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Research Article

Unbounded Solutions of Asymmetric Oscillator

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We obtain sufficient conditions for the existence of unbounded solutions of the following nonlinear differential equation $(\varphi_p(x'))' + (p-1)[\alpha\varphi_p(x^+) - \beta\varphi_p(x^-)] = (p-1)f(t,x,x')$, where $\varphi_p(u) = |u|^{p-2}u$, $p > 1$, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, α, β are positive constants, and f is continuous, bounded, and T-periodic in t for some $T > 0$.

1. Introduction

In 1960s, Lazer and Leach [1] considered the existence of 2π-periodic solution of the following differential equation:

$$x'' + n^2 x = g(x) + e(t), \tag{1}$$

where $n \in \mathbb{N}$, e is continuous and 2π-periodic, and g is bounded and continuous. If $g(\pm\infty) = \lim_{x\to\pm\infty} g(x)$ exist, they showed that (1) has at least one 2π-periodic solution provided:

$$2\left|g(+\infty) - g(-\infty)\right| > \left|\int_0^{2\pi} e(t)\, e^{\mathrm{int}} dt\right|. \tag{2}$$

Assume α, β are positive constants satisfying

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n} \in \mathbb{Q}. \tag{3}$$

Fabry and Mawhin [2] considered the following asymmetric oscillator:

$$x'' + \alpha x^+ - \beta x^- = f(x) + g(x) + e(t), \tag{4}$$

where f, g are continuous and bounded, e is continuous and 2π-periodic, $x^+ = \max\{x, 0\}$, and $x^- = \max\{-x, 0\}$. Moreover, the limits $\lim_{x\to\pm\infty} f(x) =: f(\pm\infty)$ exist; g has a sublinear primitive; that is,

$$\frac{G(x)}{x} = \frac{\int_0^x g(s)\,ds}{x} \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \tag{5}$$

They introduced the function

$$\phi(\theta) = n\left(\frac{f(+\infty)}{\alpha} - \frac{f(-\infty)}{\beta}\right) + \int_0^{2\pi} e(t)\,\psi(t+\theta)\,dt, \tag{6}$$

where ψ is the solution of the following initial value problem:

$$x'' + \alpha x^+ - \beta x^- = 0, \quad x(0) = 0, \quad x'(0) = 1, \tag{7}$$

and gave approximations of solutions of (4) with large initial values. Moreover, they proved that (4) has unbounded solutions if the function $\phi(\theta)$ has only simple zeros.

Later, Kunze et al. [3] considered the following resonant oscillator:

$$x'' + f(x)\,x' + n^2 x + g(x) = p(t), \tag{8}$$

where $f, g, p \in C^0$, g is bounded, p is 2π-periodic, and $n \in N$. They showed that if

$$4n|F|_\infty + 2\left(\sup g - \inf g\right) < \left|\int_0^{2\pi} p(t)\, e^{\mathrm{int}} dt\right|, \tag{9}$$

where $F(x) = \int_0^x f(s)ds$ is bounded, $|F|_\infty = \sup_{x\in\mathbb{R}} |F(x)|$, then every solution of (8) is unbounded.

Wang [4] considered the unboundedness of solutions of the equation:

$$x'' + f(x)\,x' + \alpha x^+ - \beta x^- = p(t), \tag{10}$$

where $\alpha^{-1/2} + \beta^{-1/2} \in \mathbb{R}^+ \setminus \mathbb{Q}$ and p is continuous and 2π-periodic. By using the well-known Birkhoff Ergodic theorem [5], he showed that if the function $F(x) = \int_0^x f(s)ds$ is bounded and the limits $\lim_{x\to\pm\infty} F(x) = F(\pm\infty)$ exist, then $F(+\infty)F(-\infty) < 0$ is a sufficient condition for the existence of unbounded solutions of (10).

Liu [6] discussed the boundedness of all solutions of the following nonlinear equation:

$$\left(\varphi_p\left(x'\right)\right)' + \alpha\varphi_p\left(x^+\right) - \beta\varphi_p\left(x^-\right) = f(t,x), \tag{11}$$

where

$$\frac{\pi_p}{\alpha^{1/p}} + \frac{\pi_p}{\beta^{1/p}} = \frac{2\pi}{n}, \tag{12}$$

with $\pi_p = 2\pi/p\sin(\pi/p)$, $n \in \mathbb{N}$. Let $n = 1$; he defined a 2π-periodic function $Z(\theta)$ as

