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GLOBAL CONTINUATION FOR BOUNDED SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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(*Submitted by J. Mawhin*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

The concepts and results of the Leray–Schauder degree and index theories have been very effectively applied to prove global continuation theorems for the existence of solutions to nonlinear equations in a Banach space. Indeed, the Leray–Schauder continuation theorem itself is the premier result of this type. In this paper we use the homotopy index of Conley to prove a global continuation theorem for the existence of full bounded solutions of ordinary differential equations. This extends a small parameter result originally proven in [13]. We study parameter dependent families of ordinary differential equations of the form

$$\frac{dx}{dt} = \mu F(x, t, \mu) \tag{1}$$

where F is a continuous function of $(x, t, \mu) \in \mathbf{E} = \mathcal{D} \times \mathbf{R} \times [0, 1]$, $\mathcal{D} \subset \mathbf{R}^m$ is an open set and $\mu \in [0, 1]$ is a parameter. By a (full) bounded solution of (1) we mean

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a solution $x = x(t)$ satisfying (1) for all $t \in \mathbf{R}$ and such that

$$\|x\| := \sup_{t\in\mathbf{R}} |x(t)| < \infty.$$

Our main result is analogous to one of Mawhin for periodic solutions. Mawhin's result is intrinsically related to degree theory. Suppose there is a number $\omega > 0$ such that $F(x, t + \omega, \mu) = F(x, t, \mu)$ for all $(x, t, \mu) \in \mathbf{E}$, and let

$$F_0(x) = \frac{1}{\omega} \int_0^{\omega} F(x, t, 0) dt$$

denote the average of $F(x, t, 0)$.

THEOREM 1 (Mawhin). *Suppose:* (i) *There is a bounded open set Ω with $\overline{\Omega} \subset \mathcal{D}$ such that if $x = x(t)$ is an ω–periodic solution of* (1) *for some $0 < \mu < 1$ satisfying $x(t) \in \overline{\Omega}$ for all t, then $x(t) \in \Omega$ for all t.* (ii) *Suppose also that $F_0(x) \neq 0$ for all $x \in \partial\Omega$ and the Brouwer degree $d_B(F_0, \Omega, 0) \neq 0$.*

Then for each $\mu \in [0, 1)$, (1) *has an ω-periodic solution with values in Ω, and at $\mu = 1$ there is an ω-periodic solution with values in $\overline{\Omega}$.*

For a proof of this (and many other, and more general, results) see [7], or [8].

Herein we study the existence of bounded solutions to (1) under the assumption that $F(x, t, \mu)$ is almost periodic in $t \in \mathbf{R}$. Condition (i) in Mawhin's result is assumed to hold for all full bounded solutions with values in $\overline{\Omega}$. Condition (ii) is altered to the assumption that $\overline{\Omega}$ is an isolating neighborhood for the flow generated by the (appropriately defined) averaged equation

$$\frac{dx}{dt} = F_0(x) \tag{2}$$

and that the maximal invariant set in $\overline{\Omega}$ has non-trivial Conley (homotopy) index. Proofs are based upon applying Conley index theory to a family of skew-product flows associated with (1). Notice that the Conley index cannot be applied directly to nonautonomous differential equations, since the solutions of such equations do not define a flow (dynamical system) on the space of initial values; see [11]. For this reason we use the associated skew product flows. For the properties of the homotopy index we refer the reader to [2] or [9], and for skew product flows to [10] or [5]; the use of these in the present context is discussed in [12] and [13]. Our main result is proven in Section 2, and in Section 3 it is applied to prove the existence of bounded solutions to differential equations having *bound sets.*

We will make use of a weak topology on our families of differential equations. This topology was studied by Artstein [1] in much greater depth and generality than will be needed here. Let $\mathcal{D} \subset \mathbf{R}^m$ be an open set and $f : \mathcal{D} \times \mathbf{R} \to \mathbf{R}^m$. Suppose f satisfies the Carathéodory conditions: for each $x \in \mathcal{D}$ the function $t \to f(x,t)$ is Lebesgue measurable, and for almost all $t \in \mathbf{R}$ (in the sense of Lebesgue measure) the function $x \to f(x,t)$ is continuous on $\mathcal{D}$. Suppose

(C1) *For every compact set $A \subset \mathcal{D}$ there exist two locally L^1 functions $m_A(t)$ and $k_A(t)$ such that if $x, y \in A$ and $t \in \mathbf{R}$ then:*
  (1) $|f(x,t)| \leq m_A(t)$,
  (2) $|f(x,t) - f(y,t)| \leq k_A(t)|x-y|$,
  (3) *for every $\varepsilon > 0$ there exists a $\mu = \mu_A(\varepsilon) > 0$ such that if $E \subset \mathbf{R}$ is measurable, contained in an interval $[t, t+1]$, and with measure less than μ then $\int_E m_A(t)dt \leq \varepsilon$, and*
  (4) *there exists a number N_A such that $\int_t^{t+1} k_A(s)ds \leq N_A$ for all $t \in \mathbf{R}$.*

Given a function f satisfying (C1) one can define an associated set of functions $\mathcal{G}$ on $\mathcal{D} \times \mathbf{R}$ that contains the time translates of f, defined for $\tau \in \mathbf{R}$ by $f_\tau(x,t) = f(x, \tau + t)$ for all $(x,t) \in \mathcal{D} \times \mathbf{R}$. With an appropriate topology on $\mathcal{G}$, one can then define skew-product flows on $\mathbf{R}^m \times \mathcal{G}$.

