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A CONTINUATION APPROACH TO FOURTH ORDER SUPERLINEAR PERIODIC BOUNDARY VALUE PROBLEMS

J. MAWHIN — F. ZANOLIN

Dedicated to the memory of Juliusz Schauder

Consider the fourth order boundary value problem

$$\text{(P)} \qquad \begin{cases} u'''' = g(u) + e(t), \\ u(\cdot) \quad T\text{ - periodic,} \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\text{(i)} \qquad \lim_{|x|\to+\infty} g(x)\text{sign}(x) = +\infty$$

and $e : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic (more general conditions for the function $e(t)$ will be considered in Section 5).

Various results for the solvability of problem (P) have been obtained in cases when asymptotically the ratio $g(x)/x$ does not interfere with the eigenvalues of the differential operator u'''' in the space of T-periodic functions. With this respect, we refer to the articles of Omari and Zanolin [10], De Coster, Fabry and Habets [5] and Gupta and Mawhin [7]. The possibility of a function g which grows faster than linear at $+\infty$ (or at $-\infty$) has been considered by Ward [11] and Afuwape, Mawhin and Zanolin [1]. In these latter papers, however, a rather strong restriction for the growth of g at $-\infty$ (respectively at $+\infty$) has to be assumed in order to obtain the a priori bounds for the solutions.

In this work we consider an example that, as far as we know, is new in the study of the periodic problem for higher order ordinary differential equations. Namely, we examine the case when g is superlinear at infinity, that is

$$\lim_{|x|\to+\infty} \frac{g(x)}{x} = +\infty, \tag{j}$$

holds, and prove the following:

THEOREM 1. *Under condition* (j), *problem* (P) *has at least one solution, for any function* $e(t)$.

Clearly, (j) implies (i), but during the next discussion, we prefer to examine separately the effect of these conditions on the behaviour of the solutions of (P).

Condition (j) in connection with the periodic problem for the second order differential equation

$$-u'' = g(u) + e(t),$$

has been considered by many authors and the approaches used to obtain existence of solutions in this case are of different nature. In a recent paper [4], jointly with A. Capietto, we have developed a new continuation theorem in order to deal with abstract problems in which there are no a priori bounds for the solutions, and we have applied this result to the solvability of second order differential equations with superlinear nonlinearities.

Here we apply the abstract result in [4] to problem (P). To this end, we follow the steps indicated below.

1. Abstract setting

We embed problem (P) into a one-parameter family of problems of the form

$$\begin{cases} u'''' = g(u) + p(t, u', \lambda), & \lambda \in [0,1], \\ u(\cdot) \quad T\text{ - periodic}, \end{cases} \tag{P_λ}$$

where

$$p(t, y, \lambda) := (1-\lambda)\zeta(y) + \lambda e(t), \qquad \text{for } \lambda \in [0,1],$$

and $\zeta : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\zeta(y)y < 0, \quad \forall\, y \neq 0, \quad \text{and} \quad |\zeta(y)| \leq 1, \quad \forall\, y \in \mathbb{R}.$$

For notational convenience, we also define

$$f(t, x, y, \lambda) := g(u) + p(t, u', \lambda),$$

so that problem (P_λ) can be written also as

$$(\mathrm{P}_\lambda) \qquad \begin{cases} u'''' = f(t,u,u',\lambda), & \lambda \in [0,1], \\ u(\cdot) \quad T \text{ - periodic.} \end{cases}$$

By the above positions, it follows that problem (P) corresponds to (P_1), while for $\lambda = 0$, we have

$$(\mathrm{P}_0) \qquad \begin{cases} u'''' - \zeta(u') = g(u), \\ u(\cdot) \quad T \text{ - periodic.} \end{cases}$$

We introduce now the following notations.

For any T-periodic function v, we denote by $|v|_q$, $(1 \le q \le +\infty)$ its L^q-norm with respect to the interval $[0,T]$. Let X be the space of the T-periodic real valued functions of class C^3 endowed with the norm

$$\|u\| := |u|_\infty + \sum_{i=1}^{3} |u^{(i)}|_\infty$$

and write (P_λ) as the equivalent operator equation

$$(1_\lambda) \qquad Lu = N(u,\lambda) := N_\lambda u, \qquad \lambda \in [0,1],$$

where $L : u \mapsto u''''$ is a linear Fredholm mapping of index zero defined in $\operatorname{dom} L = \{u \in X : u \text{ is of class } C^4\} \subset X$ with values in the space Z of continuous and T-periodic functions with the $|\cdot|_\infty$ - norm, and $N : X \times [0,1] \to Z$ is the superposition operator defined by the right hand side of the equation in (P_λ). From classical facts (see [8]) it follows that N is L-completely continuous and hence the coincidence degree $D_L(L - N_\lambda, \Omega)$ is defined for any open bounded set $\Omega \subset X$ such that $Lu \neq N_\lambda u$ for $u \in \partial\Omega$.

