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Research Article

On Certain Inequalities for Neuman-Sándor Mean

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We present several new sharp bounds for Neuman-Sándor mean in terms of arithmetic, centroidal, quadratic, harmonic root square, and contraharmonic means.

1. Introduction

A binary map $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+ := (0, +\infty)$ is the set of positive numbers) is said to be a *bivariate mean* if the following statements are satisfied for all $a, b, \lambda > 0$: (i) $m(a, a) = a$ (reflexivity property); (ii) $m(a, b) = m(b, a)$ (symmetry property); (iii) $m(\lambda a, \lambda b) = \lambda m(a, b)$ (homogeneous of order one); (iv) $m(a, b)$ is continuous and strictly increasing with respect to a and b.

Let u, v, and w be the bivariate means such that $u(a, b) < w(a, b) < v(a, b)$ for all $a, b > 0$ with $a \neq b$. The problems to find the best possible parameters α and β such that the inequalities $\alpha u(a, b) + (1-\alpha)v(a, b) < w(a, b) < \beta u(a, b) + (1-\beta)v(a, b)$ and $u^{\alpha}(a, b)v^{1-\alpha}(a, b) < w(a, b) < u^{\beta}(a, b)v^{1-\beta}(a, b)$ hold for all $a, b > 0$ with $a \neq b$ have attracted the interest of many mathematicians.

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a, b)$ [1] is defined by

$$M(a, b) = \frac{a-b}{2 \operatorname{arcsinh}[(a-b)/(a+b)]}, \qquad (1)$$

where $\operatorname{arcsinh}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the bounds for the Neuman-Sándor mean in terms of other bivariate means have been the subject of intensive research.

Let $\overline{H}(a, b) = \sqrt{2}ab/\sqrt{a^2+b^2}$, $H(a, b) = 2ab/(a+b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a-b)/(\log a - \log b)$, $P(a, b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $I(a, b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a+b)/2$, $T(a, b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $E(a, b) = 2(a^2+ab+b^2)/[3(a+b)]$, $Q(a, b) = \sqrt{(a^2+b^2)/2}$, and $C(a, b) = (a^2+b^2)/(a+b)$ be the harmonic root square, harmonic, geometric, logarithmic, first Seiffert, identric, arithmetic, second Seiffert, centroidal, quadratic, and contraharmonic means of two distinct positive real numbers a and b, respectively. Then, it is well known that the inequalities

$$\begin{aligned} \overline{H}(a, b) &< H(a, b) < G(a, b) < L(a, b) \\ &< P(a, b) < I(a, b) \\ &< A(a, b) < M(a, b) < T(a, b) \\ &< E(a, b) < Q(a, b) < C(a, b) \end{aligned} \qquad (2)$$

hold for all $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$A(a, b) < M(a, b) < \frac{A(a, b)}{\log(1+\sqrt{2})},$$

$$\frac{\pi}{4}T(a, b) < M(a, b) < T(a, b),$$

$$M(a, b) < \frac{2A(a, b) + Q(a, b)}{3},$$

$$M(a, b) < \frac{A^2(a, b)}{P(a, b)},$$

$$\sqrt{A(a,b)T(a,b)} < M(a,b) < \sqrt{\frac{A^2(a,b)+T^2(a,b)}{2}} \quad (3)$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1-a$, and $b' = 1-b$. Then, the Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \quad (4)$$

can be found in [1].

Li et al. [3] proved that $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ for all $a, b > 0$ with $a \neq b$, where $L_p(a,b) = [(b^{p+1}-a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = I(a,b)$ and $L_{-1}(a,b) = L(a,b)$, is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [4], the author proved that the double inequalities

$$\begin{aligned} &\alpha Q(a,b) + (1-\alpha)A(a,b) \\ &\quad < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b), \\ &\lambda C(a,b) + (1-\lambda)A(a,b) \\ &\quad < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b) \end{aligned} \quad (5)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots$, $\beta \geq 1/3$, $\lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$, and $\mu \geq 1/6$.

In [5, 6], the authors proved that $\alpha = 4$, $\beta = \log 2/[2\log(1+\sqrt{2})]$, $\lambda = 1$, and $\mu = e/[2\log(1+\sqrt{2})]$ are the best possible constants such that the inequalities $M(a,b) > \overline{L}_\alpha(a,b)$, $M(a,b) > M_\beta(a,b)$, and $\lambda I(a,b) < M(a,b) < \mu I(a,b)$ hold for all $a, b > 0$ with $a \neq b$. In here, $\overline{L}_r(a,b) = [(b^r-a^r)/(r(\log b-\log a))]^{1/r}$ and $M_p(a,b) = [(a^p+b^p)/2]^{1/p}$ are the rth generalized logarithmic and pth power means of a and b, respectively.

