

Research Article

On Bloch-Type Functions with Hadamard Gaps

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We give some sufficient and necessary conditions for an analytic function f on the unit ball B with Hadamard gaps, that is, for $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ (the homogeneous polynomial expansion of f) satisfying $n_{k+1}/n_k \geq \lambda > 1$ for all $k \in \mathbb{N}$, to belong to the space $\mathcal{B}_p^\alpha(B) = \{f \mid \sup_{0 < r < 1} (1 - r^2)^\alpha \|\mathcal{R}f_r\|_p < \infty, f \in H(B)\}$, $p = 1, 2, \infty$ as well as to the corresponding little space. A remark on analytic functions with Hadamard gaps on mixed norm space on the unit disk is also given.

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1. Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball of \mathbb{C}^n , $\partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ its boundary, \mathbb{D} the unit disk in \mathbb{C} , dv the normalized Lebesgue measure of B (i.e., $v(B) = 1$), and $d\sigma$ the normalized rotation invariant Lebesgue measure of S satisfying $\sigma(\partial B) = 1$. We denote the class of all holomorphic functions on the unit ball by $H(B)$.

For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\beta| \geq 0} a_\beta z^\beta$, let $\mathcal{R}f(z) = \sum_{|\beta| \geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f , where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. It is well known that $\mathcal{R}f(z) = \sum_{j=1}^n z_j (\partial f / \partial z_j)(z) = \sum_{k=0}^{\infty} k P_k(z)$, if $f(z) = \sum_{k=0}^{\infty} P_k(z)$.

As usual, we write

$$\|f_r\|_p = \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} \quad (1.1)$$

if $p \in (0, \infty)$, and where $f_r(\zeta) = f(r\zeta)$. If $p = \infty$, then $\|f\|_\infty = \sup_{z \in B} |f(z)|$.

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Let $\alpha > 0$. The α -Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$ is the space of all holomorphic functions f on B such that

$$b_\alpha(f) = \sup_{z \in B} (1 - |z|^2)^\alpha |\mathcal{R}f(z)| < \infty. \quad (1.2)$$

It is clear that \mathcal{B}^α is a normed space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, and $\mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}$ for $\alpha_1 < \alpha_2$. Let \mathcal{B}_0^α denote the subspace of \mathcal{B}^α consisting of those $f \in \mathcal{B}^\alpha$ for which $(1 - |z|^2)^\alpha |\mathcal{R}f(z)| \rightarrow 0$ as $|z| \rightarrow 1$. This space is called the little α -Bloch space. For $\alpha = 1$, the α -Bloch space and the little α -Bloch space become Bloch space \mathcal{B} and the little Bloch space \mathcal{B}_0 . Some characterizations of these spaces can be found, for example, in the following papers [1–6].

We say that an analytic function f on the unit disk \mathbb{D} has Hadamard gaps if $f(z) = \sum_{k=1}^\infty a_k z^{n_k}$ where $n_{k+1}/n_k \geq \lambda > 1$, for all $k \in \mathbb{N}$.

In [7], Yamashita proved the following result.

THEOREM 1.1. *Assume that f is an analytic function on \mathbb{D} with Hadamard gaps. Then for $\alpha > 0$, the following two propositions hold:*

- (a) $f \in \mathcal{B}^\alpha(\mathbb{D})$ if and only if $\limsup_{k \rightarrow \infty} |a_k| n_k^{1-\alpha} < \infty$;
- (b) $f \in \mathcal{B}_0^\alpha(\mathbb{D})$ if and only if $\lim_{k \rightarrow \infty} |a_k| n_k^{1-\alpha} = 0$.

An analytic function on B with the homogeneous expansion $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ (here, P_{n_k} is a homogeneous polynomial of degree n_k) is said to have Hadamard gaps if $n_{k+1}/n_k \geq \lambda > 1$, for all $k \in \mathbb{N}$. In [8], among others, Choa generalizes the main result in [9], proving the following result.

