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Research Article

Euler Numbers and Polynomials Associated with Zeta Functions

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For $s \in \mathbb{C}$, the Euler zeta function and the Hurwitz-type Euler zeta function are defined by $\zeta_E(s) = 2\sum_{n=1}^{\infty}((-1)^n/n^s)$, and $\zeta_E(s,x) = 2\sum_{n=0}^{\infty}((-1)^n/(n+x)^s)$. Thus, we note that the Euler zeta functions are entire functions in whole complex s-plane, and these zeta functions have the values of the Euler numbers or the Euler polynomials at negative integers. That is, $\zeta_E(-k) = E_k^*$, and $\zeta_E(-k,x) = E_k^*(x)$. We give some interesting identities between the Euler numbers and the zeta functions. Finally, we will give the new values of the Euler zeta function at positive even integers.

1. Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of rational integers, the field of rational numbers, the field of complex numbers, the ring p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ such that $|p|_p = p^{-v_p(p)} = p^{-1}$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Hence, $\lim_{q\to 1}[x]_q = 1$, for any x with $|x|_p \le 1$ in the present p-adic case.

Let p be a fixed odd prime. For $d(=odd)$, a fixed positive integer with $(p,d)=1$, let

$$\begin{gathered} \mathrm{X} = \mathrm{X}_d = \lim_{\overleftarrow{N}} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \qquad \mathrm{X}_1 = \mathbb{Z}_p, \qquad \mathrm{X}^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p) \\ a + dp^N\mathbb{Z}_p = \{x \in \mathrm{X} \mid x \equiv a \,(\mathrm{mod}\; dp^n)\}, \end{gathered} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

In [1], we note that

$$\mu_{-q}\left(a+dp^N\mathbb{Z}_p\right)=(1+q)\frac{(-1)^a q^a}{1+q^{dp^N}}=\frac{(-q)^a}{\left[dp^N\right]_{-q}} \tag{1.3}$$

is distribution on X for $q\in\mathbb{C}_p$ with $|1-q|_p<1$. This distribution yields an integral as follows:

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{\left[p^N\right]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad \text{for } f\in UD(\mathbb{Z}_p), \tag{1.4}$$

which has a sense as we see readily that the limit is convergent (see [1]). Let $q=1$. Then, we have the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1.5}$$

(cf. [1–5]). For any positive integer N, we set

$$\mu_q\left(a+lp^N\mathbb{Z}_p\right)=\frac{q^a}{\left[lp^N\right]_q} \tag{1.6}$$

(cf. [1–3, 6–20]) and this can be extended to a distribution on X. This distribution yields p-adic bosonic q-integral as follows (see [11, 20]):

$$I_q(f)=\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\int_X f(x)d\mu_q(x), \tag{1.7}$$

where $f\in UD(\mathbb{Z}_p)=$ the space of uniformly differentiable function on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$, (cf. [2, 11, 16–20]). In view of notation, I_{-1} can be written symbolically as $I_{-1}(f)=\lim_{q\to-1}I_q(f)$. If we take $f(x)=q^{-x}[x]_q^n$, then we can derive the q-extension of Bernoulli numbers and polynomials from p-adic q-integrals on $\mathbb{Z}_p$ as follows:

$$\beta_{n,q}=\int_{\mathbb{Z}_p} q^{-x}[x]_q^n d\mu_q(x), \qquad \beta_{n,q}(x)=\int_{\mathbb{Z}_p} q^{-y}[y+x]_q d\mu_q(y) \tag{1.8}$$

(cf. [11, 20]). Thus, we note that

$$\beta_{0,q}=\frac{q-1}{\log q}, \qquad \beta_{m,q}=\frac{1}{(q-1)^m}\sum_{i=0}^{m}\binom{m}{i}\frac{i}{[i]_q}, \tag{1.9}$$

(cf. [11, 14, 20]). In the complex plane, the ordinary Bernoulli numbers are a sequence of signed rational numbers that can be defined by the identity

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}, \quad |t|<2\pi \tag{1.10}$$

(cf. [1–33]).

