REGULAR ITERATION OF REAL AND COMPLEX FUNCTIONS

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1. Introduction

We shall deal with the iteration of analytic functions of a complex variable and analytic solutions of Schroeder's functional equation

$$\chi(f(z)) = \alpha\, \chi(z). \tag{1}$$

More specifically, we shall be concerned with the iteration of functions of the form

$$f(z) = \sum_{p=m}^{\infty} a_p z^p, \; m \geqslant 1, \; a_m > 0 \tag{2}$$

where all coefficients a_p are real; in certain cases the series (2) will not converge for any $z \neq 0$ but represent $f(z)$ asymptotically in a specified neighbourhood of the positive real axis. We shall also have opportunity to consider the iteration of continuous functions of a real variable without assuming analyticity.

A great deal is known about analytic solutions of Schroeder's equation in the neighbourhood of a fixpoint ξ where $f(\xi) = \xi$; most results refer to the case when both $f(z)$ and the Schroeder function $\chi(z)$ are analytic at $z = \xi$. The classical result of Koenigs [5] states that if $f(z)$ has a convergent power series

$$f(z) = \xi + \sum_{p=1}^{\infty} a_p (z-\xi)^p, \; |z-\xi| < R_1, \; R_1 > 0 \tag{3}$$

and $a_1 \neq 0$, $|a_1| \neq 1$, then Schroeder's equation has a solution

$$\chi(z) = \sum_{q=1}^{\infty} b_q (z-\xi)^q, \; b_1 = 1, \; |z-\xi| < R > 0. \tag{4}$$

For any determination of a_1^σ which satisfies the condition

$$a_1^1 = a_1, \quad a_1^\sigma a_1^\tau = a_1^{\sigma+\tau} \tag{5}$$

the functions

$$f_\sigma(z) = \chi_{-1}(a_1^\sigma \chi(z)), \quad \chi_{-1}(\chi(z)) = z \tag{6}$$

form a family of iterates of $f(z)$, i.e.

$$f_1(z) = f(z), \quad f_\sigma(f_\tau(z)) = f_{\sigma+\tau}(z) \tag{7}$$

for every real (and even complex) σ, τ. Clearly each $f_\sigma(z)$ is analytic at $z = \xi$,

$$f_\sigma(z) = \xi + \sum_{p=1}^{\infty} a_p^{(\sigma)} (z-\xi)^p, \quad |z-\xi| < R_\sigma > 0. \tag{8}$$

Note that the coefficients a_1^σ are uniquely determined from the condition

$$a_1^{(\sigma)} = a_1^\sigma \tag{9}$$

and the commutation relation

$$f_\sigma(f(z)) = f(f_\sigma(z)). \tag{10}$$

Every determination of a_1^σ gives a family of formal power series (8) and the theorem of Koenigs asserts that each of these series converges for some $z \neq 0$.

In the case of $|f'(\xi)| = |a_1| = 1$ the behaviour of Schroeder's equation depends quite sensitively on the arithmetical character of the amplitude of a_1. In the "irrational" case, i.e. if a_1 is not a root of unity, the coefficients b_q in (4) and $a_p^{(\sigma)}$ in (8) can be calculated uniquely from (1) and (6) (or (9), (10)); but this is generally not possible if a_1 is a root of unity. Also, in the irrational case the convergence of the formal solution (4) seems to depend on the arithmetic nature of the amplitude of a_1. For instance if a_1 is such that

$$\liminf_{n=1,2,\cdots} \left| \sqrt[n]{a_1^n - 1} \right| = 0$$

then there exist functions (3) with divergent Schroeder series (4) (Cremer [2]); on the other hand, if

$$\log |a_1^n - 1| = O(\log n) \quad (n \to \infty)$$

then the series (4) converges for every $f(z)$ with $f(\xi) = \xi$, $f'(\xi) = a_1$ (Siegel [8]).

If a_1 is an nth root of unity, a Schroeder function of the form (4) exists if and only if the nth iterate of $f(z)$ is the identity function, $f_n(z) = z$. For instance, if $f(z)$ has a power series

$$f(z) = z + \sum_{p=m}^{\infty} a_p (z-\xi)^p, \ 1 < m < \infty, \ a_m \neq 0, \ |z-\xi| < R_1 > 0, \tag{11}$$

Schroeder's equation has not even a formal (non-convergent) solution of the form (4). This case is particularly interesting from the point of view of iteration because the commutation relations (10) have a unique formal solution

$$f_\sigma(z) = z + \sum_{p=m}^{\infty} a_p^{(\sigma)} (z-\xi)^p, \quad a_m^{(\sigma)} = \sigma a_m, \tag{12}$$

and it can be expected that the $f_\sigma(z)$, if they exist at all, form a family of iterates of $f(z)$. An obvious difficulty arises from the fact that the convergence of (11) does not necessarily imply the convergence of (12). In the particular case of

$$f(z) = e^z - 1 = \sum_{p=1}^{\infty} \frac{1}{p!} z^p \quad (\xi = 0)$$

I. N. Baker has proved(1) that $f_\sigma(z)$ diverges for every $z \neq 0$ and real σ except when σ is an integer(2). The main result of this paper is that if $\xi = 0$ and all coefficients in (11) are real then $f(z)$ has exactly one family of analytical iterates $f_\sigma(z)$ which has an asymptotic expansion (12) when 0 is approached along the positive real axis.

The proof is based on an algorithm due to P. Lévy [6] which leads to an analytic solution of Abel's equation

$$\lambda(f(z)) = \lambda(z) - 1; \tag{13}$$

from $\lambda(z)$ one obtains a solution of Schroeder's equation by taking $\chi(z) = e^{\lambda(z)}$. It turns out that $\lambda(z)$ is asymptotically equal to $-1/z$, so that the corresponding Schroeder functions behaves like $1/\log(1/z)$ when $z \to 0$ from the right. The result clarifies the significance of the expansion (12) and shows that from the point of view of iteration it is not very natural to require that the Schroeder function be analytic at the fixpoint itself.

Another instance when it is clear that the Schroeder function cannot be holomorphic at the fixpoint is when $a_1 = 0$ in (3). Let m be a positive integer, $m > 1$, and suppose that

$$f(z) = z^m \sum_{p=0}^{\infty} a_p z^p, \ a_0 \neq 0, \ |z| < R_1 > 0. \tag{14}$$

(1) I. N. Baker, Zusammensetzungen ganzer Funktionen, Math. Z. 69, 121–163.

(2) There are cases in which $f_\sigma(z)$ converges for every real or complex σ; the simplest example is $f_\sigma(z) = \dfrac{z}{1+\sigma z}$. I owe this remark to I. N. Baker and N. G. de Bruijn.

For the sake of clarity we have assumed that $\xi = 0$. The expected form of the fractional iterates is

$$f_\sigma(z) = z^{m^\sigma} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{pq}^{(\sigma)} z^{pm^\sigma + q} \tag{15}$$

where

$$a_{00}^{(\sigma)} = a_0^{(m^\sigma - 1)/(m-1)} \tag{16}$$

Again, the coefficients $a_{pq}^{(\sigma)}$ can be calculated formally from the commutation relation (10) and condition (16); but the series (15) will usually not converge. We shall find, as in the previous case, that there exists a family of analytical iterates which has an asymptotic expansion (15) when $z \to 0$. The precise form of the statement will be formulated later.

2. Schroeder Iterates

Suppose that $f(z)$ is analytic in a domain D which has the property that $z \in D$ implies $f(z) \in D$. We define the natural iterates of $f(z)$ relatively to D [1] by the inductive relations

$$f_0(z) = z, \quad f_{n+1}(z) = f(f_n(z)), \quad n = 0, 1, 2, \ldots . \tag{1}$$

Generally let σ be a continuous real variable; we say that the functions $f_\sigma(z)$ form a family of fractional iterates of $f(z)$ relatively to the domain D if $f_\sigma(z)$ is defined for every $\sigma \geqslant 0$, $z \in D$, and the following conditions are satisfied:

(i) $f_\sigma(z)$ is continuous in σ, analytic in z, and $f_\sigma(z) \in D$ for $\sigma \geqslant 0$, $z \in D$.

(ii) $f_0(z) = z$, $f_1(z) = f(z)$ for $z \in D$.

(iii) $f_\sigma(f_\tau(z)) = f_{\sigma+\tau}(z)$

for every $\sigma \geqslant 0$, $\tau \geqslant 0$, $z \in D$.

The notation is clearly consistent with the definition of natural iterates and $f_\sigma(z)$ interpolates the sequence $f_n(z)$, $n = 0, 1, 2, \ldots$.

Given an arbitrary (real or complex) "multiplier" $\alpha \neq 0$ and $z_0 \in D$, define the function $\varphi(t)$ by

$$\varphi(\alpha^\sigma) = f_\sigma(z_0), \quad \sigma \geqslant 0.$$

Then

$$f(\varphi(\alpha^\sigma)) = f_{1+\sigma}(z_0) = \varphi(\alpha^{1+\sigma}) = \varphi(\alpha \cdot \alpha^\sigma),$$

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[1] The definition is relative to D. It can be shown by suitable examples that $f(f(z))$ as an analytic function is not determined unambiguously by $f(z)$. The definition can be easily extended to cases when D is a domain on a Riemann surface rather than on the complex plane. We shall later be interested in the case when D is on the Riemann surface of $\log z$.

i.e. $\varphi(t)$ satisfies the functional relation

$$f(\varphi(t)) = \varphi(\alpha t) \tag{2}$$

and its inverse $\chi(z) = \varphi_{-1}(z)$ (if it exists) satisfies Schroeder's equation [7]

$$\chi(f(z)) = \alpha\chi(z). \tag{3}$$

Thus under very general conditions the fractional iteration of $f(z)$ gives rise to a solution of Schroeder's functional equation.

Conversely, one can use Schroeder's equation to obtain a family of fractional iterates. For a given multiplier α let $\varphi(t)$ be a solution of (2) such that $\varphi(t)$ is analytic at $t=0$, $\varphi'(0) \neq 0$, and denote by $\chi(z) = \varphi_{-1}(z)$ the inverse of $\varphi(z)$; then $\chi(z)$ is analytic at $z = \xi$,

$$\xi = \varphi(0) \tag{4}$$

and it satisfies the equation (3). Furthermore,

$$f_\sigma(z) = \chi_{-1}(\alpha^\sigma \chi(z)) = \varphi(\alpha^\sigma \varphi_{-1}(z)) \tag{5}$$

is analytic at ξ for every real or complex σ and it satisfies the functional relation

$$f_\sigma(f_\tau(z)) = f_\tau(f_\sigma(z)) = f_{\sigma+\tau}(z), \quad f_1(z) = f(z) \tag{6}$$

in a suitable neighbourhood of ξ, provided that α^σ is determined so that

$$\alpha^1 = \alpha, \quad \alpha^\sigma \alpha^\tau = \alpha^{\sigma+\tau} \quad \text{for every } \sigma, \tau.$$

This can always be achieved in infinitely many ways; if

$$\alpha = a\, b^{ib}, \quad \sigma = s + it, \quad a, b, s, t \text{ real}, \quad -\pi < b \leqslant \pi,$$

and if k, l are arbitrary but fixed integers, then

$$a^\sigma = a^{s-bt}\, e^{i(t \log a + bs) + 2\pi(ksi + lt)} \tag{7}$$

is a suitable determination. Therefore in general there are infinitely many distinct families of iterates which can be derived from a given Schroeder function (i.e. solution of Schroeder's equation) by the process (5). Note that by (2) and (4), ξ is a fixpoint of $f(z)$;

$$f(\xi) = \xi. \tag{8}$$

For the iterates (5) to be meaningful it is not necessary to assume that $\varphi(z)$ is holomorphic at 0. Suppose that the multiplier α in (2) is real, positive and $\alpha \neq 1$;

suppose also that the solution $\varphi(z)$ is holomorphic and schlicht on an angular domain,

$$S: z=\varrho e^{i\psi}, \quad -\theta<\psi<\theta, \quad 0<\varrho<r(\psi) \tag{9}$$

where $0<\theta\leqslant\pi$, and $r(\psi)$ is a continuous positive valued function of r. If $z\to\varphi(z)$ maps S onto D (where D is possibly a domain on the Riemann surface of log z) and $0<\alpha<1$ then the function defined by formula (5) is holomorphic on D for every real $\sigma>0$, and the $f_\sigma(z)$ form a family of fractional iterates of $f(z)$ relatively to D. If $\alpha>1$, the same is true for the inverse function $f_{-1}(z)$. If, furthermore,

$$\xi=\lim \varphi(z) \tag{10}$$

exists when $z\to 0$ on S, then by (3) and (10),

$$\lim f(z)=\xi \tag{11}$$

when $z\to\xi$ on D; hence ξ is a fixpoint of $f(z)$ relatively to the domain D (i.e. with respect to approach from within D). We shall say that ξ is a singular fixpoint of $f(z)$ if (11) is true for some domain D of which ξ is an accessible boundary point; the definition includes the case when ξ is an ordinary fixpoint (8).

Henceforth we shall call $\chi(z)$ a Schroeder function of $f(z)$ if and only if it is the inverse of a solution of (2) which is either holomorphic at 0 itself (with $\varphi'(0)\neq 0$) or is holomorphic and schlicht on a domain (9) and has the properties stated in connection with (9) (including (11)).(1) In particular the Schroeder iterates (5) always refer to a fixpoint of $f(z)$ in the sense of (11).

We note that if $\chi(z)$ is a Schroeder function belonging to the multiplier α and c, β are positive numbers then

$$\chi^*(z)=c(\chi(z))^\beta \tag{12}$$

is also a Schroeder function with multiplier α^β; but the iterates derived from $\chi^*(z)$ are identical with those derived from $\chi(z)$.(2) A more general transformation, which leads to essentially new solutions of equation (3), is

$$\chi^*(z)=\chi(z)\,g(\lambda(z)), \quad g(t+1)=g(t) \tag{13}$$

where

$$\lambda(z)=\log\chi(z)/\log\alpha \tag{14}$$

(1) It is a matter of convention whether we prefer to call $\chi(z)$ or $\varphi(z)$ a Schroeder function of $f(z)$; for later convenience we have chosen $\chi(z)$. In the real variable case it will be necessary also to admit non-analytic (real) Schroeder functions.

(2) In case of an ordinary fixpoint c and β are allowed to be complex.

(with a suitable interpretation of log) and $g(t)$ is an arbitrary periodic (analytic) function with period 1. Combination of (12) and (13) gives the most general transformation of Schroeder functions; for Schroeder functions with the same multiplier α are easily seen to be related according to (13).

It follows from the above that the existence of a single analytic solution of Schroeder's equation usually involves the existence of an infinity of such solutions. This raises the question whether it is possible to determine a "best" analytical solution relatively to a given fixpoint. We shall find that in the case of an ordinary fixpoint such an exceptional solution does in fact exist, provided that $f(z)$ satisfies certain reality conditions; the property which distinguishes the "good" iterates is their asymptotic behaviour near the fixpoint.

We conclude this section with an observation of Schroeder which can frequently be used to transform $f(z)$ in a more convenient form. Let $g(z)$ be schlicht on $D=\varphi(S)$, and write $f^*(z)=g(f(g_{-1}(z)))$; then $\xi^*=g(\xi)$ is a fixpoint of $f^*(z)$ and $\chi^*(z)=\chi(g_{-1}(z))$ is a Schroeder function of $f^*(z)$ relatively to ξ^*. The corresponding family of Schroeder iterates is $f^*_\sigma(z)=g(f_\sigma(g_{-1}(z)))$. Therefore, every solution for $f^*(z)$ supplies automatically a solution for $f(z)$.

We can use this remark to transfer the fixpoint to the origin. If ξ is finite, we take $g(z)=z-\xi$, i.e. $f^*(z)=f(z+\xi)-\xi$; if the fixpoint is at infinity, we take $g(z)=\frac{1}{z}$, $f^*(z)=1/f\left(\frac{1}{z}\right)$. In all future work it will be assumed that the fixpoint is at 0.

3. Koenigs Iterates

In this and the following section we shall merely state results; proofs will be supplied in sections 5 to 9.

The fundamental theorem of Koenigs [5], applied to the case when the fixpoint is at 0, states that if

$$f(z)=\sum_{n=1}^{\infty} a_n z^n, \quad |z|<R_1, \tag{1}$$

$$0<|a_1|<1, \tag{2}$$

then

$$\chi(z)=\lim_{n\to\infty} a_1^{-n} f_n(z) \tag{3}$$

exists and is analytic in a suitable neighbourhood of 0. Furthermore, $\chi'(0)=1$, and $\chi(z)$ satisfies Schroeder's equation

$$\chi(f(z)) = a_1 \chi(z) \tag{4}$$

with multiplier a_1.

The effectiveness of the algorithm (3) is not restricted to analytic functions. This was observed by H. Kneser [3] who showed that for the Koenigs function (3) to exist it is sufficient to know that $f(z)$ is defined in a neighbourhood of 0 and that for some posive δ,

$$f(z) = a_1 z + O(|z|^{1+\delta}), \quad 0 < |a_1| < 1 \tag{5}$$

when $z \to 0$. Kneser's extension of the Koenigs theorem is only significant in the case that $f(z)$ is not an analytic function; its most important application therefore is to real functions of a real variable. Its usefulness for the purposes of iteration is limited by the fact that condition (5) does not by itself ensure the existence of the inverse of $\chi(z)$. This is shown by the example of

$$f(z) = \frac{z}{2} + \frac{1}{3\pi} z^2 \sin \frac{\pi}{|z|}, \quad z \neq 0, \quad f(0) = 0,$$

which satisfies the Kneser condition $f(z) = \frac{1}{2} z + O(|z|^2)$. Clearly $f(z)$ is continuous and strictly monotone increasing for $x \geq 0$, also $\chi(x) = \lim_{n\to\infty} 2^n f_n(x)$ is monotone non-decreasing. We show that $\chi(x)$ is not strictly increasing.

For any positive integer m, $f(2^{-m}) = 2^{-m-1}$, hence $f_n(2^{-m}) = 2^{-m-n}$, $\chi(2^{-m}) = \lim_{n\to\infty} 2^n f_n(2^{-m}) = 2^{-m}$. We prove that

$$\chi(x) = \lim_{n\to\infty} 2^n f_n(x) = 2^{-m} \quad \text{for } 2^{-m} \leq x < 2^{-m} + \tfrac{1}{3} 2^{-2m}. \tag{6}$$

Write $x = 2^{-m} + 2^{-2m}\varepsilon$, $0 \leq \varepsilon < \frac{1}{3}$. We have

$$\sin\frac{\pi}{x} = \sin(2^m \pi/(1 + 2^{-m}\varepsilon)) = -\sin(\pi\varepsilon/(1 + 2^{-m}\varepsilon)),$$

$$f(x) = \frac{x}{2} + \frac{1}{3\pi} x^2 \sin\frac{\pi}{x} = 2^{-m-1} + 2^{-2m-1}\varepsilon - \frac{1}{3\pi} 2^{-2m}(1 + 2^{-m}\varepsilon)^2 \sin(\pi\varepsilon/(1 + 2^{-m}\varepsilon))$$

$$< 2^{-m-1} + 2^{-2m-1}\varepsilon - \frac{\sqrt{3}}{2\pi} 2^{-2m}\varepsilon$$

$$< 2^{-m-1} + 2^{-2m-2}\varepsilon,$$

$$> 2^{-m-1},$$

$$f(x) = 2^{-m-1} + 2^{-2m-2}\varepsilon_1, \quad 0 \leq \varepsilon_1 < \varepsilon.$$

By induction $f_n(x) = 2^{-m-n} + 2^{-2m-2n}\varepsilon_n$, $0 \leq \varepsilon_n < \frac{1}{3}$, which implies (6).