DEFINITION 2. *Let f satisfy* (C1) *and for every compact set $A \subset \mathcal{D}$ and $\varepsilon > 0$ let N_A and $\mu_A(\varepsilon)$ be given by* (C1). *The family $\mathcal{G} = \mathcal{G}(f)$ consists of all Carathéodory functions $g : \mathcal{D} \times \mathbf{R} \to \mathbf{R}^m$ satisfying: For every compact $A \subset \mathcal{D}$ there exist two locally L^1 functions $M_{A,g}$ and $K_{A,g}$ such that if $x, y \in A$ and $t \in \mathbf{R}$ then*

(1) $|g(x,t)| \leq M_{A,g}(t)$,
(2) $|g(x,t) - g(y,t)| \leq K_{A,g}(t)|x-y|$, *and the functions $M_{A,g}$ and $K_{A,g}$ satisfy:*
(3) *if $E \subset [t, t+1]$ and the Lebesgue measure of E is less than $\mu_A(\varepsilon)$ then $\int_E M_{A,g}(s)ds \leq \varepsilon$ and*
(4) $\int_t^{t+1} K_{A,g}(s)ds \leq N_A$ *for all $t \in \mathbf{R}$.*

If $g \in \mathcal{G}(f)$ then so is g_τ for any $\tau \in \mathbf{R}$. Moreover, for each $x_0 \in \mathcal{D}$ the initial value problem

$$\frac{dx}{dt} = g(x,t), \quad x(0) = x_0$$

has a unique solution $x(t; x_0, g)$ defined on a maximal interval of existence $I(x_0, g) = (\alpha(x_0, g), \beta(x_0, g))$. Artstein in [1] gives the space $\mathcal{G}$ a weak metrizable topology which we will impose. This is given by

DEFINITION 3. *Let $\{g_k\}$ be a sequence in $\mathcal{G}$. We say $\{g_k\}$ converges (weakly) to $g \in \mathcal{G}$ provided for every $x \in \mathcal{D}$ and $t \in \mathbf{R}$ the sequence $\{\int_0^t g_k(x,s)ds\}$ converges in $\mathbf{R}^m$ to $\int_0^t g(x,s)ds$.*

Convergence in $\mathcal{G}$ is induced by a metric, which is explicitly given in [1]. The topological space $\mathcal{G}$ is compact and is closed under time translations. A local flow π can be defined on $\mathcal{D} \times \mathcal{G}$ by $\pi(x_0, g, t) = (x(t; x_0, g), g_t)$ for $t \in I(x_0, g)$. The flow (dynamical system) π is a *skew product flow* (see [10] or [5]).

We are not interested in flows on all of $\mathcal{D} \times \mathcal{G}$ however. Instead we take the closure in $\mathcal{G}$ of the time translates of f, the (weak) hull of f, which we denote by $H_w(f)$. Since $\mathcal{G}$ is compact and $H_w(f)$ is closed, it follows that $H_w(f)$ is compact. The flow π is locally invariant on $\mathcal{D} \times H_w(f)$ and we restrict our attention entirely to this local flow, which we also denote by π. If $f(x,t)$ is a continuous and uniformly almost periodic in $t \in \mathbf{R}$ then the usual hull of f is the closure of the time translates of f in the topology of uniform convergence on compact sets. We will call this the strong hull of f, and denote it by $H_s(f)$. Now if $\{f_{t_n}\}$ is a sequence of translates of the uniformly almost periodic function $f(x,t)$ converging in the weak topology to g, every subsequence of that sequence has in turn a subsequence converging uniformly on compact sets to some h in the strong hull; thus this subsequence converges to h in the weak topology also. Hence for each $x \in \mathcal{D}$, $g(x,t) = h(x,t)$ a.e., and in fact $\{f_{t_n}\}$ converges to h in the strong topology. Thus in the uniform almost periodic case the topologies are equivalent if we identify the equivalence classes of functions in $H_w(f)$ with their continuous representatives. Moreover, uniform convergence on compact sets of a sequence of uniformly almost periodic functions is equivalent to its uniform convergence on sets of the form $K \times \mathbf{R}$, $K \subset \mathcal{D}$ compact [3]. Thus our weak topology is equivalent in this case to this latter topology . Nevertheless, the weak topology will be useful to us in the study of parameter dependence. Since the hull of f is essentially independent of these topologies, we will simply denote it by $H(f)$. Recall that if $g \in H(f)$ then g is also uniformly almost periodic and $f \in H(g)$. The space $\mathcal{D} \times H(f)$ is a locally compact metric space in any of these topologies.

We will apply the following abstract continuation result regarding the Conley index; see [2] or [9].

THEOREM 4. *Let $\mathcal{M}$ be a locally compact metric space and suppose for each $\mu \in [0,1]$ that π_μ is a local flow on $\mathcal{M}$. Suppose:* (a) *The map $\mu \to \pi_\mu$ is continuous in the sense that if $\{\mu_n\} \subset [0,1]$, $\{x_n\} \subset \mathcal{M}$, and $\{t_n\} \subset \mathbf{R}$ are sequences with $\mu_n \to \mu$, $x_n \to x$, $t_n \to t$, and $\pi_\mu(x,t)$ is defined, then $\pi_{\mu_n}(x_n, t_n)$ is defined* *for all large n and $\pi_{\mu_n}(x_n, t_n) \to \pi_\mu(x,t)$ as $n \to \infty$.* (b) *There is a compact set N in $\mathcal{M}$ such that, for each $\mu \in [0,1]$, N is an isolating neighborhood for π_μ. Let $I(\mu) = \{x \in N : \pi_\mu(x,t) \in N$ for all $t \in \mathbf{R}\}$.*

Then the Conley (homotopy) index $h(\pi_\mu, I(\mu))$ is defined and its value is independent of $\mu \in [0,1]$.

