (2.7) we have

$$d_\lambda^2 = 2 \operatorname{Im} \lambda \int_0^\infty |b_\lambda(x)|^2 dx = 2 \operatorname{Im} \lambda \int_{-\infty}^\infty |B_\lambda(z)|^2 dz = |F(-\lambda^*)|^2. \tag{2.12}$$

This formula is already contained in Beurling, l.c.; a special case of it goes back to Müntz [10]. Take namely l as the space spanned by the set of exponentials $\{e^{i\lambda_j x}\}$. R, the Fourier transform of its orthogonal complement, has the form FH, where F is the Blaschke product

$$F(z) = \prod \frac{z + \lambda_j^*}{z + \lambda_j}.$$

To show this we note: an element of h is orthogonal to l if and only if it is orthogonal to every exponential function $e^{i\lambda_j x}$, which means that its Fourier transform vanishes at $z = -\lambda_j^*$. So R consists of those elements of H which vanish at $z = -\lambda_j^*$, $j = 1, 2, \ldots$. Clearly, any function of the form FH_1, H_1 in H and F the above Blaschke product, does vanish at $z = -\lambda_j^*$. Conversely, it is well known that any function in H which vanishes at $z = -\lambda_j^*$ can be factorized as FH_1, H_1 in H. Therefore, according to formula (2.12) the distance d of the normalized exponential function $e^{i\lambda x}(2\operatorname{Im}\lambda)^{\frac{1}{2}}$ to l is

$$|d| = \prod \left| \frac{\lambda_j - \lambda^*}{\lambda_j - \lambda} \right|.$$

For a finite set of exponentials, this formula was derived by Müntz by representing the distance as the ratio of two Gram determinants and evaluating the determinants explicitly. For an infinite set of exponentials the formula was derived by Müntz through a passage to the limit, leading to his celebrated criterion for completeness:

A set of exponentials is complete if and only if the Blaschke product formed of them diverges.

Müntz considered real exponentials only; the analogous treatment of complex exponentials is due to Szász [15].

3.1. In this section F, subscripted possibly by some index, will denote a regular analytic function in the upper half-plane, $|F(z)| \leqslant 1$ there, and $|F(z)| = 1$ for z real. If two such functions differ by a constant multiple, they shall be regarded as equivalent. The functions F form a semigroup; we shall now discuss, with the aid of the representation theorem, division in this semigroup and subsequently in the ring of all bounded analytic functions.

F_1 is divisible by F_2 if $F_1 = F_2 F_3$.

THEOREM 3.1. *F_1 is divisible by F_2 if and only if F_1H is contained in F_2H.*

Proof. Since $F_3H \subset H$, $F_1H = F_2F_3H \subset F_2H$. Conversely, assume that $F_1H \subset F_2H$, i.e., that to any H_1 in H there exists an H_2 in H such that $F_1H_1 = F_2H_2$. This relation is valid for z real and therefore for z complex and can be expressed as follows: Multiplication by $F_2^{-1}F_1$ maps H into itself. It follows then that multiplication by any power of $F_2^{-1}F_1$ maps H into H. Since the L_2 norm on the real axis is preserved in this multiplication, it follows from the corollary of the Paley–Wiener theorem that it cannot be increased on any line parallel to the real axis. Clearly, this is the case if and only if $F_2^{-1}F$ has modulus $\leqslant 1$, i.e., belongs to the semigroup.

The intersection, linear combination and closure of translation invariant spaces is likewise translation invariant. Given F_1 and F_2 it follows from the representation theorem that the spaces $F_1H \cap F_2H$ and $\overline{F_1H \oplus F_2H}$ are of the form F_3H and F_4H. We shall denote F_3 by $\{F_1, F_2\}$ and F_4 by (F_1, F_2). An immediate consequence of the divisibility criterion in Theorem 3.1 is

THEOREM 3.2. *$\{F_1, F_2\}$ is the least common multiple, (F_1, F_2) the greatest common divisor of F_1 and F_2.*

If $(F_1, F_2) = 1$, F_1 and F_2 are called relatively prime. We shall show now that the relation between the greatest common divisor and the least common multiple is the usual one:

THEOREM 3.3. $(F_1, F_2)\{F_1, F_2\} = F_1F_2$.