THEOREM 1.2. *Assume that $p \in (0, \infty)$ and $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ is an analytic function on B with Hadamard gaps. Then the following statements are equivalent:*

- (a) $\|f\|_{X_p} = \left(\int_B |\mathcal{R}f(z)|^p (1 - |z|^2)^{p-1} d\nu(z) \right)^{1/p} < \infty$;
- (b) $\sum_{k=1}^\infty \|P_{n_k}\|_p^p < \infty$.

This result motivates us to find some characterizations for certain function spaces of analytic functions on the unit ball, in terms of the sequence $(\|P_{n_k}\|_p)_{k \in \mathbb{N}}$.

Now note that the quantity b_α in the definition of the α -Bloch spaces can be written in the following form:

$$b_\alpha(f) = \sup_{0 < r < 1} (1 - r^2)^\alpha \sup_{\zeta \in S} |\mathcal{R}f(r\zeta)| = \sup_{0 < r < 1} (1 - r^2)^\alpha M_\infty(\mathcal{R}f, r). \quad (1.3)$$

On the other hand, the quantity b_α can be considered as the limit case of the following quantities:

$$\|f\|_{\mathcal{B}_p^\alpha} = \sup_{0 < r < 1} (1 - r^2)^\alpha \|\mathcal{R}f_r\|_p, \quad (1.4)$$

as $p \rightarrow \infty$. Note that for every $f \in H(B)$ and $p \in (0, \infty)$,

$$\sup_{0 < r < 1} (1 - r^2)^\alpha \|\mathcal{R}f_r\|_p \leq \sup_{0 < r < 1} (1 - r^2)^\alpha \|\mathcal{R}f_r\|_\infty. \quad (1.5)$$

Hence, in this paper we also consider analytic functions with Hadamard gaps on the following spaces:

$$\begin{aligned} \mathcal{B}_p^\alpha &= \left\{ f \mid \sup_{0 < r < 1} (1 - r^2)^\alpha \| \mathcal{R}f_r \|_p < \infty, f \in H(B) \right\}, \\ \mathcal{B}_{p,0}^\alpha &= \left\{ f \mid \lim_{r \rightarrow 1} (1 - r^2)^\alpha \| \mathcal{R}f_r \|_p = 0, f \in H(B) \right\}. \end{aligned} \tag{1.6}$$

Motivated by Theorem 1.1 in this paper, we study analytic functions with Hadamard gaps, which belong to \mathcal{B}_p^α or $\mathcal{B}_{p,0}^\alpha$ space when $p = 1, 2, \infty$. Some characterizations for these classes of functions on the unit ball are given in terms of the sequence $(\|P_{n_k}\|_p)_{k \in \mathbb{N}}$. The following are the main results.

THEOREM 1.3. *Assume that $\alpha > 0$, $p = 1, 2, \infty$, and $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ is an analytic function on B with Hadamard gaps. Then the following statements are equivalent:*

- (a) $f \in \mathcal{B}_p^\alpha$;
- (b) $\limsup_{k \rightarrow \infty} \|P_{n_k}\|_p n_k^{1-\alpha} < \infty$.

THEOREM 1.4. *Assume that $\alpha > 0$, $p = 1, 2, \infty$, and $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ is an analytic function on B with Hadamard gaps. Then the following statements are equivalent:*

- (a) $f \in \mathcal{B}_{p,0}^\alpha$;
- (b) $\lim_{k \rightarrow \infty} \|P_{n_k}\|_p n_k^{1-\alpha} = 0$.

Throughout this paper, constants are denoted by C , they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Proof of main results

Before proving the main results of this paper we quote two auxiliary results which are incorporated in the lemmas which follow (see [9, 10]).

LEMMA 2.1. *Assume that $p \in (0, \infty)$. If (n_k) is an increasing sequence of positive integers satisfying $n_{k+1}/n_k \geq \lambda > 1$, for all k , then there is a positive constant A depending only on p and λ such that*

$$\frac{1}{A} \left(\sum_{k=1}^\infty |a_k|^2 \right)^{1/2} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^\infty a_k e^{in_k \theta} \right|^p d\theta \right)^{1/p} \leq A \left(\sum_{k=1}^\infty |a_k|^2 \right)^{1/2} \tag{2.1}$$

for any number a_k , $k \in \mathbb{N}$.