These numbers arise in the series expansions of trigonometric functions, and are extremely important in number theory and analysis. From the generating function of Bernoulli numbers, we note that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $B_{18} = 43867/798$, $B_{20} = -174611/330, \ldots,$ and $B_{2k+1} = 0$ for $k \in \mathbb{N}$. It is well known that Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } s \in \mathbb{C}. \tag{1.11}$$

We also note that the Riemann zeta function is closely related to Bernoulli numbers at positive integer or negative integer in the complex plane. Riemann did develop the theory of analytic continuation needed to rigorously define $\zeta(s)$ for all $s \in \mathbb{C} - \{0\}$. From this zeta function, he derived the following formula (cf. [1–33]):

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}. \tag{1.12}$$

Thus, we note that $\zeta(-n) = 0$ if n is an even integer and greater than 0. These are called the trivial zeros of the zeta function. In 1859, starting with Euler's factorization of the zeta function

$$\zeta(s) = \prod_{p:\text{prime}} \frac{1}{1 - p^{-s}}, \tag{1.13}$$

he derived an explicit formula for the prime numbers in terms of zeros of the zeta function. He also posed the Riemann hypothesis: if $\zeta(z) = 0$, then either z is a trivial zero or z lies on the critical line $\operatorname{Re}(z) = 1/2$ (cf. [4, 5, 16–20, 27–33]). It is well known that

$$\frac{\sin z}{z} = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{(2\pi)^2}\right)\left(1 - \frac{z^3}{(3\pi)^2}\right)\cdots. \tag{1.14}$$

Thus,

$$1 - z\cot z = 2\sum_{m=1}^{\infty} \frac{\zeta(2m)}{\pi^{2m}} z^{2m} \tag{1.15}$$

(cf. [4, 5, 10, 16–20, 27–33]). From this, we can derive the following famous formula.

Lemma 1.1. *For $n \in \mathbb{N}$,*

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad \text{for } n \in \mathbb{N}. \tag{1.16}$$

It is easy to see that

$$z\cot z = \frac{2iz}{e^{2iz} - 1} + iz = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} z^{2k}. \tag{1.17}$$

However, it is not known the values of $\zeta(2k+1)$ for $k \in \mathbb{N}$. In the case of $k=1$, Apery proved that $\zeta(3)$ is irrational number (see [34]). The constants E_k^* in the Taylor series expansion

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \quad \text{where } |t| < \pi \tag{1.18}$$

(cf. [3–5, 10, 27]) are known as the first-kind Euler numbers. From the generating function of the first-kind Euler numbers, we note that

$$E_0^* = 1, \quad E_n^* = -\sum_{l=0}^{n} \binom{n}{l} E_l^*, \quad \text{for } n \in \mathbb{N}. \tag{1.19}$$

The first few are $1, -1/2, 0, 1/4, \ldots$, and $E_{2k}^* = 0$ for $k = 1, 2, \ldots$. The Euler polynomials are also defined by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n^*(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} E_k^* x^{n-k} \right) \frac{t^n}{n!}. \tag{1.20}$$

For $s \in \mathbb{C}$, the Euler zeta function and Hurwitz's type Euler zeta function are defined by

$$\zeta_E(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \qquad \zeta_E(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s} \tag{1.21}$$

(cf. [2, 4, 5, 9, 10, 27]). Thus, we note that Euler zeta functions are entire functions in the whole complex s-plane and these zeta functions have the values of the Euler numbers or Euler polynomials at negative integers. That is,

$$\zeta_E(-k) = E_k^*, \qquad \zeta_E(-k,x) = E_k^*(x) \tag{1.22}$$

(cf. [2, 4, 5, 9, 10, 27]).