The following is a real variable version of the Koenigs-Kneser theorem which gives a sufficient condition for the existence of $\chi_{-1}(x)$.

THEOREM 1 a. *Suppose that $f(x)$ is continuous, strictly monotone increasing for $0 < x \leqslant d$ and $0 < f(x) < x$;*(1) *suppose further that $f'(x)$ exists for $0 < x < d$ and*

$$f'(x) = a + O(x^{\delta}), \quad 0 < a < 1, \quad \delta > 0 \quad (x \downarrow 0). \tag{7}$$

Then

$$\chi(x) = \lim_{n\to\infty} a^{-n} f_n(x) \tag{8}$$

exists, is continuous and strictly monotone increasing for $0 < x \leqslant d$ and differentiable with respect to x for $0 < x < d$; furthermore,

$$f_{\sigma}(x) = \chi_{-1}(a^{\sigma}\chi(x)) \tag{9}$$

satisfies

$$f_1(x) = f(x), \quad f_{\sigma}(f_{\tau}(x)) = f_{\sigma+\tau}(x) \tag{10}$$

for every real σ, τ, and

$$f_{\sigma}(x) = a^{\sigma} x + O(x^{1+\delta}) \quad (x \downarrow 0). \tag{11}$$

The theorem suggests a modification of the definition of Schroeder iterates for real variable functions. Suppose that $f(x)$ satisfies the conditions stated in Theorem 1 a. We shall call $\chi(x)$ a *real* Schroeder function of $f(x)$ (or briefly a Schroeder function of $f(x)$ if it is clear from the context that we are not dealing with the analytical case) if $\chi(x)$ is positive valued, continuous, strictly monotone (increasing or decreasing) for $0 < x \leqslant d$, and

$$\chi(f(x)) = \alpha\chi(x) \tag{12}$$

for some positive constant $\alpha \neq 1$. Since $0 < f(x) < x$, $\chi(x)$ is monotone increasing if $0 < \alpha < 1$ and monotone decreasing if $\alpha > 1$. Furthermore,

$$\lim_{x\downarrow 0} \chi(x) = 0 \quad \text{if } 0 < \alpha < 1, \quad \lim_{x\downarrow 0} \chi(x) = \infty \quad \text{if } \alpha > 1. \tag{13}$$

The real Schroeder iterates corresponding to $\chi(x)$ are obtained from

$$f_{\sigma}(x) = \chi_{-1}(\alpha^{\sigma}\chi(x)); \tag{14}$$

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(1) The assumption $f(x) < x$ is necessary to make the fixpoint "attractive". In the case of $f(x) > x$, $0 < x \leqslant d$, we replace $f(x)$ by its inverse in all future considerations. The case when $f(x) - x$ has infinitely many zeros in every neighbourhood of 0 will not be considered.

they are clearly continuous with respect to σ, strictly monotone decreasing with σ and

$$\lim_{\sigma\to\infty} f_\sigma(x) = 0 \tag{15}$$

for every given x, $0 < x \leqslant d$. It follows from this that conversely, given $\alpha > 0$, $\alpha \neq 1$, the relation

$$\chi_0(f_\sigma(d)) = \alpha^\sigma \tag{16}$$

determines the Schroeder function $\chi_0(x)$ completely, provided that the iterates $f_\sigma(x)$ are known. Any other Schroeder function associated with the family $f_\sigma(x)$ is obtained by the transformation

$$\chi(x) = c\,(\chi_0(x))^\beta, \quad c > 0, \quad \beta \neq 0; \tag{17}$$

the particular representative (16) is normalized by the conditions $\chi(d) = 1$, multiplier α.

Condition (7) of Theorem 1 a is sufficient, but, of course, not necessary for the existence of the inverse of the Koenigs limit (8). The following theorem is therefore of interest as it shows that the existence of a family of real Schroeder iterates with the asymptotic property (11) necessarily implies the existence of the Koenigs iterates (with a slightly modified definition of the Koenigs functions (8)) and the two families are identical. In particular, the asymptotic property (11) characterizes the Koenigs iterates uniquely.

THEOREM 2 a. *Let $f(x)$ be continuous, strictly monotone increasing for $0 < x \leqslant d$ and $0 < f(x) < x$. Suppose that $f(x)$ has a family of real Schroeder iterates which satisfy the asymptotic relation*

$$\lim_{x\downarrow 0} \frac{1}{x} f_\sigma(x) = a^\sigma, \quad 0 < a < 1 \tag{18}$$

for every positive σ. Let $\chi(x)$ be a real Schroeder function with multiplier a from which the family $f_\sigma(x)$ has been derived; then

$$\chi(x) = \chi(d) \lim_{n\to\infty} \{f_n(x)/f_n(d)\}. \tag{19}$$

The last limit is, of course, identical with the Koenigs limit (8) whenever the latter exists, but it is slightly more general than (8). The theorem can be interpreted as stating that the Koenigs iterates behave more "regularly" in the neighbourhood of 0 than any other family of iterates. This is not an empty statement; for the real

Schroeder equation (12) has infinitely many continuous strictly increasing solutions for every positive multiplier α, $0<\alpha<1$. In fact we can define $\chi(x)$ as an arbitrary continuous strictly increasing positive valued function in the interval $f(d)\leqslant x\leqslant d$, subject to the condition that $\chi(f(d))=\alpha\chi(d)$; $\chi(x)$ is then uniquely determined for $0<x\leqslant d$ by the functional equation (12). The particular Schroeder function

$$\chi(x)=\lim_{n\to\infty}\{f_n(x)/f_n(d)\} \tag{20}$$

if it exists for $0<x\leqslant d$ will be called a *principal* Schroeder function of $f(x)$; its multiplier is $a=\lim_{x\downarrow 0}\frac{1}{x}f(x)$ and it is normalized so that $\chi(d)=1$.

We shall call $f(x)$ *regular* with respect to iteration (or briefly regular, if there is no danger of misunderstanding) if (and in the case of $0<a=\lim_{x\downarrow 0}\frac{1}{x}f(x)<1$ only if) $f(x)$ has a family of real Schroeder iterates which satisfies the condition (18) of Theorem 2 a for every positive σ. Clearly it then also satisfies the relation for negative values of σ. The fractional iterates which satisfy the asymptotic relation (18) are themselves regular, and the theorem shows that they are uniquely determined by $f(x)$.

Regularity for the cases $a=0$ and $a=1$ will be formulated later.

From the point of view of analytic functions the Koenigs–Kneser theorem has another significant extension which refers to the singular case of Schroeder iterates. There are various possibilities for such an extension, and the one given below refers to the case when $f(z)$ has an asymptotic expansion at 0. Briefly, the theorem states that if $f(z)$ has an asymptotic expansion

$$f(z)\sim\sum_{p=0}^{\infty}a_p z^{1+\beta p},\quad \beta>0,\quad 0<a_0<1,\quad (z\downarrow 0) \tag{21}$$

with real coefficients a_p, then its Koenigs iterates have an expansion of similar character,

$$f_\sigma(z)\sim\sum_{p=0}^{\infty}a_p^{(\sigma)}z^{1+\beta p},\quad (z\downarrow 0) \tag{22}$$

where $a_0^{(\sigma)}=(a_0)^\sigma$.

Note that the coefficients $a_p^{(\sigma)}$ can be determined uniquely from the commutation relation

$$f(f_\sigma(z))=f_\sigma(f(z)),$$

i.e. from the formal relation

$$\sum_{p=0}^{\infty} a_p \left[\sum_{q=0}^{\infty} a_q^{(\sigma)} z^{1+\beta q} \right]^{1+\beta p} = \sum_{\sigma=0}^{\infty} a_q^{(\sigma)} \left[\sum_{p=0}^{\infty} a_p z^{1+\beta p} \right]^{1+\beta q},$$

and the condition $a_0^{(\sigma)} = (a_0)^\sigma$; clearly each $a_p^{(\sigma)}$ is real.

THEOREM 3 a. *Suppose that $f(z)$ is real for $z = x$, $0 < x \leqslant r$, and holomorphic on an angular (semi-closed) domain*

$$A(\theta, r) = \{z;\ z = x + iy,\quad 0 < x \leqslant r,\quad -\theta x \leqslant y \leqslant \theta x\},\quad r > 0,\quad \theta > 0. \tag{23}$$

Suppose further that for every fixed positive integer k and for certain real coefficients a_p,

$$f(z) = \sum_{p=0}^{k} a_p z^{1+\beta p} + O(|z|^{1+\beta k}),\quad \beta > 0,\quad 0 < a_0 < 1 \tag{24}$$

when $z \to 0$ in $A(\theta, r)$ (so that $f(z)$ is asymptotically differentiable at 0 in the sense of § 7). Then there exist positive numbers θ_0, r_1 so that the following is true;

(i) $$f_n(z) \in A(\theta, r) \text{ for } z \in A(\theta_1, r_1)$$

and every non-negative integer n.

(ii) $$\chi(z) = \lim_{n \to \infty} a_0^{-n} f_n(z)$$

exists and is holomorphic and schlicht on $A(\theta_1, r_1)$.

(iii) $$f_\sigma(z) = \chi_{-1}(a_0^\sigma \chi(z))$$

exists for every $\sigma > 0$ and $z \in A(\theta_1, r_1)$, and forms a family of Schroeder iterates which satisfies

$$f_\sigma(z) = \sum_{p=0}^{k} a_p^{(\sigma)} z^{1+\beta p} + O(|z|^{1+\beta k}),\quad a_0^{(\sigma)} = a_0^\sigma \tag{25}$$

when $z \in A(\theta_1, r_1)$.

In particular $f(x)$, $0 < x \leqslant d$ for a suitable $0 < d \leqslant r$ is regular with respect to iteration so that the family $f_\sigma(z)$ is uniquely characterized by the asymptotic property (24).

The algorithm of Koenigs is not applicable directly to the case when $f'(0) = a_1 = 0$ in (1), i.e. if

$$f(z) = \sum_{p=m}^{\infty} a_p z^p,\ m > 1,\ a_m \neq 0 \quad |z| < R_1 > 0; \tag{26}$$

but by making use of remark at the end of § 2, it can be easily transformed into a form which comes under the range of the Koenigs algorithm. First, by a transformation $g(z)=z/b$ where $b^{m+1}=a_m$, we achieve that $a_m=1$ in (26). Next, by applying the transformation $g(z)=1/\log \frac{1}{z}$ we transform $f(z)$ into

$$f^*(z)=1/\log(1/f(e^{-1/\zeta})). \tag{27}$$

We have for $\zeta \in A(1, r)$ (definition (23)) where r is a suitable positive number,

$$f(e^{-1/\zeta})=e^{-m/\zeta}\ (1+O(|e^{-1/\zeta}|))$$
$$\log(1/f(e^{-1/\zeta}))=m/\zeta+O(1);$$

the O-symbols refer to $\zeta\to 0$ in $A(1, r)$. Hence by (27), $f^*(\zeta)=\zeta/m+O(|\zeta|^2)$. This almost comes under Theorem 3 a, except that $f^*(\zeta)$ has no asymptotic expansion of the form required by that theorem. This could be rectified with some effort, but we prefer to give independent formulations (and proofs) of the analogues of Theorems 1 a, 2 a and 3 a. The algorithm (29) below has been obtained by a combination of (8) and (27).

THEOREM 1 b. *Suppose that $f(x)$ is continuous, strictly increasing for $0<x\leqslant d$ and $0<f(x)<x$; suppose further that $f'(x)$ exist for $0<x<d$ and that for suitable real numbers $\mu>1$, $a>0$ and $\delta>0$,*

$$f'(x)=\mu a x^{\mu-1}+O(x^{\mu-1+\sigma}) \quad (x\downarrow 0). \tag{28}$$

Then

$$\chi(x)=\lim_{n\to\infty}\mu^{-n}\log(1/f_n(x)) \tag{29}$$

exists, is continuous and strictly decreasing for $0<x\leqslant d$ and differentiable with respect to x for $0<x<d$; furthermore,

$$f_\sigma(x)=\chi_{-1}(\mu^\sigma\chi(x)) \tag{30}$$

satisfies the relation (10) *and*

$$f_\sigma(x)=a^{\frac{\mu^\sigma-1}{\mu-1}}x^{\mu^\sigma}(1+O(x^\delta)) \quad (x\downarrow 0) \tag{31}$$

for $\sigma>0$.

The existence of the limit (29) already follows from

$$f(x)=a x^\mu+O(x^\mu) \quad (x\downarrow 0); \tag{32}$$

the stronger hypothesis (28) was made to ensure the existence of the inverse of $\chi(x)$. It can be shown by counterexamples that even

$$f'(x)=\mu a x^{\mu-1}+O(x^{\mu-1})$$

is not sufficient for the existence of $\chi_{-1}(x)$. Notice that $\chi(x)$ is a real Schroeder function of $f(x)$ with multiplier μ.

THEOREM 2 b. *Let $f(x)$ be continuous, strictly monotone increasing for $0 < x \leqslant d$ and $0 < f(x) < x$. Suppose that $f(x)$ has a family of real Schroeder iterates $f_\sigma(x)$ which satisfy the asymptotic relation*

$$\lim_{x \downarrow 0} x^{-\mu^\sigma} f_\sigma(x) = a^{\frac{\mu^\sigma - 1}{\mu - 1}}, \quad \mu > 1, \quad a > 0 \tag{33}$$

for every positive σ. Let $\chi(x)$ be a real Schroeder function with multiplier μ from which the family $f_\sigma(x)$ has been derived; then

$$\chi(x) = \chi(d) \lim_{n \to \infty} \{\log f_n(x) / \log f_n(d)\}. \tag{34}$$

Clearly (34) is identical with (29) whenever the latter exists. The limit

$$\chi(x) = \lim_{n \to \infty} \{\log f_n(x) / \log f_n(d)\}$$

is again called a principal Schroeder function of $f(x)$.

We shall call $f(x)$ regular with respect to iteration if $f(x)$ has a family of Schroeder iterates which satisfy the condition (33) of Theorem 2 b for every positive σ; these iterates are themselves regular and are uniquely determined by the asymptotic relation.

THEOREM 3 b. *Suppose that $f(z)$ is analytic at 0 and*

$$f(z) = \sum_{p=0}^{\infty} a_p z^{p+m}, \ m > 1, \ a_0 > 1, \ |z| < R_1 > 0, \tag{35}$$

where m is a positive integer and all coefficients a_p are real. Denote by $Q(R)$, $R > 0$, the domain

$$Q(R) = \{z; \ z = \varrho e^{i\psi}, \ 0 < \varrho < R, \ -\infty < \psi < \infty\}$$

on the Riemann surface of $\log z$. Then there is a positive number R so that

$$\chi(x) = -\lim_{n \to \infty} m^{-n} \log f_n(z) \tag{36}$$

exists for all $z \in Q(R)$ and is holomorphic and schlicht on $Q(r)$; $\log f_n(z)$ is specified by the condition that it is real valued if $\operatorname{am} z = 0$. Furthermore,

$$f_\sigma(z) = \chi_{-1}(m^\sigma \chi(z))$$

exists for every $\sigma > 0$, $z \in Q(R)$ and forms a family of Schroeder iterates such that

$$f_\sigma(z) = \sum_{(p+1)m^\sigma+q<k} \sum a_{pq}^{(\sigma)} z^{(p+1)m^\sigma+q} + O(|z|^k), \quad a_{00}^{(\sigma)} = 1 \tag{37}$$

for every fixed $k > 1$.

In particular, $f(x)$ is regular with respect to iteration so that the family $f(z)$ is uniquely determined by the asymptotic property (37).

Note that $f(z)$ itself is analytic at 0 and so are, of course, its natural iterates; but $f_\sigma(z)$ for non-integral σ has a logarithmic singularity at 0 and the expansion (37) is not necessarily convergent.

4. Lévy Iterates

We consider now the case that $a_1 = 1$ in (3.1) (1) and all coefficients a_n are real. This case was treated by P. Lévy [6] in connection with the problem of regular growth of real functions. Lévy has shown that if $f(x)$ is strictly monotone increasing and continuous for $0 \leqslant x \leqslant d$, $f(0) = 0$, $f(x) < x$, for $0 < x < d$; furthermore, if $f'(x)$ exists and is of bounded variation in the interval $0 < x < d$ and $\lim_{x \downarrow 0} f'(x) = 1$, then for every given x, y in the interval $0 < x \leqslant d$, $0 < y \leqslant d$,

$$\lambda_y(x) = \lim_{n\to\infty} \frac{x_n - y_n}{y_{n-1} - y_y}, \quad x_n = f_n(x), \ y_n = f_n(y) \tag{1}$$

exists. Also

$$\lim_{n\to\infty} \frac{x_{n-1} - x_n}{y_{n-1} - y_n} = 1, \tag{2}$$

so that $\lambda_x(y) = -\lambda_y(x)$. For a fixed y, $\lambda(x) = \lambda_y(x)$ is a (not necessarily strictly) monotone increasing function of x and it satisfies Abel's equation

$$\lambda(f(x)) = \lambda(x) - 1. \tag{3}$$

It follows that if $\lambda(x)$ is continuous and strictly monotone increasing, $\chi(x) = e^{\lambda(x)}$ is a real Schroeder function belonging to the multiplier e^{-1}.

It is more convenient to operate directly with the Abel function $\lambda(x)$; if $\lambda(x)$ is strictly monotone and continuous, so that its inverse $\lambda_{-1}(x)$ exists, then

$$f_\sigma(x) = \lambda_{-1}(\lambda(x) - \sigma) \tag{4}$$

is a family of Schroeder iterates of $f(x)$. Following Lévy, we shall call every continuous strictly monotone solution of Abel's equation a *logarithm of iteration* of $f(x)$.

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(1) (3.1) refers to formula (1) of § 3.

