THE NUMBER OF HOMEOMORPHICALLY IRREDUCIBLE TREES, AND OTHER SPECIES

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1. Introduction

Our object is to augment the already rich literature on the enumeration of trees by the addition of several previously uncounted species. Interest is moreover derived from the fact that we use variations of one general method in each of these cases; a method which is also applicable to numerous counting problems not treated in this paper (e.g., see Riordan

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[14]). The art of enumerating trees appears to have originated with Cayley [1], who obtained formulas for labeled trees, rooted trees, and also (unrooted) trees. A significant contribution was made by Pólya [13], who with the help of a powerful general enumerating procedure, solved a host of counting problems for trees of considerable interest in organic chemistry. A theorem of Otter [12] on the dissimilarity characteristic for trees enabled him to formulate an elegant equation giving the number of trees in terms of the known number of rooted trees. The dissimilarity characteristic was generalized to Husimi graphs and to arbitrary graphs by Harary and Norman [7, 8]. These results led to the enumeration of rooted Husimi graphs by Harary and Uhlenbeck [9] and unrooted Husimi graphs by Norman [11], whose method also extended to the counting of graphs with any given collection of blocks; cf. Ford, Norman and Uhlenbeck [2, paper II]. Among generalizations of trees, enumerating formulas were obtained for labeled Husimi graphs in Ford and Uhlenbeck [2, paper I] (who extended Otter's asymptotic results in [2, papers III and IV]), for forests in Harary [5], and for labeled colored and chromatic trees and oriented trees in a recent paper of Riordan [14]. In addition enumerations of graphs and directed graphs appear in Harary [5], and of labeled graphs and labeled directed graphs in Gilbert [3].

The general plan for enumerating each tree species has three parts. First, a functional equation is obtained for the generating function for trees rooted at an endpoint. Then rooted trees are expressed in terms of these using Pólya's Theorem. Finally the generating function for unrooted trees is given in terms of the function for rooted trees by an appropriate combination of the theorems of Otter and Pólya. For the sake of completeness these theorems are reviewed briefly in the succeeding two sections. Each of the remaining seven sections serves to enumerate either a new species of trees, or ordinary trees in terms of a new parameter. We include three appendices which present what is to our knowledge the most exhaustive collection of tree diagrams available. They have served as a valuable collection of data for the testing of conjectures. The diagrams have been thoroughly checked and are believed to be error free.

2. Pólya's Theorem

We shall state Pólya's Theorem (the Hauptsatz of [13]) in the form which uses two variables. In deriving the counting series for the various kinds of trees, sometimes the one variable form is used, and at one point the theorem is used for an infinite number of variables.

Let *figure* be an undefined term. To each figure there is assigned an ordered pair of non-negative integers called its *content*. Let φ_{mn} denote the number of different figures of content (m, n). Then the figure counting series $\varphi(x, y)$ is defined by

$$\varphi(x, y) = \sum_{m, n=0}^{\infty} \varphi_{mn} x^m y^n$$

Let G be a permutation group of degree s and order h. A *configuration* of length s is a sequence of s figures. The content of a configuration is the vector sum of the contents of its figures. Two configurations are *G-equivalent* if there is a permutation of G sending one into the other. Let F_{mn} denote the number of F-inequivalent configurations of content (m, n). The configuration counting series $F(x, y)$ is defined by

$$F(x, y) = \sum_{m, n=0}^{\infty} F_{mn} x^m y^n$$

We shall call G the *configuration group* henceforth.

The object of Pólya's Theorem is to express $F(x, y)$ in terms of $\varphi(x, y)$ and G. This is accomplished using the cycle index of G, defined as follows. Let $h_{(j)}$ denote the number of elements of G of type $(j) = (j_1, j_2, \ldots, j_s)$, i.e., having j_k cycles of length k for $k = 1, 2, \ldots, s$. Thus

$$1j_1 + 2j_2 + \cdots + sj_s = s \tag{1}$$

Let $f_1, f_2, \ldots, f_s$ be s indeterminates. Then $Z(G)$, the cycle index of G, is defined, as in Pólya [13, p. 157], by:

$$Z(G) = \frac{1}{h} \sum_{(j)} h_{(j)} f_1^{j_1} f_2^{j_2} \ldots f_s^{j_s}, \tag{2}$$

where the sum is taken over all partitions (j) of s satisfying (1). For any power series $f(x, y)$, let $Z(G, f(x, y))$ denote the function obtained from $Z(G)$ by replacing each indeterminate f_k by $f(x^k, y^k)$. Using these definitions, we are able to give a concise statement of:

Pólya's Theorem. The configuration counting series is obtained by substituting the figure counting series into the cycle index of the configuration group. Symbolically,

$$F(x, y) = Z(G, \varphi(x, y)).$$

This theorem reduces the problem of finding the configuration counting series to the determination of the figure counting series and the cycle index of the configuration group.

We frequently require the configuration group S_n, the symmetric group of degree n. It is well known that $Z(S_n)$ may be obtained from (2) by setting $s = n$, $h = n!$, and

$$h_{(j)} = \frac{n!}{1^{j_1} j_1! \, 2^{j_2} j_2! \ldots n^{j_n} j_n!}$$

We illustrate Pólya's Theorem while reviewing the enumeration of rooted trees. See König [10] as a general reference on graph theory. A *tree* is a connected graph with no cycles. A *rooted tree* is a tree with one distinguished point, called the *root*. Following the terminology of Riordan [14], a *planted tree* is a rooted tree in which the root is an *endpoint*, that is, a point incident to only one line. Two trees are *isomorphic* if there is one-to-one correspondence between their point sets which preserves adjacency, and two rooted trees are isomorphic if there is a tree isomorphism between them which maps one root onto the other root.

A *path* joining points a_1 and a_n in a tree is a collection of lines of the form a_1a_2, a_2a_3, $\ldots$, $a_{n-1}a_n$ where the points $a_1, a_2, \ldots, a_n$ are distinct. In any tree there is a unique path between each pair of points. The *length* of a path is the number of lines in it. The *distance* between two points is the length of the path joining them. The *diameter* of a tree is the greatest distance between any two points. The *root-diameter* of a rooted tree is the greatest distance between the root and all other points. The *branch* $\langle a, b\rangle$ of a tree determined by a point a and a line ab is that subtree containing a and all the points reachable by paths from a whose first line is ab. A *main branch* of a rooted tree is a branch at the root. Define

$$Z(S_\infty, f(x)) = \sum_{n=0}^{\infty} Z(S_n, f(x)),$$

where we take $Z(S_0, f(x)) = 1$. It is implicitly shown in [13] that

$$Z(S_\infty, f(x)) = \exp \sum_{r=1}^{\infty} \frac{1}{r} f(x^r).$$

This equation can also be verified by comparison of corresponding coefficients of x, as in Norman [11].