In this paper, we give some interesting identities between Euler numbers and zeta functions. Finally, we will give the values of the Euler zeta function at positive even integers.

2. Preliminaries/Euler numbers associated with p-adic fermionic integrals

Let $f_1(x)$ be the translation defined by $f_1(x) = f(x+1)$. Then we have

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0). \tag{2.1}$$

If we take $f(x) = e^{(x+y)t}$, then we can derive the first-kind Euler polynomials from the integral equation of $I_{-1}(f)$ as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = e^{xt} \frac{2}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n^*(x) t^n}{n!}. \tag{2.2}$$

That is,

$$\int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = E_n^*, \qquad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n^*(x). \tag{2.3}$$

For $n \in \mathbb{N}$, we have the following integral equation:

$$\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) + (-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2\sum_{l=0}^{n-1}(-1)^{n-1+l}f(l). \tag{2.4}$$

From this we note that

$$\begin{aligned} E_k^*(n) - E_k^* &= 2\sum_{l=0}^{n-1}(-1)^{l-1}l^k, \quad \text{if } n \equiv 0 \pmod 2, \\ E_k^*(n) + E_k^* &= 2\sum_{l=0}^{n-1}(-1)^{l}l^k, \quad \text{if } n \equiv 1 \pmod 2. \end{aligned} \tag{2.5}$$

Let $f(x) = \sin ax$ (or $f(x) = \cos ax$). By using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\begin{aligned} 0 &= \int_{\mathbb{Z}_p} \sin ax \, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \sin ax \, d\mu_{-1}(x) \\ &= (\cos a + 1)\int_{\mathbb{Z}_p} \sin ax \, d\mu_{-1}(x) + \sin a \int_{\mathbb{Z}_p} \cos ax \, d\mu_{-1}(x), \end{aligned} \tag{2.6}$$

see [12],

$$2 = (\cos a + 1)\int_{\mathbb{Z}_p} \cos ax \, d\mu_{-1}(x) - \sin a \int_{\mathbb{Z}_p} \sin ax \, d\mu_{-1}(x). \tag{2.7}$$

Thus, we obtain

$$\int_{\mathbb{Z}_p} \cos ax \, d\mu_{-1}(x) = 1, \qquad \int_{\mathbb{Z}_p} \sin ax \, d\mu_{-1}(x) = -\frac{\sin a}{\cos a + 1}, \tag{2.8}$$

see [12]. From this we note that

$$\tan\frac{a}{2} = -\int_{\mathbb{Z}_p} \sin ax \, d\mu_{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}a^{2n+1}}{(2n+1)!}E_{2n+1}^*. \tag{2.9}$$

By the same motivation, we can also observe that

$$\frac{a}{2}\cot\frac{a}{2} = \int_{\mathbb{Z}_p} \cos ax \, d\mu_{1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n}B_{2n}}{(2n)!}a^{2n}, \tag{2.10}$$

see [12]. These formulae are also treated in Section 3.

Let $f(x) = e^{t(2x+1)}$. Then we can derive the generating function of the second-kind Euler numbers from fermionic p-adic integral equation as follows:

$$\int_{\mathbb{Z}_p} e^{t(2x+1)}d\mu_{-1}(x) = \frac{2}{e^t + e^{-t}} = \frac{1}{\cosh t} = \sum_{n=0}^{\infty}E_n\frac{t^n}{n!}. \tag{2.11}$$

Thus, we have

$$(E+1)^n + (E-1)^n = 2\delta_{0,n}, \tag{2.12}$$

where we have used the symbolic notation E_n for E^n. The first few are $E_0 = 1$, $E_1 = 0$, $E_2 = -1$, $E_3 = 0$, $E_4 = 5, \ldots, E_{2k+1} = 0$ for $k \in \mathbb{N}$. In particular,

$$E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}. \tag{2.13}$$

Recently, Simsek, Ozden, Cangül, Cenkci, Kurt, and others have studied the various extensions of the first kind Euler numbers by using fernionic p-adic invariant q-integral on $\mathbb{Z}_p$, see [2–5, 16, 20, 27]. It seems to be also interesting to study the q-extensions of the second-kind Euler numbers due to Simsek et al. (see [4, 5, 16]).