Zhao et al. [7, 8] found the least values $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and the greatest values $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ such that the double inequalities

$$\begin{aligned} &\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) \\ &\quad < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ &\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) \\ &\quad < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b), \\ &\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) \\ &\quad < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b), \\ &I^{\alpha_4}(a,b)Q^{1-\alpha_4}(a,b) < M(a,b) < I^{\beta_4}(a,b)Q^{1-\beta_4}(a,b), \\ &I^{\alpha_5}(a,b)C^{1-\alpha_5}(a,b) < M(a,b) < I^{\beta_5}(a,b)C^{1-\beta_5}(a,b) \end{aligned} \quad (6)$$

hold for all $a, b > 0$ with $a \neq b$.

In [9], the authors proved that if $\lambda, \mu \in (1/2, 1)$, then the double inequality $C(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < M(a,b) < C(\mu a + (1-\mu)b, \mu b + (1-\mu)a)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq [1+\sqrt{1/\log(1+\sqrt{2})-1}]/2$ and $\mu \geq (6+\sqrt{6})/12$.

The aim of this paper is to present the sharp bounds for Neuman-Sándor mean in terms of the combinations of either arithmetic and centroidal means, or quadratic and harmonic root square means, contraharmonic and harmonic root square means. Our main results are shown in Theorems 1–4.

Theorem 1. *The double inequality*

$$\begin{aligned} &\alpha_1 E(a,b) + (1-\alpha_1)A(a,b) \\ &\quad < M(a,b) < \beta_1 E(a,b) + (1-\beta_1)A(a,b) \end{aligned} \quad (7)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq (3-3\log(1+\sqrt{2}))/\log(1+\sqrt{2}) = 0.4037$ and $\beta_1 \geq 1/2$.

Theorem 2. *The double inequality*

$$E^{\lambda_1}(a,b)A^{1-\lambda_1}(a,b) < M(a,b) < E^{\mu_1}(a,b)A^{1-\mu_1}(a,b) \quad (8)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq \log[\log(1+\sqrt{2})]/(\log 3 - 2\log 2) = 0.4389\cdots$ and $\mu_1 \geq 1/2$.

Theorem 3. *The double inequality*

$$\begin{aligned} &\alpha_2 Q(a,b) + (1-\alpha_2)\overline{H}(a,b) \\ &\quad < M(a,b) < \beta_2 Q(a,b) + (1-\beta_2)\overline{H}(a,b) \end{aligned} \quad (9)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq \sqrt{2}/[2\log(1+\sqrt{2})] = 0.8022\cdots$ and $\beta_2 \geq 5/6$.

Theorem 4. *The double inequality*

$$\begin{aligned} &\lambda_2 C(a,b) + (1-\lambda_2)\overline{H}(a,b) \\ &\quad < M(a,b) < \mu_2 C(a,b) + (1-\mu_2)\overline{H}(a,b) \end{aligned} \quad (10)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_2 \leq 1/[2\log(1+\sqrt{2})] = 0.5672\cdots$ and $\mu_2 \geq 2/3$.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 5 (see [10, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on*

(a,b), and let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}. \tag{11}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 6 (see [11, Lemma 1.1]). *Suppose that the power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r>0$ and $b_n>0$ for all $n\in\{0,1,2,\ldots\}$. Let $h(x)=f(x)/g(x)$. Then,*

(1) *if the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$;*

(2) *if the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0<n\leq n_0$ and (strictly) decreasing (increasing) for $n>n_0$, then there exists $x_0\in(0,r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0,x_0)$ and (strictly) decreasing (increasing) on (x_0,r).*

Lemma 7. *The function*

$$h(t)=\frac{t\cosh(3t)+11t\cosh(t)-\sinh(3t)-9\sinh(t)}{2t[\cosh(3t)-\cosh(t)]} \tag{12}$$

is strictly decreasing on $(0,\log(1+\sqrt{2}))$, where $\sinh(t)=(e^{t}-e^{-t})/2$ and $\cosh(t)=(e^{t}+e^{-t})/2$ are the hyperbolic sine and cosine functions, respectively.