We note that if $\lambda(x)$ is a logarithm of iteration of $f(x)$ and x is any real number then $\lambda^*(x) = \lambda(x) + \alpha$ is also a logarithm of iteration, but the family of iterates derived from $\lambda^*(x)$ is identical with those derived from $\lambda(x)$; the two are therefore equivalent. A logarithm of iteration will be called *principal* if it is obtained by the Lévy algorithm (1); the iterates themselves will also be called principal. More precisely, $\lambda(x)$ is called a principal logarithm of iteration of $f(x)$ if (i) $\lambda(x)$ is a logarithm of iteration, and (ii) there exists a positive monotone decreasing sequence γ_n such that

$$\lambda(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(d)}{\gamma_n}. \tag{5}$$

This normalizes $\lambda(x)$ so that $\lambda(d) = 0$. Clearly (3) and (5) give

$$\lim_{n\to\infty} \frac{f_n(x) - f_{n+1}(x)}{\gamma_n} = 1$$

for every $0 < x \leqslant d$, from which (1) and (2) follow with

$$\lambda_y(x) = \lambda(x) - \lambda(y).$$

Two principal logarithms of iteration differ only in the choice of d and are easily seen to be equivalent. From (1) it is clear that the principal iterates if they exist are uniquely determined by $f(x)$.

THEOREM 1 c. *Suppose that $f(x)$ is continuous, strictly monotone increasing for $0 < x \leqslant d$ and $0 < f(x) < x$; suppose further that $f'(x)$ exists for $0 < x < d$ and that for some finite real $a > 0$, β, δ, $0 < \delta < \beta$,*

$$f'(x) = 1 - a(\beta+1)x^{\beta} + O(x^{\beta+\delta}) \quad (x \downarrow 0). \tag{6}$$

then if $y = y_0$ is any fixed point in the interval $0 < y \leqslant d$,

$$\lambda(x) = \lim_{n\to\infty} a^{1/\beta}(\beta n)^{1+1/\beta}(f_n(x) - f_n(y)) \tag{7}$$

exists and is continuous and strictly increasing for $0 < x \leqslant d$, differentiable for $0 < x < d$; furthermore,

$$f_\sigma(x) = \lambda_{-1}(\lambda(x) - \sigma) \tag{4}$$

satisfies the relations (3.10) *and*

$$f_\sigma(x) = x - a\sigma x^{\beta+1} + O(x^{\beta+1}) \quad (x \downarrow 0). \tag{8}$$

Clearly $\lambda(x)$ is a principal logarithm of iteration with

$$\gamma_n = a^{-1/\beta}(\beta n)^{-1-1/\beta}$$

in (5). The existence of the limit (7) already follows from

$$f(x) = x - a x^{\beta+1} + O(x^{\beta+1}) \quad (x \downarrow 0), \tag{9}$$

but not the existence of $\lambda_{-1}(x)$. It does not seem to be possible to deduce the stronger error term $O(x^{\beta+1+\delta})$ in (8) from the assumption (6).

THEOREM 2c. *Let $f(x)$ be continuous, strictly monotone increasing for $0 < x \leqslant d$ and $0 < f(x) < x$. Suppose that $f(x)$ has a family of real Schroeder iterates $f_\sigma(x)$ which satisfy the asymptotic relation*

$$\lim_{x \downarrow 0} x^{-\beta-1}(x - f_\sigma(x)) = \sigma a, \; a > 0, \; \beta > 0 \tag{10}$$

for every positive σ. Let $\lambda(x)$ be a logarithm of iteration from which the family $f_\sigma(x)$ has been derived according to (4); *then $\lambda(x)$ is a principal logarithm of iteration of $f(x)$.*

The theorem shows that the iterates are uniquely characterized by the asymptotic property (10). Again we shall call $f(x)$ regular with respect to iteration if $f(x)$ has a family of Schroeder iterates which satisfies the condition (10) of Theorem 2c.

THEOREM 3c. *Suppose that $f(z)$ is real for $z = x$, $0 < x \leqslant r$, and that for certain positive numbers β and α_0,*

$$\lim_{x \downarrow 0} \frac{x - f(x)}{x^{1+\beta}} = \alpha_0. \tag{11}$$

Suppose further that $f(z)$ is asymptotically differentiable at 0 in the sense of § 7, i.e. for every $\theta > 0$ there is an $r = r(\theta) > 0$ so that $f(z)$ is holomorphic in the (semi-closed) domain

$$B(\theta, r) = \{z;\; z = x + iy,\; 0 < x \leqslant r,\; -\theta x^{1+\beta} \leqslant y \leqslant \theta x^{1+\beta}\} \tag{12}$$

and for every fixed positive integer k,

$$f(z) = z - \sum_{p=0}^{k} \alpha_p z^{1+\beta+\beta p/m} + O(|z|^{1+\beta+\beta k/m}),\; \alpha_0 > 0,\; z \in B(\theta, r) \tag{13}$$

where m is a positive integer and the coefficients α_p are real.

Then $f(z)$ is regular with respect to iteration and the principal iterates $f_\sigma(z)$ are asymptotically differentiable at 0.

The meaning of the statement of the theorem is that the restriction of $f(z)$ to the positive real axis is regular and the principal iterates, which exist according to Theorem 2 c, [1] are analytic functions which satisfy the conditions of asymptotic differentiability. More precisely, it will be shown that

$$\lim_{x \downarrow 0} \frac{x - f_\sigma(x)}{x^{1+\beta}} = \sigma a_0 \tag{14}$$

and for every $\theta > 0$ there is an $r = r(\theta, \sigma) > 0$ so that

$$f_\sigma(z) = z - \sum_{p=0}^{k} \alpha_p^{(\sigma)} z^{1+\beta+\beta p/m} + O(|z|^{1+\beta+\beta k/m}), \tag{15}$$

$$\alpha_0^{(\sigma)} = \sigma \alpha_0,$$

when $z \in B(\theta, r)$. The case (1.11), mentioned in the introduction, when $f(z)$ is analytic at 0 and

$$f(z) = z - \sum_{p=0}^{\infty} \alpha_p z^{m+1+p}, \quad \alpha_0 > 0,$$

is a particular case of the theorem corresponding to $\beta = m$.

5. Regular Iteration

Throughout this and the following section we assume that $f(x)$ is continuous, strictly monotone increasing for $0 \leqslant x \leqslant d$, $f(0) = 0$, $f(x) < x$ for $0 < x \leqslant d$. It follows from these conditions that for every given x, $0 < x \leqslant d$, the sequence $x_0 = x$, $x_n = f(x_{n-1})$, $n = 1, 2, \ldots$ is strictly monotone decreasing and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} f_n(x) = 0. \tag{1}$$

We note that $0 < x < y \leqslant d$ implies $0 < x_n < y_n < d$ for every $n > 0$.

We begin with the proofs of Theorem 1 a, 1 b and 1 c.

(a) The assumption of Theorem 1 a is

$$f'(x) = a + O(x^\delta), \; 0 < a < 1, \; \delta > 0 \; (x \downarrow 0) \tag{2}$$

which clearly implies

$$f(x) = a x + O(x^{1+\delta}) \quad (x \downarrow 0). \tag{3}$$

Hence there exists a positive number c so that

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[1] Strict monotonity of $f(x)$ follows from $\lim_{x \downarrow 0} f'(x) = 1$ which is a consequence of Theorem 6, § 7.

$$|f(x)-ax|<cx^{1+\delta} \quad \text{for } 0<x\leqslant d. \tag{4}$$

We show that

$$a^{-n} f_n(x)=x+O(x^{1+\delta}), \quad n=1, 2, \dots, \quad (x\downarrow 0). \tag{5}$$

Let $b=\frac{1}{2}(1+a)$ and choose d_0, $0<d_0\leqslant d$ so that

$$b^{\delta} a>(a+cx^{\delta})^{1+\delta} \quad \text{for } 0<x\leqslant d_0. \tag{6}$$

By induction we prove that for $0<x\leqslant d_0$,

$$f_n(x)<a^n x+a^{n-1}c(1+b^{\delta}+\cdots+b^{(n-1)\delta})x^{1+\delta}, \quad n=1, 2, \dots . \tag{7}$$

For $n=1$ the inequality follows from (4) and for $n>1$ from

$$\begin{aligned} f_{n+1}(x)=f_n(f(x))<a^n(ax+cx^{1+\delta}) \\ +a^{n-1}c(1+b^{\delta}+\cdots+b^{(n-1)\delta})(ax+cx^{1+\delta})^{1+\delta} \\ <a^{n+1}x+a^n c(1+b^{\delta}+\cdots+b^{n\delta})x^{1+\delta} \end{aligned}$$

by (6) and (7). Similarly we can prove that

$$f_n(x)>a^n x-a^{n-1}c(1+b^{\delta}+\cdots+b^{(n-1)\delta})x^{1+\delta}, \quad 0<x\leqslant d_0.$$

From (3) and (5) we get

$$\left|\frac{x_{n+1}}{a^{n+1}}-\frac{x_n}{a^n}\right|=\left|\frac{f(x_n)-ax_n}{a^{n+1}}\right|=O(a^{\delta n})$$

which proves the existence of

$$\chi(x)=\lim a^{-n} f_n(x). \tag{8}$$

To prove the existence and positivity of $\chi'(x)$, we first have formally

$$\chi'(x)=\lim_{n\to\infty} a^{-n} f_n'(x)=\lim_{n\to\infty} a^{-n}\prod_{p=0}^{n-1} f'(x_p),$$

$$\begin{aligned} \log \chi'(x) &= \lim_{n\to\infty}\left(\sum_{p=0}^{n-1}\log f'(x_p)-n\log a\right) \\ &= \lim_{n\to\infty}\left(\sum_{p=0}^{n-1} O(x_p^{\delta})+\log a\right) \end{aligned}$$

by (2); but the last expression converges uniformly for $0<x\leqslant d$ by (5).

Finally, (5) and (8) give

$$\chi(x)=x+O(x^{1+\delta}), \ \chi_{-1}(x)=x+O(x^{1+\delta}), \quad (x\downarrow 0). \tag{9}$$

Relation (3.11) now follows from (3.9) and (9). Relation (3.10) is trivial.

(b) The assumption of Theorem 1 b is

$$f'(x) = \mu a x^{\mu-1} + O(x^{\mu+\delta-1}) \quad (x \downarrow 0), \tag{10}$$
$$a > 0, \ \mu > 1, \ \delta > 0,$$

which implies
$$f(x) = a x^{\mu} + O(x^{\mu+\delta}) \quad (x \downarrow 0). \tag{11}$$

Hence there exists a positive number c so that

$$|\log f(x) - \log a - \mu \log x| < c x^{\delta} \quad \text{for } 0 < x \leqslant d. \tag{12}$$

By induction we prove that for every positive integer n,

$$\left|\log f_n(x) - \frac{\mu^n - 1}{\mu - 1} \log a - \mu^n \log x\right| < c \frac{\mu^n - 1}{\mu - 1} x^{\delta} \tag{13}$$

for $0 < x \leqslant d$.

For $n = 1$ the inequality follows from (12). Assuming its validity for n, we get

$$\left|\log f_n(f(x)) - \frac{\mu^n - 1}{\mu - 1} \log a - \mu^n \log f(x)\right| < c \frac{\mu^n - 1}{\mu - 1} (f(x))^{\delta} \tag{14}$$

and from (12),

$$|\mu^n \log f(x) - \mu^n \log a - \mu^{n+1} \log x| < c \mu^n x^{\delta}. \tag{15}$$

By adding (14) and (15) we get

$$\left|\log f_{n+1}(x) - \frac{\mu^{n+1} - 1}{\mu - 1} \log a - \mu^{n+1} \log x\right| < c\left(\frac{\mu^n - 1}{\mu - 1}(f(x))^{\delta} + \mu^n x^{\delta}\right) < c \frac{\mu^{n+1} - 1}{\mu - 1} x^{\delta}$$

which proves (13) for every n. From (13) we get for $0 < x \leqslant d_0 < \operatorname{Min}\left\{d, \frac{1}{a}\right\}$

$$x_n = O(\mu^{-n}) \tag{16}$$

and from (12)

$$|\mu^{-n-1} \log (1/f_{n+1}(x)) - \mu^{-n} \log (1/f_n(x))|$$
$$= |\mu^{-n-1} \log (1/f(x_n)) - \mu^{-n} \log (1/x_n)| < \mu^{-n-1} |\log a| + c x_n^{\delta} = O(\mu^{-n}),$$

which proves that

$$\chi(x) = \lim_{n \to \infty} \mu^{-n} \log (1/f_n(x)) \tag{17}$$

exists and is finite for $0 < x \leqslant d_0$. Since for a suitable m $0 < f_m(x) \leqslant d_0$ when $0 < x \leqslant d$, the limit (17) exists for every $0 < x \leqslant d$.

To prove the existence and negativity of $\chi'(x)$ we note that by (10) and (11),

$$\frac{x f'(x)}{f(x)} = \mu + O(x^\delta)$$

hence

$$\begin{aligned} \log \frac{f_n'(x)}{f_n(x)} &= -\log x + \sum_{p=0}^{n-1} \log \frac{x_p f'(x_p)}{f(x_p)} \\ &= n \log \mu - \log x + \sum_{p=0}^{n-1} O(x_p^\delta). \end{aligned} \tag{18}$$

But from (17) first formally,

$$\log(-\chi'(x)) = \lim_{n\to\infty} \left\{ -n \log \mu + \log \frac{f_n'(x)}{f_n(x)} \right\}$$

and this is equal to

$$-\log x + \lim_{n\to\infty} \sum_{p=0}^{n-1} O(x_p^\delta)$$

by (18). The uniform convergence of $\sum_{p=0}^{\infty} x_p^\delta$ is obvious from (16) for $0 < x \leqslant d_0$, hence $-\chi'(x)$ exists and is positive.

Finally (13) and (17) give

$$\chi(x) = \lim_{n\to\infty} \mu^{-n} \log(1/f_n(x)) = -\left\{ \log x + \frac{1}{\mu - 1} \log a + O(x^\delta) \right\},$$

$$\chi_{-1}(y) = a^{-\frac{1}{\mu-1}} e^{-y} (1 + O(e^{-\delta y})) \quad (y \to \infty)$$

from which (3.31) follows easily by (3.30).

(c) The assumption of Theorem 1 c is

$$f'(x) = 1 - (\beta + 1) a x^\beta + O(x^{\beta+\delta}) \quad (x \downarrow 0), \tag{19}$$
$$a > 0, \ \beta > \delta > 0,$$

which implies

$$f(x) = x - a x^{1+\beta} + O(x^{\beta+\delta+1}) \quad (x \downarrow 0). \tag{20}$$

We first show that

$$\lim n \beta a x_n^\beta = 1, \quad x_n = f_n(x); \tag{21}$$

in fact (21) follows already from

$$f(x) = x - a x^{\beta+1} + o(x^{\beta+1}) \quad (x \downarrow 0). \tag{22}$$

To prove the relation, write

$$x_{n+1} = f(x_n) = x_n - a_n x_n^{\beta+1} \tag{23}$$

and

$$x_n^{-\beta} = \varrho_n + \cdots + \varrho_n, \quad n = 1, 2, \ldots . \tag{24}$$

By (1), (22) and (23),

$$a_n \to a, \quad x_{n+1}/x_n \to 1 \quad (n \to \infty); \tag{25}$$

hence

$$a_n = \frac{x_n - x_{n+1}}{x_n^{\beta+1}} = \varrho_{n+1} \frac{x_{n+1}}{x_n} \frac{x_n - x_{n+1}}{x_n^\beta - x_{n+1}^\beta} \to a,$$

$\varrho_{n+1} \to \beta a$ by (23), (24), (25), and

$$\frac{1}{n}(\varrho_1 + \cdots + \varrho_n) \to \beta a \quad \text{when} \quad n \to \infty.$$

This with (24) gives (21).

From the formula we get, by (22),

$$\lim_{n\to\infty} a^{1/\beta} (n\beta)^{1+1/\beta} (x_n - x_{n+1}) = 1, \tag{26}$$

which shows that we must have $\gamma_n = a^{-1/\beta} (\beta n)^{-1-1/\beta}$ in (4.5), provided that $f(x)$ possesses a principal logarithm of iteration at all.

If the stronger assumption (20) is used instead of (22), formula (21) can be further improved. First we have, instead of (25), the sharper result

$$a_n = a + O(x_n^\delta), \quad x_{n+1}/x_n = 1 + O(x_n^\delta),$$

hence

$$a_n = \varrho_{n+1} \beta^{-1} (1 + O(x_n^\beta)) = a + O(x_n^\delta),$$

$$\varrho_{n+1} = \beta a + O(x_n^\delta) = \beta a + O_x(n^{-\delta/\beta})$$

by (21), where the suffix x with the O-symbol indicates that the constants in the symbol may depend on x. Hence

$$x_n^{-\beta} = \varrho_1 + \cdots + \varrho_n = \beta a n + O_x(n^{1-\delta/\beta}). \tag{27}$$

We show that if x is in the fixed interval $0 < c \leqslant x \leqslant d$ then O_x can be replaced in (27) by O_c. In other words

$$x_n^{-\beta} = \beta a n + O(n^{1-\delta/\beta}) \tag{28}$$

uniformly for $c \leqslant x \leqslant d$, where $0 < c < d$.

Choose a positive integer m so that $c \geqslant f_m(d)$. We have for any fixed pair of numbers x, y for which $c \leqslant x < y \leqslant d$,

$$y_n^{-\beta} < x_n^{-\beta} \leqslant y_{n+m}^{-\beta}.$$

But by (27),
$$y_n^{-\beta} = \beta\, a\, n + O_y\,(n^{1-\delta/\beta}),$$
$$y_{n+m}^{-\beta} = \beta\, a\,(m+n) + O_y\,((m+n)^{1-\delta/\beta})$$
$$= \beta\, a\, n + O_y\,(n^{1-\delta\beta})$$

since m is fixed and $0<\delta<\beta$. Hence

$$x_n^{-\beta} = \beta\, a\, n + O_y\,(n^{1-\delta/\beta}).$$

By choosing $y=d$ the result follows.