Let T_i be the number of rooted trees with i points and

$$T(x) = \sum_{i=1}^{\infty} T_i x^i$$

be the generating function for rooted trees. Let $t(x)$ and $\overline{T}(x)$ be the corresponding generating functions for trees and planted trees respectively. The following proof of the Cayley–Pólya formula for $T(x)$ is different from previous proofs in the literature, and is based on the root-diameter of a rooted tree.

THEOREM 1. (Pólya [13], p. 197)

$$\overline{T}(x) = x\,T(x) \tag{3 a}$$

$$\bar{T}(x) = x^2 Z(S_\infty, \bar{T}(x)/x) \tag{3 b}$$

$$T(x) = xZ(S_\infty, T(x)) \tag{3 c}$$

Proof. Consider a planted tree $\bar{T}$ with root point p. Let q be the point adjacent to p. Consider the tree T obtained from $\bar{T}$ by deleting p, and rooting the resulting tree at q. Consider further the set $\bar{S}$ of the main branches of T, and the set S of rooted trees obtained from the planted trees of $\bar{S}$ in the same way as we obtained T from $\bar{T}$. See, for example, Figure 1. Clearly any one of T, $\bar{T}$, S, $\bar{S}$, determines the others uniquely. If $\bar{T}$ has m points, then T has $m-1$ points. This proves equation (3 a).

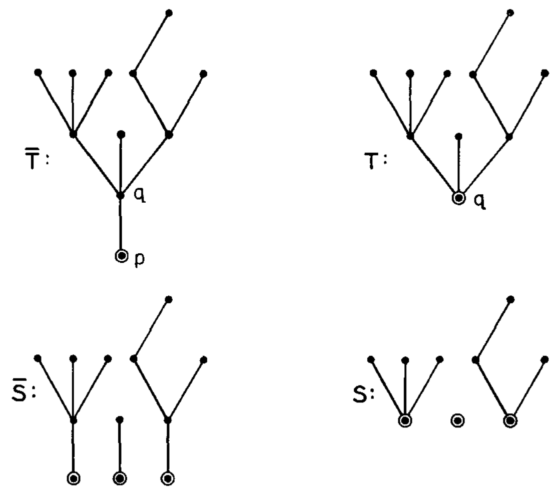

Figure 1.

If $\bar{T}$ has root-diameter n, then T has root-diameter $n-1$, and the maximum root-diameter among the trees of $\bar{S}$ and S are $n-1$ and $n-2$ respectively. Also T has one more point than S, and if $\bar{S}$ has k planted trees, then $\bar{S}$ has $k-2$ more points than $\bar{T}$.

Now let $\bar{T}^{(n)}(x)$ and $T^{(n)}(x)$ be the counting series for planted trees and rooted trees with root-diameter $\leqslant n$. The counting series for all sets (configurations) of planted trees with root-diameter $\leqslant n$ is then $Z(S_\infty, \bar{T}^{(n)}(x))$, and that for sets of rooted trees is $Z(S_\infty, T^{(n)}(x))$. From the considerations of the last paragraph, we then find:

$$\bar{T}^{(n+1)}(x) = x^2 Z\left(S_\infty, \frac{\bar{T}^{(n)}(x)}{x}\right)$$

$$T^{(n+1)}(x) = xZ(S_\infty, T^{(n)}(x))$$

As the counting series for planted trees with root-diameter $\leqslant 1$ is x^2 and the counting series for rooted trees with root-diameter 0 is x, we obtain the functional equations (3 b) and (3 c). We note that equation (3 c) can also be obtained by substituting (3 a) into (3 b).

When further on we shall consider other species of rooted trees, we shall find that whenever we are not able to obtain a functional equation for these rooted trees directly by methods analogous to the above, we shall obtain one by finding equations analogous to (3 a) and (3 b).

Explicitly,

$$T(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + 115x^8 + \cdots$$

3. Otter's Theorem

In this section we review Otter's results [12] on the number of trees. It will be seen that his approach is convenient in deriving a functional equation for counting a species of trees in terms of the counting series for the same species of rooted trees.

An *automorphism* of a tree is an isomorphism of a tree with itself. Two points a and b of a tree are *similar* if there is an automorphism sending a onto b. Similarity of two lines of a tree is defined analogously. Since the set of all automorphisms of a tree is a permutation group, similarity is an equivalence relation. The number of *dissimilar* points or lines of a tree is the number of similarity classes of points or lines. A *symmetry line* of a tree is one whose two points are similar. A *symmetric tree* is one which contains a symmetry line. Obviously any tree has either 0 or 1 symmetry line. We are now able to state Otter's *dissimilarity characteristic* for trees.

Otter's Theorem [12]. In any tree, the number of dissimilar points minus the number of dissimilar lines plus the number of symmetry lines equals 1.

With the help of this refinement of Euler's characteristic, it is possible to derive an elegant functional equation for $t(x)$ in terms of $T(x)$. A *line-rooted tree* is one in which there is a distinguished line. Using the following two lemmas, we have a proof of Theorem 2 different from that of Otter.

LEMMA 1. *The counting series for line-rooted trees is* $Z(S_2, T(x))$.

Proof. There is a one-to-one correspondence between line-rooted trees and unordered pairs of rooted trees, rooted at the points on the distinguished line.

LEMMA 2. *The counting series for symmetric trees is* $T(x^2)$.

Proof. There is a one-to-one correspondence between symmetric trees and pairs of isomorphic rooted trees, rooted at the points on the symmetry line.

THEOREM 2. (Otter [12])

$$t(x) = T(x) - \tfrac{1}{2}[T^2(x) - T(x^2)]. \tag{4}$$

Proof. First we sum the dissimilarity characteristic equation for trees over all trees with n points. This becomes: the number of trees with n points is equal to the number of rooted trees minus the number of line-rooted trees plus the number of symmetric trees with n points. Hence by Lemmas 1 and 2,

$$t(x) = T(x) - Z(S_2, T(x)) + T(x^2), \tag{5}$$

which gives equation (4).

Otter's formula (4) for the number of trees has subsequently been reproved twice without making use of his dissimilarity characteristic by Harary [6] and Riordan [14] by building up from Pólya's results [13] on the number of centered and bicentered trees.

Let A_n be the alternating group of degree n. It was shown in [7] and [9], using a theorem of Pólya [13] on configurations with no repeated figures, that

$$t(x) = T(x) - [Z(A_2, T(x)) - Z(S_2, T(x)].$$

This last equation (identical in content with (5)) is often written, by an abuse of notation, in the form

$$t(x) = T(x) - Z(A_2 - S_2, T(x)). \tag{5'}$$

We shall have occasion to use the form of equation (5′) in enumerating other species of trees.

Explicitly, we find

$$t(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 + 23x^8 + 47x^9 + 106x^{10} + 235x^{11} + 551x^{12} + \cdots$$

Diagrams of all trees with $\leqslant 10$ points are given in Appendix I.