It is easy to show that the above proposition is equivalent with the following one: *If F_1 and F_2 are relatively prime and if F_1 divides FF_2, then F_1 divides F.* This may be proved as follows:

If F_1 divides F_2F, then according to Theorem 3.1

$$FF_2H \subset F_1H.$$

Since $F_1FH \subset F_1H$, we have also

$$FF_2H \oplus F_1FH \subset F_1H. \tag{3.1}$$

The left side is equal to $F(F_2H \oplus F_1H)$; since we have assumed that F_1 and F_2 are relatively prime it follows that $F_2H \oplus F_1H$ is a dense subset of H. Therefore the closure of the space on the left in (3.1) is FH. Since the space on the right is closed, we have $FH \subset F_1H$, i.e., F_1 divides F.

An immediate *corollary* of Theorem 3.3 is that if

$$F_3 F_4 = F_1 F_2,$$

then
$$F_4 = \tilde{F}_1 \tilde{F}_2,$$

where $\tilde{F}_1$ divides F_1, $\tilde{F}_2$ divides F_2.

If F_1 and F_2 are relatively prime, then the linear combination of $R_1 = F_1 H$ and $R_2 = F_2 H$ is dense in H. This is equivalent with the assertion that L_1 and L_2, the orthogonal complement of R_1 and R_2 respectively have only zero in common. The orthogonal complement of $L_1 \oplus L_2$ is $R_1 \cap R_1$ which, according to Theorem 3.3, is $F_1 F_2 H$. Using formula (2.12) for the distance of normalized exponentials from l-spaces we have

THEOREM 3.4. *Let l_1 and l_2 denote two left translation invariant spaces whose intersection is the zero function. Then the distance of $(2\operatorname{Im}\lambda)^{\frac{1}{2}} e^{i\lambda x}$ from $l_1 \oplus l_2$ is the product of its distances from l_1 and from l_2.*

There seems to be no obvious geometric interpretation of this result.

We turn now to the ring of analytic functions bounded in the upper half-plane. We shall denote elements of this ring by C, possibly subscribed by some index.

It follows from the Paley–Wiener theorem that for any function C, $CH \subset H$. Furthermore since $e^{izs} H \subset H$, also $e^{izs} CH \subset CH$ for $s > 0$, i.e., CH is the Fourier transform of a right invariant subspace of h. Therefore the closure of CH is an R-space, and so can be represented as FH. So by construction $CH \subset FH$, i.e., multiplication by $F^{-1}C$ maps H into H; from this it follows, just as in the proof of Theorem 3.1, that $F^{-1}C = G$ is regular and bounded in the upper half-plane. So we have shown that every bounded analytic function has a unique factorization

$$C = FG.$$

In the terminology of Beurling, F is the inner factor, G the outer factor of C. It follows from our construction that if G is an outer factor, then GH is a *dense* subspace of H.

THEOREM 3.5. *The inner factor of $C_1 C_2$ is the product of the inner factors of C_1 and of C_2.*

Proof. We have to show that $C_1 C_2 H = F_1 F_2 H$. Clearly, $C_1 C_2 H = F_1 F_2 G_1 G_2 H$ is a subspace of $F_1 F_2 H$. Since G_1 and $G_2 H$ are outer factors, multiplication by them—and therefore by their product—maps H into a dense subset of H.

It follows from Theorem 3.5 that divisibility of two bounded analytic functions is equivalent to the divisibility of their inner and outer factors. Concerning divisibility by an outer factor we have

THEOREM 3.6. *C is divisible by G if and only if CG^{-1} is bounded on the real axis.*

Proof. We have to show that if CG^{-1} is bounded on the real axis, it is bounded in the upper half-plane. First we note that multiplication by CG^{-1} maps GH into H. Secondly, since CG^{-1} is bounded on the real axis, this operation is bounded and so can be extended to the closure of GH. Third, since G is an outer factor, the closure of GH is H. Therefore we have the result: *Multiplication by* CG^{-1} *maps* H *into* H. From this the boundedness of CG^{-1} in the upper half-plane can be deduced as in the proof of Theorem 3.1.