Proof. Let

$$\begin{gathered} h_1(t)=t\cosh(3t)+11t\cosh(t)-\sinh(3t)-9\sinh(t), \\ h_2(t)=2t[\cosh(3t)-\cosh(t)]. \end{gathered} \tag{13}$$

Then, making use of power series formulas, we have

$$\begin{aligned} h_1(t) &= t\sum_{n=0}^{\infty}\frac{(3t)^{2n}}{(2n)!}+11t\sum_{n=0}^{\infty}\frac{(t)^{2n}}{(2n)!} \\ &\quad -\sum_{n=0}^{\infty}\frac{(3t)^{2n+1}}{(2n+1)!}-9\sum_{n=0}^{\infty}\frac{(t)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty}\frac{(2n+3)\left(3^{2n+2}+11\right)-\left(3^{2n+3}+9\right)}{(2n+3)!}t^{2n+3}, \\ h_2(t) &= \sum_{n=0}^{\infty}\frac{2\left(3^{2n+2}-1\right)}{(2n+2)!}t^{2n+3}. \end{aligned} \tag{14}$$

It follows from (12)–(14) that

$$h(t)=\frac{\sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}}, \tag{15}$$

where

$$\begin{aligned} a_n &= \frac{(2n+3)\left(3^{2n+2}+11\right)-\left(3^{2n+3}+9\right)}{(2n+3)!}, \\ b_n &= \frac{2\left(3^{2n+2}-1\right)}{(2n+2)!}. \end{aligned} \tag{16}$$

Equation (16) leads to

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=\frac{18c_n}{2(2n+3)(2n+5)\left(3^{2n+2}-1\right)\left(3^{2n+4}-1\right)}, \tag{17}$$

where

$$c_n=3\cdot 9^{n}\left(81\cdot 9^{n}-64n^2-224n-166\right)-1. \tag{18}$$

It is not difficult to verify that $x\to 81\cdot 9^{x}-64x^2-224x-166$ is positive and strictly increasing in $[1,\infty)$. Then, from (18), we get that

$$c_n>0 \tag{19}$$

for $n\geq 1$. Note that

$$c_0=-256, \qquad c_1=7424. \tag{20}$$

Equations (17) and (20) together with inequality (19) lead to the conclusion that the sequence $\{a_n/b_n\}$ is strictly decreasing for $0\leq n\leq 1$ and strictly increasing for $n\geq 2$. Then, from Lemma 6(2) and (15), we clearly see that there exists $t_0\in(0,\infty)$ such that $h(t)$ is strictly decreasing on $(0,t_0)$ and strictly increasing on (t_0,∞).

Let $t^{*}=\log(1+\sqrt{2})$. Then, simple computations lead to

$$\begin{aligned} \sinh(t^{*}) &= 1, \qquad \cosh(t^{*})=\sqrt{2}, \\ \sinh(3t^{*}) &= 7, \qquad \cosh(3t^{*})=5\sqrt{2}. \end{aligned} \tag{21}$$

Differentiating (12) yields

$$\begin{aligned} h'(t) &= \frac{3t\sinh(3t)+11t\sinh(t)-2\cosh(3t)+2\cosh(t)}{h_2(t)} \\ &\quad -\frac{2t[3\sinh(3t)-\sinh(t)]+2[\cosh(3t)-\cosh(t)]}{h_2(t)^2}h_1(t). \end{aligned} \tag{22}$$

From (13) together with (21) and (22), we get

$$h'(t^{*})=\frac{-3\sqrt{2}t^{*2}+2t^{*}+\sqrt{2}}{t^{*2}}=-0.1529\cdots<0. \tag{23}$$

From the piecewise monotonicity of $h(t)$ and inequality (23) we clearly see that $t_0>t^{*}=\log(1+\sqrt{2})$, and the proof of Lemma 7 is completed. □

Lemma 8. *The function*

$$g(t) = \frac{t\cosh(3t) + 3t\cosh(t) - 2\sinh(2t)}{t[\cosh(3t) + 2\cosh(2t) + 3\cosh(t) - 6]} \quad (24)$$

is strictly increasing from $(0, \log(1+\sqrt{2}))$ *onto* $(1/3, 1 - 1/[2\log(1+\sqrt{2})])$.