LEMMA 2.2. *Assume that $\alpha > 0$, $p > 0$, $n \in \mathbb{N}_0$, $(a_n)_{n \in \mathbb{N}_0}$ is the sequence of nonnegative numbers, $I_n = \{k \mid 2^n \leq k < 2^{n+1}, k \in \mathbb{N}\}$, $t_n = \sum_{k \in I_n} a_k$, and $g(x) = \sum_{n=1}^\infty a_n x^{2^n}$. Then there is a positive constant K depending only on p and α such that*

$$\frac{1}{K} \sum_{n=0}^\infty \frac{t_n^p}{2^{n\alpha}} \leq \int_0^1 (1-x)^{\alpha-1} g^p(x) dx \leq K \sum_{n=0}^\infty \frac{t_n^p}{2^{n\alpha}}. \tag{2.2}$$

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Proof of Theorem 1.3. (a) \Rightarrow (b) (Case $p = 1$). Let $f \in \mathcal{B}_1^\alpha$. Let $f_\zeta(w) = f(\zeta w)$, $\zeta \in S$, where ζ is fixed and $w \in \mathbb{D}$, be a slice function. By some calculation we see that

$$f'_\zeta(w) = \zeta_1 \frac{\partial f}{\partial z_1}(w\zeta) + \cdots + \zeta_n \frac{\partial f}{\partial z_n}(w\zeta) = \frac{1}{w} \mathcal{R}f(w\zeta). \quad (2.3)$$

From (2.3) and since $f'_\zeta(w) = \sum_{k=1}^\infty n_k P_{n_k}(\zeta) w^{n_k-1}$, we have that

$$\begin{aligned} \int_S n_k |P_{n_k}(\zeta)| d\sigma(\zeta) &= \int_S \left| \frac{1}{2\pi i} \int_{\partial r\mathbb{D}} \frac{\eta f'_\zeta(\eta)}{\eta^{n_k+1}} d\eta \right| d\sigma(\zeta) \\ &\leq \frac{1}{2\pi} \int_{\partial r\mathbb{D}} \int_S \frac{|\mathcal{R}f(\zeta\eta)|}{|\eta^{n_k+1}|} d\sigma(\zeta) |d\eta| \\ &\leq \frac{\|f_r\|_{\mathcal{B}_1^\alpha}}{(1-r)^\alpha r^{n_k}}, \end{aligned} \quad (2.4)$$

which implies that

$$n_k r^{n_k} \|P_{n_k}\|_1 \leq \frac{\|f\|_{\mathcal{B}_1^\alpha}}{(1-r)^\alpha}, \quad (2.5)$$

for every $k \in \mathbb{N}$ and $r \in (0, 1)$. Choosing $r = 1 - (1/n_k)$, we obtain $n_k^{1-\alpha} \|P_{n_k}\|_1 \leq C$, as desired.

(b) \Rightarrow (a) (Case $p = 1$). Assume $\limsup_{k \rightarrow \infty} \|P_{n_k}\|_1 n_k^{1-\alpha} < \infty$. We have that

$$\begin{aligned} \|f\|_{\mathcal{B}_1^\alpha} &= \sup_{0 < r < 1} (1-r^2)^\alpha \int_S |\mathcal{R}f(r\zeta)| d\sigma(\zeta) \\ &= \sup_{0 < r < 1} (1-r^2)^\alpha \int_S \left| \sum_{k=1}^\infty n_k P_{n_k}(\zeta) r^{n_k} \right| d\sigma(\zeta) \\ &\leq \sup_{0 < r < 1} (1-r^2)^\alpha \sum_{k=1}^\infty n_k \|P_{n_k}\|_1 r^{n_k} \\ &\leq \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty \left(\sum_{n_k \leq n} n_k \|P_{n_k}\|_1 \right) r^n \\ &\leq C \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty \left(\sum_{n_k \leq n} n_k^\alpha \right) r^n \\ &\leq C \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty n^\alpha r^n \leq C, \end{aligned} \quad (2.6)$$

where we have used the fact that there is a positive constant C independent of n such that $\sum_{n_k \leq n} n_k^\alpha \leq Cn^\alpha$ (here is used the assumption that $n_{k+1}/n_k \geq \lambda > 1$) and the following well-known estimate:

$$\sum_{n=1}^\infty n^\alpha r^n \leq C(1-r)^{-(\alpha+1)}, \quad (2.7)$$

$\alpha > 0$, $r \in [0, 1)$; see, for example, [11].

