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CONTROL PROBLEMS ON CLOSED SUBSETS OF $\mathbf{R}^n$ VIA FEEDBACK CONTROLS

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Dedicated to the memory of Juliusz Schauder

1. Introduction

The paper deals with a controllability problem for a nonlinear control process described by a system of non autonomous ordinary differential equations. Specifically, given a proximate retract K of $\mathbf{R}^n$, which consists in a closed subset of $\mathbf{R}^n$ satisfying a suitable property, we look for feedback controls such that the corresponding dynamics admit trajectories belonging to K, called viable solutions.

The paper is organized as follows. In Sections 2 and 3 we review several results concerning both the theory of proximate retracts and the theory of differential inclusions on proximate retracts; such results will be the tools for proving the later results. In Section 4, in the spirit of [1], [8] and the references therein, the required feedback controls are seen as selectors of a set-valued feedback map $C(t, x)$ suitably defined by means of the Bouligand tangent map associated to K. Here, a problem connected with the regularity of these feedback controls arises; in fact, conditions

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ensuring the existence of continuous selectors (with respect to the state x) of a set-valued map are very restrictive. On the other hand, such controls allow to consider classical solutions of the resulting dynamics, i.e. absolutely continuous solutions. Therefore, one is often obliged to deal with solutions which are only measurable with respect to both the variables (t, x). In this case we cannot expect to have solutions in the classical sense; thus different notions of solutions are in order. We consider here two generalized notions of solutions for the dynamical system: the Krasowskii's and the Filippov's definitions of solutions for a differential equation with discontinuous right-hand side.

In Section 5 we give conditions ensuring the existence of viable solutions both in the case when there exists a continuous feedback control and in the case when the feedback control is only measurable.

Finally, in Section 6, the approach developed in the previous Sections is applied to show the existence of periodic solutions which are also viable, and in Section 7, the existence of viable solutions for implicit control problems is proved.

As in the present paper, the theory of set-valued maps (and the related differential inclusions) plays an important role in many other controllability problems, for instance in the case of nonlinear boundary value control problems (see e.g. [5]).

2. Proximate retracts

We recall the notion of a class of subsets of $\mathbf{R}^n$, the so-called proximate retracts, introduced in [3] and [9] under the different name "sets with property ρ" (compare also [6]).

Let K be a nonempty, closed subset of $\mathbf{R}^n$. We define

$$\operatorname{dist}(u, K) = \inf\{|u - x|;\ x \in K\}.$$

Definition 2.1. *A nonempty, closed subset $K \subset \mathbf{R}^n$ is called a proximate retract if there exists an open neighbourhood $\mathcal{U}$ of K in $\mathbf{R}^n$ and a continuous map $r : \mathcal{U} \to K$ (called metric retraction) such that the following two conditions are verified:*

$$r(x) = x \quad \text{for all } x \in K; \tag{2.1}$$

$$|r(u) - u| = \operatorname{dist}(u, K) \quad \text{for all } u \in \mathcal{U}. \tag{2.2}$$

Note that any closed, convex subset of $\mathbf{R}^n$, as well as any C^2-submanifold of $\mathbf{R}^n$ is a proximate retract, so the class of such sets is quite large (see [3]).

Let K be a nonempty, closed subset of $\mathbf{R}^n$. We recall from [1] and [3] that the subset $T_K(x) \subset \mathbf{R}^n$, $x \in K$, defined by

$$T_K(x) = \left\{ y \in \mathbf{R}^n \ : \ \liminf_{\tau \to 0^+} \frac{1}{\tau} \operatorname{dist}(x + \tau y, K) = 0 \right\}$$

is called the Bouligand tangent cone to K at x.

The following proposition is important:

PROPOSITION 2.1. (cf. [1], [3] and [9]). *Let $K \subset \mathbf{R}^n$ be a proximate retract. Let $T : K \to \mathbf{R}^n$ be the Bouligand map defined as follows:*

$$T(x) = T_K(x), \qquad \text{for every } x \in K.$$

Then T is a lower semicontinuous (l.s.c.) map with closed and convex values.

Let us remark (comp. [1] and [3]) that for an arbitrary closed set $K \subset \mathbf{R}^n$ Proposition 2.1 is no longer true. We would like to add also that, in general, the graph Γ_T of T is not a closed subset of $K \times \mathbf{R}^n$ (even in the case where K is a proximate retract).