Proof. Let

$$g_1(t) = t\cosh(3t) + 3t\cosh(t) - 2\sinh(2t),$$
$$g_2(t) = t[\cosh(3t) + 2\cosh(2t) + 3\cosh(t) - 6]. \quad (25)$$

Then, making use of power series formulas, we have

$$g_1(t) = t\sum_{n=0}^{\infty}\frac{(3t)^{2n}}{(2n)!} + 3t\sum_{n=0}^{\infty}\frac{(t)^{2n}}{(2n)!} - 2\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty}\frac{(2n+3)\left(3^{2n+2}+3\right) - 2^{2n+4}}{(2n+3)!}t^{2n+3}, \quad (26)$$
$$g_2(t) = \sum_{n=0}^{\infty}\frac{3^{2n+2}+2^{2n+3}+3}{(2n+2)!}t^{2n+3}.$$

It follows from (24)–(26) that

$$h(t) = \frac{\sum_{n=0}^{\infty} d_n t^{2n}}{\sum_{n=0}^{\infty} e_n t^{2n}}, \quad (27)$$

where

$$d_n = \frac{(2n+3)\left(3^{2n+2}+3\right) - 2^{2n+4}}{(2n+3)!},$$
$$e_n = \frac{3^{2n+2}+2^{2n+3}+3}{(2n+2)!}. \quad (28)$$

Note that

$$\frac{d_n}{e_n} = \frac{(2n+3)\left(3^{2n+2}+3\right) - 2^{2n+4}}{(2n+3)\left[3^{2n+2}+2^{2n+3}+3\right]}$$
$$= \frac{1 - \left(16\cdot 4^n/(2n+3)\left(9^{n+1}+3\right)\right)}{1 + \left(8\cdot 4^n/\left(9^{n+1}+3\right)\right)}. \quad (29)$$

It is not difficult to verify that the function $x \to 4^x/(9^{x+1}+3)$ is strictly decreasing in $(0,\infty)$. Then from (29), we know that the sequence $\{d_n/e_n\}$ is strictly increasing for $n = 0, 1, 2, \ldots$. Hence, from Lemma 6(1), (24), and (27) the monotonicity of $g(t)$ follows. Moreover, $g(\log(1+\sqrt{2})) = 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$ and

$$\lim_{t\to 0} g(t) = \frac{d_0}{e_0} = \frac{1}{3}. \quad (30)$$

□

3. Proofs of Theorems 1–4

Proof of Theorem 1. Since $M(a,b)$, $E(a,b)$, and $A(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \operatorname{arcsinh}(x)$. Then, $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$, and

$$\frac{M(a,b) - A(a,b)}{E(a,b) - A(a,b)}$$
$$= \frac{x/\operatorname{arcsinh}(x) - 1}{\left(3+x^2\right)/3 - 1} = \frac{3[\sinh(t) - t]}{t\sinh^2(t)} = \frac{6[\sinh(t) - t]}{t[\cosh(2t) - 1]}. \quad (31)$$

Let

$$F(t) = \frac{6[\sinh(t) - t]}{t[\cosh(2t) - 1]}. \quad (32)$$

Then, simple computations lead to

$$F(t) = \frac{\sum_{n=0}^{\infty} a_n^* t^{2n}}{\sum_{n=0}^{\infty} b_n^* t^{2n}}, \quad (33)$$

where $a_n^* = 6/(2n+3)!$ and $b_n^* = 2^{2n+2}/(2n+2)!$. Note that $a_n^*/b_n^* = 6/[(2n+3)2^{2n+2}]$ is strictly decreasing for $n = 0, 1, 2, \ldots$. Hence, from Lemma 6(1) and (33), we know that $F(t)$ is strictly decreasing in $(0, \log(1+\sqrt{2}))$. Moreover,

$$\lim_{t\to 0} F(t) = \frac{a_0^*}{b_0^*} = \frac{1}{2},$$
$$\lim_{t\to\log(1+\sqrt{2})} F(t) = \frac{3 - 3\log\left(1+\sqrt{2}\right)}{\log\left(1+\sqrt{2}\right)} = 0.4037\cdots. \quad (34)$$

Therefore, Theorem 1 follows from (31), (32), and (34) together with the monotonicity of $F(t)$. □

Proof of Theorem 2. Without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \operatorname{arcsinh}(x)$. Then, $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$, and

$$\frac{\log[M(a,b)] - \log[A(a,b)]}{\log[E(a,b)] - \log[A(a,b)]}$$
$$= \frac{\log[x/\operatorname{arcsinh}(x)]}{\log(1 + (x^2/3))} \quad (35)$$
$$= \frac{\log[\sinh(t)/t]}{\log\left[1 + \sinh^2(t)/3\right]}.$$