For later reference we state the above-mentioned theorem of Pólya completely.

THEOREM 3. (Pólya [13], p. 161). *The counting series for all S_n-inequivalent configurations of figures whose counting series is $f(x)$, where each figure appears at most once in any configuration is*

$$Z(A_n - S_n, f(x)).$$

COROLLARY. The corresponding counting series for all configurations regardless of length is

$$Z(A_\infty - S_\infty, f(x)) = \sum_{n=0}^{\infty} Z(A_n - S_n, f(x)).$$

4. Trees with a given partition

The *degree* of a point of a tree is the number of lines incident to it. The *partition* of a tree is the sequence of non-negative integers $(a_1, a_2, a_3, \dots)$ where a_m is the number of points of degree m. Let

$$\overline{P}(x, t_1, t_2, \dots) = \sum_{j, i_1, i_2, \dots = 0}^{\infty} \overline{P}_{j i_1 i_2 \dots} x^j t_1^{i_1} t_2^{i_2} \dots$$

where $\overline{P}_{j i_1 i_2} \dots$ is the number of planted trees with i_m points of degree m and a total of j points. Let $P(x, t_1, t_2, \dots)$ and $p(x, t_1, t_2, \dots)$ be the corresponding counting series for rooted trees and trees respectively.

THEOREM 4. *The counting series for planted trees, rooted trees, and trees with a given partition are given by:*

$$\overline{P}(x, t_1, t_2, \dots) = \sum_{m=0}^{\infty} x^2 t_1 t_{m+1} Z\left(S_m, \frac{1}{x t_1} \overline{P}(x, t_1, t_2, \dots)\right) \tag{6 a}$$

$$P(x, t_1, t_2, \dots) = \sum_{m=0}^{\infty} x t_m Z\left(S_m, \frac{1}{x t_1} \overline{P}(x, t_1, t_2, \dots)\right) \tag{6 b}$$

$$p(x, t_1, t_2, \dots) = P(x, t_1, t_2, \dots) - \frac{1}{x^2 t_1^2} Z(A_2 - S_2, \overline{P}(x, t_1, t_2, \dots)) \tag{6 c}$$

Proof of (6a). Let $\overline{T}$ be a planted tree with root p and let q be the point adjacent to p. By a *q-branch* we mean a branch of the form $\langle q, r \rangle$ for some r. Consider the sets of all q-branches not containing p. This set of branches determines $\overline{T}$, and any set of planted trees each of root diameter $\leqslant n-1$ uniquely determines a planted tree of root diameter $\leqslant n$ in this way.

Now let $\bar{S}$ be a set of m planted trees each with root diameter $\leqslant n$. Then the planted tree $\overline{T}$ determined by S as in Figure 1 includes two points not in $\bar{S}$, corresponding to p and q, one of degree 1 and the other of degree $m+1$. On the other hand, the root points of the planted trees of $\bar{S}$, which have degree 1, are not included in this planted tree. Hence if we let $\overline{P}^{(n)}(x, t_1, t_2, \dots)$ be the counting series for planted trees with root diameter $\leqslant n$ and given partition, we find

$$\overline{P}^{(1)}(x, t_1, t_2, \dots) = x^2 t_1^2$$

and
$$\overline{P}^{(n+1)}(x, t_1, t_2, \dots) = \sum_{m=0}^{\infty} x^2 t_1 t_{m+1} Z\left(S_m, \frac{1}{x t_1} \overline{P}^{(n)}(x, t_1, t_2, \dots)\right)$$

from which (6a) follows.

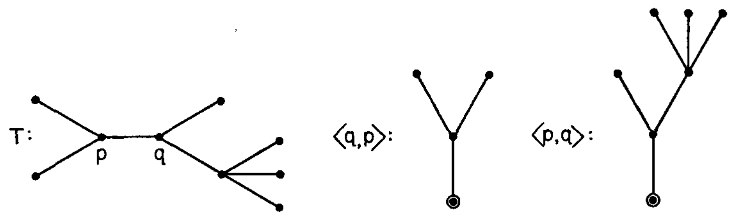

Figure 2.

Proof of (6 b). The same set $\bar{S}$ of m planted trees also determines a rooted tree T as in Figure 1 by identifying the roots of the planted trees in $\bar{S}$, whose root diameter is $\leqslant n$. Its root has degree m and the m root points of $\bar{S}$ do not appear. Hence

$$P^{(1)}(x, t_1, t_2, \dots) = x t_1$$

and
$$P^{(n+1)}(x, t_1, t_2, \dots) = \sum_{m=0}^{\infty} x t_m Z\left(S_m, \frac{1}{x t_1} \bar{P}^{(n)}(x, t_1, t_2, \dots)\right),$$

proving (6 b).

Proof of (6 c). A tree T rooted at a line pq is uniquely determined by the two planted trees formed by the branches $\langle p, q\rangle$ and $\langle q, p\rangle$ of T as illustrated in Figure 2.

Conversely any two planted trees uniquely determine a line-rooted tree. The two planted trees together have two points more than T, each extra point having degree 1. Hence the counting series for line-rooted trees with given partition is

$$\frac{1}{x^2 t_1^2} Z(S_2, \bar{P}(x, t_1, t_2, \dots)).$$

Similarly the counting series for symmetric trees with given partition is

$$\frac{1}{x^2 t_1^2} \bar{P}(x^2, t_1^2, t_2^2, \dots),$$

which is, of course, equal to

$$\frac{1}{x^2 t_1^2} Z(A_2, \bar{P}(x, t_1, t_2, \dots)).$$

Applying Otter's Theorem, equation (6 c) results.

Explicitly we obtain

$$\begin{aligned} p(x, t_1, t_2, \dots) = {} & x t_1 + x^2 t_1^2 + x^3 t_1^2 t_2 + x^4 t_1^3 t_3 + x^4 t_1^2 t_2^2 + x^5 t_1^4 t_4 + x^5 t_1^3 t_2 t_3 + x^5 t_1^2 t_2^3 \\ & + x^6 t_1^5 t_5 + x^6 t_1^4 t_2 t_4 + x^6 t_1^4 t_3^2 + 2x^6 t_1^3 t_2^2 t_3 + x^6 t_1^2 t_2^4 + x^7 t_1^6 t_6 + x^7 t_1^5 t_2 t_5 + x^7 t_1^5 t_3 t_4 \\ & + 2x^7 t_1^4 t_2^2 t_4 + 2x^7 t_1^4 t_2 t_3^2 + 3x^7 t_1^3 t_2^3 t_3 + x^7 t_1^2 t_2^5 + \cdots \end{aligned}$$