DEFINITION 2.2. *Let K be a proximate retract and $\varphi : [0, a] \times K \to K$ be a set-valued map with compact, nonempty values. We shall say that φ is strongly tangent to K if*

$$\varphi(t, x) \subset T_K(x) \quad \text{for all } x \text{ and almost all (a.a) } t; \tag{2.3}$$

φ is called tangent to K if

$$\varphi(t, x) \cap T_K(x) \neq \emptyset \quad \text{for all } x \text{ and a.a.t.} \tag{2.4}$$

Note that if $\varphi = f$ is a single-valued map then the above two notions coincide.

The construction we present below is very useful.

Let $\mathcal{U}$ be a fixed open neighbourhood of K and let $r : \mathcal{U} \to K$ be a metric retraction. Assume also that $\lambda : \mathbf{R}^n \to [0, 1]$ is an Uryshon function for K and $\mathcal{U}$, i.e., $\lambda(x) = 1$ if $x \in K$, $\lambda(x) = 0$ if $x \notin \mathcal{U}$ and λ continuous.

We define $\widetilde{\varphi} : [0, a] \times \mathbf{R}^n \to \mathbf{R}^n$ to be an extension of φ by means of the formula:

$$\widetilde{\varphi}(t, x) = \begin{cases} \lambda(x) \cdot \varphi(t, r(x)) & \text{if } x \in \mathcal{U} \\ \{0\} & \text{if } x \notin \mathcal{U}. \end{cases} \tag{2.5}$$

We need the following lemma (compare [3], [6] and [9]).

LEMMA 2.1. *Let $K \subset \mathbf{R}^n$ be a proximate retract and $\varphi : [0, a] \times K \to \mathbf{R}^n$ be a map strongly tangent to K. If $x : [0, a] \to \mathbf{R}^n$ is an absolutely continuous function such that $\dot{x}(t) \in \widetilde{\varphi}(t, x(t))$ for a.a. $t \in [0, a]$ and $x(0) \in K$ then $x(t) \in K$ for a.a. $t \in [0, a]$.*

To prove Lemma 2.1 we consider the absolutely continuous function $d : [0,a] \to \mathbf{R}$ defined by

$$d(t) = \operatorname{dist}(x(t), K).$$

Since $d(0) = 0$ and $d(t) \geq 0$, it is sufficient to show that d is nonincreasing. In fact, using (2.3) and (2.5), it is easy to show that $\dot{d}(t) \leq 0$ for a.a. $t \in [0,a]$ (see [6] for details).

We shall use also the notion of R_δ-set (comp. [1], [3] and [4])

Definition 2.3. *A compact nonempty space A is called an R_δ-set provided there exists a sequence $\{A_n\}$ of compact spaces such that:*

$$A_n \supset A_{n+1} \quad \textit{for every } n \tag{2.6}$$

$$A_n \textit{ is contractible} \quad \textit{for every } n \tag{2.7}$$

$$A = \bigcap_n A_n. \tag{2.8}$$

Observe that in particular any R_δ-set is a compact, nonempty and connected space. One can show that any R_δ-set is an acyclic space in the sense of an arbitrary continuous homology theory. Finally, let us add that the notion of R_δ-set was used, for the first time, in the theory of ordinary differential equations by N. Aronszajn in 1942 (compare [1], [3] and [4]).

We end this section by formulating the following:

Proposition 2.2. (cf. [3] and [9]). *The intersection of a decreasing sequence of R_δ-sets is an R_δ-set.*

3. Differential inclusions on proximate retracts

By K we will denote a proximate retract. The results presented in this Section are related to the theory developed in [3], [6] and [9].

Definition 3.1. *Let $\varphi : [0,a] \times K \to \mathbf{R}^n$ be a set-valued map with compact, nonempty values. We will say that φ is of* u.s.c. *type* (l.s.c.-*type*) *if the following conditions are satisfied:*

- (φ_1) $\varphi(t,\cdot) : K \to \mathbf{R}^n$ *is upper semicontinuous* (u.s.c.) *(resp. lower semicontinuous* (l.s.c.) *for a.a. $t \in [0,a]$).*
- (φ_2) $\varphi(\cdot,x) : [0,a] \to \mathbf{R}^n$ *is measurable for any $x \in K$.*
- (φ_3) φ *is integrably bounded, i.e. for any $\rho > 0$ there exists $h_\rho \in L^1([0,a],\mathbf{R}_+)$ such that $|y| \leq h_\rho(t)$ for a.a. $t \in [0,a]$, any $x \in K$ such that $|x| \leq \rho$ and $y \in \varphi(t,x)$.*

Note that if $\varphi = f$ is a single-valued map then the above two notions coincide and we will say that f is a Carathéodory map.