3. Some relationships between Euler numbers and zeta functions

In this section, we also consider Bernoulli and the second Euler numbers in the complex plane. The second-kind Euler numbers E_k are defined by the following expansion:

$$sech\, x = \frac{1}{\cosh x} = \frac{2e^x}{e^{2x}+1} = \sum_{k=0}^{\infty} E_k \frac{x^k}{k!}, \quad \text{for } |x| < \frac{\pi}{2} \tag{3.1}$$

(cf. [10]). From (1.18) and (3.1), we can derive the following equation:

$$E_k = \sum_{l=0}^{k} \binom{k}{l} 2^l E_l^*, \quad \text{where } \binom{k}{l} \text{ is binomial coefficient.} \tag{3.2}$$

By (3.2) and (1.18), we easily see that $E_0 = 1$, $E_1 = 0$, $E_2 = -1$, $E_3 = 0$, $E_4 = 5$, $E_6 = 61, \ldots,$ and $E_{2k+1} = 0$ for $k = 1, 2, 3, \ldots$. As Euler formula, it is well known that

$$e^{ix} = \cos x + i \sin x, \quad \text{where } i = (-1)^{1/2}. \tag{3.3}$$

From (3.3), we note that $\cos x = (e^{ix} + e^{-ix})/2$. Thus, we have

$$\begin{aligned} \sec x &= \frac{2}{e^{ix}+e^{-ix}} = sech(ix) = \sum_{n=0}^{\infty} \frac{i^n E_n}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n+1}}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}. \end{aligned} \tag{3.4}$$

From (3.4), we derive

$$x \sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n+1}, \quad \text{for } |x| < \frac{\pi}{2}. \tag{3.5}$$

The Fourier series of an odd function on the interval $(-p, p)$ is the sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right), \tag{3.6}$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx. \tag{3.7}$$

Let us consider $f(x) = \sin ax$ on $[-\pi, \pi]$. From (3.6) and (3.7), we note that

$$\sin ax = \sum_{n=1}^{\infty} b_n \sin nx, \tag{3.8}$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\left[\frac{\cos(n-a)x - \cos(n+a)x}{2}\right]dx \\ &= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi} = (-1)^{n-1}\frac{2}{\pi}\sin a\pi\left(\frac{n}{n^2-a^2}\right). \end{aligned} \tag{3.9}$$

In (3.8), if we take $x = \pi/2$, then we have

$$\begin{aligned} \sin\frac{\pi a}{2} &= \sum_{n=1}^{\infty} b_{2n-1}(-1)^{n-1} = \frac{2}{\pi}\sin a\pi \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n-1}{(2n-1)^2 - a^2} \\ &= \frac{2}{\pi}\sin a\pi \sum_{n=1}^{\infty}\frac{(2n-1)(-1)^{n-1}}{(2n-1)^2\left(1-\left(a/(2n-1)\right)^2\right)} = \frac{2}{\pi}\sin a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\sum_{k=0}^{\infty}\frac{a^{2k}}{(2n-1)^{2k}} \\ &= \frac{2}{\pi}\sin a\pi\sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k+1}}\right)a^{2k} \end{aligned} \tag{3.10}$$

From (3.10), we note that

$$\frac{\pi a}{2}\sec\left(\frac{\pi a}{2}\right) = \sum_{k=0}^{\infty}\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k+1}}\right)a^{2k+1}. \tag{3.11}$$

In (3.5), it is easy to see that

$$\frac{\pi a}{2}\sec\left(\frac{\pi a}{2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n+1} a^{2n+1}. \tag{3.12}$$

By (3.11) and (3.12), we obtain the following.