3.2. The Convolution Theorem. Let $c_1(x)$ and $c_2(x)$ be a pair of functions which are zero for x negative, in L_1 over the positive reals. Denote their convolution by $c(x)$:

$$c = c_1 * c_2 = \int c_1(y)\, c_2(x-y)\, dy. \tag{3.2}$$

Let d_1, d_2 and d be largest numbers such that the supports of $c_1(x)$, $c_2(x)$, $c(x)$ are contained in $x \geqslant d_1$, d_2, d respectively. Clearly, $d \geqslant d_1 + d_2$. What is much less obvious is the

Convolution Theorem of Titchmarsh:

$$d = d_1 + d_2.$$

Proof. Denote the Fourier transforms of c_1, c_2, c by C_1, C_2, C. Taking the Fourier transform of (3.2) we conclude

$$C = C_1 C_2. \tag{3.3}$$

Since c_1 and c_2 are in L_1 over the positive reals, C_1, C_2, and C are bounded analytic functions in the upper half-plane. Denote their inner factors by F_1, F_2 and F. According to Theorem (3.5) it follows from (3.3) that

$$F = F_1 F_2. \tag{3.4}$$

Since the support of c is contained in $x \geqslant d$, it follows that e^{idz} divides C. Since e^{idz} is an inner factor, according to Theorem 3.5, it divides the inner factor of C:

$$F = e^{idz} F_3.$$

Combining this with (3.4) we get

$$e^{idz} F_3 = F_1 F_2.$$

According to the corollary of Theorem 3.3, F_4 being taken as e^{idz}, it follows that

$$e^{idz} = \tilde{F}_1 \tilde{F}_2$$

where $\tilde{F}_1$ divides F_1, $\tilde{F}_2$ divides F_2. We use now the

Theorem: *The only factorization of* e^{idz} *in the ring of bounded analytic functions is the trivial one*

$$e^{idz} = e^{i\tilde{d}_1 z} e^{i\tilde{d}_2 z}, \qquad \tilde{d}_1 + \tilde{d}_2 = d, \qquad \tilde{d}_1, \tilde{d}_2 \geqslant 0.$$

For the sake of completeness we include a proof of this well-known result. Denote by $h(x,y)$ the function $-\log|\tilde{F}_1(z)|$. Since both $|\tilde{F}_1|$ and $|\tilde{F}_2|$ are $\leqslant 1$, it follows from the above factorization that

$$0 \leqslant h(x,y) \leqslant dy, \quad 0 \leqslant y.$$

I.e., h is a positive harmonic function in the upper half-plane which vanishes at the boundary. We shall show now that the only such functions are constant multiples of y. We continue h into the lower half by reflection, and represent $h(x,y)$ by the Poisson integral along a circle of radius R around the origin:

$$h(x, y) = \int_0^\pi P(x, y, R, \theta)\, h\, d\theta;$$

P here is the difference between the values of the Poisson kernel at R,θ and $R, -\theta$. For y positive, P is positive and for large R it is asymptotically equal to $(2y \sin\theta)/R$. Since the integrand in the above representation is positive, it follows that $h(x,y)/h(x',y')$ is asymptotically—and thus actually—equal to y/y'.

We conclude that $\tilde{F}_1 = e^{i\tilde{d}_1 z}$, $\tilde{F}_2 = e^{i\tilde{d}_2 z}$. Since $\tilde{F}_1$ and $\tilde{F}_2$ divide F_1 and F_2, they also divide C_1 and C_2. But then according to the Paley–Wiener theorem (the L_1 variety) it follows that the supports of c_1 and c_2 are contained in $x \geqslant \tilde{d}_1$ and $x \geqslant \tilde{d}_2$. This shows that $d_1 \geqslant \tilde{d}_1$, $d_2 \geqslant \tilde{d}_2$, and so $d_1 + d_2 \geqslant \tilde{d}_1 + \tilde{d}_2 = d$. Combined with the trivial inequality $d \geqslant d_1 + d_2$, this yields the desired result $d = d_1 + d_2$.

Previous proofs of the convolution theorem such as the one by Dufresnoy [3] or Koosis [6], also make use of theorems on positive harmonic functions. For this reason the present proof cannot be called new. Its virtue lies in reducing the convolution theorem to the one about the factorization of e^{idz} swiftly and painlessly.

4. In this section we derive the representation theorem of p. 164 for translation invariant spaces of functions whose values lie in a finite-dimensional Hilbert space S.

The L_2 scalar product of two such functions f and g is defined as

$$\int_0^\infty (f(x), g(x))\, dx,$$

where (f,g) denotes the scalar product in S.

The spaces h, l and r and their Fourier transforms are defined analogous to their old definitions.

As before the proof proceeds by projecting $e^{i\lambda x}u$, where u is an arbitrary element of S, into r:

$$e^{i\lambda x}u = a_\lambda(x) + b_\lambda(x), \tag{4.1}$$

a_λ in l, b_λ in r. The value of $b_\lambda(x)$ depends on u, and this dependence is linear; therefore we can write

$$b_\lambda(x) = b_\lambda(x)u,$$

where now $b_\lambda(x)$ denotes an *operator* mapping S into itself. It is easy to show, on account of the finite-dimensionality of S, that $\|b_\lambda(x)\|$, the operator norm of $b_\lambda(x)$, is square integrable.