$$Z(\theta) = \int_{I^+} f_+(t+\theta)\, v(t)\, dt + \int_{I^-} f_-(t+\theta)\, v(t)\, dt, \tag{13}$$

where $f_\pm(t) = \lim_{x\to\pm\infty} f(t,x) \in L^\infty[0,2\pi]$ exist, $I^+ = \{t \in [0,2\pi] : v(t) > 0\}$, $I^- = \{t \in [0,2\pi] : v(t) < 0\}$, and $v(t)$ is the solution of

$$\left(\varphi_p\left(v'\right)\right)' + \alpha\varphi_p\left(v^+\right) - \beta\varphi_p\left(v^-\right) = 0, \tag{14}$$

with the initial value $(v(0), v'(0)) = (0,1)$. Under some smoothness conditions, he showed that all solutions of (11) are bounded if the function $Z(\theta)$ has no zero for all $\theta \in \mathbb{R}$.

Recently, Li and Zhang [7] studied the unboundedness of solutions of the following asymmetric oscillator:

$$x'' + f\left(x'\right) + ax^+ - bx^- = \phi(t,x), \tag{15}$$

where a, b are positive constants satisfying nonresonance condition:

$$a^{-1/2} + b^{-1/2} \in \mathbb{R}^+ \setminus \mathbb{Q}, \tag{16}$$

f and ϕ are bounded and ϕ is 2π-periodic in t. By using the similar method used in [4], they obtained some sufficient conditions for the existence of unbounded solutions of (15). For more recent results on the boundedness or unboundedness of solutions of differential equations of second order, we refer to [6, 8–18] and the references therein.

Assume α, β are positive constants. Let $S(t)$ be the solution of the following initial value problem:

$$\begin{gathered}\left(\varphi_p\left(u'\right)\right)' + (p-1)\left[\alpha\varphi_p\left(u^+\right) - \beta\varphi_p\left(u^-\right)\right] = 0, \\ u(0) = 0, \qquad u'(0) = 1;\end{gathered} \tag{17}$$

then it is well-known that S is τ-periodic with

$$\tau = \pi_p\left(\alpha^{-1/p} + \beta^{-1/p}\right), \quad \pi_p = \frac{2\pi}{p\sin(\pi/p)}. \tag{18}$$

In this paper, we consider the following asymmetric oscillator:

$$\begin{aligned}\left(\varphi_p\left(x'\right)\right)' &+ (p-1)\left[\alpha\varphi_p\left(x^+\right) - \beta\varphi_p\left(x^-\right)\right] \\ &= (p-1)\, f\left(t,x,x'\right),\end{aligned} \tag{19}$$

where $\varphi_p(u) = |u|^{p-2}u$, $p > 1$, $x^\pm = \max\{\pm x, 0\}$, f is continuous, bounded, and T-periodic in t, and the limits $\lim_{x\to\pm\infty, y\to\pm\infty} f(t,x,y) = f(t,\pm\infty,\pm\infty) \in L^\infty[0,T]$ exist. We assume that T satisfies the following resonance condition:

$$\frac{T}{\tau} = \frac{n}{m} \in \mathbb{Q}^+. \tag{20}$$

A solution $x(t)$ of (19) is called to have large initial value if $|x(0)| + |x'(0)| \gg 1$. We will give some sufficient conditions for the unboundedness of solutions of (19). Especially, if $f(t,x,y) = f(t)$, we will obtain higher order approximation and corresponding sufficient conditions for the unboundedness of solutions of (19) with large initial values. The results of this paper are new which improve some relative results on the literature in some sense. Throughout this paper, we assume $T/\tau = n/m$ with $m, n \in \mathbb{N}$.

The main results of this paper are the following.

Theorem 1. *Introduce a τ-periodic function*

$$\begin{aligned}\lambda(\theta) = &-\int_{I_\theta^{++}} f(t,+\infty,+\infty)\, S(\theta+t)\, dt \\ &-\int_{I_\theta^{+-}} f(t,+\infty,-\infty)\, S(\theta+t)\, dt \\ &-\int_{I_\theta^{-+}} f(t,-\infty,+\infty)\, S(\theta+t)\, dt \\ &-\int_{I_\theta^{--}} f(t,-\infty,-\infty)\, S(\theta+t)\, dt,\end{aligned} \tag{21}$$

where

$$\begin{aligned}I_\theta^{++} &= \left\{t \in [0,mT] : S(\theta+t) > 0, S'(\theta+t) > 0\right\}, \\ I_\theta^{+-} &= \left\{t \in [0,mT] : S(\theta+t) > 0, S'(\theta+t) < 0\right\}, \\ I_\theta^{-+} &= \left\{t \in [0,mT] : S(\theta+t) < 0, S'(\theta+t) > 0\right\}, \\ I_\theta^{--} &= \left\{t \in [0,mT] : S(\theta+t) < 0, S'(\theta+t) < 0\right\}.\end{aligned} \tag{22}$$