Recall that the homotopy index is a homotopy class of compact pointed spaces, and that if the index of a compact isolated invariant set I_0 is not the homotopy class of $\overline{0}$, the one–point pointed space, then $I_0 \neq \emptyset$.

2. A Global Continuation Theorem

Let $F : \mathbf{E} = \mathcal{D} \times \mathbf{R} \times [0,1] \to \mathbf{R}^m$ be continuous, $\mathcal{D} \subset \mathbf{R}^m$ open, and consider the parameterized family of differential equations

$$\frac{dx}{dt} = \mu F(x,t,\mu). \tag{3}$$

We will always suppose that $F(x,t,\mu)$ is locally Lipschitz continuous in x, uniformly in t and μ. In particular, initial value problems for (3) have unique solutions, and $F(\cdot,\cdot,\mu)$ satisfies (C1) for each $\mu \in [0,1]$. We also assume that $F(x,t,\mu)$ is almost periodic in $t \in \mathbf{R}$, uniformly with respect to x and μ in compact sets, and that for each compact subset $K \subset \mathcal{D}$, $F(x,t,\mu)$ is continuous in $\mu \in [0,1]$, uniformly with respect to $(x,t) \in K \times \mathbf{R}$. Thus, given $\varepsilon > 0$ and compact $K \subset \mathcal{D}$ there exists $\delta > 0$ such that $|F(x,t,\mu_1) - F(x,t,\mu_2)| < \varepsilon$ whenever $(x,t) \in K \times \mathbf{R}$ and $|\mu_1 - \mu_2| < \delta$. We let $H(\mu)$ denote $H(F(\cdot,\cdot,\mu))$ and $F_0(x)$ denote the average of $F(x,t,0)$, so that

$$F_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(x,t,0)dt.$$

We relate (3) to the averaged equation

$$\frac{dx}{dt} = F_0(x). \tag{4}$$

It follows from our assumptions that initial value problems for (4) have unique solutions. We will represent a generic $g \in H(\mu)$ as $F^*(x,t;\mu)$. Consider the family of differential equations

$$\frac{dx}{dt} = \mu F^*(x,t;\mu) \tag{5}$$

for $F^* \in H(\mu)$.

We can now state

THEOREM 5. *Suppose:* (i) *There is a bounded open set* Ω *with* $\overline{\Omega} \subset \mathcal{D}$ *such that if* $x = x(t)$ *is a solution of* (5) *for some* $F^* \in H(\mu)$, $0 < \mu < 1$, *with* $x(t) \in \overline{\Omega}$ *for all* $t \in \mathbf{R}$ *then* $x(t) \in \Omega$ *for all* $t \in \mathbf{R}$. (ii) *If* $x = x(t)$ *is a solution of* (4) *with* $x(t) \in \overline{\Omega}$ *for all* $t \in \mathbf{R}$ *then* $x(t) \in \Omega$ *for all* $t \in \mathbf{R}$. *Moreover, if* I_0 *denotes the maximal invariant set in* Ω *for* (4) *then the Conley index* $h(I_0) \neq \overline{0}$.

Then for each $0 < \mu < 1$, (3) *has a full bounded solution with values in* Ω, *and at* $\mu = 1$ (3) *has a full bounded solution with values in* $\overline{\Omega}$.

PROOF. There are two parts to the proof. In the first part, we show that there is a $\mu_0 > 0$ such that for each $0 < \mu < \mu_0$, a skew product flow associated with (3) has a non-empty invariant set of non-trivial homotopy index. In the second part we show that these invariant sets of nontrivial indices can be continued for $0 < \mu < 1$. We can then obtain from these sets bounded solutions to (3) lying in Ω for each $0 < \mu < 1$, and a solution in $\overline{\Omega}$ when $\mu = 1$.

Part 1: The proof for small $\mu > 0$ was first given in [13]. For coherence we will include the proof here.

First notice that if $x = x(t)$ is a solution of (1) for some $\mu > 0$ then $y(t) = x(t/\mu)$ is a solution of

$$\frac{dy}{dt} = F(y, t/\mu, \mu), \tag{6}$$

and $x(t) \in \Omega$ for all $t \in \mathbf{R}$ if and only if $y(t) \in \Omega$ for all $t \in \mathbf{R}$. We will homotopy (6) for fixed small $\mu > 0$ to the equation averaged at $\mu = 0$. Let $0 < \mu < 1$ and consider the homotopy