The following two propositions are immediate:

PROPOSITION 3.1. *If φ is, respectively, of the* u.s.c. *type or of the* l.s.c. *type, then so is $\widetilde{\varphi}$.*

PROPOSITION 3.2. *If φ satisfies (φ_3), then it is locally bounded, i.e. it maps bounded sets into bounded sets.*

For a given map $\varphi : [0,a] \times K \to \mathbf{R}^n$ of the u.s.c. type (l.s.c. type) and $x_0 \in K$ we will consider the following Cauchy problem

$$\begin{cases} \dot{x}(t) \in \varphi(t, x(t)), & t \in [0,a] \\ x(0) = x_0, \end{cases} \tag{3.1}$$

where the solution $x : [0,a] \to K$ is an absolutely continuous function such that $x(0) = x_0$, and $\dot{x}(t) \in \varphi(t, x(t))$ for a.a. $t \in [0,a]$.

We put

$$S(\varphi; x_0) = \{x : [0,a] \to K \ : \ x \text{ is a solution of } (3.1)\}.$$

We have the following (compare [3] and [6]):

THEOREM 3.1. *Let $\varphi : [0,a] \times K \to \mathbf{R}^n$ be strongly tangent to K. We have the following facts:*

(a) *If φ is of the* u.s.c. *type with convex values, then $S(\varphi; x_0)$ is an R_δ-set.*
(b) *If φ is of the* l.s.c. *type, then $S(\varphi, x_0) \neq \emptyset$.*

PROOF. Consider the extension $\widetilde{\varphi} : [0,a] \times \mathbf{R}^n \to \mathbf{R}^n$ defined in (2.5). Then by Lemma 2.1 we derive that

$$S(\varphi; x_0) = S(\widetilde{\varphi}; x_0) \qquad \text{for any } x_0 \in K.$$

Thus (a) follows from a result proved in [4]. On the other hand it is well known, in view of Proposition 3.1, that $S(\widetilde{\varphi}; x_0) \neq \emptyset$ (see [1]).

At this point the following question arises naturally: does Theorem 3.1 remain true if φ is only tangent to K?

We are able to give a positive answer only in the u.s.c. type case. In fact the following result holds.

THEOREM 3.2. *If $\varphi : [0,a] \times K \to \mathbf{R}^n$ is an* u.s.c. *type map with convex values and tangent to K, then $S(\varphi; x_0)$ is an R_δ-set.*

The main idea for the proof of Theorem 3.2 is to use the metric retraction in order to approximate, for any $\varepsilon > 0$, the map φ by a strongly tangent, convex-valued, u.s.c. type map $\varphi_\varepsilon : [0,a] \times U_\varepsilon \to \mathbf{R}^n$ in such a way that $S(\varphi; x_0) = \bigcap_{n\in\mathbf{N}} S(\varphi_{1/n}; x_0)$.

Here U_ε is an open neighbourhood of K. Finally, the application of Proposition 2.2 gives the result. All the details can be found in [6](see also [3] and [9]).

4. Feedback controls

In this Section $K \subset \mathbf{R}^n$ will denote a proximate retract and $U \subset \mathbf{R}^m$ a compact set. Let $f : [0,a] \times K \times U \to \mathbf{R}^n$ be a single-valued map.

We will say that f is tangent to K provided that

$$\forall\, x \in K \quad \forall\, t \in [0,a] \quad \exists\, u = u(t,x) \in U \;:\; f(t,x,u) \in T_K(x). \tag{4.1}$$

Following [1], [8] and [11] we introduce the concept of feedback control as a selector of a set-valued map. For this we give the following:

DEFINITION 4.1. *Let $f : [0,a] \times K \times U \to \mathbf{R}^n$ be a map tangent to K. We associate to f a set-valued map $C = C(f) : [0,a] \times K \to \mathbf{R}^m$, the so-called feedback control map, as follows*

$$C(t,x) = \{u \in U : f(t,x,u) \in T_K(x)\}. \tag{4.2}$$

We are interested in considering single-valued selectors of C. Therefore we let

$$\mathcal{C}(C(f)) = \{v : [0,a] \times K \to \mathbf{R}^m : v \text{ is continuous}$$
$$\text{and } v(t,x) \in C(t,x) \text{ for any } t \text{ and } x\}.$$

$$\mathcal{M}(C(f)) = \{u : [0,a] \times K \to \mathbf{R}^m : u \text{ is measurable}$$
$$\text{and } u(t,x) \in C(t,x) \text{ for any } t \text{ and } x\}.$$

Obviously, we want to know when

$$\mathcal{C}(C(f)) \neq \emptyset \quad \text{and} \quad \mathcal{M}(C(f)) \neq \emptyset.$$