Then all solutions of (19) *with large initial values are unbounded provided that the function $\lambda(\theta)$ has only finite number of zeros in $[0,\tau]$ and all its zeros are simple.*

Corollary 2. *Assume $f(t,x,y) = f(t) \in L^\infty[0,T]$ is T-periodic and define a τ-periodic function $\lambda(\theta)$ as*

$$\lambda(\theta) = -\int_0^{mT} S(\theta+t)\, f(t)\, dt. \tag{23}$$

If $\lambda(\theta)$ has only finite number of zeros in $[0,\tau]$ and all its zeros are simple, then all solutions of (19) *with large initial values are unbounded.*

Theorem 3. *Consider* (19) *with* $f(t,x,y) = f(t) \in L^\infty[0,T]$ *is* T*-periodic. Let* $\lambda(\theta)$ *be given in* Corollary 2 *and assume* $\lambda(\theta) \equiv 0$. *Define* τ*-periodic functions* $\overline{\lambda}(\theta)$ *and* $\overline{\mu}(\theta)$ *as*

$$\begin{aligned}\overline{\lambda}(\theta) &= (p-2)\int_0^{mT} S(\theta+t)\, f(t) \\ &\quad\times \int_0^t S'(\theta+u)\, f(u)\, du\, dt, \\ \overline{\mu}(\theta) &= -(p-1)\int_0^{mT} S''(\theta+t)\, f(t) \\ &\quad\times \int_0^t S(\theta+u)\, f(u)\, du\, dt.\end{aligned} \tag{24}$$

Then all solutions of (19) *with large initial values are unbounded provided one of the following conditions holds:*

(i) $p \neq 2$ *and* $\overline{\lambda}(\theta)$ *has only finite number of zeros in* $[0,mT] = [0,n\tau]$ *and all of them are simple;*

(ii) $p = 2$ *and the function* $\overline{\mu}(\theta)$ *has no zero in* $[0,mT] = [0,n\tau]$.

2. Generalized Polar Coordinate Transformation

Let $u = u(t)$ be the solution of the following initial value problem:

$$\begin{gathered}\left(\varphi_p\left(u'\right)\right)' + (p-1)\,\varphi_p(u) = 0, \\ u(0) = 0,\ u'(0) = 1,\end{gathered} \tag{25}$$

and define a function $S(t)$ as

$$S(t) = \begin{cases}\alpha^{-1/p} u\left(\alpha^{1/p} t\right), & t \in \left[0, \alpha^{-1/p}\pi_p\right], \\ -\beta^{-1/p} u\left[\beta^{1/p}\left(t-\alpha^{-1/p}\pi_p\right)\right], & t \in \left[\alpha^{-1/p}\pi_p, \tau\right];\end{cases} \tag{26}$$

then it is easy to verify that $S \in C^2$; is τ-periodic and satisfies the identity:

$$\left|S'(t)\right|^p + \alpha(S^+(t))^p + \beta(S^-(t))^p \equiv 1, \quad \forall t \in \mathbb{R}. \tag{27}$$

Moreover, one can verify that S is the solution of (17).

For $r > 0$, $\theta \in \mathbb{R}$, define the generalized polar coordinate transformation $T_S : (r,\theta) \to (x,x')$ by

$$T_S : x = r^{1/(p-1)} S(\theta), \qquad x' = r^{1/(p-1)} S'(\theta). \tag{28}$$

Under the mapping T_S and by using (27), it is not difficult to verify that (19) can be transformed into the following planar system:

$$\begin{aligned} r' &= (p-1)\, S'(\theta)\, f\left(t,x,x'\right), \\ \theta' &= 1 - r^{-1} S(\theta)\, f\left(t,x,x'\right),\end{aligned} \tag{29}$$

where x, x' are given by (28).

Lemma 4. *Let* $(r(t),\theta(t))$ *be the solution of* (29) *satisfying initial condition* $(r(0),\theta(0)) = (r_0,\theta_0)$, *then for* $r_0 \gg 1$; *the Poincaré mapping* $P : (r_0,\theta_0) \to (r_1,\theta_1) =: (r(mT),\theta(mT))$ *satisfies*

$$\begin{aligned} r_1 &= r_0 - (p-1)\,\lambda'(\theta_0) + o(1), \\ \theta_1 &= \theta_0 + mT + r_0^{-1}\lambda(\theta_0) + o\left(r_0^{-1}\right),\end{aligned} \tag{30}$$

where $\lambda(\theta)$ *is given by* (21).