We can now prove the existence of

$$\lambda(x) = \lim_{n\to\infty} a^{1/\beta}\,(\beta\, n)^{1+1/\beta}\,(f_n(x) - f_n(y)) \tag{29}$$

and the existence and positivity of $\lambda'(x)$. We have, by (19) and (28),

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{\beta}\right)\log n + \log f_n'(x)\right\}$$
$$= \lim_{n\to\infty}\left\{\left(1+\frac{1}{\beta}\right)\log n + \sum_{p=0}^{n-1}\log f'(x_p)\right\}$$
$$= \lim_{n\to\infty}\left\{\left(1+\frac{1}{\beta}\right)\log n - (\beta+1)\,a\sum_{p=0}^{n-1}x_p^{\beta} + \sum_{p=0}^{n-1}O\,(x_p^{\beta+\delta})\right\}$$
$$= \lim_{n\to\infty}\left\{\left(1+\frac{1}{\beta}\right)\log n - \left(1+\frac{1}{\beta}\right)\sum_{p=1}^{n-1}\frac{1}{p} - (\beta+1)\,a\,x^{\beta} + \sum_{p=1}^{n-1}O\,(p^{-1-\delta/\beta})\right\},$$

uniformly for $c\leqslant x\leqslant d$. Therefore

$$\log \lambda'(x) = \lim_{n\to\infty}\left\{\frac{1}{\beta}\log a + \left(1+\frac{1}{\beta}\right)\log(\beta\, n) + \log f_n'(x)\right\} \tag{30}$$

exists and is finite, so that $\lambda'(x)$ is positive for every $x>0$. The existence of $\lambda(x)$ now follows from the uniform convergence of (30) in every interval $c\leqslant x\leqslant d$, and from the fact that the limit (29) exists at least for the value $x=y$; in fact the convergence of (29) is uniform for $c\leqslant x\leqslant d$, ([4], p. 342). We note also that

$$\lambda(f(x) = \lim_{n\to\infty} a^{1/\beta}\,(\beta\, n)^{1+1/\beta}\,(f(x_n) - y_n)$$
$$= \lim_{n\to\infty} a^{1/\beta}\,(\beta\, n)^{1+1/\beta}\,(x_{n+1} - x_n)$$
$$+ \lim_{n\to\infty} a^{1/\beta}\,(\beta\, n)^{1+1/\beta}\,(x_n - y_n)$$
$$= -1 + \lambda(x)$$

by (26) and (29), which is the function equation (4.3). Generally

$$\lambda(f_n(x)) = \lambda(x) - n, \quad n = 1, 2, \dots . \tag{31}$$

Finally, to prove (4.8) we take $y = d$ in (29), so that $\lambda(x)$ increases from 0 to d. From (29) and (31) we get

$$\lim_{n\to\infty} a^{1/\beta}(\beta n)^{1+1/\beta}(\lambda_{-1}(\lambda(x) - n) - \lambda_{-1}(-n)) = \lambda(x)$$

or if we set $\lambda(x) = -\varrho$, $\varrho > 0$,

$$\lim_{n\to\infty} a^{1/\beta}(\beta n)^{1+1/\beta}(\lambda_{-1}(-\varrho - n) - \lambda_{-1}(-n)) = -\varrho. \tag{32}$$

This is true for every $\varrho > 0$, and the convergence is uniform for say $0 \leqslant \varrho \leqslant 1$. Write $\varrho + \sigma$ for ϱ in (31), and subtract (32) from the new relation:

$$\lim_{n\to\infty} a^{1/\beta}(\beta n)^{1+1/\beta}\{\lambda_{-1}(-\sigma - \varrho - n) - \lambda_{-1}(-\varrho - n)\} = -\sigma.$$

Again the convergence is uniform for $0 \leqslant \varrho \leqslant 1$, and we can replace $\varrho + n$ by a continuous variable t,

$$\lim_{t\to\infty} a^{1/\beta}(\beta t)^{1+1/\beta}\{\lambda_{-1}(-\sigma - t) - \lambda_{-1}(-t)\} = -\sigma, \tag{33}$$

true for every real σ.

Next we note that for $f(d) < x \leqslant d$,

$$n = (\beta a)^{-1} x_n^{-\beta} + O(x_n^{-\beta+\delta})$$

by (28), hence by (31),

$$\lambda(x_n) = \lambda(x) - n = -(\beta a)^{-1} x_n^{-\beta} + O(x_n^{-\beta+\delta}).$$

Since the points x_n, $f(d) < x = x_0 \leqslant d$ cover the whole interval $(0, d)$ we conclude that

$$\lambda(x) = -(\beta a)^{-1} x^{-\beta} + O(x^{-\beta+\delta}). \tag{34}$$

Write

$$\lambda(x) = -(\beta a)^{-1} x^{-\beta}(1 + \varphi(x)), \tag{35}$$

$$\lambda_{-1}(-y) = (\beta a)^{-1/\beta} y^{-1/\beta} + \psi(y), \tag{36}$$

where by (34),

$$\varphi(x) = O(x^{\delta}) = O(1) \quad (x \downarrow 0) \tag{37}$$

and by (35), (36),

$$x(1 + \varphi(x))^{-1/\beta} + \psi\{(\beta a)^{-1} x^{-\beta}(1 + \varphi(x))\} = x. \tag{38}$$

Furthermore, from (33) and (36),

$$\psi(\sigma+t)-\psi(t)=O(t^{-1-1/\beta}), \quad (t\to\infty). \tag{39}$$

Now (4.4) and (35), (36), (37) give

$$\begin{aligned} f_\sigma(x) &= \lambda_{-1}(\lambda(x)-\sigma) \\ &= x(1+\varphi(x))^{-1/\beta}\left\{1-\frac{a\,\sigma\cdot x^\beta}{1+\varphi(x)}+O(x^{2\beta})\right\} \\ &\quad +\psi\{(\beta a)^{-1}x^{-\beta}(1+\varphi(x))+\sigma\} \\ &= x-a\,\sigma\,x^{1+\beta}+O(x^{1+\beta}) \\ &\quad +\psi\{(\beta a)^{-1}x^{-\beta}(1+\varphi(x))+\sigma\}-\psi\{(\beta a)^{-1}x^{-\beta}(1+\varphi(x))\} \end{aligned}$$

by (38), which is equal to $x-a\,\sigma\,x^{1+\beta}+O(x^{1+\beta})$ by (39). This proves (4.8).

We conclude this section with the proofs of Theorems 2a, 2b and 2c.

(a) The assumption of Theorem 2a is

$$\lim_{\xi\downarrow 0}\frac{1}{\xi}\chi_{-1}(a^\sigma\chi(\xi))=a^\sigma$$

for every fixed positive σ; or writing $\xi=\chi_{-1}(a^n\chi(d))$, $a^\sigma=\chi(x)/\chi(d)$,

$$\lim_{n\to\infty}\frac{\chi_{-1}(a^n\chi(x))}{(a^n\chi(d))}=\frac{\chi(x)}{\chi(d)}$$

which is the statement to be proved.

(b) The assumption of Theorem 2b can be written in the form

$$\lim_{\xi\downarrow 0}[-\mu^\sigma\log\xi+\log\chi_{-1}(\mu^\sigma\chi(x))]=\frac{\mu^\sigma-1}{\mu-1}\log a,$$

hence

$$\lim_{\xi\downarrow 0}\frac{\log\chi_{-1}(\mu^\sigma\chi(\xi))}{\log\xi}=\mu^\sigma$$

for every fixed positive σ. The substitution $\xi=\chi_{-1}(\mu\chi(d))$, $\mu^\sigma=\chi(x)/\chi(d)$ gives the desired relation.

(c) The assumption of Theorem 2c is

$$\lim_{\xi\downarrow 0}\xi^{-\beta-1}[\lambda_{-1}(\lambda(\xi)-\sigma)-\xi]=-\sigma a \tag{40}$$

where the free additive constant of $\lambda(x)$ can be chosen so that

$$\lambda(d)=0. \tag{41}$$

By setting $\xi = \lambda_{-1}(-n)$, $\sigma = -\lambda(x)$ in (40) we get

$$\lim_{n\to\infty} [\lambda_{-1}(-n)]^{-\beta-1} [\lambda_{-1}(\lambda(x) - n) - \lambda_{-1}(-n)] = a\,\lambda(x),$$

or equivalently

$$\lim_{n\to\infty} (f_n(d))^{-\beta-1} [f_n(x) - f_n(d)] = a\,\lambda(x) \tag{42}$$

because of (41). Since condition (22) is satisfied, (21) is valid, and in particular

$$\lim_{n\to\infty} (n\,\beta\,a)^{1/\beta} f_n(d) = 1,$$

which, with (42), gives the desired relation

$$\lambda(x) = \lim_{n\to\infty} a^{1/\beta} (\beta\,n)^{1+1/\beta} [f_n(x) - f_n(d)]. \tag{43}$$

6. **Regular Iteration** (cont.)

The assumptions of Theorems 1 a, 1 b and 1 c are sufficient but by no means necessary for the regularity of a function. It is therefore desirable to have some further criteria and useful methods of construction of regular families. We shall only consider here functions which come under Theorems c $\left(\text{case } f'_+(0) = \lim_{x\downarrow 0} f(x)/x = 1\right)$, with one trivial exception.

Theorem 4 c. *The principal iterates $f_\sigma(x)$ of a regular $f(x)$ with $f'_+(0) = 1$ are themselves regular. If $\lambda(x)$ is a principal logarithm of iteration of $f(x)$ then $\frac{1}{\sigma}\lambda(x)$ is a principal logarithm of iteration of $f_\sigma(x)$. Furthermore, if ν is a positive number then the functions $\frac{1}{\nu} f(\nu x)$, $(f(x^\nu))^{1/\nu}$ are regular and $\lambda(\nu x)$, $\lambda(x^\nu)$ are their principal logarithms of iteration respectively.*

Proof. The existence of $f_\sigma(x)$ follows from Theorem 2 c, its regularity from the definition of regularity. We have to verify that $\lambda^*(x) = \frac{1}{\sigma}\lambda(x)$ is a logarithm of iteration of $f_\sigma(x)$. But $\lambda^*(x)$ is continuous, strictly monotone increasing, and $\lambda^*_{-1}(y) = \lambda_{-1}(\sigma y)$ so that

$$\lambda^*_{-1}(\lambda^*(x) - \tau) = \lambda_{-1}(\lambda(x) - \sigma\tau) = f_{\sigma\tau}(x) = (f_\sigma)_\tau(x).$$

To prove the second half of the theorem, put first $\lambda^*(x) = \lambda(\nu x)$; then $\lambda^*(x)$ is continuous, strictly monotone increasing for $0 < \nu \leqslant d/\nu$ and $\lambda^*_{-1}(y) = \frac{1}{\nu}\lambda_{-1}(y)$ so that

$$\lambda_{-1}^{*}(\lambda^{*}(x)-\sigma)=\frac{1}{\nu}\,\lambda_{-1}(\lambda(\nu x)-\sigma)=\frac{1}{\nu}\,f_{\sigma}(\nu x).$$

Also $$\lim_{x\downarrow 0} x^{-1-\beta}\left\{\frac{1}{\nu} f_{\sigma}(\nu x)-x\right\}=\lim_{x\downarrow 0} \nu^{\beta}(\nu x)^{-1-\beta}\{f_{\sigma}(\nu x)-\nu x\}=-a\,\nu^{\beta}\,\sigma$$

by (4.8).

Finally to deal with $(f(x^{\nu}))^{1/\nu}$, put $\lambda^{*}(x)=\lambda(x^{\nu})$; then $\lambda_{-1}^{*}(y)=(\lambda_{-1}(y))^{1/\nu}$ hence

$$\lambda_{-1}^{*}(\lambda^{*}(x)-\sigma)=\{\lambda_{-1}(\lambda(x^{\nu})-\sigma)\}^{1/\nu}=\{f_{\sigma}(x^{\nu})\}^{1/\nu}.$$

Also $$\begin{aligned}&\lim_{x\downarrow 0} x^{-1-\nu\beta}\{(f_{\sigma}(x^{\nu}))^{1/\nu}-x\}\\ &=\lim_{x\downarrow 0} x^{-1-\nu\beta}\{x(1-\sigma a x^{\nu\beta}+O(x^{\nu\beta}))^{1/\nu}-x\}\\ &=-\frac{1}{\nu}\,a\,\sigma\end{aligned}$$

which is an asymptotic relation of the form (4.8).

THEOREM 5c. *If $f(x)$ is regular, $f'_{+}(0)=1$, and $\lambda(x)$ is a principal logarithm of iteration of $f(x)$, then*

$$\lim_{t\to\infty} a^{1/\beta}(\beta t)^{1+1/\beta}\{\lambda_{-1}(-t)-\lambda_{-1}(-\sigma-t)\}=\sigma. \tag{1}$$

Conversely, given a continuous and strictly monotone increasing $\lambda(x)$ such that

$$\lim_{x\downarrow 0}\lambda(x)=-\infty,\ \lambda(d)=0,$$

and condition (1) *is satisfied for every real σ for some $a>0$, $\beta>0$, then $f(x)=\lambda_{-1}(\lambda(x)-1)$ is regular and $\lambda(x)$ is a principal logarithm of iteration of $f(x)$.*

Proof. Suppose first that $f(x)$ is regular and $\lambda(x)$ is a principal logarithm of iteration, normalized so that $\lambda(d)=0$. Regularity implies (5.22) hence (5.21) which can be written in the form

$$\lim_{n\to\infty}(n\beta a)^{1/\beta}\lambda_{-1}(\lambda(x)-n)=1.$$

Clearly we can replace here n by a continuous variable t; taking $x=d$ we get

$$\lim_{t\to\infty}(t\beta a)^{1/\beta}\lambda_{-1}(-t)=1. \tag{2}$$

On the other hand, the condition of regularity (4.8) gives, by setting $x=\lambda_{-1}(-t)$,

$$\lim_{t\to\infty} \{\lambda_{-1}(-t)\}^{-\beta-1} \{\lambda_{-1}(-t) - \lambda_{-1}(-t-\sigma)\} = \sigma a \tag{3}$$

which, combined with (2), gives (1).

Conversely, suppose that $\lambda(x)$ satisfies (1); we show that then it also satisfies (2). Put $\sigma = 1$ in (1) and choose t_0 so large that

$$|\lambda_{-1}(-t) - \lambda_{-1}(-1-t) - a^{-1/\beta}(\beta t)^{-1-1/\beta}| < \varepsilon a^{-1/\beta}(\beta t)^{-1-1/\beta}$$

for every $t \geqslant t_0$. Write

$$\lambda_{-1}(-k-t) - \lambda_{-1}(-k-1-t) = (1+\delta_k) a^{-1/\beta} \beta^{-1-1/\beta} (k+t)^{-1-1/\beta},$$
$$|\delta_k| < \varepsilon, \quad k = 1, 2, \dots ;$$

then
$$\lambda_{-1}(-t) = \lambda_{-1}(-k-1-t) + \sum_{p=0}^{k} (1+\delta_p) a^{-1/\beta} \beta^{-1-1/\beta} (p+t)^{-1-1/\beta},$$

and since $\lambda_{-1}(-k-1-t) \to 0$ when $k \to \infty$,

$$\lambda_{-1}(-t) = (1+\delta) a^{-1/\beta} \beta^{-1-1/\beta} \sum_{p=0}^{\infty} (p+t)^{-1-1/\beta}, \quad |\delta| < \varepsilon.$$

This is true for every $\varepsilon > 0$ and $t \geqslant t_0(\varepsilon)$, therefore (2) is true. The combination of (1) and (2) gives (3) which is in effect equivalent to the condition of regularity.

As an application of Theorem 5 c consider a $\lambda(x)$ which has the form

$$\lambda(x) = -cx^{-m} + \sum_{p=1}^{m} \varphi_p(\log x) x^{-m+p} + o(1), \quad (x \downarrow 0) \tag{4}$$

where m is a positive integer, $c > 0$ and $\varphi_p(u)$ is a polynomial in u. Clearly

$$\lambda_{-1}(-t) = c^{1/m} t^{-1/m} + \sum_{p=1}^{m} \psi_p(\log t) t^{-(p+1)/m} + O(t^{-1-1/m}), \quad (t \to \infty) \tag{5}$$

with certain polynomials $\psi_p(u)$ which can be expressed recursively by the $\varphi_p(u)$ from (4). Condition (1) can easily be verified to hold with $\beta = m$, $a = 1/mc$, hence $f(x) = \lambda_{-1}(\lambda(x) - 1)$ is regular, and

$$\lim_{x\downarrow 0} x^{-1-m}(x - f_\sigma(x)) = \sigma/mc. \tag{6}$$

Theorem 5 c has a trivial analogue in the case of $f'_+(0) = a$, $0 < a < 1$.

THEOREM 5 a. *If $f(x)$ is regular, $f'_+(0) = a$, $0 < a < 1$, and $\chi(x)$ is a principal Schroeder function of $f(x)$ then*

$$\lim_{x\downarrow 0} \frac{\chi(\mu x)}{\chi(x)} = \mu \tag{7}$$

for every positive number μ.[1] *Conversely, suppose that* $\chi(x)$ *is continuous, strictly monotone increasing,* $\lim_{x\downarrow 0}\chi(x)=0$, $\chi(d)=1$, *and condition* (7) *is satisfied for every* $\mu>0$; *then for every* a, $0<a<1$, $f(x)=\chi_{-1}(a\,\chi(x))$ *is regular and* $\chi(x)$ *is a principal Schroeder function of* $f(x)$.

Proof. $f(x)$ is by definition regular if and only if for every given real σ,

$$\lim_{x\downarrow 0}\frac{1}{x}\chi_{-1}(a^{\sigma}\chi(x))=a^{\sigma},$$

$$\lim_{x\downarrow 0}\frac{\chi(a^{\sigma}x)}{\chi(x)}=a^{\sigma}.$$

This is equivalent to relation (7) with $\mu=a^{\sigma}$.

7. Asymptotic Differentiability

Henceforth we shall assume that $f(z)$ is function of a complex variable z. Let $\varphi(x)$ be real valued, continuous and strictly monotone increasing for $x\geqslant 0$, $\varphi(0)=0$, and denote

$$D(\varphi)=\{z;\ z=x+iy,\ x>0,\ -\varphi(x)\leqslant y\leqslant\varphi(x)\},\tag{1}$$

$$D(\varphi,r)=\{z;\ z=x+iy,\ 0<x\leqslant r,\ -\varphi(x)\leqslant y\leqslant\varphi(x)\}.\tag{2}$$

We say, $f(x)$ has a regular asymptotic expansion in $D(\varphi)$ if

(i) $f(z)$ is defined in $D(\varphi,r)$ for some $r>0$, and

(ii) there is a positive number β such that for certain real coefficients a_p,

$$f(z)\sim\sum_{p=0}^{\infty}a_p z^{1+\beta p},\quad a_0>0\tag{3}$$

when $z\to 0$ in $D(\varphi,r)$. The meaning of the representation (3) is that for every fixed positive integer k,

$$f(z)=\sum_{p=0}^{k}a_p z^{1+\beta p}+O(|z|^{1+\beta k})\tag{4}$$

when $z\in D(\varphi,r)$, z^{β} is specified by the condition that z^{β} is real and positive when am $z=0$.

Let $f(z)$ be defined at least on the interval

$$z=x,\ 0<x\leqslant d,\ d>0,$$

---

[1] In other words $\chi(x)/x$ is a slowly growing function and $\chi(x)$ a regularly growing function in the sense of Karamata.

and suppose that
$$\lim_{x\downarrow 0} \frac{1}{x} f(x) = a_0 \tag{5}$$
exists and $0 < a_0 < \infty$. We say that $f(z)$ is *asymptotically differentiable* at 0 if the following is true:

Case $a_0 \neq 1$. There exist positive numbers β, r, θ, so that $f(z)$ is holomorphic on the angular domain
$$A(\theta, r) = \{z;\ z = x + iy,\ 0 < x \leqslant r,\ -\theta x \leqslant y \leqslant \theta x\} \tag{6}$$
and has a regular asymptotic expansion (3) in $A(\theta)$.

Case $a_0 = 1$. (i) There is a positive number β and a finite real number α_0, $\alpha_0 \neq 0$, so that
$$\lim_{x\downarrow 0} x^{-1-\beta}(x - f(x)) = \alpha_0 \tag{7}$$
(ii) There is a positive integer m and for every $\theta > 0$ an $r = r(\theta) > 0$ so that $f(x)$ is holomorphic on
$$B(\theta, r) = \{z;\ z = x + iy,\ 0 < x \leqslant r,\ -\theta x^{1+\beta} \leqslant y \leqslant \theta x^{1+\beta}\} \tag{8}$$
and has a regular asymptotic expansion of the form
$$f(z) \sim z\left(1 - \sum_{p=0}^{\infty} \alpha_p z^{\beta + \beta p/m}\right) \tag{9}$$
in $B(\theta)$.