In this regard, there are different possible results concerning $\mathcal{C}(C(f))$ and $\mathcal{M}(C(f))$. We will formulate in the sequel two of the most important results. First, observe that, in view of the Michael's Selection Theorem and Proposition 2.1, Theorem 3 in ([1], p.49) can be reformulated as follows:

THEOREM 4.1. *Let $f : [0,a] \times K \times U \to \mathbf{R}^n$ be a continuous single-valued map which is affine with respect to the last variable. Assume that*

$$\begin{aligned} &\exists\, \gamma > 0 : \forall\, (t,x) \in [0,a] \times K, \quad \forall\, (t',x') \in [0,a] \times K \\ &\exists\, u \in B \text{ such that from } |t-t'| < \gamma \text{ and } |x-x'| < \gamma \\ &\qquad \text{it follows that} f(t,x',u) \in T_K(x). \end{aligned} \tag{4.3}$$

Then $\mathcal{C}(C(f)) \neq \emptyset$.

The following example shows that Theorem 4.1 is false if we replace condition (4.3) by the weaker condition (4.1).

EXAMPLE 4.1. Let $f : [0,2] \times [0,1] \times [0,1] \to \mathbf{R}$ be the map defined as follows:

$$f(t,x,u) = \begin{cases} \max\{1-u, t\} - 1 & \text{for } t \leq 1 \\ \max\{u, 2-t\} - 1 & \text{for } t \geq 1. \end{cases}$$

Then f satisfies (4.1). Moreover, the feedback map $C = C(f)$ associated with f is defined as follows

$$C(t,x) = \begin{cases} \{0\} & \text{for } x = 0 \text{ and } t < 1 \\ \{1\} & \text{for } x = 0 \text{ and } t > 1 \\ [0,1] & \text{for } (x = 0 \text{ and } t = 1) \text{ or } x > 0. \end{cases}$$

Observe that $C = C(f)$ is convex, closed-valued, but it is not l.s.c.

Furthermore, from Corollary 1.Q ([10], p. 171), we derive the following:

THEOREM 4.2. *Assume that $f : [0,a] \times K \times U \to \mathbf{R}^n$ is a locally bounded (single-valued) map tangent to K. Assume further that f is measurable with respect to the first variable and continuous with respect to the pair (x,u). Then $\mathcal{M}(C(f)) \neq \emptyset$.*

Now, given a Carathéodory map $f : [0,a] \times K \times U \to \mathbf{R}^n$ tangent to K and given $v \in \mathcal{C}(C(f))$ and $u \in \mathcal{M}(C(f))$ we define the maps

$$f^v : [0,a] \times K \to \mathbf{R}^n \qquad \text{and} \qquad f_u : [0,a] \times K \to \mathbf{R}^n$$

as follows

$$f^v(t,x) = f(t,x,v(t,x)) \quad \text{for any } t \text{ and } x \tag{4.3bis}$$

$$f_u(t,x) = f(t,x,u(t,x)) \quad \text{for any } t \text{ and } x. \tag{4.4}$$

Clearly, the map f^v has the same regularity properties as f, but this is no longer true for f_u. Hence, if we consider the Cauchy problem

$$\begin{cases} \dot{x} = f_u(t,x) \\ x(0) = x_0, \end{cases}$$

we cannot expect, in general, that it possesses classical solutions, i.e. absolutely continuous solutions $x = x(t)$. This is the reason why we will consider in the sequel two regularizations of f_u, called Krasowskii's regularization and Filippov's regularization, respectively.

We define the multivalued maps (compare [1] and [7])

$$K(f_u) : [0,a] \times K \to \mathbf{R}^n, \qquad \text{Krasowskii's regularization of } f_u,$$
$$F(f_u) : [0,a] \times K \to \mathbf{R}^n, \qquad \text{Filippov's regularization of } f_u,$$

as follows:

$$K(f_u)(t,x) = \bigcap_{\varepsilon>0} \overline{\text{conv}} f_u(t, B(x,\varepsilon)), \tag{4.5}$$

$$F(f_u)(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} \overline{\text{conv}} f_u(t, B(x,\varepsilon) \setminus N), \tag{4.6}$$

where μ stands for the Lebesgue measure, $\overline{\text{conv}}$ for the closed, convex hull and $B(x,\varepsilon)$ for the open ball centered at x with radius ε.