Proof. For $r(0) = r_0 \gg 1$, $t \in [0,mT]$, by the boundedness of S, S' and f, we obtain from (29)

$$r(t) = r_0 + O(1), \tag{31}$$

which implies

$$r^{-1}(t) = r_0^{-1} + O\left(r_0^{-2}\right). \tag{32}$$

Substituting (32) into (29) and integrating it over $[0,t] \subset [0,mT]$, we obtain

$$\begin{aligned}\theta(t) &= \theta_0 + t + O\left(r_0^{-1}\right), \\ x(t) &= r_0^{1/(p-1)}\left[S(\theta_0+t) + O\left(r_0^{-1}\right)\right], \\ x'(t) &= r_0^{1/(p-1)}\left[S'(\theta_0+t) + O\left(r_0^{-1}\right)\right].\end{aligned} \tag{33}$$

Substituting (32) and (33) into (29) and integrating it over $[0,mT]$, we obtain

$$\begin{aligned} r_1 &= r_0 + \mu(\theta_0) + o(1), \\ \theta_1 &= \theta_0 + mT + \lambda(\theta_0)\, r_0^{-1} + o\left(r_0^{-1}\right),\end{aligned} \tag{34}$$

where

$$\begin{aligned}\mu(\theta) &= (p-1)\int_0^{mT} S'(\theta+t) \\ &\quad\times \lim_{r_0\to+\infty} f\left(t,x(t),x'(t)\right) dt \\ &= (p-1)\left[\int_{I_\theta^{++}} S'(\theta+t)\, f(t,+\infty,+\infty)\, dt \right. \\ &\quad \left. + \int_{I_\theta^{+-}} S'(\theta+t)\, f(t,+\infty,-\infty)\, dt\right] \\ &= (p-1)\left[\int_{I_\theta^{-+}} S'(\theta+t)\, f(t,-\infty,+\infty)\, dt \right. \\ &\quad \left. + \int_{I_\theta^{--}} S'(\theta+t)\, f(t,-\infty,-\infty)\, dt\right],\end{aligned} \tag{35}$$

$I_\theta^{++}, I_\theta^{+-}, I_\theta^{-+}, I_\theta^{--}$, and $\lambda(\theta)$ are given by (21). Moreover, it is easy to verify from (21) and (35) that for all $\theta \in \mathbb{R}$,

$$\mu(\theta) = -(p-1)\,\lambda'(\theta). \tag{36}$$

□

Lemma 5. *Let $f(t,x,x') = f(t) \in L^{\infty}[0,T]$ and be T-periodic. Assume $\lambda(\theta) \equiv 0$, where $\lambda(\theta)$ is given in Corollary 2 and $P : (r_0,\theta_0) \to (r_1,\theta_1) =: (r(mT),\theta(mT))$ is the Poincaré mapping of the solution of (29). Then for $r_0 \gg 1$, there exists the following higher order approximation:*

$$\begin{aligned} r_1 &= r_0 + \overline{\mu}(\theta_0) r_0^{-1} + o(r_0^{-1}), \\ \theta_1 &= \theta_0 + mT + \overline{\lambda}(\theta_0) r_0^{-2} + o(r_0^{-2}), \end{aligned} \tag{37}$$

where $\overline{\mu}(\theta)$ and $\overline{\lambda}(\theta)$ are given in Theorem 3.

Proof. Since $f(t,x,x') = f(t)$ in (29), we integrate the first equation of (29) from $t = 0$ to $t \in (0,mT]$. For $r_0 \gg 1$, we obtain

$$r(t) = r_0 + O(1), \qquad r^{-1}(t) = r_0^{-1} + O(r_0^{-2}). \tag{38}$$

Substituting (38) into the second equation of (29) and integrating it over $[0,t]$, we get

$$\theta(t) = \theta_0 + t + \lambda_1(\theta_0,t) r_0^{-1} + o(r_0^{-1}), \tag{39}$$

where

$$\lambda_1(\theta,t) = -\int_0^t S(\theta+u) f(u)\, du. \tag{40}$$

Substituting (38) and (40) into (29) and integrating over $[0,t]$ we get

$$\begin{aligned} r(t) &= r_0 + \mu_1(\theta_0,t) + o(1), \\ \theta(t) &= \theta_0 + t + \lambda_1(\theta_0,t) r_0^{-1} + o(r_0^{-1}), \end{aligned} \tag{41}$$