Making this change also in the meaning of a_λ, (4.1) can be rewritten as

$$e^{i\lambda x}u = a_\lambda(x)u + b_\lambda(x)u. \tag{4.1'}$$

Let v denote any element of S. Take the scalar product of (4.1′) with $b_\mu(x-s)v$, s non-negative, and integrate with respect to x. The resulting expression is an analogue of (2.2) and can be transformed, by Parseval's theorem, into

$$e^{-i\lambda^* s}(B_\mu(-\lambda^*)v, u) = \int_{-\infty}^{\infty} (B_\mu(z)v, B_\lambda(z)u)\, e^{izs}\, dz. \tag{4.3}$$

Take the complex conjugate of (4.3), interchange the role of λ and μ and of u and v and write $-s$ for s. We get the analogue of (2.3); from this and from (4.3) we can determine $(B_\mu(z)v,\ B_\lambda(z)u)$ for real z by Fourier inversion:

$$(B_\mu(z)v, B_\lambda(z)u) = \frac{i(\lambda^*-\mu)}{(\mu+z)(\lambda^*+z)}(B_{\lambda\mu}v, u), \tag{4.4}$$

where $B_{\lambda\mu}$ abbreviates $B_\lambda^*(-\mu^*)$. The left side of (4.4) can be written as $(B_\lambda^*(z)B_\mu(z)v, u)$; since (4.4) holds for all vectors u and v in S, we conclude that

$$B_\lambda^*(z)B_\mu(z) = \frac{i(\lambda^*-\mu)}{(\mu+z)(\lambda^*+z)}B_{\lambda\mu}. \tag{4.5}$$

Similarly, by setting $s=0$ in (4.3) and transforming the right side by shifting the operator $B_\lambda(z)$ we obtain

$$B_{\lambda\mu} = \int_{-\infty}^{\infty} B_\lambda^*(z)B_\mu(z)\,dz. \tag{4.6}$$

Setting $\lambda=\mu$ we obtain

$$B_{\lambda\lambda} = \int_{-\infty}^{\infty} B_\lambda^*(z)B_\lambda(z)\,dz, \tag{4.6'}$$

which shows that $B_{\lambda\lambda}$ is a symmetric, non-negative operator.

The *rank* of $B_{\lambda\lambda}$ has some maximum p as λ varies in the upper half-plane. For any value of λ, the nullspace of $B_{\lambda\lambda}$ is then at least $(n-p)$-dimensional, n denoting the dimension of S. Let u be a vector annihilated by $B_{\lambda\lambda}$. According to formula (4.6′),

$$0=(B_{\lambda\lambda}\,u,\,u)=\int \|\,B_\lambda(z)\,u\,\|^2\,d\,z;$$

by Parseval's relation, we have then

$$\int_0^\infty \|\,b_\lambda(x)\,u\,\|^2\,d\,x=0.$$

i.e., $b_\lambda(x)\,u\equiv 0$. Since $b_\lambda u$ is the projection of $e^{i\lambda x}u$ into r, we have

LEMMA 4.1. *If u is annihilated by $B_{\lambda\lambda}$, $e^{i\lambda x}u$ is orthogonal to r.*

Let $g(x)$ be an arbitrary element of r, $G(z)$ its Fourier transform. According to Lemma 4.1

$$0=\int_0^\infty (g(x),\ e^{i\lambda x}\,u)\,d\,x=(G(-\lambda^*),\,u).$$

In other words, the value of any function G in R at $-\lambda^*$ is orthogonal to the nullspace of $B_{\lambda\lambda}$. Since the dimension of this nullspace is at least $n-p$, we have

LEMMA 4.2. *At every point of the upper half-plane, the values of the functions in R lie in a p-dimensional linear subspace of S.*

By a passage to the limit we can deduce the following

COROLLARY. *Let $G_1(z), \ldots, G_k(z)$ be a finite set of functions in R; then for almost all z on the real axis their values lie in a p-dimensional subspace of S.*

Denote by λ a value where the rank of $B_{\lambda\lambda}$ is maximal. Putting $\mu=\lambda$ in (4.5) we get for real z

$$B_\lambda^*(z)\,B_\lambda(z)=\frac{2\,\mathrm{Im}\,\{\lambda\,B_{\lambda\lambda}\}}{|\,z+\lambda\,|^2}.$$

Denote the non-negative square root of $2\,\mathrm{Im}\{\lambda B_{\lambda\lambda}\}$ by G. Since the nullspace of $B_\lambda(z)$ includes that of G, there exists an operator $F(z)$ such that

$$B_\lambda(z)=\frac{F(z)\,G}{\lambda+z}. \tag{4.7}$$

$F(z)$ is defined on the range T of G only, and is an isometry there (see e.g. [14], p. 283).