$$\frac{dy}{dt} = (1-\lambda)F_0(y) + \lambda F^*(y, t\mu^{-1}; \mu) \tag{7}$$

for $F^* \in H(\mu) := H(F(\cdot,\cdot,\mu))$. Now $\overline{\Omega}$ is an isolating neighborhood for I_0 in the flow β generated by (4). We claim that there is a value $\mu_0 > 0$ such that, for each $0 < \mu \leq \mu_0$ each member of the family of equations (7) has $\overline{\Omega}$ as an isolating neighborhood. That is, if for some $\lambda \in [0,1]$ and $F^* \in H(\mu)$, $y = y(t)$ is a solution of (7) with $y(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}$ then in fact $y(t) \in \Omega$ for all $t \in \mathbf{R}$. If this is not the case, then for each positive integer $n \in N$ there is a function $y_n = y_n(t)$ with $y_n(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}$, and numbers $\lambda_n \in [0,1]$, $\mu_n \in (0, n^{-1}]$, $s_n \in \mathbf{R}$, and an $F_n \in H(\mu_n)$, such that for all t, $y_n(t)$ satisfies the differential equation

$$\frac{dy_n}{dt} = (1-\lambda_n)F_0(y_n) + \lambda_n F_n(y_n, t\mu_n^{-1}; \mu_n) \tag{8}$$

and $y_n(s_n) \in \partial\Omega$. Let $z_n(t) = y_n(t+s_n)$; then z_n satisfies (8) translated by s_n. Now both z_n and dz_n/dt are uniformly bounded on $\mathbf{R}$ independently of $n \in \mathbf{N}$, so there is a subsequence of $\{z_n\}$ uniformly convergent on compact subsets of $\mathbf{R}$ to some $z = z(t)$ with $z(t) \in \overline{\Omega}$ and $z(0) \in \partial\Omega$. We can assume $\lambda_n \to \lambda_0 \in [0,1]$. Now the key here is that the sequence of functions $\{F_n(\cdot, \cdot\mu_n^{-1}; \mu_n)\}$ converges in our weak topology to $F_0(\cdot)$, ([13], Lemma 4.1). From all of this it follows that z satisfies

$$\frac{dz}{dt} = (1-\lambda_0)F_0(z(t)) + \lambda_0 F_0(z(t)) = F_0(z(t))$$

for all $t \in \mathbf{R}$. Since $z(0) \in \partial\Omega$ and $z(t) \in \overline{\Omega}$ for all t, this contradicts the hypothesis that $\overline{\Omega}$ is an isolating neighborhood for (4), and proves the existence of $0 < \mu_0 \leq 1$ satisfying the claim.

We now return to the equation with $x = x(t) = y(\mu t)$, for fixed $0 < \mu \leq \mu_0$ and consider the homotopy

$$\frac{dx}{dt} = (1-\lambda)\mu F_0(x) + \lambda\mu F^*(x,t;\mu) \tag{9}$$

for $\lambda \in [0,1]$ and $F^* \in H(\mu)$. We define a family of skew product flows π_λ on $\mathcal{D} \times H(\mu)$ associated with (9) by

$$\pi_\lambda(x_0, F^*, t) = (x(t; x_0, \lambda\mu F^*), F_t^*)$$

where $x(t; x_0, \lambda\mu F^*)$ denotes the solution to (9) with $x(0) = x_0$. It follows, essentially from results of Artstein [1], that the family of flows $\{\pi_\lambda\}$ is continuous in the sense of Theorem 1. Moreover we have just shown that $\overline{\Omega} \times H(\mu)$ is a compact isolating neighborhood for each π_λ, $\lambda \in [0,1]$. Thus the Conley index of the maximal invariant set in $\overline{\Omega} \times H(\mu)$ for π_λ is independent of $\lambda \in [0,1]$. Let $I(0)$ denote this set for π_0. Now in this case, $\lambda = 0$ so (9) becomes the equation

$$\frac{dx}{dt} = \mu F_0(x). \tag{10}$$

The flow determined by (10) is the same as that of (4) except for a change of independent variable $t \to \mu^{-1}t$. We denote this flow by β_μ. By hypothesis, $h(\beta_\mu, I(0)) \neq \overline{0}$. Let γ denote the flow on $H(\mu)$ given by $\gamma(F^*, t) = F_t^*$. Then π_0 is a product flow on $\mathcal{D} \times H(\mu)$ given by

$$\pi_0 = \beta_\mu \times \gamma$$

and

$$I(0) = I_0 \times H(\mu).$$

It follows (see [1] or [9]) that

$$h(\pi_0, I(0)) = h(\beta_\mu \times \gamma, I_0 \times H(\mu)) = h(\beta_\mu, I_0) \wedge h(\gamma, H(\mu))$$

where $\wedge$ denotes the smash product of pointed topological spaces (homotopy types). By hypothesis, $h(\beta_\mu, I_0) \neq \overline{0}$, and $h(\gamma, H(\mu))$ is of the form of a compact connected topological space with separated distinguished point (since $H(\mu)$ has an empty exit set under the flow γ). It follows from a result in [12] that $h(\pi_0, I(0)) \neq \overline{0}$. Hence

$$h(\pi_1, I(1)) = h(\pi_0, I(0)) \neq \overline{0}. \tag{11}$$

Since π_1 is the skew product flow on $\mathcal{D} \times H(\mu)$ generated by the initial value problems

$$\frac{dx}{dt} = \mu F^*(x, t; \mu),\ x(0) = x_0$$

where $F^* \in H(\mu)$, we will denote this flow by Φ_μ, and $I(1)$ by I_μ. Thus we have shown that there is a number $0 < \mu_0 \leq 1$ such that for $0 < \mu \leq \mu_0$ we have