5. Homeomorphically irreducible trees

A *homeomorphically irreducible* tree is one with no points of degree 2. Let $h(x)$, $H(x)$, and $\bar{H}(x)$ be the counting series for homeomorphically irreducible trees, rooted trees, and planted trees respectively. Let $\bar{S}$ be a set of planted trees of root diameter $\leqslant n$ determining a planted tree $\bar{T}$ of root diameter $\leqslant n+1$ as in Figure 1. Then $\bar{T}$ is homeomorphically irreducible if and only if $\bar{S}$ contains at least two trees and all the trees of $\bar{S}$ are homeomorphically irreducible. The counting series for planted homeomorphically irreducible trees with root diameter $\leqslant 1$ is x^2. Hence $\bar{H}(x)$ is determined by the functional equation

$$\bar{H}(x) = x^2 \left(1 + \sum_{n=2}^{\infty} Z\left(S_n, \frac{\bar{H}(x)}{x}\right)\right). \tag{7 a}$$

There is a one-to-one correspondence between the set of all rooted trees T and all planted trees $\bar{T}$, defined by deleting the root point of $\bar{T}$ and rooting the point of the resulting tree T adjacent in $\bar{T}$ to the root of $\bar{T}$ (as in Figure 1). If T is homeomorphically irreducible, then so is $\bar{T}$ unless the root of T is an endpoint. However, not all homeomorphically irreducible planted trees are formed in this way. For if T has exactly two main branches, each homeomorphically irreducible, then $\bar{T}$ is homeomorphically irreducible even though T is not. We therefore obtain

$$\bar{H}(x) = x[(H(x) - \bar{H}(x)) + Z(S_2, \bar{H}(x))],$$

and hence

$$H(x) = \frac{x+1}{x}\bar{H}(x) - \frac{1}{x} Z(S_2, \bar{H}(x)). \tag{7 b}$$

Applying Otter's Theorem to homeomorphically irreducible trees, we find:

$$h(x) = H(x) - \frac{1}{x^2} Z(A_2 - S_2, \bar{H}(x)). \tag{7 c}$$

The proof of equation (7c) follows the method of proof of equation (6 c).

THEOREM 5. *The counting series for* $\bar{H}(x)$, $H(x)$, *and* $h(x)$ *are given by equations* (7a), (7b) *and* (7c).

Explicitly,

$$h(x) = x + x^2 + x^4 + x^5 + 2x^6 + 2x^7 + 4x^8 + 5x^9 + 10x^{10} + 14x^{11} + 26x^{12} + \cdots.$$

Diagrams of all homeomorphically irreducible trees with $\leqslant 12$ points appear in Appendix II.

6. Trees with a given diameter

We have already seen in the proof of Theorem 1 that the counting series for rooted trees with root-diameter n is $T^{(n)}(x) - T^{(n-1)}(x)$, where $T^{(n)}(x)$ is defined recursively by $T^{(0)}(x) = x$ and $T^{(n)}(x) = xZ(S_\infty, T^{(n-1)}(x))$.

The *associated number* of a point p of a tree is the greatest distance between p and all other points. Thus the root-diameter of a rooted tree is the associated number of the root and the diameter of a tree is the maximum among all the associated numbers of the points. The minimum associated number is the *radius* of a tree. The *center* of a tree is the set of all points of minimum associated number. It is well known (see König [10]) that any tree has either one or two central points and if there are two, they are adjacent. A tree is *centered* or *bicentered* according to whether it has one or two central points. Counting series for centered and bicentered trees were obtained in Riordan [14]. Other expressions for these arise in connection with trees with a given diameter. For trees with odd diameter are bicentered, while trees with even diameter are centered. Let $d_m(x)$ be the counting series for trees with diameter m.

Consider a bicentered tree with odd diameter $2n+1$ and bicenters p, q. This tree is uniquely determined by the branches $\langle p, q\rangle$ and $\langle q, p\rangle$. If we omit the endpoints of these branches (and root them at the points adjacent to these endpoints), we may consider them as rooted trees of root-diameter n. Conversely, any two rooted trees of root-diameter n uniquely determine a tree of diameter $2n+1$. Hence

$$\begin{cases} d_1(x) = Z(S_2, T^{(0)}(x)) \\ d_{2n+1}(x) = Z(S_2, T^{(n)}(x) - T^{(n-1)}(x)), \ n \geq 1. \end{cases} \tag{8_1}$$

On the other hand, a centered tree with diameter $2n$ and center p is determined by the set of its p-branches. If we again delete the endpoints of these branches, we find we have a set of rooted trees, at least two of which have root diameter $n-1$, and the rest root-diameter $\leq n-2$. Therefore,

$$\left.\begin{aligned} &d_0(x) = T^{(0)}(x) \\ &d_2(x) = x\sum_{m=2}^{\infty} Z(S_m, T^{(0)}(x)) \\ &d_{2n}(x) = x\,[Z(S_\infty, T^{(n-2)}(x))]\left[\sum_{m=2}^{\infty} Z(S_m, T^{(n-1)}(x) - T^{(n-2)}(x))\right], \ n>1 \end{aligned}\right\} \tag{8_2}$$

Explicitly we obtain:

$$
\begin{array}{lrrrrrrrrr}
d_0(x) = x & & & & & & & & & \\
d_1(x) = & x^2 & & & & & & & & \\
d_2(x) = & & x^3 + & x^4 + & x^5 + & x^6 + & x^7 + & x^8 + & x^9 + \cdots \\
d_3(x) = & & & x^4 + & x^5 + & 2x^6 + & 2x^7 + & 3x^8 + & 3x^9 + \cdots \\
d_4(x) = & & & & x^5 + & 2x^6 + & 5x^7 + & 8x^8 + & 14x^9 + \cdots \\
d_5(x) = & & & & & x^6 + & 2x^7 + & 7x^8 + & 14x^9 + \cdots \\
d_6(x) = & & & & & & x^7 + & 3x^8 + & 11x^9 + \cdots \\
d_7(x) = & & & & & & & x^8 + & 3x^9 + \cdots \\
d_8(x) & & & & & & & & x^9 + \cdots \\
\hline
\end{array}
$$

$$t(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + 11x^7 + 23x^8 + 47x^9 + \cdots$$

7. Weighted trees

A weighted tree is a tree to each of whose points is assigned a positive integer called its *weight*. The *weight* of a tree is the sum of the weights of its points. Let

$$W(x, y) = \sum_{i, j=1}^{\infty} W_{i,j}\, x^i y^j,$$

be the counting series for rooted weighted trees where W_{ij} is the number of rooted trees with i points and weight j, and let $w(x, y)$ be the series for weighted trees. By the usual methods we obtain the functional equation

$$W(x, y) = x \sum_{i=1}^{\infty} y^i Z(S_\infty, W(x, y)). \tag{9 a}$$

By Otter's Theorem, $w(x, y)$ is then given by

$$w(x, y) = W(x, y) - Z(A_2 - S_2, W(x, y)). \tag{9 b}$$

From equations (9 a) and (9 b) we obtain

$$
\begin{aligned}
w(x, y) = {} & xy + xy^2 + x^2y^2 + xy^3 + x^2y^3 \\
& + x^3y^3 + xy^4 + 2x^2y^4 + 2x^3y^4 + 2x^4y^4 \\
& + xy^5 + 2x^2y^5 + 4x^3y^5 + 4x^4y^5 + 3x^5y^5 \\
& + xy^6 + 3x^2y^6 + 6x^3y^6 + 10x^4y^6 + 9x^5y^6 \\
& + 6x^6y^6 + xy^7 + 3x^2y^7 + 9x^3y^7 + 17x^4y^7 \\
& + 24x^5y^7 + 20x^6y^7 + 11x^7y^7 + \cdots .
\end{aligned}
$$