The expansion (9) is just another form of (3) and is obtained from the latter by writing β/d instead of β and $\alpha_p = -a_{m+p}$, where m is the smallest positive integer for which $a_m \neq 0$. Note that the validity of the expansion (9) in a fixed domain $A(\theta_0)$ implies its validity in every $B(\theta)$ so that the present assumption on the domain of the expansion is weaker than the one in the previous case. However, there is no point in making the stronger assumption, as it would not improve on any of the results to be obtained. Also note that $f(z)$ has, in the case $a_0 \neq 1$, a regular asymptotic expansion in every $D(\varphi)$ for which $\limsup_{x\downarrow 0} \frac{\varphi(x)}{x} < \theta$, and in the case $a_0 = 1$, a regular asymptotic expansion in every $D(\varphi)$ for which $\limsup_{x\downarrow 0} \frac{\varphi(x)}{x^{1+\beta}} < \infty$.

The definition of asymptotic differentiability does not exclude the case when the series (3) is convergent, and for instance if $f(z)$ is analytical at 0, real valued for real z, and $f'(0) > 0$, then it is also asymptotically differentiable at 0.

Our purpose is to show that both the derivative and the inverse of an asymptotically differentiable function are asymptotically differentiable.

THEOREM 6. *If $f(z)$ is asymptotically differentiable at 0 then so is $f'(z)$ and the asymptotic expansion of $f'(z)$ is obtained by termwise differentiation of* (3).

Proof. We shall only give details for the case $a_0 = 1$; the case $a_0 \neq 1$ can be dealt with similarly and is in fact slightly easier. We shall prove that from the validity of (9) in $B(\theta)$ it follows that

$$f'(z) \sim 1 - \sum_{p=0}^{\infty} \left(1 + \beta + \beta \frac{p}{m}\right) \alpha_p z^{\beta + \beta p/m} \tag{10}$$

when $z \to 0$ in $B(\frac{1}{2}\theta)$.

Given $k \geq 0$ write

$$g(z) = f(z) - z + \sum_{p=0}^{k+2m} \alpha_p z^{1+\beta+\beta p/m}. \tag{11}$$

From (9) it follows that there exists a positive number K which depends on θ, r, k, but not on z, such that

$$|g(z)| \leq K |z|^{1+3\beta+\beta k/m} \tag{12}$$

when $z \in B(\theta, r)$.

Now let $z = x + iy \in B(\frac{1}{2}\theta, \frac{1}{2}r)$, and denote by Γ the path $\Gamma_1 + \Gamma_2 - \Gamma_3$,

$$\Gamma_1 : \zeta = t - i\theta t^{1+\beta}, \quad 0 \leq t \leq 2x,$$
$$\Gamma_2 : \zeta = 2x + it, \quad -\theta (2x)^{1+\beta} \leq t \leq \theta (2x)^{1+\beta},$$
$$\Gamma_3 : \zeta = t + i\theta t^{1+\beta}, \quad 0 \leq t \leq 2x.$$

We have, since $g(\zeta)$ is holomorphic when $\zeta \in B(\theta, r)$ and $g(\zeta) \to 0$ when $\zeta \to 0$ in $B(\theta, r)$,

$$g'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta)}{(\zeta - z)^2} d\zeta. \tag{13}$$

Now clearly there exist positive numbers c_1, c_2 and c_3 which depend only on β, θ, r, so that

$$|\zeta| \leq c_1 |z|, \quad |\zeta - z| \geq c_2 |z|^{1+\beta} \tag{14}$$

when ζ is on Γ, and

$$\text{length of } \Gamma \leq c_3 |z|. \tag{15}$$

Hence by (12), (13), (14) and (15),

$$|g'(z)| \leq \frac{1}{2\pi} K c_1^{1+3\beta+\beta k/m} c_2^{-2} c_3 |z|^{\beta+\beta k/m} = c_4 |z|^{\beta+\beta k/m} \tag{16}$$

for a suitable positive number c_4 which does not depend on z. But (11) and (16) give

$$f'(z)=1-\sum_{p=0}^{k}\left(1+\beta+\beta\frac{p}{m}\right)\alpha_p z^{\beta+\beta p/m}+O\left(|z|^{\beta+\beta k/m}\right)$$

which proves (10).

In the case of $a_0 \neq 1$ and $z=x+iy \in A(\frac{1}{2}\theta, \frac{1}{2}x)$ we take

$$g(z)=f(z)-\sum_{p=0}^{k} a_p z^{1+p\beta}$$

and the path of integration $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$,

$$\Gamma_1:\zeta=t-i\theta t,\quad 0\leqslant t\leqslant 2x$$

$$\Gamma_2:\zeta=2x+it,\quad -2\theta x\leqslant t\leqslant 2\theta x$$

$$\Gamma_3:\zeta=t+i\theta t,\quad 0\leqslant t\leqslant 2x.$$

Otherwise the proof goes like in the previous case.

Remark. The proof evidently remains valid if $f(z)$ has a more general asymptotic expansion

$$f(z)\sim z^{\gamma}\left\{a_0+\sum_{p=0}^{\infty} a_p(\log z)\, z^{\beta p}\right\},\quad a_0>0 \tag{17}$$

where γ is any real number and the $a_p(t)$, $p=1, 2, \ldots$ are polynomials in t.

THEOREM 7a. *Suppose that $f(z)$ is asymptotically differentiable at* 0 *and it has a regular asymptotic expansion*

$$f(z)\sim\sum_{p=0}^{\infty} a_p z^{1+\beta p},\quad 0<a_0<1, \tag{18}$$

which is valid in the domain $A(\theta)$, as defined under (6). *Let*

$$\varphi(x)=\frac{\theta}{1+\theta}x-\gamma x^2,\quad \psi(x)=\frac{\theta}{1+\theta}x+\gamma x^2 \tag{19}$$

where

$$\gamma=\frac{2\theta(1+|a_1|)}{(1+\theta)a_0(1-a_0)}, \tag{20}$$

and define $D(\varphi, r)$, $D(\psi, r)$ as in (2). *Then there is a positive number r_1 with the following properties.*

(i) $f(z)$ *is holomorphic and schlicht on $D(\varphi, r_1)$ and $D(\psi, r_1)$.*

(ii) $f(z)$ *maps $D(\varphi, r)$ into $D(\varphi, \frac{1}{2}(1+a_0)r_1)$.*

(iii) $f_{-1}(z)$ *exists on $D(\psi, r_1)$ and it maps $D(\psi, r_1)$ into $D\left(\psi, \frac{2}{a_0}r_1\right)$.*

(iv) $f_{-1}(z)$ *has a regular asymptotic expansion in $D(\psi, r_1)$.*

Statements (i) *and* (iv) *are still valid if $a_0>1$ in* (18).

The theorem obviously implies that $f_{-1}(z)$ is asymptotically differentiable at 0.

THEOREM 7 c. *Suppose that $f(z)$ is asymptotically differentiable at 0 and it has an expansion*

$$f(z) \sim z\left(1 - \sum_{p=0}^{\infty} \alpha_p z^{\beta+\beta p/m}\right) \tag{21}$$

which is valid in every domain $B(\theta)$, $\theta > 0$, as defined under (8). *Let*

$$\varphi_\theta(x) = \theta x^{1+\beta} - \gamma x^{1+\beta+\beta/m}, \quad \psi_\theta(x) = \theta x^{1+\beta} + \gamma x^{1+\beta+\beta/m} \tag{22}$$

where

$$\gamma = \frac{\theta}{\alpha_0}(|\alpha_1| + m(1+\beta)\alpha_0^2); \tag{23}$$

then to every $\theta > 0$ there is a positive number $r_1 = r_1(\theta)$ so that

(i) *$f(z)$ is holomorphic and schlicht on $D(\varphi_\theta, r_1)$ and $D(\psi_\theta, r_1)$.*

(ii) *$f(z)$ maps $D(\varphi_\theta, r_1)$ into $D(\varphi_\theta, r_1 - \frac{1}{2}\alpha_0 r_1^{1+\beta})$.*

(iii) *$f_{-1}(z)$ exists on $D(\psi_\theta, r_1)$ and it maps $D(\psi_\theta, r_1)$ into $D(\psi_\theta, r_1 + 2\alpha_0 r_1^{1+\beta})$.*

(iv) *$f_{-1}(z)$ has a regular asymptotic expansion in $D(\psi_\theta, r_1)$.*

Statements (i) *and* (iv) *are still true if $\alpha_0 < 0$ in* (21).

To prove Theorem 7 a, choose r_1 in a later to be specified manner, but in any case so that

$$|f(z) - a_0 z| < \tfrac{1}{2} a_0 |z|, \quad |f'(z) - a_0| < a_0 \tag{24}$$

when $z \in A(\theta, r_1)$; this is possible by (18) and Theorem 6. From the first condition in (24) it follows that

$$\tfrac{1}{2} a_0 |z| < |f(z)| < \tfrac{3}{2} a_0 |z|; \tag{25}$$

from the second condition, that $f(z)$ is schlicht on $A(\theta, r_1)$.(1) Now let

$$z = x + i\frac{\theta}{1+\theta}x - i\gamma x^2, \quad 0 < x \leq r_1$$

be a boundary point of $D(\varphi, r_1)$. From (18) we obtain

$$f(z) = a_0 x + a_1 \frac{1+2\theta}{(1+\theta)^2} x^2 + i\left[\frac{\theta}{1+\theta} a_0 x + \left(\frac{2\theta}{1+\theta} a_1 - \gamma a_0\right) x^2\right] + O(x^3);$$

hence if we write $f(z) = x_1 + i y_1$, we see that

$$\begin{aligned} y_1 &= \frac{\theta}{1+\theta} x_1 + \frac{1}{a_0^2}\left[\frac{\theta}{1+\theta} a_1 + \frac{\theta^3}{(1+\theta)^3} a_1 - \gamma a_0\right] x_1^2 + O(x_1^3) \\ &\leq \frac{\theta}{1+\theta} x_1 - \frac{1}{a_0^2}\left(\gamma a_0 - \frac{2\theta}{1+\theta}|a_1| - \frac{2\theta}{1+\theta}\right) x_1^2 \\ &= \frac{\theta}{1+\theta} x_1 - \gamma x_1^2 \end{aligned}$$

(1) See reference [1], p. 297, Satz 3.

by (20), and also $y_1 > 0$, provided that r_1 is sufficiently small. Similarly we find that if

$$z = x + i \frac{\theta}{1+\theta} x + i \gamma x^2, \quad 0 < x \leqslant r_1$$

is a boundary point of $D(\psi, r_1)$ and $f(z) = x_2 + i y_2$ then

$$y_2 \geqslant \frac{\theta}{1+\theta} x_2 + \gamma x_2^2.$$

Finally it is seen from (18) that if $z = r_1 + i y$, $|y| \leqslant \frac{\theta}{1+\theta} r_1 + \gamma r_1^2$, and if $f(z) = x_3 + i y_3$, then $\frac{1}{2} a_0 r_1 < x_3 < \frac{1}{2}(1 + a_0) r_1$ provided that r_1 is sufficiently small. Therefore $f(z)$ maps the boundary of $D(\varphi, r_1)$ upon a curve which is inside $D(\varphi, \frac{1}{2}(1 + a_0) r_1)$ and the boundary of $D(\psi, r_1)$ upon a curve which is outside $D(\psi, \frac{1}{2} a_0 r_1)$. The statements (ii) and (iii) now follow from (i).

It remains to be shown that $f_{-1}(z)$ has a regular asymptotic expansion in $D(\psi, r_1)$. We can uniquely determine coefficients b_p, where $b_0 = \frac{1}{a_0}$, so that

$$z \sim \sum_{p=0}^{\infty} b_p w^{1+\beta p}$$

is formally satisfied if $w = f(z)$ is substituted from (18). But then we get for every $k > 0$

$$\sum_{p=0}^{k} b_p w^{1+\beta p} = z + O(|z|^{1+\beta k}) = z + O(|w|^{1+\beta k}) \tag{26}$$

by (25), hence

$$z = f_{-1}(w) \sim \sum_{p=0}^{\infty} b_p w^{1+\beta p}, \quad b_0 = 1/a_0 \tag{27}$$

is valid in $D(\psi, r_1)$.

The proofs in the case of $a_0 > 1$ are similar, except that γ is now negative and the roles of φ and ψ are interchanged.

We note that

$$f'_{-1}(w) \sim \sum_{p=0}^{\infty} (1 + \beta p) b_p w^{\beta p}, \quad b_0 = 1/a_0 \tag{28}$$

by (27) and Theorem (6).

The proof of Theorem 7 c runs on very similar lines. We first secure the validity of (24) and (25) with $a_0 = 1$, hence schlichtness of $f(z)$ on $B(\theta, r_1(\theta))$. Mapping of the boundary point

$$z = x + i\theta x^{1+\beta} - i\gamma x^{1+\beta+\beta/m}$$

gives

$$f(z) = x - \sum_{p=0}^{m+1} \alpha_p x^{1+\beta+\beta p/m} + \tfrac{1}{2}\beta(1+\beta)\alpha_0 \theta^2 x^{1+3\beta}$$
$$+ i x^{1+\beta}[\theta - \gamma x^{\beta/m} - \alpha_0 \theta (1+\beta) x^{\beta} + \alpha_0 (1+\beta)\gamma x^{\beta+\beta/m}$$
$$- \alpha_1 (1+\beta+\beta/m)\theta x^{\beta+\beta/m}] + O(x^{1+2\beta+\beta/m}),$$

or if $f(z) = x_1 + i y_1$,

$$y_1 = \theta x_1^{1+\beta} - \gamma x_1^{1+\beta+\beta/m} - \frac{\beta}{m}(\alpha_0 \gamma + \alpha_1 \theta) x_1^{1+2\beta+\beta/m} - \tfrac{1}{2}\beta(1+\beta)\theta a_0^2 x_1^{1+3\beta} + O(\alpha_1^{1+2\beta+\beta/m})$$
$$< \theta x_1^{1+\beta} - \gamma x_1^{1+\beta+\beta/m} - \left[\frac{\beta}{m}\alpha_0 \gamma - \frac{\beta}{m}|\alpha_1|\theta - \beta(1+\beta)\theta \alpha_0^2\right] x_1^{1+2\beta+\beta/m}$$
$$= \theta x_1^{1+\beta} + \gamma x_1^{1+\beta+\beta/m}$$

by (23). Similarly if $z = x + i\theta x^{1+\beta} + i\gamma x^{1+\beta+\beta/m}$ and $f(z) = x_2 + i y_2$ then $y_2 > \theta x_1^{1+\beta} + + \gamma x_1^{1+\beta+\beta/m}$. Finally if $z = r_1 + iy$, $|y| \leqslant \theta r_1^{1+\beta} + \gamma r_1^{1+\beta+\beta/m}$ and if $f(z) = x_3 + i y_3$ then

$$r_1 - 2\alpha_0 r_1^{1+\beta} < x_3 < r_1 - \tfrac{1}{2}\alpha_0 r_1^{1+\beta}. \tag{29}$$

The conclusions now are similar to those in the previous case.

The asymptotic expansion of $f_{-1}(w)$ has the form

$$f_{-1}(w) \sim w\left(1 + \sum_{p=0}^{\infty} b_p z^{\beta+\beta/m}\right), \quad b_0 = \alpha_0 \tag{30}$$

and

$$f'_{-1}(w) \sim 1 + \sum_{p=0}^{\infty}\left(\beta + \beta\frac{p}{m} + 1\right) b_p z^{\beta+\beta p/m}. \tag{31}$$

8. Analytical Iterates

In this section we shall consider Theorems 3 c and 3 b of § 3; the more difficult Theorem 3 c is deferred to the last section.

Suppose that the conditions of Theorem 7 a are satisfied with $0 < a_0 < 1$; by property (ii) of this theorem, the natural iterates $f_n(z)$ relatively to $D(\varphi, r_1)$ exist for every positive integer n. We first show that for a suitable r_2, $0 < r_2 \leqslant r_1$,

$$\chi(z) = \lim_{x\to\infty} a_0^{-n} f_n(z) \tag{1}$$

exists for every $z \in D(\varphi, r_2)$, and $\chi(x)$ is holomorphic on $D(\varphi, r_2)$, The proof follows the pattern of Theorem 1 a in § 5. First we obtain from (7.18) a positive number c so that

$$|f(z) - a_0 z| < c z^{1+\beta} \tag{2}$$

for $z \in D(\varphi, r_1)$. From this we get by induction, for a suitable r_2 and for every $z \in D(\varphi, r_2)$

$$|f_n(z) - a_0^n z| < a_0^{n-1} c (1 + b^\beta + \cdots + b^{(n-1)\beta}) |z|^{1+\beta}, \quad n = 1, 2, \ldots \tag{3}$$

where $b = \frac{1}{2}(1 + a_0)$. Hence

$$z_n = f_n(z) = O(a_0^n) \tag{4}$$

and

$$\left| \frac{f_{n+1}(z)}{a_0^{n+1}} - \frac{f_n(n)}{a_0^n} \right| = a_0^{-n-1} |f(z_n) - a_0 z_n|$$

$$= O(a_0^{-n} |z_n|^{1+\beta}) = O(a_0^{\beta n})$$

which proves the existence and uniform convergence of (1) in $D(\varphi, r_2)$. In particular, $\chi(z)$ is holomorphic in $D(\varphi, r_2)$.

From (1), (3) and (4) we get

$$\chi(z) = z + O(|z|^{1+\beta}), \quad z \in D(\varphi, r_2). \tag{5}$$

By an argument similar to the one used in the proof of Theorem 6 it follows from (5) and (7.19) that

$$\chi'(z) = 1 + O(|z|^\beta) \quad \text{for } z \in A(\theta_1, \tfrac{1}{2} r_1) \tag{6}$$

where $\theta_1 = \dfrac{\theta}{2(1+\theta)}$; hence $\chi(z)$ is schlicht on $A(\theta_2, r_2)$ for a suitable θ_2, r_2. By (5) we can therefore choose θ_1, r_1 and θ_2, r_2 (which are of course not necessarily the same as the previous ones) so that:

(i) $\chi(z)$ exists and is schlicht on $A(\theta_1, r_1)$.

(ii) $z \in A(\theta_1, r_1)$ implies $\chi(z) \in A(\theta_2, r_2)$.

(iii) $\chi_{-1}(z)$ exists and is schlicht on $A(\theta_2, r_2)$.