We have the following (see [1], [7]):

THEOREM 4.3. (i) *If f_u is measurable and locally bounded, then $K(f_u)$ and $F(f_u)$) are compact, convex-valued maps, measurable in t and* u.s.c. *in x.*

(ii) *If f_u is tangent to K, then $K(f_u)$ is also tangent to K; while for any $t \in [0,a]$ we have*

$$F(f_u)(t,x) \cap T_K(x) \neq \emptyset \quad \textit{for a.a. } x \in K.$$

If we assume more conditions on the Bouligand map $T : K \to \mathbf{R}^n$ then we get a stronger version of (ii) for $F(f_u)$. In fact, we have the following:

PROPOSITION 4.1. *Assume that all the assumptions of Theorem 4.3 are satisfied. Assume further that the Bouligand map $T : K \to \mathbf{R}^n$ has a closed graph. Then $F(f_u)$ is tangent to K.*

PROOF. Assume to the contrary that

$$F(f_u)(t_0,x_0) \cap T_K(x_0) = \emptyset$$

for some $(t_0,x_0) \in [0,a] \times K$. By (ii) we can choose a sequence $\{x_n\} \subset K$ such that

$$\lim_{n\to\infty} x_n = x_0 \quad \text{and} \quad F(f_u)(t_0,x_n) \cap T_K(x_n) \neq \emptyset$$

for any $n \in \mathbf{N}$. Let $y_n \in F(f_u)(t_0,x_n) \cap T_K(x_n)$, $n \in \mathbf{N}$; since f_u is locally bounded from (4.6) we get that $F(f_u)$ is also locally bounded and thus we can assume without loss of generality, that $\lim_{n\to\infty} y_n = y_0$.

Now, by using (i) and the fact that the graph of T is closed, we obtain

$$y_0 \in F(f_u)(t_0, x_0) \cap T_K(x_0)$$

and this is a contradiction.

5. Control problems on proximate retracts.

Let $K \subset \mathbf{R}^n$ be a proximate retract, U be a compact subset of $\mathbf{R}^m$ and $f : [0,a] \times K \times U \to \mathbf{R}^n$ be a (single-valued) map satisfying the following conditions:

(f_1) $f(\cdot, x, u) : [0,a] \to \mathbf{R}^n$ is measurable for any $(x,u) \in K \times U$,

(f_2) $f(t,\cdot,\cdot) : K \times U \to \mathbf{R}^n$ is continuous for a.a. $t \in [0,a]$;

(f_3) for every $\rho > 0$ there exists $h_\rho \in L^1([0,a], \mathbf{R}_+)$ such that $|f(t,x,u)| \leq h_\rho(t)$ for a.a. $t \in [0,a]$ and any (x,u) with $|x| \leq \rho$ and $u \in U$.

We will refer to f as a Carathéodory map.

In this Section we solve the following:

Control problem. Find a feedback control $w = w(t,x)$ in such a way that the control system

$$\begin{cases} \dot{x} = f(t,x,w(t,x)) \\ x(0) = x_0 \in K \end{cases} \tag{5.1}$$

has a viable solution $x = x(t)$, i.e. $x(t) \in K$ for any $t \in [0,a]$.

On the other hand we have already seen that, depending on the regularity of the feedback control $w = w(t,x)$, the right-hand side of (5.1) can be either single-valued or set-valued. Therefore we have to distinguish two cases in order to solve the proposed problem. The first case can be formulated as follows:

I. Assume that $\mathcal{C}(C(f)) \neq \emptyset$. Does the control system (5.1), corresponding to a feedback control $v \in \mathcal{C}(C(f))$, admit a viable solution?

The positive answer is based on the properties of the solution map $S(f; x_0, v)$ given by

$$S(f; x_0, v) = \{x : [0,a] \to K : x \text{ is a solution of } (5.1)\}. \tag{5.2}$$

Indeed we can prove the following:

THEOREM 5.1. *Assume that $f : [0,a] \times K \times U \to \mathbf{R}^n$ is a Carathéodory map tangent to K. Then for every $v \in \mathcal{C}(C(f))$ the set $S(f; x_0, v)$ is an R_δ-set and so, in particular, it is nonempty.*

PROOF. Let $v \in \mathcal{C}(C(f))$. Consider the map $f^v : [0,a] \times K \to \mathbf{R}^n$ defined in (4.3bis), we have that $S(f; x_0, v) = S(f^v; x_0)$. On the other hand, $f^v : [0,a] \times K \to \mathbf{R}^n$ satisfies all the assumptions of Theorem 3.1. This concludes the proof.

Observe that the assumptions of Theorem 5.1, in general do not guarantee that the set $\mathcal{C}(C(f))$ is nonempty (compare Theorem 4.1 and Example 4.1). Therefore we are led to consider the second case.