8. Directed trees and signed trees

An *oriented tree* is a tree to each of whose lines pq is assigned exactly one of two directions, $\overrightarrow{pq}$ or $\overleftarrow{pq}$. Such a line is called a directed line. A *directed tree* is a tree in which each line is assigned either one direction or both directions. A *signed tree* [4] is one in which each line is assigned a plus or minus sign. Let $r(x)$, $d(x)$ and $s(x)$ be the counting series for oriented, directed and signed trees respectively with the usual notation for rooted and planted trees. Then $\bar{R}(x)$ is determined by the functional equation

$$\bar{R}(x) = 2x^2 Z\left(S_\infty, \frac{\bar{R}(x)}{x}\right). \tag{10 a}$$

For each set of planted oriented trees of root diameter $\leqslant n$ determines *two* planted oriented trees of root diameter $\leqslant n+1$, one for each direction which may be assigned to the additional line. Clearly,

$$R(x) = \frac{1}{2x}\bar{R}(x). \tag{10 b}$$

To obtain $r(x)$ we again apply Otter's Theorem. Every two non-isomorphic rooted oriented trees determine *two* line-rooted oriented trees, one for each direction. But two isomorphic rooted oriented trees determine only *one* line-rooted oriented tree. Hence the counting series for line-rooted oriented trees is

$$2Z(A_2 - S_2, R(x)) + R(x^2) = R^2(x).$$

Since a directed line cannot be a symmetry line, this gives

$$r(x) = R(x) - R^2(x), \tag{10 c}$$

which is found in Riordan [14, equation (42)].

Explicitly, we find

$$r(x) = x + x^2 + 3x^3 + 8x^4 + 27x^5 + 91x^6 + \cdots$$

The counting series for planted and rooted signed trees are the same as for oriented trees. But in this case not only do two *nonisomorphic* rooted signed trees determine two line-rooted signed trees, but two *isomorphic* rooted signed trees also determine *two* line-rooted signed trees. Hence the counting series for line-rooted signed trees is

$$2Z(S_2, R(x)).$$

On the other hand, the two line-rooted trees determined by two isomorphic rooted signed trees are symmetric. Hence the counting series for symmetric signed trees is $2R(x^2)$ and $s(x) = R(x) - 2Z(S_2, R(x)) + 2R(x^2)$ or

$$s(x) = R(x) - R^2(x) + R(x^2) \tag{11}$$

Explicitly, $\quad s(x) = x + 2x^2 + 3x^3 + 10x^4 + 27x^5 + 98x^6 + \cdots.$

On comparing equations (10 c) and (11) we see that the number of oriented trees of n points equals the number of signed trees of n points when n is odd, but is less than that number when n is even. This may be verified in the corresponding explicit series.

Similar arguments show that

$$\bar{D}(x) = 3x^2 Z\left(S_\infty, \frac{\bar{D}(x)}{x}\right), \tag{12 a}$$

$$D(x) = \frac{1}{3x}\bar{D}(x), \tag{12 b}$$

$$d(x) = D(x) - \tfrac{1}{2}[3D^2(x) - D(x^2)]. \tag{12 c}$$

Explicitly, $\quad d(x) = x + 2x^2 + 6x^3 + 25x^4 + 114x^5 + \cdots.$

9. Trees of given strength

For certain applications, including psychological and electrical, it has proved useful to define structures that allow more than one line between two points. A tree has *strength s* if at least one pair of points is joined by s lines, but no two points are joined by more than s lines. Graphs of strength s are counted in [5].

It is convenient to consider trees of strength $\leqslant s$ rather than trees of strength s. Let

$$\bar{A}^{(s)}(x, y) = \sum_{i,j=1}^{\infty} \bar{A}_{ij}^{(s)} x^i y^j,$$

where $\bar{A}_{ij}^{(s)}$ is the number of planted trees of strength $\leqslant s$ with i points and j lines.

THEOREM 6. *Trees of given strength are enumerated by the three equations:*

$$\bar{A}^{(s)}(x, y) = x^2 \sum_{i=1}^{s} y^i Z\left(S_\infty, \frac{\bar{A}^{(s)}(x, y)}{x}\right), \tag{13 a}$$

$$A^{(s)}(x, y) = x Z\left(S_\infty, \frac{\bar{A}^{(s)}(x, y)}{x}\right), \tag{13 b}$$

$$a^{(s)}(x, y) = A^{(s)}(x, y) - \sum_{i=1}^{s} y^i Z(A_2 - S_2, A^{(s)}(x, y)). \tag{13 c}$$

The proof is analogous to preceding ones and is omitted. The counting series for trees of strength s is then

$$a^{(s)}(x, y) - a^{(s-1)}(x, y).$$

Explicitly, the counting series for trees of strength 2 is

$$x^2y^2 + x^3y^3 + x^3y^4 + 3x^4y^4 + 3x^4y^5 + 2x^4y^6 + 6x^5y^5 + 9x^5y^6 + 6x^5y^7 + 3x^5y^8 + \cdots.$$

A tree of type 2 is one in which there are two different kinds of lines, defined in [5]; trees of type t are defined similarly. The same methods used in this paper also serve enumerate trees of type t, but we omit the details here. In his enumeration of trees with colored lines, Riordan [14] contains this result.

10. Trees whose automorphism group is the identity

For any tree T, let $\Gamma(T)$ be the automorphism group of T. Let E_n be the identity group of degree n. Then $\Gamma(T) = E_n$ if and only if T has n points, no two of which are similar. We wish to enumerate all trees whose group is E (the identity group of unspecified degree). The *absolute* $|T|$ of a rooted or line-rooted tree is the unrooted tree with the same points and adjacencies as T. It is clear that if $\Gamma(|T|) = E$, then $\Gamma(T) = E$, but not conversely.

THEOREM 7. *Let $u(x)$ and $U(x)$ be the counting series for trees and rooted trees whose automorphism group is the identity. Then*

$$U(x) = xZ(A_\infty - S_\infty, U(x)) \tag{14 a}$$

$$u(x) = U(x) - Z(S_2, U(x)). \tag{14 b}$$

Proof of equation (14a). Any set of nonisomorphic rooted trees of root-diameter $\leqslant n$ whose group is E determines a rooted tree of root-diameter $\leqslant n+1$ whose group is E. By the corollary to Theorem 3, this equation follows at once.