where

$$\mu_1(\theta,t) = (p-1)\int_0^t S'(\theta+u) f(u)\, du. \tag{42}$$

By (41) we get the following approximation

$$\begin{aligned} r^{-1}(t) &= r_0^{-1} - \mu_1(\theta_0,t) r_0^{-2} + o(r_0^{-2}), \\ S(\theta(t)) &= S(\theta_0+t) + S'(\theta_0+t)\lambda_1(\theta_0,t) r_0^{-1} + o(r_0^{-1}). \end{aligned} \tag{43}$$

Substituting (43) into (29) and integrating it over $[0,t]$, we get

$$\begin{aligned} r(t) &= r_0 + \mu_1(\theta_0,t) + \mu_2(\theta_0,t) r_0^{-1} + o(r_0^{-1}), \\ \theta(t) &= \theta_0 + t + \lambda_1(\theta_0,t) r_0^{-1} + \lambda_2(\theta_0,t) r_0^{-2} + o(r_0^{-3}), \end{aligned} \tag{44}$$

where

$$\begin{aligned} \mu_2(\theta,t) &= (p-1)\int_0^t \lambda_1(\theta,u) S''(\theta+u) f(u)\, du, \\ \lambda_2(\theta,t) &= \int_0^t \left[\mu_1(\theta,u) S(\theta+u) \right. \\ &\qquad \left. - S'(\theta+u)\lambda_1(\theta,u)\mu_1\right] f(u)\, du. \end{aligned} \tag{45}$$

Substituting (40) and (42) into (45) and then integrating it over $[0,mT]$, we obtain

$$\begin{aligned} r(mT) &= r_0 + \mu_1(\theta_0,mT) \\ &\quad + \mu_2(\theta_0,mT) r_0^{-1} + o(r_0^{-1}), \\ \theta(mT) &= \theta_0 + mT + \lambda_1(\theta_0,mT) r_0^{-1} \\ &\quad + \lambda_2(\theta_0,mT) r_0^{-2} + o(r_0^{-3}), \end{aligned} \tag{46}$$

where

$$\begin{aligned} \lambda_1(\theta,mT) &= \lambda(\theta) \equiv 0, \\ \mu_1(\theta,mT) &= -(p-1)\lambda_1'(\theta) \equiv 0, \end{aligned} \tag{47}$$

$$\begin{aligned} \mu_2(\theta,mT) &= -(p-1)\int_0^{mT} S''(\theta+t) f(t) \\ &\qquad \times \int_0^t S(\theta+u) f(u)\, du\, dt, \\ \lambda_2(\theta,mT) &= (p-1)\int_0^{mT} S(\theta+t) f(t) \\ &\qquad \times \int_0^t S'(\theta+u) f(u)\, du\, dt \\ &\qquad + \int_0^{mT} S'(\theta+t) f(t) \\ &\qquad \times \int_0^t S(\theta+u) f(u)\, du\, dt \\ &= (p-2)\int_0^{mT} S(\theta+t) f(t) \\ &\qquad \times \int_0^t S'(\theta+u) f(u)\, du\, dt + \lambda_1(\theta)\lambda_1'(t). \end{aligned} \tag{48}$$

Let

$$\begin{aligned} \lambda_1(\theta,mT) &= \lambda(\theta) \equiv 0, \\ \lambda_2(\theta,mT) &= \overline{\lambda}(\theta), \\ \mu_2(\theta,mT) &= \mu(\theta); \end{aligned} \tag{49}$$

then (46) reduces to (37). □

Lemma 6. *If $\lambda(\theta) \equiv 0$, then*

$$(p-2)\overline{\mu}(\theta) = (p-1)\overline{\lambda}'(\theta). \tag{50}$$

Proof. It follows from the second equation of (48) that

$$\begin{aligned} \lambda_2'(\theta,mT) &= \frac{p}{2}\left(\lambda_1'(\theta,mT)\right)^2 \\ &\quad + (p-1)\lambda_1(\theta,mT)\lambda_1''(\theta,mT) \\ &\quad + \frac{p-2}{p-1}\mu_2(\theta,mT), \end{aligned} \tag{51}$$

which, together with $\lambda_1(\theta, mT) \equiv 0$, $\lambda_2(\theta, mT) = \overline{\lambda}(\theta)$, $\mu_2(\theta, mT) = \overline{\mu}(\theta)$ implies (50). □

3. Unboundedness Motions of Planar Mappings

In this section, we adopt the notations used in [9]. Given $\sigma > 0$, let the set E_σ be the exterior of the open ball B_σ centered at the origin with radius σ; that is,

$$E_\sigma = \mathbb{R}^2 - B_\sigma; \tag{52}$$

then $E_\sigma = \{(\theta, r) \mid r \geq \sigma\}$.