II. Assume that $\mathcal{M}(C(f)) \neq \emptyset$. Does the control system (5.1), corresponding to a feedback control $u \in \mathcal{M}(C(f))$, admit a viable solution?

First of all observe that, under the assumptions of Theorem 5.1, we have that $\mathcal{M}(C(f)) \neq \emptyset$ by Theorem 4.2. Furthermore, as already noticed, in this case we have to consider solutions of (5.1) in a generalized sense. That is, we consider the following two Cauchy problems for differential inclusions:

$$\begin{cases} \dot{x} \in K(f_u)(t,x) \\ x(0) = x_0 \in K, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \dot{x} \in F(f_u)(t,x) \\ x(0) = x_0 \in K. \end{cases} \tag{5.4}$$

By Krasowskii's (Filippov's) solution of (5.1) we mean any solution of (5.3) (resp. (5.4)).

For a given $u \in \mathcal{M}(C(f))$ we let

$$S_K(f; x_0, u) = S(K(f_u); x_0)$$

and

$$S_F(f; x_0, u) = S(F(f_u); x_0);$$

thus $S_K(f; x_0, u)$ (resp. $S_F(f; x_0, u)$) is the set of all Krasowskii (Filippov) solutions of (5.1).

Obviously, we have that

$$S_F(f; x_0, u) \subset S_K(f; x_0, u).$$

We have the following result:

THEOREM 5.2. *Assume that $f : [0,a] \times K \times U \to \mathbf{R}^n$ is a Carathéodory map tangent to K. Then $\mathcal{M}(C(f)) \neq \emptyset$ and for every $u \in \mathcal{M}(C(f))$ the set $S_K(f; x_0, u)$ is an R_δ-set, and so, in particular, it is nonempty.*

PROOF. The fact that $\mathcal{M}(C(f)) \neq \emptyset$ follows immediately from Theorem 4.2. Let $u \in \mathcal{M}(C(f))$. Consider $f_u : [0,a] \times K \to \mathbf{R}^n$ as defined in (4.4) and its Krasowskii's regularization $K(f_u) : [0,a] \times K \to \mathbf{R}^n$.

By Theorem 4.3 one can easily show that $K(f_u)$ is tangent to K and of u.s.c. type. Hence, we can apply Theorem 3.2 to $K(f_u)$ and thus we infer that $S(K(f_u);x_0)$ is an R_δ-set. Since

$$S_K(f;x_0,u) = S(K(f_u);x_0)$$

the proof is completed.

Finally, we are in the position to prove the following:

THEOREM 5.3. *Assume that* $f : [0,a] \times K \times U \to \mathbf{R}^n$ *is a Carathéodory map tangent to* K. *Assume further that the Bouligand map* $T : K \to \mathbf{R}^n$ *has closed graph. Then* $\mathcal{M}(C(f)) \neq \emptyset$ *and for every* $u \in \mathcal{M}(C(f))$ *the set* $S_F(f;x_0,u)$ *is an* R_δ*-set, and so, in particular, it is nonempty.*

PROOF. The proof of Theorem 5.3 is analogous to that of Theorem 5.2. In fact, to obtain that $F(f_u)$ is tangent to K we use Proposition 4.1 instead of Theorem 4.3.

REMARK 5.1. It is still an open problem to prove Theorem 5.3 without the assumption on the closedness of the graph of T.

6. Periodic control problems

In this Section we will assume that $K \subset \mathbf{R}^n$ is a compact proximate retract, $U \subset \mathbf{R}^m$ a compact set and $F : [0,a] \times K \times U \to \mathbf{R}^n$ a Carathéodory map tangent to K. Then we will consider the following periodic control problem

$$\begin{cases} \dot{x} = f(t,x,v(t,x)) \\ x(0) = x(a) \in K \end{cases} \tag{6.1}$$

and we will ask as in the previous Section, for classical, Krasowskii and Filippov viable solutions of (6.1).

We can state the following:

THEOREM 6.1. *Assume that the Euler characteristic* $\chi(K)$ *of* K *is different from zero. We have the following:*

(i) *If all the assumptions of Theorem* 5.1 *are satisfied for every* $v \in C(C(f))$, *then there exists a solution of* (6.1).
(ii) *If all the assumptions of Theorem* 5.2 *are satisfied for every* $u \in \mathcal{M}(C(f))$, *then there exists a Krasowskii solution of* (6.1).
(iii) *If all the assumptions of Theorem* 5.3 *are satisfied for every* $u \in \mathcal{M}(C(f))$, *then there exists a Filippov solution of* (6.1).