Case $p = 2$. Since

$$\|f\|_{\mathfrak{B}_2^\alpha} = \sup_{0 < r < 1} (1 - r^2)^\alpha \left(\sum_{k=1}^{\infty} n_k^2 \|P_{n_k}\|_2^2 r^{2n_k} \right)^{1/2} \quad (2.8)$$

we have that

$$\sup_{0 < r < 1} (1 - r^2)^\alpha n_k \|P_{n_k}\|_2 r^{n_k} \leq \|f\|_{\mathfrak{B}_2^\alpha} \leq \sup_{0 < r < 1} (1 - r^2)^\alpha \sum_{k=1}^{\infty} n_k \|P_{n_k}\|_2 r^{n_k}, \quad (2.9)$$

from which the result follows similar to the case $p = 1$.

Now we show that (a) \Leftrightarrow (b) for case $p = \infty$. As above, the function $f_\zeta(w) = \sum_{k=1}^{\infty} P_{n_k}(\zeta) w^{n_k}$, where $w = re^{i\theta}$, is a lacunary series in \mathbb{D} and

$$(1 - r^2)^\alpha \mathcal{R}f(r\zeta) = re^{i\theta} (1 - r^2)^\alpha f'_{\zeta e^{-i\theta}}(re^{i\theta}), \quad (2.10)$$

from which by Theorem 1.1 the equivalence follows. \square

Proof of Theorem 1.4. (a) \Rightarrow (b) (Case $p = 1$). Let $f \in \mathfrak{B}_{1,0}^\alpha$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(1 - r^2)^\alpha \int_S |\mathcal{R}f(r\zeta)| d\sigma(\zeta) < \varepsilon, \quad (2.11)$$

whenever $\delta < r < 1$. From (2.4), (2.11), and rotational invariance of $d\sigma$, we have that

$$\begin{aligned} \int_S n_k |P_{n_k}(\zeta)| d\sigma(\zeta) &\leq \frac{1}{2\pi} \int_{\partial r\mathbb{D}} \int_S \frac{|\mathcal{R}f(\zeta\eta)|}{|\eta^{n_k+1}|} d\sigma(\zeta) |d\eta| \\ &\leq \frac{1}{2\pi} \int_{\partial r\mathbb{D}} \int_S \frac{(1 - r^2)^\alpha |\mathcal{R}f(\zeta\eta)|}{(1 - r^2)^\alpha r^{n_k+1}} d\sigma(\zeta) |d\eta| \\ &\leq \frac{\varepsilon}{(1 - r)^\alpha r^{n_k}}, \end{aligned} \quad (2.12)$$

which implies that

$$n_k r^{n_k} \|P_{n_k}\|_1 \leq \frac{\varepsilon}{(1 - r)^\alpha} \quad (2.13)$$

for every $k \in \mathbb{N}$ and $r \in (\delta, 1)$. Choosing $r = 1 - (1/n_k)$, we obtain

$$n_k \|P_{n_k}\|_1 \leq C\varepsilon n_k^\alpha, \quad (2.14)$$

from which (b) follows in this case.

(b) \Rightarrow (a) (Case $p = 1$). Assume that $\lim_{k \rightarrow \infty} \|P_{n_k}\|_1 n_k^{1-\alpha} = 0$, then for every $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that

$$\|P_{n_k}\|_1 \leq \varepsilon n_k^{\alpha-1}, \quad \text{for } k \geq k_0. \quad (2.15)$$

We may assume that $k_0 = 1$. From this and by the proof of Theorem 1.3, (b) \Rightarrow (a) (Case $p = 1$), we have that

$$\begin{aligned} (1-r^2)^\alpha \|\mathcal{R}f_r\|_1 &\leq \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty \left(\sum_{n_k \leq n} n_k \|P_{n_k}\|_1 \right) r^n \\ &\leq C\varepsilon \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty \left(\sum_{n_k \leq n} n_k^\alpha \right) r^n \\ &\leq C\varepsilon \sup_{0 < r < 1} (1-r^2)^{\alpha+1} \sum_{n=1}^\infty n^\alpha r^n \leq C\varepsilon, \end{aligned} \tag{2.16}$$

from which the implication follows.