Theorem 3.1. *For $n \in \mathbb{N}$,*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{2n+1}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n\frac{E_{2n}}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}. \tag{3.13}$$

It is easy to see that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}} &= 2\sum_{n=1}^{\infty}\frac{1}{(4n-3)^{2k+1}}+\sum_{n=1}^{\infty}\frac{1}{(2n)^{2k+1}}-\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}-1\\ &= \frac{1}{2^{4k+1}}\zeta\left(2k+1,\frac{1}{4}\right)-\frac{2^{2k+1}-1}{2^{2k+1}}\zeta(2k+1)-1.\end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we obtain the following.

Corollary 3.2. *For $n \in \mathbb{N}$,*

$$\zeta\left(2n+1,\frac{1}{4}\right)+2^{2n}\left(1-2^{2n+1}\right)\zeta(2n+1) = (-1)^n\frac{E_{2n}}{2(2n)!}\pi^{2n+1}2^{2n}. \tag{3.15}$$

By simple calculation, we easily see that

$$i\tan x = \frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}} = 1-\frac{2}{e^{2ix}-1}+\frac{4}{e^{4ix}-1}. \tag{3.16}$$

Thus, we have

$$x\tan x = -xi+\frac{2xi}{e^{2xi}-1}-\frac{4xi}{e^{4xi}-1} = \sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}4^n\left(1-4^n\right)}{(2n)!}x^{2n}. \tag{3.17}$$

From (3.17), we can easily derive

$$\tan x = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}4^{n+1}\left(1-4^{n+1}\right)B_{2n+2}}{(2n+2)!}x^{2n+1}. \tag{3.18}$$

By (3.3), we also see that

$$i\tan x = 1-\frac{2}{e^{2ix}+1} = i\sum_{n=0}^{\infty}\frac{E^*_{2n+1}}{(2n+1)!}2^{2n+1}(-1)^{n+1}x^{2n+1}. \tag{3.19}$$

Thus, we have

$$\tan x = \sum_{n=0}^{\infty}\frac{E^*_{2n+1}}{(2n+1)!}2^{2n+1}(-1)^{n+1}x^{2n+1}. \tag{3.20}$$

By (3.18) and (3.20), we obtain the following.

Theorem 3.3. *For $n \in \mathbb{N}$,*

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}E^*_{2n-1}}{4(2n-1)!\left(1-4^n\right)}, \tag{3.21}$$

where E^*_n are the first-kind Euler numbers.

It is easy to see that

$$\sum_{k=1}^{\infty}\frac{1}{(2k+1)^{2n}} = \left(1-\frac{1}{4^n}\right)\zeta(2n) = \frac{(-1)^n(2\pi)^{2n}}{4^{n+1}(2n-1)!}E^*_{2n-1}. \tag{3.22}$$

Therefore, we obtain the following corollary.

Corollary 3.4. *For $n \in \mathbb{N}$,*

$$\sum_{k=1}^{\infty}\frac{1}{(2k+1)^{2n}} = \frac{(-1)^n(2\pi)^{2n}}{4^{n+1}(2n-1)!}E^*_{2n-1}. \tag{3.23}$$

Now we try to give the new value of the Euler zeta function at positive integers. From the definition of the Euler zeta function, we note that

$$\zeta_E(s) = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} = -2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s} + \frac{1}{2^{s-1}}\zeta(s), \quad s \in \mathbb{C}. \tag{3.24}$$

By (3.24), Theorem 3.3, and Corollary 3.4, we obtain the following theorem.