$$h(\Phi_\mu, I_\mu) = h(\pi_1, I(1)) \neq \overline{0}.$$

It follows that there exists $(x^*, F^*) \in \Omega \times H(\mu)$ such that $\Phi_\mu(x^*, F^*, t) \in \Omega \times H(\mu)$ for all $t \in \mathbf{R}$. Thus the solution $\widetilde{x} = \widetilde{x}(t)$ to

$$\frac{dx}{dt} = \mu F^*(x, t; \mu),\ \ x(0) = x^*$$

exists and satisfies $\widetilde{x}(t) \in \Omega$ for all $t \in \mathbf{R}$. By the almost periodicity of F and our remarks following Definition 2 we can take $F^* \in H_s(F)$, that is F^* is in the usual (strong, in our sense) hull of F, and hence is continuous and uniformly almost periodic. Now $H_s(F) = H_s(F^*)$, and there is a sequence $\{s_n\} \subset \mathbf{R}$ such that $F^*_{s_n}(x, t; \mu) \to F(x, t, \mu)$ uniformly on compact sets. Let $x_n(t) = \widetilde{x}(t + s_n)$. Then $\{x_n\}$ is a uniformly bounded and equicontinuous family of functions on the real line. It follows that there exists a subsequence which converges uniformly on compact sets to a solution $z = z(t)$ of

$$\frac{dx}{dt} = \mu F(x, t, \mu)$$

with $z(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}$. By applying again hypothesis (i) we conclude that in fact $z(t) \in \Omega$ for all $t \in \mathbf{R}$.

This completes Part 1 of the proof.

Before proceeding with Part 2 of the proof, we pause for the statement and proof of a lemma.

Now $F(x,t,\mu)$ is a function mapping $\mathcal{D} \times \mathbf{R} \times [0,1]$ into $\mathbf{R}^m$, almost periodic in t uniformly with respect to x and μ in compact sets. For any $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]$ the mapping

$$(x,t) \mapsto (F(x,t,\mu_1), F(x,t,\mu_2), \ldots, F(x,t,\mu_n))$$

defines an uniformly almost periodic function with range in $\mathbf{R}^{nm}$ (see, e.g., [3]). We will denote its hull by $H(\mu_1, \mu_2, \ldots, \mu_n)$. Note that this is a subset of $H(\mu_1) \times H(\mu_2) \times \cdots \times H(\mu_n)$, but these sets are not, in general, equal.

LEMMA 6. *Let $F(x,t,\mu)$ satisfy the general conditions assumed at the beginning of this section. Let $\{\mu_n\} \subset [0,1]$ be a sequence converging to some $\mu \in [0,1]$, and let $\{(f_n, g_n)\}$ be a sequence with $(f_n, g_n) \in H(\mu_n, \mu)$ converging uniformly on compact sets to some (f,g). Then $(f,g) \in H(\mu,\mu)$ and $f = g$.*

PROOF. For each $n \in \mathbf{N}$ let $\mathcal{F}(\mu_n, \mu) = \{(F_\tau(\cdot,\cdot,\mu_n), F_\tau(\cdot,\cdot,\mu)) \, : \, \tau \in \mathbf{R}\}$ be the set of time translates of $(F(\cdot,\cdot,\mu_n),\ F(\cdot,\cdot,\mu))$, so that $H(\mu_n,\mu)$ is the closure of $\mathcal{F}(\mu_n,\mu)$. First suppose $(f_n,g_n) \in \mathcal{F}(\mu_n,\mu)$ for each $n \in \mathbf{N}$. Then there exist $\tau_n \in \mathbf{R}$ such that

$$(f_n(x,t,\mu_n), g_n(x,t,\mu)) = (F(x,t+\tau_n,\mu_n),\ F(x,t+\tau_n,\mu)).$$

Now for each $\varepsilon > 0$ and compact $K \subset \mathcal{D}$ there exists $\delta > 0$ such that $|\mu_n - \mu| < \delta$ implies

$$|F(x,t+\tau_n,\mu_n) - F(x,t+\tau_n,\mu)| < \varepsilon$$

for all $(x,t) \in K \times \mathbf{R}$. Since $\mu_n \to \mu$ and $F(x,t+\tau_n,\mu) \to g(x,t)$, we conclude that $F(x,t+\tau_n,\mu_n) \to g(x,t)$ also. Hence $f = g$ and since $g \in H(\mu)$ it follows that $(f,g) = (g,g) \in H(\mu,\mu)$. Now suppose $(f_n,g_n) \in H(\mu_n,\mu)$. Let $\{K_n\}$ be a sequence of compact sets in $\mathcal{D}$ with $K_n \subset K_{n+1}$ and

$$\mathcal{D} = \bigcup_{n=1}^{\infty} K_n.$$

For each $n \in \mathbf{N}$ there is a pair $(\widetilde{f}_n, \widetilde{g}_n) \in \mathcal{F}(\mu_n,\mu)$ with

$$\big|(f_n(x,t), g_n(x,t)) - (\widetilde{f}_n(x,t), \widetilde{g}_n(x,t))\big| < n^{-1}$$

for all $(x,t) \in K_n \times \mathbf{R}$. This implies that $(\widetilde{f}_n, \widetilde{g}_n) \to (f,g)$. But $(\widetilde{f}_n, \widetilde{g}_n) \in \mathcal{F}(\mu_n,\mu)$ and $\mu_n \to \mu$ implies by the first part of the argument that $f = g$ and $(g,g) \in H(\mu,\mu)$. This proves the lemma.