Proof of equation (14b). Let $\{T'\}$ and $\{T''\}$ be the sets of all rooted and line-rooted trees with group E. By definition the counting series for $\{T'\}$ is $U(x)$, and it follows that the counting series for $\{T''\}$ is $Z(A_2 - S_2, U(x))$. Let $U_1(x)$ and $U_2(x)$ be the respective counting series for all rooted and line-rooted trees the group of whose absolute is E. Since any tree with group E is not symmetric, it follows from Otter's Theorem that $u(x) = U_1(x) - U_2(x)$.

We find it convenient to define

$$V(x) = [U(x) - U_1(x)] - [Z(A_2 - S_2, U(x)) - U_2(x)].$$

If we can find the series $V(x)$, then we can solve for $u(x)$:

$$u(x) = U(x) - Z(A_2 - S_2, U(x)) - V(x).$$

Clearly $U(x) - U_1(x)$ and $Z(A_2 - S_2, U(x)) - U_2(x)$ enumerate respectively all rooted trees and line-rooted trees T such that $\Gamma(T) = E$ and $\Gamma(|T|) \neq E$.

Consider a non-symmetric tree T whose group is not E. We investigate how many rooted trees T' and line-rooted trees T'' with group E have T as absolute. If there exist any such trees T' or T'' at all, then $\Gamma(T)$ has exactly one element besides the identity, and this element must permute two branches at some point p of T. Each of the two similar branches at p, considered as rooted trees, has group E. If each of these branches has $n + 1$ points, then there exist exactly n rooted trees T' such that $|T'| = T$. Moreover, the line-rooted trees T'' obtained by rooting the n lines of each of these two branches also have $|T''| = T$. We conclude that for all non-symmetric trees T whose group is not E, the number of rooted trees with group E and absolute T equals the number of line-rooted trees with group E and absolute T.

Now take a symmetric tree T. The order of $\Gamma(T)$ is at least 2. If there exist rooted trees T' or line-rooted trees T'' with group E and absolute T, then $\Gamma(T)$ has order 2 and the non-identity element permutes the central points of T. The branches $\langle p, q\rangle$ and $\langle q, p\rangle$ of T at the central points p and q, considered as rooted trees, have group E. We may regard the rooted trees obtained from these branches by deleting their root points as the subgraphs of T obtained on deleting the line joining the two central points. If these branches have n points each, then there are n rooted trees T' such that $|T'| = T$ and $n - 1$ line-rooted trees T'' such that $|T''| = T$. Hence for each symmetric tree whose group has order 2, the number of rooted trees with absolute T is one greater than the number of line-rooted trees with absolute T.

The counting series for symmetric trees whose group has order 2 is clearly $U(x^2)$. Therefore, summing over all symmetric and non-symmetric absolutes T of rooted trees T' and line-rooted trees T'' with group E, the group of whose absolute is not E, we have:

$$V(x) = U(x^2).$$

Hence
$$\begin{aligned} u(x) &= U(x) - Z(A_2 - S_2, U(x)) - U(x^2) \\ &= U(x) - Z(S_2, U(x)) \\ &= U(x) - \tfrac{1}{2}[U^2(x) + U(x^2)]. \end{aligned}$$

Explicitly,
$$u(x) = x + x^7 + x^8 + 3x^9 + 6x^{10} + 15x^{11} + 29x^{12} + \cdots.$$

Diagrams of all trees with $\leqslant 12$ points whose automorphism group is the identity are presented in Appendix III.