Define $S^1 = \mathbb{R}\backslash\mathbb{Z}\tau$; then the points and the group distance in S^1 can be described, respectively, by

$$\begin{gathered}\overline{\theta} = \theta + k\tau, \quad k \in \mathbb{Z},\ \theta \in \mathbb{R}, \\ \left\|\overline{\theta}\right\| = \min\{|\theta + k\tau|, k \in \mathbb{Z}\}.\end{gathered} \tag{53}$$

Let $\overline{P} : E_\sigma \to \mathbb{R}^2$ be a one to one and continuous mapping. We denote its lift by P in the form

$$\begin{gathered}r_1 = r_0 - c_p\lambda'(\theta_0) + F(r_0, \theta_0), \\ \theta_1 = \theta_0 + n\tau + r_0^{-1}\lambda(\theta_0) + G(r_0, \theta_0),\end{gathered} \tag{54}$$

where $\lambda \in C^1[0, \tau]$ is τ-periodic, $n \in \mathbb{N}$, $c_p \neq 0$ is a constant, and F, G are continuous, τ-periodic and satisfy $F(r, \theta) = o(1)$, $G(r, \theta) = o(r^{-1})$.

Given a point $(\theta_0, r_0) \in E_\sigma$, let $\{(\theta_k, r_k)\}_{k\in I}$ be the unique solution of the initial value problem for the differential equation:

$$(r_{k+1}, \theta_{k+1}) = P(r_k, \theta_k). \tag{55}$$

This solution is defined in a maximal interval

$$I = \{k \in \mathbb{Z} \mid k_a < k < k_b\}, \tag{56}$$

where k_a, k_b are certain numbers in the set $\mathbb{Z} \cup \{+\infty, -\infty\}$ satisfying

$$-\infty \leq k_a < 0 < k_b \leq +\infty. \tag{57}$$

The solution $\{(\theta_k, r_k)\}$ is said to be defined in the future if $k_b = +\infty$ and is said to be defined in the past if $k_a = -\infty$.

Let $\{\theta_i\}_{i=1}^k$ be the ordered sequence of zeros of $\lambda(\theta)$ in $[0, \tau)$ such that

$$\begin{gathered}0 \leq \theta_1 < \theta_2 < \cdots < \theta_k < \tau, \\ \Omega = \{\theta_1, \ldots, \theta_k\} \neq \emptyset.\end{gathered} \tag{58}$$

Lemma 7. *If λ has a simple zero θ_j; that is, $\lambda'(\theta_j) \neq 0$, $\theta_j \in \Omega$, then there exists orbits of* (54) *which are to be defined in the future and satisfy*

$$r_n \longrightarrow \infty \quad as\ n \longrightarrow +\infty, \tag{59}$$

or they are defined in the past and satisfy

$$r_n \longrightarrow \infty \quad as\ n \longrightarrow -\infty. \tag{60}$$

Moreover, if $\Omega \neq \emptyset$ and all zeros of $\lambda(\theta)$ are simple, then every orbit of (54) *with large initial value is either to be defined in the future satisfying*

$$r_n \longrightarrow \infty \quad as\ n \longrightarrow +\infty, \tag{61}$$

or is defined in the past satisfying

$$r_n \longrightarrow \infty \quad as\ n \longrightarrow -\infty. \tag{62}$$

The proof of the above lemma is similar to that of Proposition 3.1 in [9].

Lemma 8. *Assume that for $r_0 \gg 1$, $\theta_0 \in \mathbb{R}$, P has the following expression:*

$$\begin{gathered}r_1 = r_0 + \mu(\theta_0) r_0^{-k} + F(r_0, \theta_0), \\ \theta_1 = \theta_0 + n\tau + \lambda(\theta_0) r_0^{-m} + G(r_0, \theta_0),\end{gathered} \tag{63}$$

where $k, m, n \in \mathbb{N}$, $\mu, \lambda \in C[0, \tau]$, and F, G are continuous, τ-periodic in θ and satisfy $F(r, \theta) = o(r^{-k})$, $G(r, \theta) = o(r^{-m})$ uniformly in $\theta \in S^1$. Moreover If $\mu(\theta) > 0$, for all $\theta \in \mathbb{R}$, then all orbits of (63) *with large initial values are defined in the future and satisfy* $\lim_{n\to+\infty} r_n = \infty$. *If $\mu(\theta) < 0$, for all $\theta \in \mathbb{R}$, then all orbits of* (63) *with large initial values are defined in the past and satisfy* $\lim_{n\to-\infty} r_n = \infty$.