Part 2 *of the proof of Theorem* 5:

In this part we will change our notation slightly. Let

$$G(x,t,\mu) := F(x,t\mu^{-1},\mu), \quad \text{for } 0<\mu\le 1, \quad \text{and } G(x,t,0) := F_0(x).$$

Fix $\mu^* \in [0,1)$. For $\mu \in [0,1)$ and $\lambda \in [0,1]$ let us consider the family of differential equations

$$\frac{dx}{dt} = (1-\lambda)G^*(x,t;\mu) + \lambda G^*(x,t;\mu^*) \tag{12}$$

for $(G^*(\cdot,\cdot;\mu),\ G^*(\cdot,\cdot;\mu^*)) \in H(\mu,\mu^*)$. We claim that $0 < \mu^* < 1$ there exist numbers

$$0 \le \nu_1 = \nu_1(\mu^*) < \mu^* < \nu_2 = \nu_2(\mu^*) \le 1,$$

such that any solutions to (12) with $x = x(t) \in \overline{\Omega}$ for all t must satisfy $x(t) \in \Omega$ for all t. If not, there exist $\mu_n \to \mu^*$, $\lambda_n \to \lambda_0 \in [0,1]$, $(g_n,h_n) \in H(\mu_n,\mu^*)$, and solutions x_n to

$$\frac{dx_n}{dt} = (1-\lambda_n)g_n(x_n,t) + \lambda_n h_n(x_n,t)$$

satisfying $x_n(t) \in \overline{\Omega}$ for all t, and $x_n(t_n) \in \partial\Omega$ for some t_n. Let $y_n(t) = x_n(t+t_n)$, then

$$\frac{dy_n}{dt} = (1-\lambda_n)g_n(y_n,t+t_n) + \lambda_n h_n(y_n,t+t_n).$$

Since $\{y_n\}$ and $\{dy_n/dt\}$ are uniformly bounded, we may assume that $\{y_n\}$ converges uniformly on compact sets to some y with $y(t) \in \overline{\Omega}$ for all t, and $y(0) \in \partial\Omega$. Moreover by hypothesis $(G(x,t,\mu),\ G(x,t,\mu^*))$ is almost periodic in t, uniformly with respect to x and μ, μ^* in compact sets. It follows that without loss of generality we may assume that $(g_n(x,t+t_n),\ h_n(x,t+t_n))$ converges uniformly on compact sets to some $(g^*,\ h^*)$. By the lemma, $g^* = h^*$ and $(g^*,g^*) \in H(\mu^*,\mu^*)$, and

$$\frac{dy}{dt} = (1-\lambda_0)g^*(y,t) + \lambda_0 g^*(y,t) = g^*(y,t)$$

where $g^* \in H(\mu^*)$. This contradicts hypothesis (i). Thus $\nu_1 < \mu^* < \nu_2$ exist, as claimed. When $\mu^* = 0$ the existence of such an interval $[0,\nu_2(0))$ was proven in Part 1. Fix $\overline{\mu} \in (0,1)$, and for each $\mu \in [0,1)$ let $J(\mu) = (\nu_1(\mu),\nu_2(\mu))$; then

$$\{J(\mu) \,:\, \mu \in [0,1)\}$$

forms an open cover of $[0,\overline{\mu}]$. Thus it has a finite subcover

$$\{J(\mu_0), J(\mu_1), \ldots, J(\mu_N)\}.$$

We can assume that $0 = \mu_0 < \mu_2 < \ldots < \mu_N = \overline{\mu}$ and

$$J(\mu_n) \cap J(\mu_{n+1}) \neq \emptyset \tag{13}$$

for $n = 0, 1, \ldots, N-1$.

We will show that

$$\frac{dx}{dt} = G(x, t, \overline{\mu})$$

has a full bounded solution with values in Ω. Let $k \in \{0, 1, \ldots, N\}$ be the least integer such that $\overline{\mu} \in J(\mu_k)$. If $k = 0$, we are done by Part 1. Assume $k > 0$. By (13) there is a number $\alpha_n \in J(\mu_{n-1}) \cap J(\mu_n)$, for $n = 1, 2, \ldots, k$. Consider the following homotopies of families of equations:

$$\begin{aligned}
\frac{dx}{dt} &= (1-\lambda)G^*(x, t; \overline{\mu}) + \lambda G^*(x, t; \mu_k),\\
\frac{dx}{dt} &= (1-\lambda)G^*(x, t; \mu_k) + \lambda G^*(x, t; \alpha_k),\\
\frac{dx}{dt} &= (1-\lambda)G^*(x, t; \alpha_k) + \lambda G^*(x, t; \mu_{k-1}),\\
\frac{dx}{dt} &= (1-\lambda)G^*(x, t; \mu_{k-1}) + \lambda G^*(x, t; \alpha_{k-1}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{dx}{dt} &= (1-\lambda)G^*(x, t; \alpha_1) + \lambda G^*(x, t; 0).
\end{aligned}$$

where, letting $G^*(\mu) := G^*(\cdot, \cdot; \mu)$,

$$(G^*(\overline{\mu}), G^*(\mu_k), G^*(\alpha_k), G^*(\mu_{k-1}), \ldots, G^*(\alpha_1), G^*(0)) \\ \in H(\overline{\mu}, \mu_k, \alpha_k, \mu_{k-1}, \ldots, \alpha_1, 0).$$