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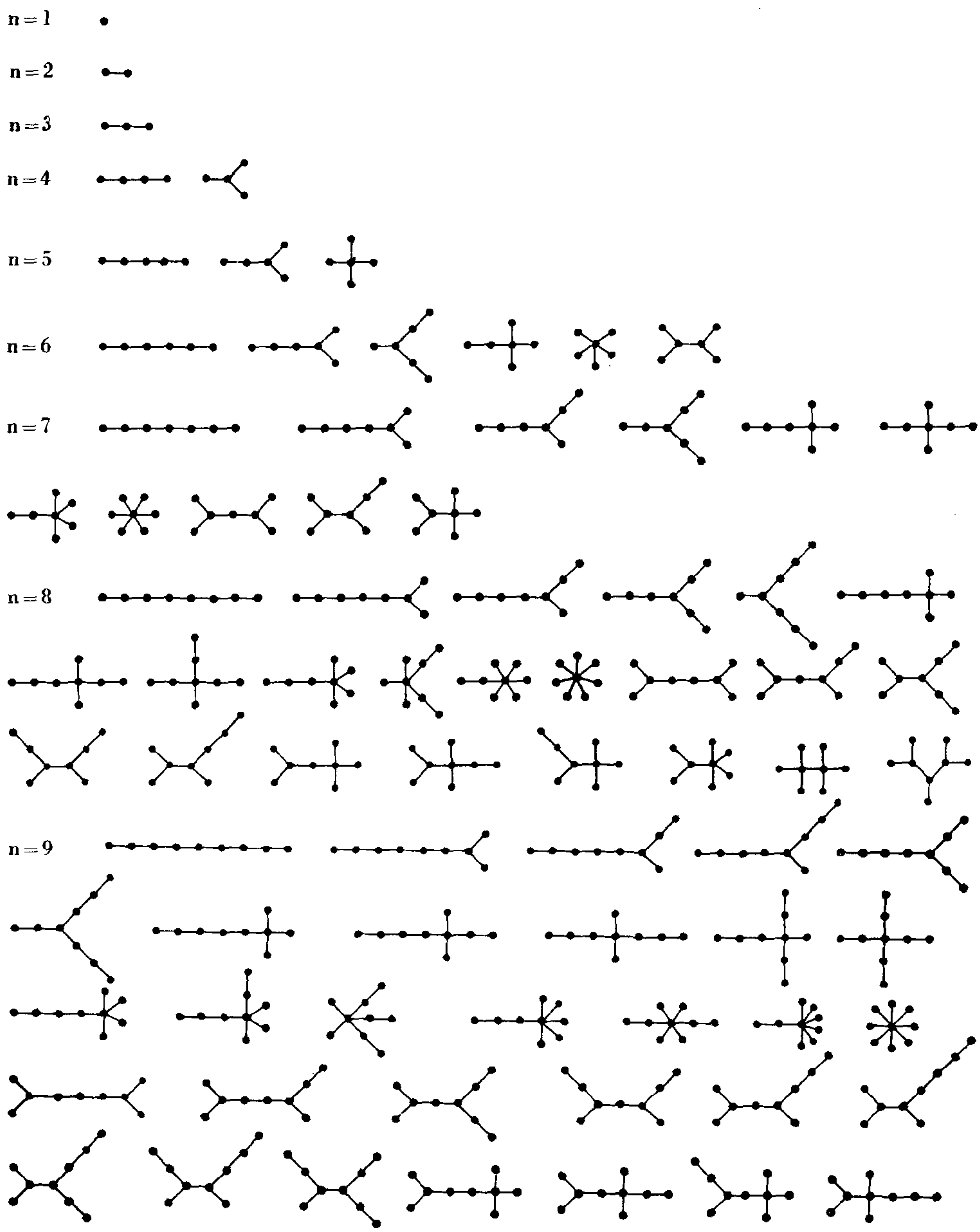
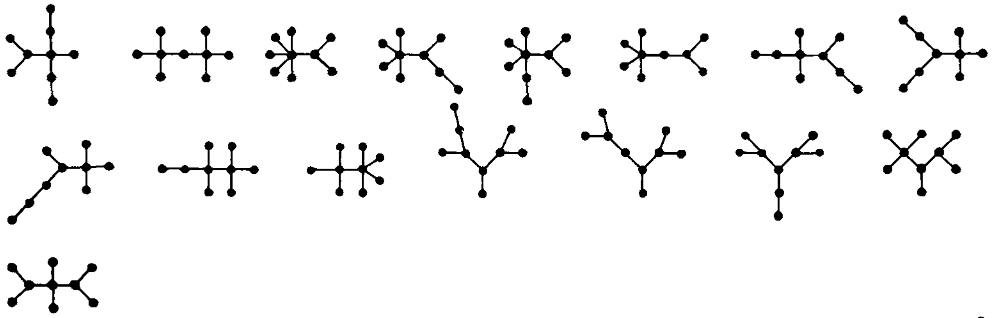
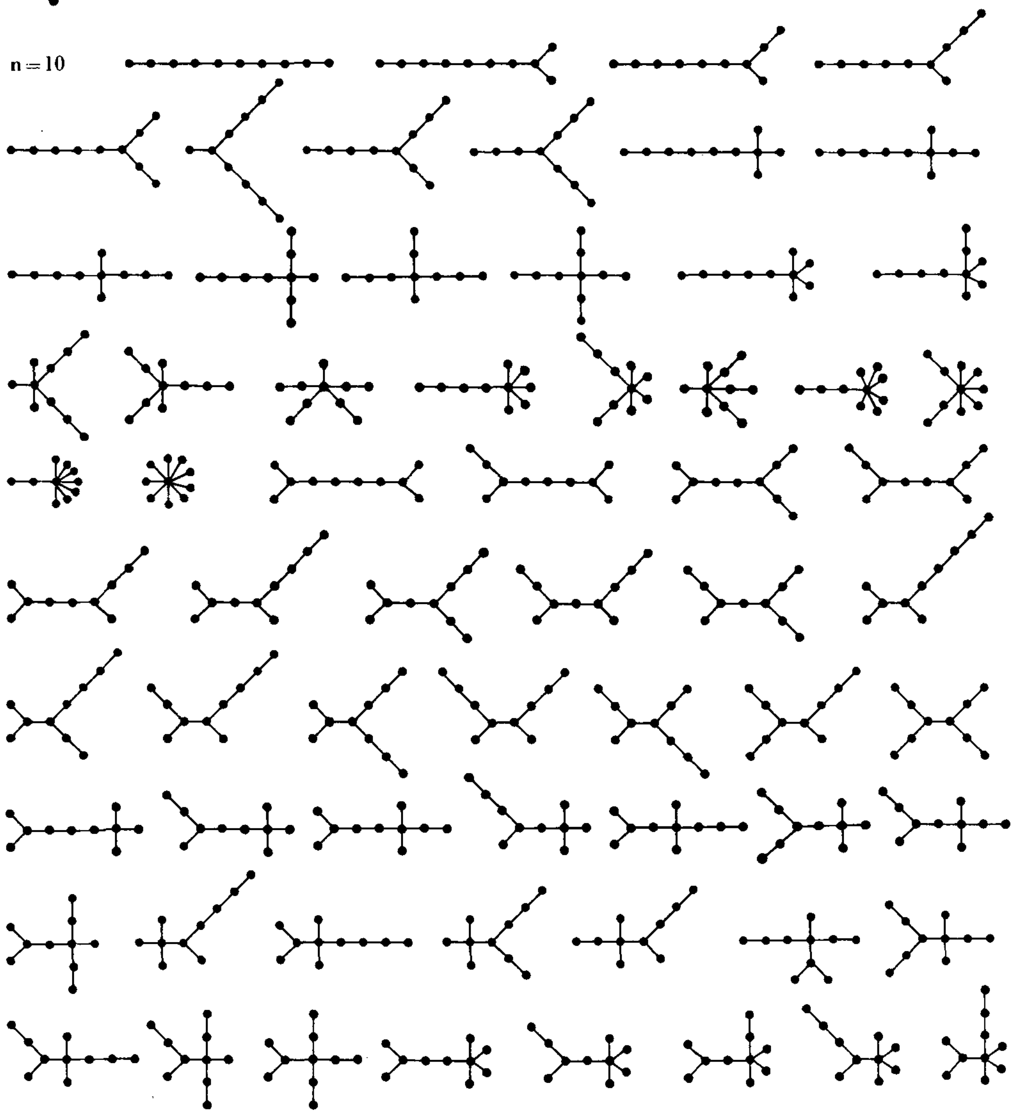
Appendix I

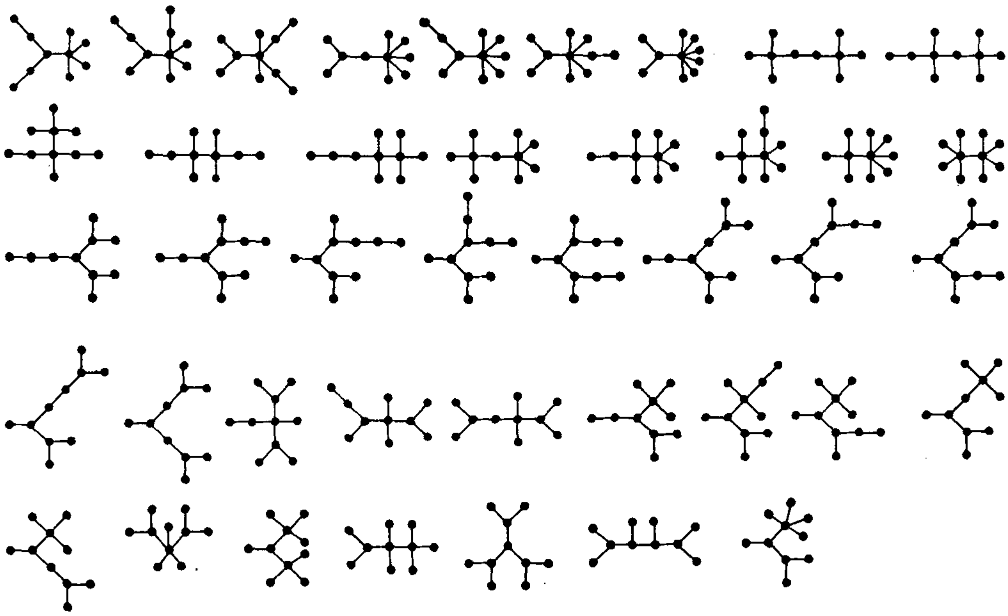
Diagrams of all Trees with $n \leqslant 10$ Points