Theorem 3.5. *For $n \in \mathbb{N}$,*

$$\zeta_E(2n) = \frac{(-1)^{n-1}\pi^{2n}(2-4^n)}{2(2n-1)!(1-4^n)}E^*_{2n-1}. \tag{3.25}$$

Remark 3.6. We note that $\zeta(2) = \pi^2/6$, $\zeta_E(2) = -\pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta_E(4) = -7\pi^4/360\ldots$. For $q \in \mathbb{C}$ with $|q| < 1$, $s \in \mathbb{C}$, q-ζ-function is defined by

$$\zeta_q(s) = \sum_{n=1}^{\infty}\frac{q^n}{[n]_q^s} - \frac{1}{s-1}\frac{(1-q)^s}{\log q} \tag{3.26}$$

(cf. [10, 14]). Note that $\zeta_q(s)$ is analytic continuation in $\mathbb{C}$ with only one simple pole at $s = 1$, and

$$\zeta_q(1-k) = -\frac{\beta_{k,q}}{k}, \quad \text{where } k \text{ is a positive integer} \tag{3.27}$$

(cf. [14]). By simple calculation, we easily see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{[n]_q^{2k+1}}\sum_{j=0}^{\infty}\frac{\theta^{2j+1}[n]_q^{2j+1}}{(2j+1)!} + \frac{1}{\log q}\sum_{j=0}^{k-1}\frac{(1-q)^{2k-2j}}{(2k-2j-1)(2j+1)!} - \frac{1}{\log q}\sum_{j=0}^{k-1}\frac{(1-q)^{2k-2j}}{(2k-2j-1)(2j+1)!} \\
&= \sum_{j=0}^{k-1}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\left(-\zeta_q(2k-2j)+\zeta_{q^2}(2k-2j)\frac{2}{[2]_q^{2k-2j}}\right) - \frac{q}{1+q}\frac{\theta^{2k+1}}{(2k+1)!}(-1)^k + \sum_{j=k+1}^{\infty}\frac{\theta^{2j+1}(-1)^j}{(2j+1)!}\frac{H_{2j-2k,q}(-q^{-1})}{1+q},
\end{aligned} \tag{3.28}$$

where $H_{n,q}(-q)$ are Carlitz's q-Euler numbers with $\lim_{q\to 1} H_{n,q}(-q) = E_n^*$ (cf. [6, 21, 22]). If $q\to 1$, then we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k+1}}\sin(n\theta)=\sum_{j=0}^{k-1}\frac{(-1)^j}{(2j+1)!}\theta^{2j+1}\left(\frac{2}{2^{2k-2j}}-1\right)\times(-1)^{k-j+1}\frac{(2\pi)^{2k-2j}}{2\cdot(2k-2j)!}B_{2k-2j}-\frac{1}{2}\frac{\theta^{2k+1}}{(2k+1)!}(-1)^k. \tag{3.29}$$

For $k\in\mathbb{N}$, and $\theta=\pi/2$, it is easy to see that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^{2k+1}}=\sum_{j=0}^{k-1}\frac{(-1)^k\pi^{2k+1}(2^{2k-2j}-2)B_{2k-2j}}{(2j+1)!(2k-2j)!2^{2j+2}}-\frac{\pi^{2k+1}(-1)^k}{(2k+1)!2^{2k+2}}. \tag{3.30}$$

From (3.30) and Theorem 3.1, we can also derive the following equation:

$$\sum_{j=0}^{k-1}\frac{(-1)^{k-1}\pi^{2k+1}(2^{2k-2j}-2)B_{2k-2j}}{(2j+1)!(2k-2j)!2^{2j+2}}+\frac{\pi^{2k+1}(-1)^k}{(2k+1)!2^{2k+2}}=(-1)^k\frac{E_{2k}}{2(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}. \tag{3.31}$$

Thus, we have

$$\sum_{j=0}^{k-1}\frac{(2^{2k-2j}-2)B_{2k-2j}}{(2j+1)!(2k-2j)!2^{2j+2}}=\frac{1}{(2k+1)!2^{2k+2}}-\frac{E_{2k}}{2^{2k+2}(2k)!}. \tag{3.32}$$

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