Proof. Let $\mu(\theta) > 0$, for all $\theta \in \mathbb{R}$, then $2C_0 = \max_{\theta\in\mathbb{R}}\mu(\theta) \geq \min_{\theta\in\mathbb{R}}\mu(\theta) = 2c_0 > 0$ and it follows from (63) that for $r_0 \gg 1$,

$$r_1 > r_0 + c_0 r_0^{-k}, \tag{64}$$

$$r_1 < r_0 + 3C_0 r_0^{-k} < r_0 + 3C_0. \tag{65}$$

By induction, we get from (65)

$$r_n < r_0 + 3nC_0, \tag{66}$$

and replacing r_0 by r_n, r_1 by r_{n+1} we get from (64)

$$r_{n+1} > r_n + c_0 r_n^{-k}. \tag{67}$$

Obviously, (66) implies that the orbit r_n is defined in the future. Next we claim that the orbit r_n is unbounded in the future. In fact, (67) implies that r_n is monotone increasing, hence the limit $\lim_{n\to+\infty} r_n = r^* \leq \infty$ exists. If $r^* < \infty$, taking the limit on both sides of (67) we obtain

$$r^* \geq r^* + \frac{c_0}{r^{*k}} > r^*, \tag{68}$$

which is a contradiction. Hence the orbit of (63) is defined in the future and satisfies $\lim_{n\to+\infty} r_n = \infty$. Similarly, if $\mu(\theta) < 0$, for all $\theta \in \mathbb{R}$, we can prove that the orbit of (63) is defined in the past and satisfies $\lim_{n\to-\infty} r_n = \infty$. □

4. Proofs of Theorems and an Example

Proof of Theorem 1. Since $mT = n\tau$, by Lemma 4, for $r_0 \gg 1$, $\theta_0 \in \mathbb{R}$, the mapping $P : (r_0, \theta_0) \to (r_1, \theta_1) := (r(mT), \theta(mT))$ has the approximate expression of (30) with $mT = n\tau$, where $\lambda(\theta)$ is given by Theorem 1 and $\mu(\theta) = -(p-1)\lambda'(\theta)$. Now Lemma 7 implies that all solutions of (30) with $r_0 \gg 1$ go to infinity either in the future or in the past, which implies that the solutions of (19) with large initial values are unbounded. □

Proof of Theorem 3. By Lemma 5, for $r_0 \gg 1$, $\theta_0 \in \mathbb{R}$, the mapping $P' : (r_0, \theta_0) \to (r_1, \theta_1) = (r(mT), \theta(mT))$ has the form (37) with $mT = n\tau$ and $\overline{\lambda}$ and $\overline{\mu}$ are given by Theorem 3. If $p \neq 2$, then it follows from Lemma 6 that

$$\overline{\mu}(\theta) = \frac{p-1}{p-2}\overline{\lambda}'(\theta). \tag{69}$$

Now, Lemma 7 implies that all orbits of (37) with $r_0 \gg 1$ go to infinity either in the future or in the past, which means that all solutions of (19) with large initial values are unbounded.

If $p = 2$, then by assumption of Theorem 3, $\overline{\lambda}(\theta) \equiv 0$ and the function $\overline{\mu}(\theta)$ has no sign-changing. Lemma 8 implies that all orbits of (37) with $r_0 \gg 1$ go to infinity either in the future or in the past, which implies that all solutions of (19) with large initial values are unbounded. □

Example 1. For (19), we assume $p > 1$, $\tau = T = 2\pi_p$, $m = n = 1$, $\alpha \neq \beta$, and $f(t, x, y) = f(t) = \sin(k\pi t/\pi_p)$ for some $k \in \mathbb{N}$. We write $S(t)$ in Fourier series form as

$$S(t) = \frac{a_0}{2} + \sum_{k=1}^{+\infty}\left[a_k \cos\frac{k\pi t}{\pi_p} + b_k \sin\frac{k\pi t}{\pi_p}\right], \tag{70}$$

where

$$\begin{aligned} a_0 &= \frac{2\int_0^{2\pi_p} S(t)\,dt}{\pi_p}, \\ a_k &= \frac{1}{\pi_p}\int_0^{2\pi_p} S(t)\cos\frac{k\pi t}{\pi_p}dt, \\ b_k &= \frac{1}{\pi_p}\int_0^{2\pi_p} S(t)\sin\frac{k\pi t}{\pi_p}dt, \quad k \in \mathbb{N}. \end{aligned} \tag{71}$$