It follows that

$$f_\sigma(z) = \chi_{-1}(a_0^\sigma \chi(z)) \tag{7}$$

exists and is holomorphic on $A(\theta_1, r_1)$ for every real positive σ. In particular, $z_n = f_n(z)$ exists for $z \in A(\theta_1, r_1)$ and

$$\chi_n = \chi(z_n) = a_0^n \chi(z). \tag{8}$$

We show that

$$z_n \sim \sum_{q=0}^{\infty} c_q \chi_n^{1+\beta q}, \quad c_0 = 1 \quad (n \to \infty) \tag{9}$$

for certain real coefficients c_q, where the interpretation of (9) is

$$z_n - \sum_{q=0}^{k} c_q \chi_n^{1+\beta q} = O(\chi_n^{1+\beta k}) = O(a_0^{n+\beta k n})$$

for every fixed positive integer k and $z \in A(\theta_1, r_1)$. The coefficients c_q can be determined formally from

$$z_{n+1} \sim \sum_{p=0}^{\infty} a_p z_n^{1+\beta p},$$

i.e. from
$$\sum_{q=0}^{\infty} c_q (a_0 \chi_n)^{1+\beta q} \sim \sum_{p=0}^{\infty} a_q \left(\sum_{q=0}^{\infty} c_q \chi_n^{1+\beta q} \right)^{1+\beta p};$$

and the formal relation is turned into a true relation by observing that both z_n/χ_n and χ_n/z_n are bounded on $A(\theta_1, r_1)$, because of (5).

A trivial inversion of (9) gives

$$\chi(z_n) = \chi_n \sim z_n + \sum_{q=1}^{\infty} d_q z_n^{1+\beta q} \quad (n \to \infty)$$

for certain real coefficients d_q which can be determined uniquely from the c_q, and since this is valid uniformly for $z \in A(\theta_1, r_1)$ we conclude that

$$\chi(z) \sim z + \sum_{q=1}^{\infty} d_q z^{1+\beta q} \tag{10}$$

when $z \to 0$ in $A(\theta_1)$. Incidentally, the coefficients d_q can also be determined directly from the functional relation

$$\chi(f(z)) = a_0 \chi(z).$$

Corresponding to (10) we also have an expansion of the form

$$\chi_{-1}(w) \sim w + \sum_{q=1}^{\infty} d_q^* w^{1+\beta q} \tag{11}$$

for $w = \chi(z)$, $z \in A(\theta, r_2)$; this gives easily, by (7) and (10),

$$f_\sigma(z) \sim \sum_{p=0}^{\infty} a_p^{(\sigma)} z^{1+\beta p}, \quad a_p^{(\sigma)} = (a_p)^\sigma, \tag{12}$$

and Theorem 3 a is proved.

To prove Theorem 3 b we follow the pattern of Theorem 1 b in § 5. The assumption of Theorem 3 b is

$$f(z) = \sum_{p=0}^{\infty} a_p z^{p+m}, \; m > 1, \; a_0 = 1, \; |z| < R_1 > 0. \tag{13}$$

We can clearly choose a positive $R_0 < \text{Min}\{R_1, 1\}$ so that $|f(z)| < |z|$ whenever $0 < |z| \leq R_0$ and

$$|\log f(z) - m \log z| < c|z|, \quad 0 < |z| \leq R_0 \tag{14}$$

for a suitable positive number c. This inequality is satisfied on the Riemann surface of $\log z$, irrespective of the amplitude of z, and we have for instance

$$|\log f_{n+1}(z) - m \log f_n(z)| < c\,|f_n(z)| \tag{15}$$

for any z on the Riemann surface of $\log z$ such that $0<|z|\leqslant R_0$, provided that $\log f_n(z)$ is specified as in Theorem 3 b.

From (14) it follows by induction, as in § 5, that

$$|\log f_n(z) - m^n \log z| < c\,\frac{m^n-1}{m-1}\,|z|. \tag{16}$$

This gives
$$z_n = O(m^{-n}) \tag{17}$$
and
$$|m^{-n}\log f_n(z) - m^{-n-1}\log f_{n+1}(z)| = O(m^{-n})$$
by (15) which proves the existence and uniform convergence of

$$\chi(z) = -\lim_{n\to\infty} m^{-n}\log f_n(z) \tag{18}$$

for
$$0<|z|\leqslant R_0,\quad -\infty<\operatorname{am} z<\infty.$$

Hence $\chi(z)$ is holomorphic for

$$0<|z|<R_0,\quad -\infty<\operatorname{am} z<\infty.$$

From (16) and (18) we get

$$\chi(z) = -\log z + O(|z|) \tag{19}$$

This implies—provided that R_0 is sufficiently small—that $\chi(z)$ is schlicht (¹) on the annulus

$$Q(R_0) = \{z;\ z = \varrho\, e^{i\psi},\ 0<\varrho<R_0,\ -\infty<\psi<\infty\} \tag{20}$$

and that it maps $Q(R_0)$ upon a domain T such that $S(r_2)\subset D\subset S(r_1)$ where $S(r)$ is the strip

$$S(r) = \{w;\ w = u + i v,\ r<u<\infty,\ -\infty<v<\infty\}$$

and
$$r_1>0,\ \log\frac{1}{R_0} - c_1 R_0 < r_1 < r_2 < \log\frac{1}{R_0} + c_1 R_0$$

for a positive c_1 which is independent of R_0. Consequently there exist positive numbers R, r with the following properties:

(¹) Details of proof in a similar but slightly more difficult case will be given in § 9, Lemma 7.

(i) $\chi(z)$ is schlicht on $Q(R)$.

(ii) $z \in Q(R)$ implies $\chi(z) \in S(r)$.

(iii) $\chi_{-1}(z)$ exists and is schlicht on $S(r)$.

It follows that
$$f_\sigma(z) = \chi_{-1}(m^\sigma \chi(z)) \tag{21}$$
exists and is holomorphic on $Q(R)$ for every real positive σ. In particular, $z_n = f_n(z)$ exists for $z \in Q(R)$ and
$$\chi_n = \chi(z_n) = m^n \chi(z). \tag{22}$$

Now the proof proceeds as in the previous case. First we show that
$$z_n \sim \sum_{q=0}^{\infty} c_q e^{-(q+1)\chi_n}, \quad c_0 = 1 \quad (n \to \infty) \tag{23}$$
for certain real coefficients c_q, uniformly for $z \in Q(R)$. The coefficients c_q are obtained formally from
$$z_{n+1} = \sum_{p=0}^{\infty} a_p z_n^{m+p}$$
i.e. by equating coefficients in
$$\sum_{q=0}^{\infty} c_q e^{-(q+1)m\chi_n} \sim \sum_{p=0}^{\infty} a_p \left(\sum_{q=0}^{\infty} c_q e^{-(q+1)\chi_n} \right)^{m+p};$$
and the formal relation is turned into a true asymptotic relation by observing that both $z_n e^{\chi_n}$ and $z_n^{-1} e^{-\chi_n}$ are bounded on $Q(R)$, because of (19).

Inverting (23) we get
$$\chi(z_n) = \chi_n \sim -\log z_n + \sum_{q=1}^{\infty} d_q z_n^p \quad (n \to \infty)$$
for certain real coefficients d_q which can be determined uniquely from (23). Since this is valid uniformly for $z \in Q(R)$, we conclude that
$$\chi(z) \sim -\log z + \sum_{q=1}^{\infty} d_q z^q \tag{24}$$
when $z \to \infty$ on $Q(R)$. From (24) we get an expansion
$$\chi_{-1}(w) \sim e^{-w} + \sum_{q=1}^{\infty} d_q^* e^{-(q+1)w} \tag{25}$$
for $w = \chi(z)$, $z \in Q(R)$; this gives with (21) and (24)
$$f_\sigma(z) \sim \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{pq}^{(\sigma)} z^{(p+1)m^\sigma + q}, \quad a_{00}^{(\sigma)} = 1, \tag{26}$$
as required.

9. Analytical Iterates (cont.)

We come now to the proof of Theorem 3 c; the main steps will be formulated as separate Lemmas. The present assumption is that $f(z)$ is asymptotically differentiable at 0 and the regular asymptotic expansion of $f(z)$ has the form

$$f(z) \sim z\left(1-\sum_{p=1}^{\infty} \alpha_p z^{\beta+\beta p/m}\right), \quad \alpha_0>0, \ \beta>0. \tag{1}$$

As a preparatory step we show that without loss in generality it can be assumed that $\alpha_0=1$, $\beta=1$.

If $\beta \neq 1$, consider the function $f^*(z)=(f(z^{1/\beta}))^\beta$. Clearly

$$f^*(z) \sim z\left(1-\sum_{p=0}^{\infty} \alpha_p z^{1+p/m}\right)^\beta$$

and this is an asymptotic expansion of the form

$$f^*(z) \sim z\left(1-\sum_{p=0}^{\infty} \alpha_p^* z^{1+p/m}\right) \tag{2}$$

which is valid in

$$B=\{z;\ z=(x+iy)^\beta,\ 0<x\leqslant r,\ -\theta x^{1+\beta} \leqslant y \leqslant \theta x^{1+\beta}\}$$

if (1) is valid in

$$B(\theta, r)=\{z;\ z=x+iy,\ 0<x\leqslant r,\ -\theta x^{1+\beta} \leqslant y \leqslant \theta x^{1+\beta}\}. \tag{3}$$

Clearly, B includes a domain $D(\varphi, r_1)$ (definition (7.2)) with

$$\lim_{x \downarrow 0} \frac{\varphi(x)}{x^2}=\beta\theta;$$

hence $f^*(z)$ is asymptotically differentiable at 0. By Theorem 4 c, § 6, $f(x)=(f^*(x^\beta))^{1/\beta}$ is regular if $f^*(x)$ is so, and also $f_\sigma(z)=(f^*(z^\beta))^{1/\beta}$ is asymptotically differentiable at 0 if $f_\sigma^*(z)$ is so. Therefore, if Theorem 3 c is true for $f^*(z)$, then it is also true for $f(z)$.

If $\alpha_0^* \neq 1$ in (2), we replace $f^*(z)$ by $f^{**}(z)=\alpha_0^* f^*(z/\alpha_0^*)$. Again, $f^{**}(z)$ is asymptotically differentiable at 0 since

$$f^{**}(z) \sim z\left\{1-\sum_{p=0}^{\infty} \alpha_0^* (z/\alpha_0^*)^{1+p/m}\right\}$$

which is an asymptotic expansion of the form

$$f^{**}(z) \sim z\left(1-\sum_{p=0}^{\infty} a_p z^{1+p/m}\right), \quad a_0=1;$$

the expansion is valid in $D(\varphi/\alpha_0^*, \alpha_0^* r_1)$. Also by Theorem 4 c, $f^*(z) = \frac{1}{\alpha_0^*} f^{**}(\alpha_0^* z)$ is regular if $f^{**}(z)$ is so and $f_\sigma^*(z) = \frac{1}{\alpha_0^*} f_\sigma^{**}(\alpha_0^* z)$ is asymptotically differentiable at 0 if $f_0^{**}(z)$ is so.

Henceforth we shall assume that

$$f(z) \sim z \sum_{p=0}^{\infty} a_p z^{2+p/m}, \quad a_0 = 1, \tag{4}$$

and the expansion is valid in

$$B(\theta, r_1(\theta)) = \{z;\ z = x + iy,\ 0 < x \leqslant r_1(\theta),\ -\theta x^2 \leqslant y \leqslant \theta x^2\}. \tag{5}$$

It is assumed that $r_1(\theta)$ is chosen so small that the conditions (i), (ii), (iii) and (iv) of Theorem 7 c are fulfilled in $D(\varphi_\theta, r_1)$, $D(\psi_\theta, r_1)$, where

$$\varphi_\theta(x) = \theta x^2 - \gamma x^{2+1/m}, \quad \psi_\theta(x) = \theta x^2 + \gamma x^{2+1/m}, \tag{6}$$
$$\gamma = \theta(|a_1| + 2m).$$

By Theorem 6, § 5,

$$f'(z) \sim 1 - \sum_{p=0}^{\infty} \left(2 + \frac{p}{m}\right) a_p z^{1+p/m}, \quad a_0 = 1. \tag{7}$$

We note that the regular asymptotic expansion of $f_\sigma(z)$, if it exists at all, is uniquely determined by the expansion of $f(z)$. Suppose that

$$f_\sigma(z) \sim z - \sum_{p=0}^{\infty} a_p^{(\sigma)} z^{2+p/m}, \quad a_p^{(\sigma)} = \sigma, \tag{8}$$

and $a_p^{(1)} = a_p$, $p = 0, 1, 2, \ldots$. If $f_\sigma(z)$ is a fractional iterate of $f(z)$, we must have

$$f(f_\sigma(z) = f_\sigma(f(z)), \tag{9}$$

i.e.

$$z - \sum_{q=0}^{\infty} a_q^{(\sigma)} z^{2+q/m} - \sum_{p=0}^{\infty} a_p \left(z - \sum_{q=0}^{\infty} a_q^{(\sigma)} z^{2+q/m}\right)^{2+p/m}$$
$$\sim z - \sum_{p=0}^{\infty} a_p z^{2+p/m} - \sum_{q=0}^{\infty} a_q^{(\sigma)} \left(z - \sum_{p=0}^{\infty} a_p z^{2+p/m}\right)^{2+q/m}.$$

The coefficient of $z^{3+p/m}$ gives that $\left(2 + \frac{q}{m}\right) a_q^{(\sigma)} = \left(2 + \frac{q}{m}\right) \sigma a_q +$ terms composed of a_p, $a_p^{(\sigma)}$ with $p < q$. Thus the relation allows us to calculate each $a_q^{(\sigma)}$ in a perfectly definite manner.

By property (ii) of Theorem 7 c, every $z \in D(\varphi_\theta, r_1)$ determines a sequence z_n defined inductively by

$$z_0 = z, \; z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots .$$

Each member of the sequence is in $D(\varphi_\theta, r_1)$.

LEMMA 1. *Given* $\theta > 0$ *there is a positive* $r_2 = r_2(\theta) \leqslant r_1(\theta)$ *such that for every fixed* ξ, $0 < \xi \leqslant r_2$ *and every* $z \in D(\varphi_\theta, r_2)$, $|z - \xi| \leqslant \theta \xi^2$,

$$n z_n = 1 + O_\xi(n^{-1/m}). \tag{10}$$

The index ξ with the O-symbol indicates, as in § 5, that the constant appearing in the symbol is allowed to depend on ξ. This convention (both with o and O symbols) will be used throughout the section. A consequence of the Lemma is that $n z_n \to 1$ uniformly for $z \in D(\varphi_\theta, r_2)$, $|z - \xi| \leqslant \theta \xi^2$.

Proof. For $0 < \xi \leqslant r_1$ write

$$\sigma(\xi) = \sup \{n \xi_n; \quad n = 1, 2, \ldots\}.$$

We first show that
$$\lim_{\xi \downarrow 0} \sigma(\xi) = 1. \tag{11}$$

By formula (5.27) $\left(\text{with } \beta = 1, \; a = 1, \; \delta = \dfrac{1}{m}\right)$

$$n \xi_n = 1 + O_\xi(n^{-1/m}) \tag{12}$$

so that $\sigma(\xi)$ is finite and $\sigma(\xi) \geqslant 1$. Also if we set $\sigma(\eta, h) = \sup \{n \eta_n; \; n \geqslant h\}$, then $\lim\limits_{h \to \infty} \sigma(\eta, h) = 1$. Let $h = h(\xi)$ be defined

$$\eta_h \leqslant \xi < \eta_{h-1}, \quad \eta = \eta_0 = r_1;$$

then $\eta_{h+n} \leqslant \xi_n < \eta_{h+n-1}$ and $n \xi_n < n \eta_{h+n-1} \leqslant \dfrac{n}{h+n-1} \sigma(\eta, h-1)$, $\sigma(\xi) \leqslant \sigma(\eta, h-1)$. But $\lim\limits_{\xi \downarrow 0} h(\xi) = \infty$, therefore $\lim\limits_{\xi \downarrow 0} \sigma(\xi) \leqslant \lim\limits_{h \to \infty} \sigma(\eta, h-1) = 1$. This proves (11).

Now determine a positive integer n_0 so that

$$1 + (n + n_0)^{-1-1/3m} < \left(1 + \frac{1}{n + n_0}\right)^{1/3m} \quad \text{for } n \geqslant n_0; \tag{13}$$

and choose $r_2 \leqslant 1$ so that the following conditions be fulfilled for $z \in D(\varphi_\theta, r_2)$, $0 < \xi \leqslant r_2$ and $n \geqslant 0$:

$$|f'(z) - 1 + 2z| < |z|^{1+1/2m} \tag{14}$$

$$f(\xi) > \tfrac{1}{2} \xi \tag{15}$$

$$|\xi^2 - 2\xi^3 - f^2(\xi)| < \xi^{3+1/2m} \tag{16}$$

$$\theta\left(1+\frac{n}{n_0}\right)^{1/3m} < \xi_n^{-1+1/2m} \tag{17}$$

$$17\,\xi_n^{1+1/2m} < (n+n_0)^{-1-1/3m}, \tag{18}$$

where n_0 is the integer determined from (13). The first three conditions are possible by (4) and (7), the last two conditions by (11).

Let $0<\xi\leqslant r_2$, $z\in D(\varphi_\theta, r_2)$, $|z-\xi|\leqslant\theta\,\xi^2$, and write

$$|z_n-\xi_n|=\theta_n\,\xi_n^2 \tag{19}$$

so that $\theta_0\leqslant\theta$. We have for $n\geqslant 0$

$$z_{n+1}-\xi_{n+1}=f(z_n)-f(\xi_n)=\int_{\xi_n}^{z_n} f'(\zeta)\,d\zeta$$

where the integration it taken along a straight path. By (14),

$$\left|\int_{\xi_n}^{z_n} f'(\zeta)\,d\zeta-(z_n-\xi_n)+(z_n^2-\xi_n^2)\right|<|z_n-\xi_n|\,(\xi_n+|z_n-\xi_n|)^{1+1/2m},$$

$$|z_{n+1}-\xi_{n+1}-(z_n-\xi_n)(1-2\,\xi_n)|<|z_n-\xi_n|\,\{(\xi_n+|z_n-\xi_n|)^{1+1/2m}+|z_n-\xi_n|\},$$

$$\left|\frac{z_{n+1}-\xi_{n+1}}{\xi_{n+1}^2}-\frac{(z_n-\xi_n)}{\xi_n^2}\,\frac{(1-2\,\xi_n)\,\xi_n^2}{\xi_{n+1}^2}\right|<\frac{4\,|z_n-\xi_n|}{\xi_n^2}\{(\xi_n+|z_n-\xi_n|)^{1+1/2m}+|z_n-\xi_n|\}$$

by (15), hence by (16) and (19)

$$\left|\frac{z_{n+1}-\xi_{n+1}}{\xi_{n+1}^2}-\frac{z_n-\xi_n}{\xi_n^2}\right|<\frac{4\,|z_n-\xi_n|}{\xi_n^2}\{\xi_n^{1+1/2m}+(\xi_n+|z_n-\xi_n|)^{1+1/2m}+|z_n-\xi_n|\},$$

$$\theta_{n+1}<\theta_n\{1+4\,\xi_n^{1+1/2m}+4\,(1+\theta_n\,\xi_n)^{1+1/2m}\,\xi_n^{1+1/2m}+\theta_n\,\xi_n^2\}\cdot \tag{20}$$

We shall prove that

$$\theta_n\leqslant\theta\left(\frac{n}{n_0}+1\right)^{1/3m}\qquad\text{for } n\geqslant 0. \tag{21}$$

The statement is true for $n=0$ since $\theta_0\leqslant\theta$; suppose therefore that it is true for some $n\geqslant 0$. We have, by (17) and (21), since $\xi_n\leqslant r_2\leqslant 1$,

$$\theta_n\,\xi_n\leqslant\theta\left(\frac{n}{n_0}+1\right)^{1/3m}\xi_n<\xi_n^{1/2m}\leqslant 1,$$

$$\theta_n\,\xi_n^2<\xi_n^{1+1/2m},$$

hence by (20) and (18),

$$\theta_{n+1} < \theta_n (1 + 17\, \xi_n^{1+1/2m}) < \theta_n \{1 + (n+n_0)^{-1-1/3m}\}. \tag{22}$$

Finally, by (13), (21) and (22)

$$\theta_{n+1} < \theta_n \left(1 + \frac{1}{n+n_0}\right)^{1/3m} \leqslant \theta \left(\frac{n}{n_0} + 1\right)^{1/3m} \left(1 + \frac{1}{n+n_0}\right)^{1/3m} = \theta \left(\frac{n+1}{n_0} + 1\right)^{1/3m}.$$

Thus (21) and therefore also (22) is true for every $n \geqslant 0$; but (22) clearly implies that θ_n is bounded, $\theta_n \leqslant K$ for every n, hence

$$|z_n - \xi_n| \leqslant K\, \xi_n^2 \tag{23}$$

for every $n \geqslant 0$ and every $z \in D(\varphi_\theta, r_2)$ with $|z - \xi| \leqslant \theta\, \xi^2$.