For the proof of Theorem 6.1 we need two preliminary lemmas. For this, let $C([0,a],\mathbf{R}^n)$ be the space of continuous functions in the usual supremum norm. For $K \subset \mathbf{R}^n$ we let

$$C([0,a],K) = \{x \in C([0,a],\mathbf{R}^n) \ : \ x(t) \in K \text{ for all } t \in [0,a]\}.$$

We shall consider also the evaluation map

$$e : C([0,a],K) \times [0,1] \to K$$

defined as follows

$$e(x,\lambda) = x(\lambda a) \quad \text{for every } x \in C([0,a],K).$$

The following fact is well known (see [1], (compare also [3] and [9])).

Lemma 6.1. *Let $\varphi : [0,a] \times K \to \mathbf{R}^n$ be an* u.s.c. *type map tangent to the proximate retract K. Let $P : K \to C([0,a],K)$ be defined as follows:*

$$P(x) = S(\varphi;x) \quad \textit{for every } x \in K.$$

Then P is an u.s.c. *map with R_δ-values.*

The following lemma follows from the Lefschetz fixed point theorem for set-valued mappings so-called admissible (for details see [3] and the references therein).

Lemma 6.2. *Let K be a compact proximate retract with $\chi(K) \neq 0$. Assume that $\psi : K \times [0,1] \to K$ is a set-valued map such that*

(i) *there exists a metric space X and two* u.s.c. *mappings $\psi_1 : K \times [0,1] \to X$ and $\psi_2 : X \to K$ with R_δ-values such that $\psi = \psi_2 \circ \psi_1$;*
(ii) *the map $\psi(\cdot,0)$ is the identity on K, i.e. $\psi(\cdot,0) = \mathrm{Id}_K$.*

Then the map $\psi(\cdot,1) : K \to K$ has a fixed point.

Now we are able to prove Theorem 6.1.

Proof of Theorem 6.1. We define the following set-valued maps

$$\begin{aligned}
P &: K \to C([0,a],K); & P(x) &= S(f^v;x) & &\text{for } v \in \mathcal{C}(C(f)).\\
P_K &: K \to C([0,a],K); & P_K(x) &= S(K(f_u);x) & &\text{for } u \in \mathcal{M}(C(f)).\\
P_F &: K \to C([0,a],K); & P_F(x) &= S(F(f_u);x) & &\text{for } u \in \mathcal{M}(C(f)).
\end{aligned}$$

Then from Theorems 4.3, 5.1, 5.2, 5.3 and Lemma 6.1 we obtain that P, P_K and P_F are u.s.c. maps with R_δ-values.

Observe that the single-valued evaluation map

$$e : C([0,a],K) \times [0,1] \to K; \qquad e(x,\lambda) = x(\lambda a)$$

is continuous. We define

$$\psi, \psi_K, \psi_F : K \times [0,1] \to K$$

as follows:

$$\begin{aligned} \psi(x,\lambda) &= e(P(x),\lambda) \\ \psi_K(x,\lambda) &= e(P_K(x),\lambda) \\ \psi_F(x,\lambda) &= e(P_F(x),\lambda), \end{aligned}$$

for every $x \in K$ and $\lambda \in [0,1]$.

We have the following three commutative diagrams:

$$\begin{tikzcd} K \times [0,1] \arrow[rr, "\widetilde{P}_F, \widetilde{P}_K, \widetilde{P}"] \arrow[d, "\psi_F, \psi_K, \psi"'] & & C([0,a],K) \times [0,1] \arrow[dll, "e"] \\ K & & \end{tikzcd}$$

where $\widetilde{P}(x,\lambda) = (P(x),\lambda)$, $\widetilde{P}_K(x,\lambda) = (P_K(x),\lambda)$, $\widetilde{P}_F(x,\lambda) = (P_F(x),\lambda)$; i.e.,

$$\psi = e \circ \widetilde{P}, \quad \psi_K = e \circ \widetilde{P}_K, \quad \text{and} \quad \psi_F = e \circ \widetilde{P}_F.$$

Now observe that $\psi(\cdot,0) = \psi_K(\cdot,0) = \psi_F(\cdot,0) = \mathrm{Id}_K$, $\psi(\cdot,1) = P$, $\psi_K(\cdot,1) = P_K$ and $\psi_F(\cdot,1) = P_F$.

So, in view of Lemma 6.2, there are $x_0, x_1, x_2 \in K$ such that $x_0 \in P(x_0)$, $x_1 \in P_K(x_1)$ and $x_2 \in P_F(x_2)$. It means that there are solutions $y_0 \in S(f^v; x_0)$, $y_1 \in S(K(f_u); x_1)$ and $y_2 \in S(F(f_u); x_2)$ such that

$$y_0(a) = x_0, \quad y_1(a) = x_1, \quad \text{and} \quad y_2(a) = x_2.$$

Finally, we have

$$\begin{aligned} y_0(0) &= y_0(a) = x_0, \\ y_1(0) &= y_1(a) = x_1, \\ y_2(0) &= y_2(a) = x_2 \end{aligned}$$

and the proof of Theorem 6.1 is completed.