By the preceding argument, none of these homotopies have any solutions in $\overline{\Omega}$ which assume a value on $\partial\Omega$. Let

$$\widetilde{H} := H(\overline{\mu}, \mu_k, \alpha_k, \ldots, \alpha_1, 0)$$

We define a finite sequence of $2k+1$ skew product homotopies on the space $\mathcal{D} \times \widetilde{H}$, denoted by the starting point of each homotopy, as $\pi_\lambda[\overline{\mu}]$, $\pi_\lambda[\mu_k]$, $\pi_\lambda[\alpha_k], \ldots, \pi_\lambda[\alpha_1]$. These are given by

$$\pi_\lambda[\overline{\mu}](x_0, (f^1, f^2, \ldots, f^{2k+1}), t) = (x(t; x_0, (1-\lambda)f^1 + \lambda f^2),\ (f^1_t, f^2_t, \ldots, f^{2k+1}_t)),$$

$$\pi_\lambda[\mu_k](x_0, (f^1, f^2, \ldots, f^{2k+1}), t) = (x(t; x_0, (1-\lambda)f^2 + \lambda f^3),\ (f^1_t, f^2_t, \ldots, f^{2k+1}_t)),$$

and so on, for $\alpha_k, \mu_{k-1}, \ldots, \alpha_1, \mu_0 = 0$, and $x_0 \in \mathcal{D}$ and $(f^1, f^2, \ldots, f^{2k+1}) \in \widetilde{H}$. Of course, these flows are only defined locally, that is, as long as the solutions of the defining differential equations exist. We have actually defined a homotopy joining $\pi_0[\overline{\mu}]$ and $\pi_1[0]$, which could be made explicit by reparameterizing λ. Now

$$\overline{\Omega} \times \widetilde{H}$$

is a compact isolating neighborhood for each flow in the homotopy, and the results of [1] show that the continuous dependence conditions of Theorem 4 hold. Thus if $I(0)$ is the maximal invariant set in $\overline{\Omega} \times \widetilde{H}$ for the flow $\pi_0[\overline{\mu}]$ and $I(1)$ is the maximal invariant set in $\overline{\Omega} \times \widetilde{H}$ for the flow $\pi_1[0]$ then we have that the Conley indices of these sets are defined and

$$h(\pi_0[\overline{\mu}], I(0)) = h(\pi_1[0], I(1)).$$

Now

$$\pi_1[0] = \beta \times \widetilde{\gamma}$$

where β is the flow defined by the autonomous equation (4) and $\widetilde{\gamma}$ is the flow on $\widetilde{H}$ given by

$$\widetilde{\gamma}((f^1, f^2, \ldots, f^{2k+1}), t) = (f^1_t, f^2_t, \ldots, f^{2k+1}_t).$$

Now $I(1) = I_0 \times \widetilde{H}$, so that, like in Part 1 of this proof,

$$h(\pi_0[\overline{\mu}], I(0)) = h(\beta \times \widetilde{\gamma}, I_0 \times \widetilde{H}) = h(\beta, I_0) \wedge h(\widetilde{\gamma}, \widetilde{H}) \neq \overline{0}.$$

Thus $I(0) \neq \emptyset$. Now, arguing as at the end of Part 1, we can conclude that

$$\frac{dx}{dt} = \overline{\mu} F(x, t, \overline{\mu})$$

has a full bounded solution taking values in Ω. This proves there is a solution to (3) with values in Ω for each $0 < \mu < 1$. To obtain a solution at $\mu = 1$ let $\{\mu_n\}$ be a sequence in $(0,1)$ converging to 1, and $x_n = x_n(t)$ corresponding full bounded solutions with values in Ω. Standard arguments show that a subsequence of $\{x_n\}$ will converge to a full bounded solution of (3) with $\mu = 1$, and this solution will take all its values in the closed bounded set $\overline{\Omega}$. This proves the theorem.

REMARK 1. It is natural to suppose that one could simply use the parameter μ in $F(x, t, \mu)$ to define a homotopy to prove Theorem 5, and avoid introducing the piecewise linear approach used here. The difficulty is that we don't seem to have enough information on how $F(x, t, \mu)$ (or $H(\mu)$) varies with μ. By making stronger assumptions we can apparently simplify the proof. Recall that if f is an almost periodic function, then the module of f, Mod (f), is the smallest additive group of real numbers that contains the Fourier exponents of f (see [3]). If Mod $(F(\cdot,\cdot,\mu))$ is independent of $0 < \mu \leq 1$ then, by applying Theorem 4.5 of [3] one can show that $F^*(x,t;\mu) \in H(\mu)$ can be followed continuously (in a unique way) as μ varies. It should now be possible to prove the conclusion of Theorem 5 using the homotopy defined naturally by the skew product flows associated with

$$\frac{dx}{dt} = \mu F^*(x,t;\mu)$$

on the space $\mathcal{D} \times H(1)$.

REMARK 2. If the bounded solution found in Theorem 5 is locally unique then it will be almost periodic; see [3] for a proof of this.

COROLLARY 7. *Let F be as in Theorem 5, except suppose $\mathcal{D} = \mathbf{R}^m$. Suppose:*

(i) *There is a number $r_0 > 0$ such that if $x = x(t)$ is a bounded solution of (4) then $\|x\| < r_0$.*

(ii) *If I_0 denotes the maximal compact invariant set for (4) then $h(I_0) \neq \overline{0}$.*

Then either for each $\mu \in [0,1]$ there is a full bounded solution of (3), or else for each $r > r_0$ there is a pair $(\mu, F^*) \in (0,1] \times H(\mu)$ such that (5) has a full bounded solution x with $\|x\| = r$.