Hence $|n z_n - n \xi_n| \leqslant K (n \xi_n) \xi_n = O_\xi \left(\frac{1}{n}\right)$ which by (12) proves (10).

Remark. It can be assumed that $r_2(\theta)$ (and all future functions $r_\nu(\theta)$, $\nu = 3, 4, \ldots$) are monotonically non-increasing functions of θ.

LEMMA 2. *Given $\theta > 0$ there exists a positive number $r_3 = r_3(\theta)$ with the following properties:*

(i) $r_3(\theta) < r_2(3\theta)$.

(ii) *If $0 < \xi < r_3(\theta)$ and $T(\theta, \xi)$ denotes the union of the sets*

$$S_n(\theta, \xi) = \{z;\ z = \zeta_n,\ \zeta = x + iy,\ \xi - \xi^2 \leqslant x \leqslant \xi + \xi^2,$$
$$-\varphi_{2\theta}(x) \leqslant y \leqslant \varphi_{2\theta}(x)\}, \qquad n = 0, 1, 2, \ldots$$

then
$$T(\theta, \xi) \supset B(\theta, \xi),$$

(iii) *Given ξ such that $0 < \xi \leqslant r_3(\theta)$,*

$$n z_n = 1 + O(n^{-1/m}) \tag{24}$$

uniformly for $z \in S_0(\theta, \xi)$.

Here $r_2(3\theta)$ is the function obtained in Lemma 1, $\varphi_{2\theta}(x)$ is defined by (6) and $B(\theta, \xi)$ by (3).

Proof. If r_3 satisfies condition (i) and is sufficiently small, we can achieve that (with the notations (6))

$$\varphi_{3\theta}(x) > \psi_{2\theta}(x) > \varphi_{2\theta}(x) > \psi_\theta(x) \quad \text{for } 0 < x \leqslant r_3 + r_3^2. \tag{25}$$

Also we can achieve, by Theorem 7 c, § 7 and in particular by the inequalities (7.29), that the following be true: To every $z \in B(\theta, \xi)$, where $0 < \xi \leqslant r_3$, there is a non-negative integer n so that $\zeta = f_{-n}(z)$ is in the region

$$\{\zeta;\ \zeta = x+iy,\ \ \xi-\xi^2 \leqslant x \leqslant \xi+\xi^2,\ \ -\psi_\theta(x) \leqslant y \leqslant \psi_\theta(x)\}.$$

Then by (25), $f_{-n}(z)$ is in $S_0(\theta, \xi)$, $z = \zeta_n$ is in $S_n(\theta, \xi)$, so that property (ii) is true, and the only thing that remains to be proved is the uniformity of (24). But clearly we can select a finite number of points $\xi^{(p)}$, $p = 0, 1, \dots, h$,

$$\xi - \xi^2 = \xi^{(0)} < \xi^{(1)} < \dots < \xi^{(h)} = \xi + \xi^2$$

so that the sets

$$U_p = \{z;\ \ z \in D(\varphi_{3\theta}, r_2(3\theta)),\ \ |z - \xi^{(p)}| \leqslant 3\theta(\xi^{(p)})^2\}$$

cover $S_0(\theta, \xi)$; and in each U_p (24) is uniformly valid by Lemma 1. Therefore it is also uniform in $S_0(\theta, \xi)$.

LEMMA 3. *Let the numbers A_p, $p = 0, 1, \dots, m$ be calculated recursively from*

$$h^2(t)\, g(t^m h(t)) - h(t) - \frac{1}{m} t h'(t) + (1 + A_m) t^m \equiv 0 \pmod{t^{m+1}} \tag{26}$$

where
$$g(t) = \sum_{p=0}^{m} a_p t^{p/m}, \quad a_0 = 1, \tag{27}$$

$$h(t) = \sum_{p=0}^{m-1} A_p t^p, \quad A_0 = 1. \tag{28}$$

For $z \in D(\varphi_\theta, r_3(\theta))$ define $\lambda_n = \lambda_n(z)$ by

$$z_n = \sum_{p=0}^{m-1} A_p n^{-1-p/m} + A_m n^{-2} \log n + n^{-2} \lambda_n; \tag{29}$$

then
$$\lim_{n\to\infty} \lambda_n = \lambda^*(z) \tag{30}$$

exists for every $z \in D(\varphi_\theta, r_3(\theta))$ and the convergence is uniform for $z \in S_0(\theta, \xi)$, where ξ is any fixed positive number, $0 < \xi \leqslant r_3(\theta)$.

Remark.

1. The coefficients a_p in (27) are those from (4); $r_3(\theta)$ is the function obtained in Lemma 2.

2. The numbers A_p are uniquely determined by the recursive relation. In fact, the coefficient of t^p, $1 \leqslant p \leqslant m-1$ is equal to $\left(1 - \frac{p}{m}\right) A_p +$ terms composed of A_q with $q < p$, and the coefficient of t^m is equal to $A_m +$ terms composed of A_q with $q < m$.

In the particular case of $m=1$ the recursion gives $A_1 = -(1+a_1)$, hence

$$\lambda^*(z) = \lim_{n\to\infty} n^2 \left\{ z_n - \frac{1}{n} + (1+a_1)\frac{\log n}{n^2} \right\}. \tag{31}$$

3. It follows from (30) that $\lambda^*(z)$ is bounded on $S_0(\theta, \xi)$.

4. Let $d>0$ and such that $f(z)$ is holomorphic for $z=x$, $0<x\leqslant d$ and $f(x)<x$ for $0<x\leqslant d$. If $r_3(\theta)<\eta\leqslant d$, there is a smallest positive integer k so that $\eta_k=\xi\leqslant r_3(\theta)$; furthermore, there is a positive number $\varrho=\varrho(\theta,\eta)$ so that $t=z_k=f_k(z)$ exists and is holomorphic and schlicht on the disk $|z-\eta|\leqslant\varrho(\theta,\eta)$, and $t\in S_0(\theta,\tau)$. But then if we define

$$\lambda_n(z) = n^2 z_n - \sum_{p=0}^{m-1} A_p n^{1-p/m} - A_m \log n, \tag{32}$$

we get for $n>k$, by (29),

$$\begin{aligned}
\lambda_n(z) &= n^2 t_{n-k} - \sum_{p=0}^{m-1} A_p n^{1-p/m} - A_m \log n \\
&= (n-k)^2 t_{n-k} + 2k(n-k)t_{n-k} + k^2 t_{n-k} \\
&\quad - \sum_{p=0}^{m-1} A_p (n-k)^{1-p/m} - k - A_m \log(n-k) + O(n^{-1/m}) \\
&= \lambda_{n-k}(t) + k + O_k(n^{-1/m}).
\end{aligned}$$

Thus $\lambda^*(z) = \lim_{n\to\infty} \lambda_n(z)$ exists, the convergence is uniform for $|z-\eta|\leqslant\varrho(\varphi,\eta)$, and

$$\lambda^*(z) = \lambda^*(z_k) + k. \tag{33}$$

In particular, $\lambda^*(\eta)$ exists for $0<\eta\leqslant d$.

Proof of Lemma 3. Let ξ, $0<\xi\leqslant r_3(\theta)$ be fixed and $z\in S_0(\theta,\xi)$; then by Lemma 2 and (29),

$$\lambda_n(z) = 0_\xi(n) \qquad (n\to\infty). \tag{34}$$

We have, by (4),

$$z_{n+1} = f(z_n) = z_n - \sum_{q=0}^{m} a_q z_n^{2+q/m} + O(|z_n|^{3+1/m})$$

hence, by (29), (34) and Lemma 2,

$$\begin{aligned}
&\sum_{p=0}^{m-1} A_p (n+1)^{-1-p/m} + A_m (n+1)^{-2}\log(n+1) + (n+1)^{-2}\lambda_{n+1} \\
&\qquad = \sum_{p=0}^{m-1} A_p n^{-1-p/m} + A_m n^{-2}\log n + n^{-2}\lambda_n \\
&\qquad\quad - \sum_{p=0}^{m} a_q \left[\sum_{p=0}^{m-1} A_p n^{-1-p/m} + A_m n^{-2}\log n + n^{-2}\lambda_n \right]^{2+q/m} + O_\xi(n^{-3-1/m}).
\end{aligned} \tag{35}$$

The λ-free terms in the first member of this relation become, if for brevity we write

$$t = n^{-1/m}, \tag{36}$$

$$\sum_{p=0}^{m-1} A_p t^{p+m} \left(1 - \left(1 + \frac{p}{m}\right) t^m\right) + t^{3m} + A_m t^{2m} \log n - 2 A_m t^{3m} \log n + A t^{3m}. \tag{37}$$

If the λ-terms are deleted from the second member of (35), the expression can be written in the form

$$\sum_{p=0}^{m-1} A_p t^{p+m} + A_m t^{2m} \log n - \sum_{q=0}^{m} a_q \left[\sum_{p=0}^{m-1} A_p t^{p+m}\right]^{2+q/m} - 2 A_m t^{3m} \log n + O_\xi(n^{-3-1/2m})$$

$$= t^m h(t) + A_m t^{2m} \log n - t^{2m} h^2(t) g(t^m h(t)) - 2 A_m t^{3m} \log n + O_\xi(n^{-3-1/2m})$$

with the notations (27), (28). But the last expression is, by the recursive relation (26), equal to

$$t^m h(t) + A_m t^{2m} \log n - t^{2m} h(t) - \frac{1}{m} t^{2m+1} h'(t) + (1 + A_m) t^{3m} - 2 A_m t^{3m} \log n + O_\xi(n^{-3-1/2m})$$

which is identical with (37). Thus the λ-free terms in (35) cancel and we obtain, by observing (34),

$$(n+1)^{-2} \lambda_{n+1} = n^{-2} \lambda_n - 2 n^{-3} \lambda_n - n^{-4} \lambda_n^2 + \lambda_n O_\xi(n^{-3-1/2m}) + \lambda_n^2 O_\xi(n^{-4-1/2m}) + O_\xi(n^{-3-1/2m}),$$

i.e. $$\lambda_{n+1} = \lambda_n [1 - n^{-2} \lambda_n + O_\xi(n^{-1-1/2m}) + \lambda_n O_\xi(n^{-2-1/2m})] + O_\xi(n^{-1-1/2m}). \tag{38}$$

This implies

$$|\lambda_{n+1}| \leqslant |\lambda_n| (1 + c_1 n^{-2} |\lambda_n| + c_2 n^{-1-1/2m}) + c_3 n^{-1-1/2m} \tag{39}$$

where c_1, c_2, c_3 are suitable positive numbers; they depend on ξ but not on z.

Now suppose that there is a positive n_0 so that $|\lambda_n| > n^{-1-1/2m}$ for $n \geqslant n_0$; then by (39),

$$|\lambda_{n+1}| < |\lambda_n| (1 + c_4 n^{-2} |\lambda_n|) \quad \text{for } n \geqslant n_1$$

where $n_1 \geqslant n_0$, hence by (34)

$$|\lambda_{n+1}| (n+1)^{-1+1/2m} < |\lambda_n| n^{-1+1/2m} (1 + c_4 n^{-2} |\lambda_n|) \left(1 + \frac{1}{n}\right)^{-1+1/2m}$$

$$< |\lambda_n| n^{-1+1/2m} \left(1 - \frac{1}{3n}\right) \quad \text{for } n \geqslant n_2.$$

Hence eventually $|\lambda_n| n^{-1+1/2m} < 1$ for some $n \geqslant n_0$, which contradicts our assumption. We conclude that $|\lambda_n| \leqslant n^{1-1/2m}$ for infinitely many n. But if this inequality holds for sufficiently large n then by (39),

$$|\lambda_{n+1}| < n^{1-1/2m} \{1 + (c_1 + c_2 + c_3) n^{-1-1/2m}\} < (n+1)^{1-1/2m},$$

hence $|\lambda_n| < n^{1-1/2m}$ for $n \geqslant n_3$, and

$$|\lambda_{n+1}| < |\lambda_n| \{1 + (c_1 + c_2) n^{-1-1/2m}\} + c_3 n^{-1-1/2m}.$$

This clearly implies

$$\lambda_n = O_\xi (1). \tag{40}$$

Going back to the relation (38) we find

$$\lambda_{n+1} = \lambda_n + O_\xi (n^{-1-1/2m}) \tag{41}$$

which proves the existence of (30) and the uniformity of convergence for $z \in S_0(\theta, \xi)$.

Let us write now

$$\lambda(z) = \lambda^*(z) - \lambda^*(d) \tag{42}$$

where d is as in Remark 4; $\lambda(z)$ is defined in a suitable neighbourhood of the positive interval $0 < \eta \leqslant d$ and $\lambda(d) = 0$. Also by Remark 4 and formula (32),

$$\lambda(z) = \lim_{n\to\infty} n^2 (z_n - d_n) \tag{43}$$

and in particular

$$\lambda(x) = \lim_{n\to\infty} n^2 (f_n(x) - f_n(d)) \tag{44}$$

for $0 < x \leqslant d$. By comparing this with (4.5) we see that $\lambda(x)$ is a principal logarithm of iteration of $f(x)$, provided that it is a logarithm of iteration, i.e. that it satisfies

$$\lambda(f(z)) = \lambda(z) - 1 \tag{45}$$

and the inverse $\lambda_{-1}(x)$ exists for $0 < x \leqslant \lambda(d)$. The relation (45) is trivial from (33) and (42); the existence of $\lambda_{-1}(x)$ is assured if we can show that $\lambda'(x)$ exists, is continuous and $\lambda'(x) \neq 0$ for $0 < x \leqslant d$.

LEMMA 4. *Let $T(\theta)$ denote the union of every $S_\theta(\theta, \xi)$, $0 < \xi \leqslant r_3(\theta)$ and every disk*

$$K(\theta, \xi) = \{z;\ |z - \xi| \leqslant \varrho(\theta, \xi)\}, \quad r_3(\theta) < \xi \leqslant d$$

where $\varrho(\theta, \xi)$ is the number defined in Lemma 3, Remark 4: let $\lambda(z)$ be defined by (30), (32) and (42). Then $\lambda(z)$ is holomorphic on $T(\theta)$ and $\lambda'(z) \neq 0$ for $z \in T(\theta)$. In particular, $\lambda(x)$ is a principal logarithm of iteration of $f(x)$.

Proof. By (29), (30) and Remark 4,

$$\lambda^*(z) = \lim_{n\to\infty} \left\{ n^2 f(z) - \sum_{p=0}^{m-1} n^{-1-p/m} A_p - A_m \log n \right\} \tag{46}$$

and the convergence is uniform on $T(\theta)$. But $f_n(z)$ is holomorphic on $T(\theta)$, therefore $\lambda^*(z)$ hence also $\lambda(z)$ is holomorphic on $T(\theta)$. Thus the Lemma is proved if we can show that $\lambda'(z) \neq 0$ for $z \in T(\theta)$. Since

$$\lambda'(z) = f'(z)\,\lambda'(f(z)) \tag{47}$$

by (45) and $f'(z) \neq 0$ on $T(\theta)$ by construction, it is sufficient to prove the statement when $z \in S_0(\theta, \xi)$, $0 < \xi \leqslant r_3(\theta)$.

From (43)

$$\lambda'(z) = \lim_{n\to\infty} n^2 f_n'(z), \tag{48}$$

hence

$$\log \lambda'(z) = \lim_{n\to\infty} \left\{ 2 \log n + \sum_{p=0}^{n-1} \log f'(z_p) \right\}. \tag{49}$$

But by (7) and (24),

$$\log f'(z_p) = -2 z_p + O(|z_p|^{1+1/m})$$

$$= -\frac{2}{p} + O_z(p^{-1-1/m}),$$

$$\sum_{p=0}^{n-1} \log f'(z_p) = -2 \log n + \gamma + O_z(n^{-1/m})$$

for some finite γ, which implies finiteness of $\log \lambda'(z)$ in (49) hence $\lambda'(z) \neq 0$.

From Lemma 4 it follows that

$$f_\sigma(z) = \lambda_{-1}(\lambda(z) - \sigma) \tag{50}$$

exists and is holomorphic for $z = x$, $0 < x \leqslant d$; to complete the proof of Theorem 3c, we have to show that $f_\sigma(z)$ is asymptotically differentiable at 0. The proof requires a substantial refinement of Lemma 3.

LEMMA 5. *For every $z \in D(\varphi_\theta, r_3(\theta))$, z_n has an asymptotic representation*

$$z_n \sim \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} B_{pq}\, n^{-1-p/m} \left(\frac{\log n}{n}\right)^q \tag{51}$$

where

$$B_{00} = 1, \quad B_{p0} = A_p \quad \text{for } p = 1, \ldots, m-1,$$

$$B_{m0} = \lambda^*(z), \quad B_{01} = A_m; \tag{52}$$

the representation is valid uniformly for $z \in S_0(\theta, \xi)$ where ξ is a fixed positive number,

$0 < \xi \leqslant r_3(\theta)$. *The notations are those of Lemma* 3, *and the meaning of the representation* (51) *is that*

$$z_n = \sum_{p+qm \leqslant k} B_{pq}\, n^{-1-p/m} \left(\frac{\log n}{n}\right)^q + O_\xi(n^{-1-k/m}) \tag{53}$$

for every fixed $k \geqslant m$.