It follows from Corollary 2 that

$$\begin{aligned} \lambda_1(\theta) &= \lambda(\theta) \\ &= \int_0^{2\pi_p} S(\theta + t)\sin\frac{k\pi t}{\pi_p}dt \\ &= \pi_p\left[a_k \sin\frac{k\pi\theta}{\pi_p} - b_k\cos\frac{k\pi\theta}{\pi_p}\right], \end{aligned} \tag{72}$$

and it is not difficult to show that $a_k \cdot b_k \neq 0$. Then $\lambda(\theta)$ has only finite number of zeros in $[0, 2\pi_p]$ and all of them are simple. Corollary 2 implies that all solutions of (19) with large initial values are unbounded.

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References

[1] A. C. Lazer and D. E. Leach, "Bounded perturbations of forced harmonic oscillators at resonance," *Annali di Matematica Pura ed Applicata*, vol. 82, pp. 49–68, 1969.

[2] C. Fabry and J. Mawhin, "Oscillations of a forced asymmetric oscillator at resonance," *Nonlinearity*, vol. 13, no. 3, pp. 493–505, 2000.

[3] M. Kunze, T. Küpper, and B. Liu, "Boundedness and unboundedness of solutions for reversible oscillators at resonance," *Nonlinearity*, vol. 14, no. 5, pp. 1105–1122, 2001.

[4] Z. Wang, "Irrational rotation numbers and unboundedness of solutions of the second order differential equations with asymmetric nonlinearities," *Proceedings of the American Mathematical Society*, vol. 131, no. 2, pp. 523–531, 2003.

[5] P. Walters, *An Introduction to Ergodic Theory*, vol. 79, Springer, New York, NY, USA, 1982.

[6] B. Liu, "Boundedness of solutions for equations with *p*-Laplacian and an asymmetric nonlinear term," *Journal of Differential Equations*, vol. 207, no. 1, pp. 73–92, 2004.

[7] X. Li and Z. Zhang, "Unbounded solutions and periodic solutions for second order differential equations with asymmetric nonlinearity," *Proceedings of the American Mathematical Society*, vol. 135, no. 9, pp. 2769–2777, 2007.

[8] J. M. Alonso and R. Ortega, "Unbounded solutions of semilinear equations at resonance," *Nonlinearity*, vol. 9, no. 5, pp. 1099–1111, 1996.

[9] J. M. Alonso and R. Ortega, "Roots of unity and unbounded motions of an asymmetric oscillator," *Journal of Differential Equations*, vol. 143, no. 1, pp. 201–220, 1998.

[10] C. Fabry, "Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities," *Journal of Differential Equations*, vol. 116, no. 2, pp. 405–418, 1995.

[11] C. Fabry and A. Fonda, "Nonlinear resonance in asymmetric oscillators," *Journal of Differential Equations*, vol. 147, no. 1, pp. 58–78, 1998.

[12] C. Fabry and A. Fonda, "Unbounded motions of perturbed isochronous Hamiltonian systems at resonance," *Advanced Nonlinear Studies*, vol. 5, no. 3, pp. 351–373, 2005.

[13] B. Liu, "Boundedness in nonlinear oscillations at resonance," *Journal of Differential Equations*, vol. 153, no. 1, pp. 142–174, 1999.

[14] R. Ortega, "Asymmetric oscillators and twist mappings," *Journal of the London Mathematical Society*, vol. 53, no. 2, pp. 325–342, 1996.

[15] R. Ortega, "Boundedness in a piecewise linear oscillator and a variant of the small twist theorem," *Proceedings of the London Mathematical Society*, vol. 79, no. 2, pp. 381–413, 1999.

[16] X. Yang, "Boundedness in nonlinear asymmetric oscillations," *Journal of Differential Equations*, vol. 183, no. 1, pp. 108–131, 2002.

[17] X. Yang, "Unbounded solutions of asymmetric oscillator," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 137, no. 2, pp. 487–494, 2004.

[18] X. Yang, "Unboundedness of the large solutions of an asymmetric oscillator," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 1, pp. 304–314, 2005.