Formula (4.7) serves to extend $F(z)$ as a regular analytic function to the upper half-plane. As in Section 2, it follows that $F(z)$ is bounded there.(1)

Since, for z real, $F(z)$ is an isometry on T, we have

$$F^*(z)F(z) = I. \tag{4.8}$$

Substitute (4.7) into (4.5) and into the expression $B_{\lambda\mu} = B^*_\lambda(-\mu^*)$. We obtain the relation

$$G^* F^*(z) B_\mu(z) = i\, G^* \frac{F^*(-\mu^*)}{\mu+z}, \tag{4.9}$$

where G is regarded as mapping S into T. Since G is a non-negative hermitean operator $G^* = G$ does not annihilate any element of T and so can be cancelled from both sides of (4.9):

$$F^*(z) B_\mu(z) = i\frac{F^*(-\mu^*)}{\mu+z}. \tag{4.10}$$

Using (4.8), (4.10) can be written as

$$F^*(z)\left\{B_\mu(z) - i\frac{F(z)F^*(-\mu^*)}{\mu+z}\right\} = 0. \tag{4.11}$$

Let $u_1, u_2, \ldots, u_n$ be a set of n elements spanning S. The functions $B_\lambda(z)u_j$, $j = 1, \ldots, n$ belong to R and, as formula (4.7) shows, they span the range of $F(z)$. By our choice of λ the range of $F(z)$ has the maximal dimension p. According to the corollary of Lemma 4.2 the range of $B_\mu(z)$ belongs to the range of $F(z)$. But F^* does not annihilate any element on the range of F; therefore the factor $F^*(z)$ can be dropped on the left in (4.11), leaving

$$B_\mu(z) = i\frac{F(z)F^*(-\mu^*)}{\mu+z}. \tag{4.12}$$

There remains to be shown that the space $R' = FH$ is R itself. We shall show this as before by verifying that the orthogonal projection of $e^{i\mu x}u$ into r' is $b_\mu(x)u$, or what is the same, that the projection of $iu/(\mu+z)$ into R' is $B_\mu(z)u$, with B_μ given by formula (4.12). This means that

$$\frac{I - F(z)F^*(-\mu^*)}{\mu+z}\,u$$

is orthogonal to FH for all u, i.e., that

$$F^*(z)\frac{I - F(z)F^*(-\mu^*)}{\mu+z}\,u$$

belongs to H^*_T. Using (4.8), this last expression can be rewritten, for z real, as

(1) One can show the boundedness of $(F(z)v, u)$ for all vectors u, v in T.

$$\frac{F^*(z^*) - F^*(-\mu^*)}{\mu + z}.$$

This function can be continued as a regular analytic function into the lower half-plane and its square integral along any line parallel to the real axis is uniformly bounded. Therefore, by the Paley–Wiener theorem, it belongs to H^*. This completes the proof of the representation theorem.

Those parts of the division theory developed in Section 3 which do not use commutativity remain valid in the vector-valued case. In particular, every matrix-valued analytic function can be written as the product of an inner and an outer factor, in this order. Even further splitting of inner factors F is possible. Take the Fourier inverse r of $R = FH$ and form its orthogonal complement l. Take the set of all exponential polynomials contained in l and form their orthogonal complement r'. r' is a closed invariant space, and it contains r. Its Fourier transform R' then contains R; according to the representation theorem, R' is of the form BH. According to Theorem 3.1, F is divisible by B on the left:

$$F = BE.$$

In the scalar case, B is a Blaschke product and E the exponential of the Poisson integral with respect to a singular measure. Just how useful this factorization is in the matrix case remains to be seen.

It is already known through the researches of Wiener [17], Wiener and Masani [18], and Helson and Lowdenslager [19], that square matrix valued analytic functions whose determinant does not vanish identically can be written as products of an inner and outer factor.

In [20], Potapov shows that bounded analytic matrix functions with determinant $\neq 0$ can be factored as a Blaschke product times a multiplicative integral of the exponential of the Poisson kernel.

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