The coefficients B_{pq} depend on $\lambda^*(z)$; specifically, B_{pq} is a polynomial in $\lambda^*(z)$ of degree $\leqslant p/m$.

Proof. Let the coefficients A_{pq} be calculated recursively from the generating relations

$$A_{00} = 1, \quad A_{m0} = 0,$$

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq}\, t^p (1+t^m)^{1-q-p/m} (u + t^m \log(1+t^m))^q - \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq}\, t^p u^q$$
$$+ \sum_{s=0}^{\infty} a_s t^{m+s} \left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} A_{pq}\, t^p u^q \right)^{2+s/m} \equiv 0. \tag{54}$$

It is seen easily that the coefficients of $t^p u^q$, $p = 0, 1, \dots, m-1$, $q \geqslant 0$ and of $u t^m$ are identically zero, the coefficient of t^{2m} is A_{01} + terms composed of A_{p0}, $p < m$; finally, the coefficient of $u^q t^{p+m}$ when $q + \frac{p}{m} \neq 1$ is $\left(1 - q - \frac{p}{m}\right) A_{pq}$ + terms composed of $A_{p'q'}$ with $p' \leqslant p - m$ or $p' \leqslant p$, $q' < q$, or $p' < p$, $q' = q$. Therefore, every A_{pq} can be determined uniquely from the recursion; it is also seen, by comparing (54) with (26), that the coefficient A_{p0}, $p = 0, 1, \dots, m-1$ and A_{01} are identical with $A_0, \dots, A_{m-1}$ and A_m respectively.

Now having fixed a positive integer $k \geqslant m$, let us write

$$z_n = \sum_{p+qm \leqslant k} A_{pq}\, n^{-1-p/m} \left(\frac{\log n}{n}\right)^q + n^{-2} \lambda_n. \tag{55}$$

Lemma 3 implies that
$$\lim_{n\to\infty} \lambda_n = \lambda^*(z) \tag{56}$$

and by Remark 3 after Lemma 3,

$$\lambda_n = \lambda_n(z) = O_\xi(1) \tag{57}$$

if $z \in S_0(\theta, \xi)$.

We get from (4)

$$z_{n+1} = f(z_n) = z_n - \sum_{s=0}^{k} a_s z_n^{2+s/m} + O(|z_m|^{2+(k+1)/m})$$

hence by (55) and (57),

$$\sum_{p+qm\leqslant k} A_{pq}(n+1)^{-1-p/m}\left(\frac{\log(n+1)}{n+1}\right)^q+(n+1)^{-2}\lambda_{n+1}$$

$$=\sum_{p+qm\leqslant k} A_{pq}n^{-1-p/m}\left(\frac{\log n}{n}\right)^q+n^{-2}\lambda_n$$

$$-\sum_{s=0}^{k} a_s\left[\sum_{p+qm\leqslant k} A_{pq}n^{-1-p/m}\left(\frac{\log n}{n}\right)^q+n^{-2}\lambda_n\right]^{2+s/m}+O_\xi(n^{-2-(k+1)/m}). \quad (58)$$

The λ-free terms in the first member of this relation can be written in the form

$$\sum_{p+qm\leqslant k} A_{pq}t^{p+m}(1+t^m)^{-1-q-p/m}(u+t^m\log(1+t^m))^q$$

with
$$t=n^{-1/m},\quad u=\frac{\log n}{n} \quad (59)$$

and these cancel, by (54), the λ-free terms in the second member of (58), apart from terms of order $O_\xi(n^{-2-(k+1)/m}\log^k n)$. Hence we are left with

$$(n+1)^{-2}\lambda_{n+1}=n^{-2}\lambda n-\sum_{s=0}^{k} a_s\sum_{r=1}^{k}\binom{2+s/m}{r}n^{-2r}\lambda_n^r$$

$$\left[\sum_{p+qm\leqslant k} A_{pq}n^{-1-p/m}\left(\frac{\log n}{n}\right)^q\right]^{2-r+s/m}+O_\xi(n^{-2-(k+1)/m}\log^k n),$$

i.e.
$$\lambda_{n+1}=\left(1+\frac{1}{n}\right)^2\lambda_n\left\{1-\sum_{s=0}^{k} a_s\sum_{r=1}^{k}\binom{2+s/m}{r}n^{-r-s/m}\lambda_n^{r-1}\right.$$

$$\left.\left[\sum_{p+qm\leqslant k} A_{pq}n^{-p/m}\left(\frac{\log n}{n}\right)^q\right]^{2-r+s/m}\right\}+O_\xi(n^{-(k+1)/m}\log^k n). \quad (60)$$

On multiplying out, the term $n^{-1}\lambda_n$ drops out, and the expression takes the form

$$\lambda_{n+1}=\lambda_n\left\{1+\sum_{1\leqslant p+qm\leqslant k-m} C_{pq}^{(1)}n^{-1-p/m}\left(\frac{\log n}{n}\right)^q\right\}$$

$$+\sum_{r=2}^{k}\lambda_n^r\sum_{p+qm\leqslant k-rm} C_{pq}^{(r)}n^{-r-p/m}\left(\frac{\log n}{n}\right)^q+O_\xi(n^{-(k+1)/m}\log^k n) \quad (61)$$

for certain coefficient $C_{pq}^{(r)}$ which can be calculated uniquely from (60). From (61) we get, if $k\geqslant m$,

$$\lambda_{n+1}-\lambda_n=O_\xi(n^{-1-1/m}\log^m n),$$

hence by (56),
$$\lambda^*(z)-\lambda_n=\sum_{r=n}^{\infty}(\lambda_{r+1}-\lambda_r)=O_\xi(n^{-1/m}\log^m n),$$

$$\lambda_n=\lambda^*(z)+O_\xi(n^{-1/m}\log^m n).$$

Suppose now that for some j, $0 \leqslant j \leqslant k-m$, and for certain coefficients D_{pq} which are polynomials in $\lambda^*(z)$ of degree $\leqslant 1+\dfrac{p}{m}$, we have proved that

$$\lambda_n = \lambda^*(z) + \sum_{1 \leqslant p+qm \leqslant j} D_{pq} n^{-p/m} \left(\frac{\log n}{n}\right)^q + O_\xi(n^{-(j+1)/m} \log^{m+j} n); \tag{62}$$

if this is put back into (61), we get

$$\lambda_{n+1} - \lambda_n = \sum_{1 \leqslant p+qm \leqslant j+1} E_{pq} n^{-1-p/m} \left(\frac{\log n}{n}\right)^q + O_\xi(n^{-1-(j+2)/m} \log^{m+j+1} n)$$

hence

$$\lambda_n - \lambda^*(z) = \sum_{r=n}^{\infty} (\lambda_r - \lambda_{r+1}) = \sum_{1 \leqslant p+qm \leqslant j+1} D_{pq} n^{-p/m} \left(\frac{\log n}{n}\right)^q + O_\xi(n^{-(j+2)/m} \log^{m+j+1} n).$$

By induction, (62) is true for $j=k-m$,

$$\lambda_n = \sum_{p+qm \leqslant k-m} D_{pq} n^{-p/m} \left(\frac{\log n}{n}\right)^q + O_\xi(n^{1-(k+1)/m} \log^k n) \tag{63}$$

with $D_{00} = \lambda^*(z)$ and D_{pq} of degree $\leqslant 1+p/m$ in $\lambda^*(z)$. Finally, if (63) is substituted into (55), we obtain (51), (52) with

$$B_{pq} = A_{pq} \qquad \text{for } p < m,$$
$$B_{pq} = A_{pq} + D_{p-m,q} \quad \text{for } p \geqslant m.$$

This concludes the proof of the Lemma.

Lemma 6. *$\lambda(z)$ has an asymptotic expansion*

$$\lambda(z) \sim -\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} F_{pq} z^{q-1+p/m} \log^q z, \quad F_{00} = 1 \tag{64}$$

when $z \to 0$ in $B(\theta, r_3(\theta))$, $\theta > 0$.

Proof. By an obvious inversion we get from (53)

$$\frac{1}{n} = \sum_{p+qm \leqslant k} \breve{A}_{pq} z_n^{1+q+p/m} \log^q z_n + O_\xi(|z_n|^{1+k/m}) \tag{65}$$

where the O symbol refers to $n \to \infty$ and formula is valid for $z \in S_0(\theta, \xi)$. From (52) we see that $\breve{A}_{00} = 1$, and $\breve{A}_{pq}$ is a polynomial in $\lambda^*(z)$ of degree $\leqslant p/m$; in particular, $\breve{A}_{m0} = \lambda^*(z) + A^*_{m0}$ where A^*_{m0} does not depend on $\lambda^*(z)$. Hence

$$n = \lambda^*(z) + \sum_{p+qm \leqslant k} F_{pq}(\lambda^*(z)) z_n^{q-1+p/m} \log^q z_n = O_\xi(|z_n|^{-1+k/m}) \tag{66}$$

where $F_{00}(\lambda^*(z))=1$ and $F_{pq}(\lambda^*(z))$ is a polynomial in $\lambda^*(z)$ of degree $\leqslant p/m$. It turns out that F_{pq} does not actually depend on $\lambda^*(z)$; for if we write $t=z_1$ so that $z_n=t_{n-1}$, and apply the formula with $n-1$ instead of n, we get

$$n-1=\lambda^*(t)+\sum_{p+qm\leqslant k} F_{pq}(\lambda^*(t))\, t_{n-1}^{q-1+p/m}\log^q t_{n-1}+O_{\xi_1}(|t_{n-1}|^{-1+k/m}) \tag{67}$$

$$=\lambda^*(t)+\sum_{p+qm\leqslant k} F_{pq}(\lambda^*(t))\, z_n^{q-1+p/m}\log^q z_n+O_{\xi}(|z_n|^{-1+k/m});$$

we have replaced here O_{ξ_1} by O_{ξ} which is clearly permissible. But $\lambda^*(t)=\lambda^*(z)-1$, hence by comparison with (66), $F_{pq}(\lambda^*(z)-1)=F_{pq}(\lambda^*(z))$. This is only possible if F_{pq} is of degree 0 in $\lambda^*(z)$. Hence

$$n\sim\lambda^*(z)+\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} F_{pq}\, z_n^{q-1+p/m}\log^q z_n, \tag{68}$$

uniformly for $z\in S_0(\theta,\xi)$. By choosing $\xi=r_3(\theta)$ and noticing that $\lambda^*(z_n)=\lambda^*(z)-n$, the Lemma now follows from the remark that every $t\in B(\theta, r_3(\theta))$ has the form $t=z_n$, $z\in S_0(\theta, r_3(\theta))$ by condition (ii) in Lemma 2.

An immediate consequence of Lemma 6 is, by Theorem 5 c, § 6 and the remark (6.4), that $f(x)$ is regular with respect to iteration; this will, of course, also follow from the asymptotic differentiability of $f_\sigma(z)$.

LEMMA 7. *Given $\theta>0$ there is a positive number $r_4=r_4(\theta)\leqslant r_3(2\theta)$ such that*

$$\lambda'(z)\sim\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left\{\left(1-q-\frac{p}{m}\right)\log^q z-q\log^{q-1} z\right\}F_{pq}\, z^{q-2+p/m} \tag{69}$$

when $z\to 0$ on $B(\theta, r_4(\theta))$, and $\lambda(z)$ is schlicht on $B(\theta, r_4(\theta))$.

Proof. The condition that (69) be valid on $B(\theta, r_4(\theta))$ can be fulfilled by the remark after the proof of Theorem 6, § 7. Subject to this condition, choose $r_4(\theta)$ so small that also

$$r_4(\theta)<(4(4+\theta))^{-2m} \tag{70}$$

$$|\lambda'(z)-z^{-2}|<|z|^{-2+1/2m}\quad\text{for } z\in B(\theta, r_4(\theta)). \tag{71}$$

Take any two points $z_1=\xi_1+i\eta_1$, $z_2=\xi_2+i\eta_2$ on $B(\theta, r_4(\theta))$; clearly

$$|\eta_1|\leqslant\theta\xi_1^2,\quad |\eta_2|\leqslant\theta\xi_2^2,\quad \xi_1>\tfrac{2}{3}|z_1|,\quad \xi_2>\tfrac{2}{3}|z_2|. \tag{72}$$

We have
$$\lambda(z_2)-\lambda(z_1)=\int_{z_1}^{z_2}\lambda'(\zeta)\,d\zeta=\frac{1}{z_1}-\frac{1}{z_2}+\int_{z_1}^{z_2}(\lambda'(\zeta)-\zeta^{-2})\,d\xi \tag{73}$$

where the integration is taken over any path connecting z_1 and z_2 inside $B(\theta, r_4(\theta))$.

Suppose first that $2\xi_1 \leqslant \xi_2 \leqslant r_4$, and take the integral in (73) over the path $\Gamma_1 + \Gamma_2$ where Γ_1 is a straight line from z_1 to ξ_1 and Γ_2 a straight line from ξ_1 to z_2. By (71), (72),

$$\left|\int_{\Gamma_1} (\lambda'(\zeta) - \zeta^{-2})\, d\zeta\right| \leqslant \theta\, \xi_1^2\, \xi_1^{-2+1/2m} \leqslant \theta\, |z_1|^{1/2m} < \theta\, |z_1|^{-1+1/2m},$$

$$\left|\int_{\Gamma_2} (\lambda'(\zeta) - \zeta^{-2})\, d\zeta\right| < \frac{3}{2}\int_{\xi_1}^{\infty} x^{-2+1/2m}\, dx \leqslant 3\, \xi_1^{-1+1/2m} < 4\, |z_1|^{1+1/2m},$$

$$\left|\frac{1}{z_1} - \frac{1}{z_2}\right| \geqslant \frac{1}{|z_1|} - \frac{1}{\xi_2} \geqslant \frac{1}{|z_1|} - \frac{1}{2\xi_1} > \frac{1}{4}|z_1|^{-1},$$

hence by (70) and (73),

$$\begin{aligned}|\lambda(z_2) - \lambda(z_1)| &> \tfrac{1}{4}|z_1|^{-1}\{1 - 4(4+\theta)|z_1|^{1/2m}\} \\ &> \tfrac{1}{4}|z_1|^{-1}\{1 - 4(4+\theta)\, r_4^{1/2m}\} > 0.\end{aligned}$$

Suppose next that $\xi_1 < \xi_2 < 2\xi_1$, and choose the path of integration in (73) so that the length of path shall not exceed $2|z_2 - z_1|$. By (71),

$$\left|\int_{z_1}^{z_2} (\lambda'(\xi) - \zeta^{-2})\, d\xi\right| \leqslant 2\,|z_2 - z_1|\, \xi_1^{-2+1/2m}, \quad \left|\frac{1}{z_1} - \frac{1}{z_2}\right| = \frac{|z_2 - z_1|}{|z_1 z_2|} \geqslant |z_2 - z_1|\, \xi_1^{-2}$$

hence by (70) and (73)

$$|\lambda(z_2) - \lambda(z_1)| \geqslant |z_2 - z_1|\, \xi_1^{-2}(1 - 2\,\xi_1^{1/2m}) > 0$$

provided that $z_1 \neq z_2$. Thus $z_1 \in B(\theta, r_4)$, $z_2 \in B(\theta, r_4)$, $z_1 \neq z_2$ imply $\lambda(z_1) \neq \lambda(z_2)$ and $\lambda(z)$ is schlicht on $B(\theta, r_4(\theta))$.

We can now complete the proof of Theorem 3 c. By Lemmas 6 and 7, the curve $z = x + i\theta x^2$, $x > 0$ is mapped by $w = -\lambda(z)$ upon a curve in the w-plane which has the line $w = u - i\theta$, $u > 0$ for an asymptote. Therefore, if $d(\theta)$ denotes the domain in the w-plane upon which $B(\theta, r_4(\theta))$ is mapped by $-\lambda(z)$ and $r_5(\theta)$ is sufficiently small, the domain $D(2\theta)$ contains in its interior all points $w = -\lambda(z) - \sigma$, $z \in B(\theta, r_5(\theta))$, $|\sigma| \leqslant \frac{1}{2}\theta$. Hence $f_\sigma(z) = \lambda_{-1}(\lambda(z) - \sigma)$ is uniquely defined and is holomorphic on $B(\theta, r_5(\theta))$. Now it can be shown similarly to Theorem 7 c, § 7, that $\lambda_{-1}(-w)$ has an asymptotic expansion of the form

$$\lambda_{-1}(-w) \sim \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} G_{pq}\, w^{-1-q-p/m} \log^q w, \quad G_{00} = 1 \tag{74}$$

when $w \to 0$ on $D(\theta)$ (or even on the strip $w = u + iv$, $u \geqslant R(\theta)$, $-\theta \leqslant v \leqslant \theta$, where

$R(\theta)$ is a suitable (large) positive number). In fact the coefficients of the expansion (74) can uniquely be determined so that if $w = -\lambda(z)$ is substituted from (64), it should satisfy formally the relation (74). We only have to note that

$$\log(-\lambda(z)) \sim -\log z + \log\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} F_{pq} z^{q+p/m} \log^q z\right)$$

$$\sim -\log z + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} F^*_{pq} z^{q+p/m} \log^q z, \quad F^*_{00} = 0.$$

The formal expansion is converted in a true asymptotic expansion by choosing $r_5(\theta)$ so small that

$$\frac{1}{2|z|} < |w| < \frac{3}{2|z|} \quad \text{when } z \in B(\theta, r_5(\theta)).$$

If we substitute $w = -(\lambda(z) - \sigma)$ from (64) into (74), where $|\sigma| \leqslant \frac{1}{2}\theta$, we obtain an expansion of the form

$$f_\sigma(z) = \lambda_{-1}(\lambda(z) - \sigma) \sim z \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a^{(\sigma)}_{pq} z^{q+p/m} \log^q z \tag{75}$$

where $a^{(\sigma)}_{pq}$ is a polynomial of degree $\leqslant p/m$ in σ. By assigning θ a large positive value we see that the expansion (75) is valid for every (real or complex) value of σ. But $a^{(\sigma)}_{pq}$ is a polynomial in σ, and clearly

$$\begin{aligned} a^{(\sigma)}_{p0} &= 0 \quad \text{for } 0 < p < m \\ a^{(\sigma)}_{pq} &= 0 \quad \text{for } q > 0 \end{aligned} \tag{76}$$

when σ is a positive integer, therefore these relations are valid for every σ. Hence

$$f_\sigma(z) \sim z - \sum_{p=0}^{\infty} a^{(\sigma)}_p z^{2+p/m}, \tag{77}$$

as required. Incidentally, the relations (76) can also be verified directly from the commutation relation (9).

The validity of (77) has so far only been confirmed on $B(\theta, r_5(\theta))$ if $|\sigma| \leqslant \frac{1}{2}\theta$; but if k is any positive integer, then $f_{k\sigma}(z)$ has an expansion of the form (77) in $B(\frac{1}{2}\theta, r_6(\theta))$, say, where now $r_6(\theta)$ may also depend on k. Therefore, $f_\sigma(z)$ is asymptotically differentiable at 0, at least for every real σ.

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