Let $G_1(t) = \log[\sinh(t)/t]$, $G_2(t) = \log[1 + \sinh^2(t)/3]$, and

$$G(t) = \frac{\log[\sinh(t)/t]}{\log\left[1 + \sinh^2(t)/3\right]}. \quad (36)$$

Then, $G_1(0^+) = G_2(0) = 0$, $G(t) = G_1(t)/G_2(t)$, and

$$\frac{G_1'(t)}{G_2'(t)} = \frac{[t\cosh(t) - \sinh(t)]\left[\sinh^2(t) + 3\right]}{2t\sinh^2(t)\cosh(t)} = \frac{[t\cosh(t) - \sinh(t)][\cosh(2t) + 5]}{2t\sinh(2t)\sinh(t)} = h(t), \tag{37}$$

where $h(t)$ is defined as in Lemma 7.

It follows from Lemmas 5 and 7, (36), and (37) that $G(t)$ is strictly decreasing in $(0, \log(1+\sqrt{2}))$. Moreover,

$$\lim_{t\to 0} G(t) = \lim_{t\to 0}\frac{G_1'(t)}{G_2'(t)} = \frac{1}{2}, \qquad \lim_{t\to\log(1+\sqrt{2})} G(t) = \frac{\log\left[\log\left(1+\sqrt{2}\right)\right]}{\log 3 - 2\log 2} = 0.4389\cdots. \tag{38}$$

Therefore, Theorem 2 follows easily from (35), (36), and (38) together with the monotonicity of $G(t)$. □

Proof of Theorem 3. Since $M(a,b)$, $Q(a,b)$, and $\overline{H}(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \operatorname{arcsinh}(x)$. Then, $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$, and

$$\frac{M(a,b) - Q(a,b)}{\overline{H}(a,b) - Q(a,b)} = \frac{x/\operatorname{arcsinh}(x) - \sqrt{1+x^2}}{(1-x^2)/\sqrt{1+x^2} - \sqrt{1+x^2}} = \frac{\sinh(t)/t - \cosh(t)}{\left[1-\sinh^2(t)\right]/\cosh(t) - \cosh(t)} = \frac{t\cosh(2t) + t - \sinh(2t)}{2t[\cosh(2t) - 1]}. \tag{39}$$

Let

$$\varphi(t) = \frac{t\cosh(2t) + t - \sinh(2t)}{2t[\cosh(2t) - 1]}. \tag{40}$$

Then, simple computations lead to

$$\varphi(t) = \frac{\sum_{n=0}^{\infty} d_n^* t^{2n}}{\sum_{n=0}^{\infty} e_n^* t^{2n}}, \tag{41}$$

where $d_n^* = (2n+1)\cdot 2^{2n+2}/(2n+3)!$, and $e_n^* = 2^{2n+3}/(2n+2)!$. Note that $d_n^*/e_n^* = [1 - 2/(2n+3)]/2$ is strictly increasing for $n = 0, 1, 2, \ldots$. Hence, from Lemma 6(1) and (41), we know that $\varphi(t)$ is strictly increasing in $(0, \log(1+\sqrt{2}))$. Moreover,

$$\lim_{t\to 0}\varphi(t) = \frac{d_0^*}{e_0^*} = \frac{1}{6}, \qquad \lim_{t\to\log(1+\sqrt{2})}\varphi(t) = 1 - \frac{\sqrt{2}}{2\log\left(1+\sqrt{2}\right)} = 0.1977\cdots. \tag{42}$$

Therefore, Theorem 3 follows from (39), (40), and (42) together with the monotonicity of $\varphi(t)$. □

Proof of Theorem 4. Since $M(a,b)$, $C(a,b)$, and $\overline{H}(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \operatorname{arcsinh}(x)$. Then, $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$, and

$$\frac{M(a,b) - C(a,b)}{\overline{H}(a,b) - C(a,b)} = \frac{x/\operatorname{arcsinh}(x) - \left(1+x^2\right)}{(1-x^2)/\sqrt{1+x^2} - \left(1+x^2\right)} = \frac{\sinh(t)/t - \cosh^2(t)}{\left[1-\sinh^2(t)\right]/\cosh(t) - \cosh^2(t)} = \frac{t\cosh^3(t) - \sinh(t)\cosh(t)}{t\left[\cosh^3(t) + \sinh^2(t) - 1\right]} = g(t), \tag{43}$$

where $g(t)$ is defined as in Lemma 8.

Therefore, Theorem 4 follows from (43) and Lemma 8. □

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