7. Control problems for implicit equations

In this Section we will show that the method for solving implicit differential equations developed in [2] can be adapted to deal with implicit control problems. Given a function

$$f : [0,a] \times K \times \mathbf{R}^n \times U \to \mathbf{R}^n,$$

where K is a proximate retract and $U \subset \mathbf{R}^m$ is a compact set, we define a set-valued function $\sigma : [0,a] \times K \times U \to \mathbf{R}^n$ associated with f, as follows:

$$\sigma(t,x,u) = \{y \in \mathbf{R}^n : y = f(t,x,y,u)\} = \operatorname{Fix} f(t,x,\cdot,u). \tag{7.1}$$

Obviously, the control problem

$$\dot{x} = f(t,x(t),\dot{x}(t),v(t,x)) \tag{7.2}$$

can be rewritten in the following form:

$$\dot{x}(t) \in \sigma(t,x(t),v(t,x)). \tag{7.3}$$

However, since σ may have empty values it is almost never of l.s.c. type and rarely it is of u.s.c. type. We will present below some situations where the results of Section 5 can be applied by means of more sophisticated arguments.

PROPOSITION 7.1. (i) *If $f : [0,a] \times K \times \mathbf{R}^n \times U \to \mathbf{R}^n$ is contractive with respect to the third variable, i.e., there is $k \in [0,1]$ such that for every $t \in [0,a]$, $x \in K$, $y,y' \in K$, $u \in U$ we have*

$$|f(t,x,y,u) - f(t,x,y',u)| \le k|y-y'|,$$

then $\sigma : [0,a] \times K \times U \to \mathbf{R}^n$ is a single-valued map.

(ii) *If $f : [0,a] \times K \times \mathbf{R}^n \times U \to \mathbf{R}^n$ is a bounded map such that there exists $k \in [0,1]$ for which*

$$\langle f(t,x,y,u) - f(t,x,y',u)\rangle \le k|y-y'|^2$$

for every t,x,y,y',u and $< \cdot,\cdot >$ stands for the inner product in $\mathbf{R}^n$, then $\sigma : [0,a] \times K \times U \to \mathbf{R}^n$ is a single-valued map.

PROOF. Part (i) follows immediately from the Banach contraction principle. To prove (ii) first observe that, in view of Schauder's fixed point theorem, the set $\operatorname{Fix} f(t,x,\cdot,u)$ is nonempty for every t,x,u.

Assume now that for given t,x,u we have $y, y_1 \in \operatorname{Fix} f(t,x,\cdot,u)$. Then

$$\langle f(t,x,y,u) - f(t,x,y_1,u), y-y_1\rangle = \langle y-y_1, y-y_1\rangle \le k|y-y_1|^2,$$

so we get

$$|y-y_1|^2 \le |y-y_1|^2$$

and this implies that $y = y_1$. The proof is completed.

PROPOSITION 7.2. *Assume that* $f : [0,a] \times K \times \mathbf{R}^n \times U \to \mathbf{R}^n$ *is continuous. We have that*

(i) *If* f *is nonexpansive in* y, *i.e.*

$$|f(t,x,y,u) - f(t,x,y_1,u)| \leq |y - y_1|$$

for any $(t,x,u) \in [0,a] \times K \times U$, *then* $\sigma : [0,a] \times K \times U \to \mathbf{R}^n$ *is* u.s.c. *with convex values.*

(ii) *If* f *is bounded and for every* (t,x,u) *we have that*

$$\dim \operatorname{Fix} f(t,x,\cdot,u) = 0,$$

where dim *stands for the topological covering dimension, then* σ *has a* l.s.c. *selector* J *with compact values.*

Since $f(t,x,\cdot,u)$ is nonexpansive on $\mathbf{R}^n$, the set of fixed points is convex and so the proof of (i) is straightforward. For the proof of (ii) we refer to [2]. Observe that under the assumption that K and U are C^1-manifolds, one can show that the set of all continuous maps satisfying (ii) is dense in the set of all continuous maps from $[0,a] \times K \times \mathbf{R}^n \times U$ into $\mathbf{R}^n$ (see [2]).

Finally, Proposition 7.1 and 7.2 can be now applied to the control problems considered in Section 5 (compare [6]).

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