PROOF. If for some $r > r_0$ there is no pair $(\mu, F^*) \in (0,1]$ satisfying the second alternative then the hypotheses of Theorem 5 are met with $\Omega = \{x \in \mathbf{R}^m : |x| < r\}$.

3. An Application

Let $F : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}$ be a continuous function, $(x,t) \mapsto F(x,t)$, uniformly almost periodic in $t \in \mathbf{R}$ and locally Lipshitz continuous in x, uniformly in t. We will consider the existence of bounded solutions to

$$\frac{dx}{dt} = F(x,t). \tag{14}$$

Recall that by a bounded solution we mean a function $x \in C^1(\mathbf{R}, \mathbf{R}^m)$ which solves (14) at all $t \in \mathbf{R}$ and such that

$$\|x\| := \sup_{t \in \mathbf{R}} |x(t)| < \infty.$$

DEFINITION 8. *An open bounded set $\Omega \subset \mathbf{R}^m$ is called a strict bound set for* (14) *if for each $y \in \partial\Omega$ there exists $V_y \in C^1(\mathbf{R}^m, \mathbf{R})$ such that*

(i) $\Omega \subset \{x \in \mathbf{R}^m : V_y(x) < 0\}$.

(ii) $V_y(y) = 0$.

(iii) *For each $y \in \partial\Omega$ and $F^* \in H(F)$ one has $V_y'(y) \cdot F^*(y,t) \neq 0$ for all $t \in \mathbf{R}$.*

Here V_y' denotes the gradient of V_y and $H(F)$ is the hull of F, defined in Section 1. Let $F_0(x)$ denote the average of F, that is

$$F_0(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T F(x,t)dt.$$

We will relate (14) to

$$\frac{dx}{dt} = F_0(x). \tag{15}$$

Let β denote the (local) flow in $\mathbf{R}^m$ generated by (15). We have

THEOREM 9. *Suppose:* (i) *There is a strict bound set $\Omega \subset \mathbf{R}^m$ for* (14)*; it follows that Ω is also a strict bound set for* (15) *and $\overline{\Omega}$ is a compact isolating neighborhood for the associated flow β.* (ii) *The homotopy index $h(I_0, \beta) \neq \overline{0}$, where I_0 denotes the maximal invariant set in Ω for* (15).

Then (14) *has a full bounded solution taking all its values in Ω.*

PROOF. We apply Theorem 5. Let $\mu \in (0,1]$ and consider the family of equations

$$\frac{dx}{dt} = \mu F^*(x,t) \tag{16}$$

for $F^* \in H(F)$. First, by (i) if $y \in \partial\Omega$ is fixed then since $F(y,t)$ is continuous in $t \in \mathbf{R}$ we have that $V_y'(y) \cdot F(y,t)$ is of one sign for all t; to be definite, suppose it is positive. Then

$$V_y'(y) \cdot F_0(y) = V_y'(y) \cdot \lim_{T \to \infty} \frac{1}{T} \int_0^T F(y,t)dt > 0$$

so that we see $V_y'(y) \cdot F_0(y) \neq 0$ for all $y \in \partial\Omega$, and Ω is a bound set for (15), and hence $\overline{\Omega}$ is a compact isolating neighborhood for the flow β generated by (15), all as claimed. We need only show that if $x = x(t)$ is a solution to (14) for some $\mu \in (0,1]$ and $F^* \in H(F)$, with $x(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}$, then in fact $x(t) \in \Omega$ for all $t \in \mathbf{R}$; that is, $x(t)$ assumes no values in $\partial\Omega$. Suppose this is not the case; then there are $F^* \in H(F)$, $\mu_0 \in (0,1]$, and a solution $x_0 = x_0(t)$ to (14) with $\mu = \mu_0$ such that $x_0(t) \in \overline{\Omega}$ for all t, and $x_0(t_0) \in \partial\Omega$ for some $t_0 \in \mathbf{R}$. Let $y = x_0(t_0)$, and $v(t) = V_y(x_0(t))$. Then $v(t) \leq 0$ for all $t \in \mathbf{R}$ and $v(t_0) = V_y(x_0(t_0)) = V_y(y) = 0$. Thus $v'(t_0) = 0$. But since Ω is a strict bound set for (14) we must have

$$v'(t_0) = V_y'(y) \cdot F^*(y, t_0) \neq 0.$$

Thus we have a contradiction, and Theorem 5 now implies the conclusion.

REMARK 3. Bound sets have been applied in the study of both periodic and bounded solutions of ordinary differential equations, generally in connection with degree theory; see [4], [6]. Bounded solutions can be obtained by first solving the periodic problem on the intervals $[-n, n]$, $n \in \mathbf{N}$, and then obtaining a subsequence of these solutions convergent to a solution bounded on $(-\infty, \infty)$. The result of Theorem 9 apparently cannot be obtained in that way, since one can have a bound set Ω for $F_0(x)$ when the Brouwer degree $d(F_0, \Omega, 0) = 0$, but the homotopy index of the maximal invariant set in Ω is not $\overline{0}$, so that the conditions of Theorem 9 would be satisfied. See the examples constructed (for other purposes) in [9], p. 135, or in [13].

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