Case $p = 2$. By using (2.9) the result follows similar to the Case $p = 1$. The proof is omitted.

Finally, in view of (2.10) and employing Theorem 1.1(b) it is easy to see that (a) \Leftrightarrow (b) for case $p = \infty$. \square

3. The case of mixed norm space

In this section, we give a note concerning analytic functions with Hadamard gaps on the mixed norm space. The mixed norm space $H_{p,q,\alpha}(B)$, $p, q > 0$, and $\alpha \in (-1, \infty)$, consists of all $f \in H(B)$ such that

$$\|f\|_{p,q,\alpha} = \left(\int_0^1 \|f(r\zeta)\|_p^q (1-r)^\alpha dr \right)^{1/q} < \infty. \tag{3.1}$$

From [12, Theorem 4] the following result holds.

THEOREM 3.1. *Assume that $p \in (0, \infty)$, $\alpha > -1$ and $f(z) = \sum_{k=1}^\infty a_k z^{n_k}$ is an analytic function on \mathbb{D} with Hadamard gaps. Then $f^{(m)} \in H_{p,q,\alpha}(\mathbb{D})$ if and only if $\sum_{k=0}^\infty n_k^{qm-\alpha-1} |a_k|^q < \infty$.*

Proof. First we consider the case $m = 0$. Similar to the proof of [12, Theorem 4] and by Lemmas 2.1 and 2.2, we have that

$$\begin{aligned} \|f\|_{H_{p,q,\alpha}}^q &= \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^\infty a_k r^{n_k} e^{in_k\theta} \right|^p d\theta \right)^{q/p} (1-r)^\alpha dr \\ &\asymp \int_0^1 \left(\sum_{k=1}^\infty |a_k|^2 r^{2n_k} \right)^{q/2} (1-r)^\alpha dr \\ &\asymp \int_0^1 \left(\sum_{k=1}^\infty |a_k|^2 \rho^{n_k} \right)^{q/2} (1-\rho)^\alpha d\rho \\ &\asymp \sum_{k=0}^\infty \frac{1}{2^{(\alpha+1)k}} \left(\sum_{m \in I_k} |a_m|^2 \right)^{q/2} \asymp \sum_{k=0}^\infty \frac{|a_k|^q}{n_k^{\alpha+1}}, \end{aligned} \tag{3.2}$$

from which the result follows in this case.

Since f has Hadamard gaps and $f^{(m)}(z) = \sum_{k=1}^{\infty} a_k n_k (n_k - 1) \cdots (n_k - m + 1) z^{n_k - m}$, it follows that $f^{(m)}$ has Hadamard gaps too. Applying the just proved result to the function $f^{(m)}$, we obtain that $f^{(m)} \in H_{p,q,\alpha}(\mathbb{D})$ if and only if

$$\sum_{k=0}^{\infty} \frac{|n_k (n_k - 1) \cdots (n_k - m + 1) a_k|^q}{n_k^{\alpha+1}} \asymp \sum_{k=0}^{\infty} \frac{|a_k|^q}{n_k^{\alpha+1-mq}} < \infty, \tag{3.3}$$

finishing the proof. □

Remark 3.2. Motivated by [12, Theorems 3 and 4], we can conjecture that if $p \in (0, \infty)$, $\alpha > -1$, and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ is an analytic function on B with Hadamard gaps, then $\mathcal{R}^{(m)} f \in H_{p,q,\alpha}(B)$ if and only if $\sum_{k=0}^{\infty} n_k^{qm-\alpha-1} \|P_{n_k}\|_p^q < \infty$. Note that the result is true for the case of the weighted Bergman space, that is, when $p = q$, see [12, Corollary 1]. It is also expected that Theorems 1.3 and 1.4 hold for every $p \in [1; \infty]$ (for the case $n = 1$, see [13]).

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