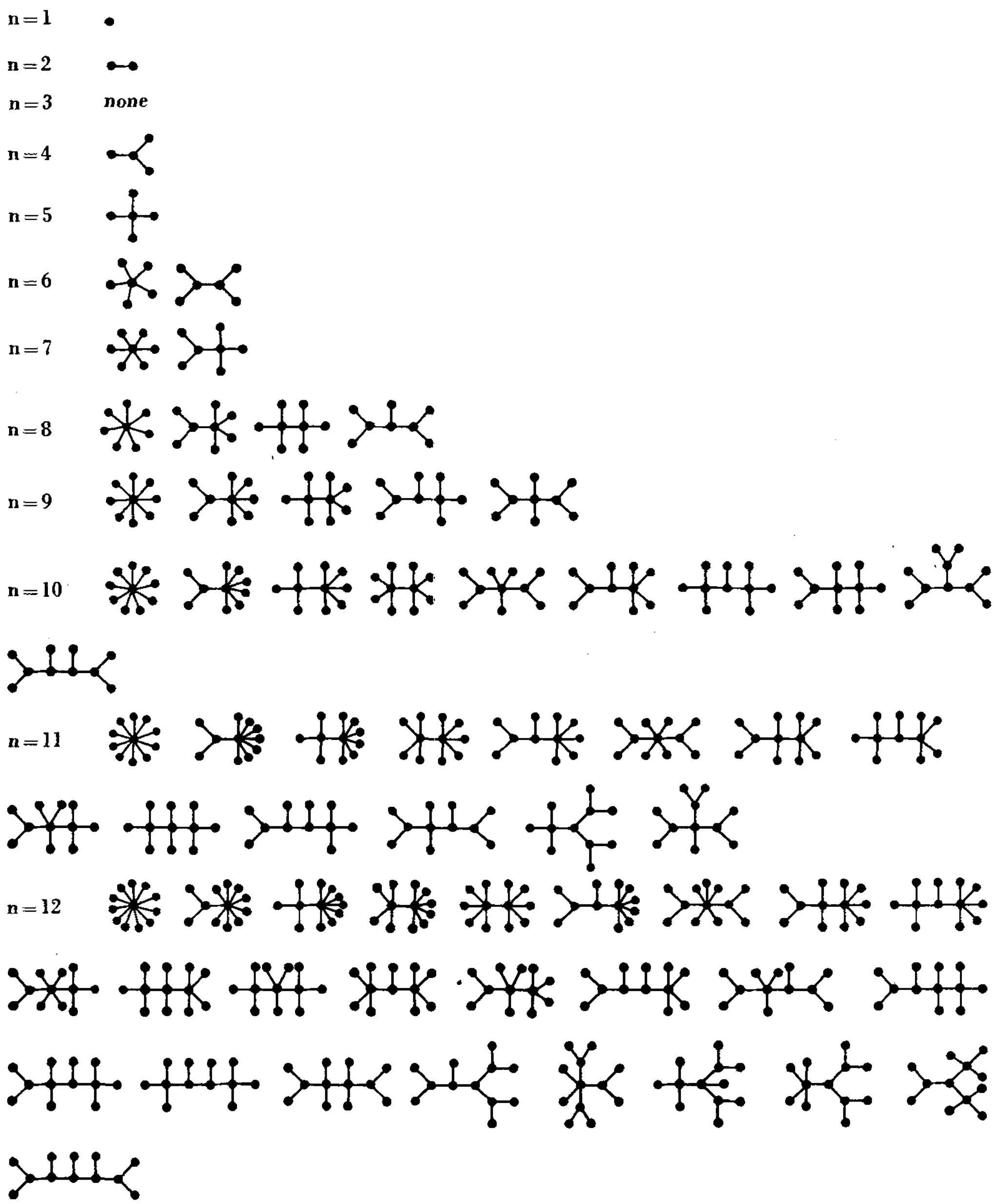
n = 10





Appendix II

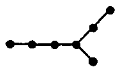
Diagrams of all Homeomorphically Irreducible Trees with $n \leqslant 12$ Points

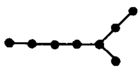


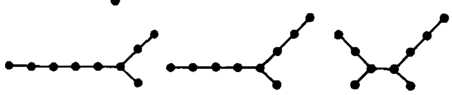
Appendix III

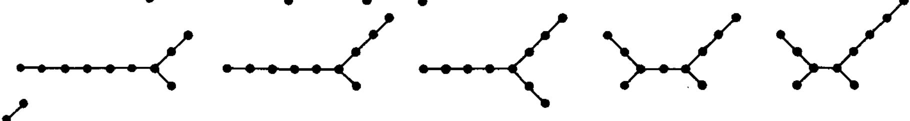
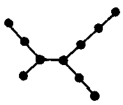
Diagrams of all Trees with $n \leqslant 12$ Points whose Automorphism Group is the Identity

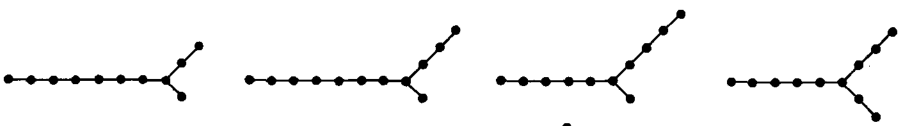
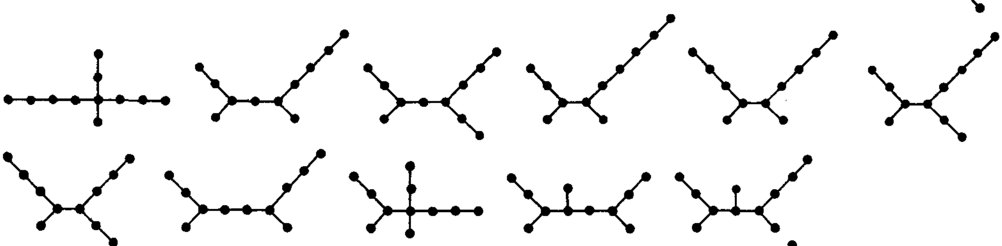
We see from Appendix I that there are no trees with $2 \leqslant n \leqslant 6$ points whose automorphism group is the identity. The tree with 1 point obviously has the identity group.

n = 7


n = 8


n = 9


n = 10



n = 11



n = 12