We also denote by

$$\Sigma \subset X \times [0,1]$$

the set of solutions (u,λ) of (1_λ) and define, for any $\lambda \in [0,1]$,

$$\Sigma_\lambda := \{u \in \operatorname{dom} L : \ (u,\lambda) \in \Sigma\}.$$

In other terms, Σ_λ is the set of the solutions u of (P_λ) for a fixed $\lambda \in [0,1]$.

Now we observe that from (i) and the choice of the function ζ, it follows that

$$(2) \qquad \Sigma_0 \text{ is bounded and } D_L(L - N_0, \Omega) \neq 0,$$

for any open bounded set $\Omega \subset X$ with

$$\Sigma_0 \subset \Omega.$$

Indeed, if u is a solution of (P_0), then multiplying the equation in (P_0), by u' and integrating on $[0,T]$, we obtain that $\int_0^T \zeta(u'(t))u'(t)\,dt = 0$ and thus $u(t) \equiv k \in \mathbb{R}$, for a suitable constant k. Then we have $g(k) = 0$ and therefore, by (i) it follows that $k \in (-d_0, d_0)$ where $d_0 > 0$ is chosen such that $g(x)x > 0$ for $|x| \geq d_0$. Hence Σ_0 is a bounded subset of X made by constant functions. Moreover, for any Ω as above, it follows from [3] that

$$|D_L(L - N_0, \Omega)| = |d_B(g, (-d_0, d_0), 0)| = 1$$

where d_B is the Brouwer degree in $\mathbb{R}$. In the sequel, $B(0, R)$ denotes an open ball in X of center 0 and radius $R > 0$.

2. Continuation theorem

According to [4], and having proved (2), we can now use the following result:

LEMMA 1. *Suppose that there exists a continuous functional*

$$\phi : X \times [0,1] \to \mathbb{R}^+ = [0 + \infty)$$

which satisfies the following conditions:

$$(\mathrm{k}_1) \qquad \exists R^* > 0 : \ \phi(u, \lambda) \in \mathbb{N}, \qquad \forall (u, \lambda) \in \Sigma \setminus (B(0, R^*) \times [0,1]);$$

$$(\mathrm{k}_2) \qquad \Sigma \cap \phi^{-1}(n) \ \ \textit{is bounded,} \ \ \forall n \in \mathbb{N}.$$

Then equation (1_1) *and thus problem* (P) *has at least one solution.*

For our problem, we define the continuous functional ϕ as follows:

$$\phi(u, \lambda) := \frac{1}{2\pi} \left| \int_0^T (u'(t)^2 - u(t)u''(t))\delta(u(t), u'(t))\, dt \right|,$$

where

$$\delta(x, y) := \min \left\{ 1, \frac{1}{x^2 + y^2} \right\}$$

(see [4] for a similar definition).

From now on, all our work will be that to show that the functional ϕ that we have chosen satisfies the conditions (k_1) and (k_2) with respect to the set of solutions Σ of problem (P_λ). Since the above conditions are vacuously satisfied if Σ is bounded, we can assume henceforth that

$$\Sigma \text{ is unbounded in } X \times [0, 1],$$

that is there are solutions u to (P_λ) with arbitrarily large norm in X. Then we prove some qualitative results concerning such solutions.

3. Analysis of the solutions

Firt of all, using (i), we fix a constant $d > 0$ and a constant M, such that

(3) $$g(x)\mathrm{sign}(x) \geq M > 1 + |e|_\infty, \qquad \text{for all } |x| \geq d.$$

Note that by the definition of p it follows that

(4) $$|p(t, y, \lambda)| \leq 1 + |e|_\infty < M, \qquad \forall (t, y) \in \mathbb{R}^2, \lambda \in [0, 1],$$

and therefore,

(5) $$f(t, x, y, \lambda)\mathrm{sign}(x) \geq \varepsilon > 0, \qquad \forall\, |x| \geq d,\ (t, y) \in \mathbb{R}^2,\ \lambda \in [0, 1].$$

Let u be a solution of (P_λ) for some $\lambda \in [0, 1]$. Define also $u^+(t) := \max\{u(t), 0\}$ and $u^-(t) := u^+(t) - u(t)$. Throughout this section, only condition (i) is assumed.

LEMMA 2. *If any of the following quantities: $|u^+|_\infty$, $|u^-|_\infty$, $|u|_\infty$, $|u'|_\infty$, $|u''|_\infty$, $|u'''|_\infty$, is bounded (by a constant independent of u and λ) then $\|u\|$ is bounded as well.*

PROOF. First of all, we observe that from (5) and taking the mean value at both the members of the equation in (P_λ), we easily obtain that there is $\widetilde{t} \in [0, T)$ such that

(6) $$|u(\widetilde{t})| \leq d.$$

This clearly implies that if any of the quantities $|u'|_\infty$, $|u''|_\infty$, $|u'''|_\infty$, is bounded, then $|u|_\infty$ is bounded as well and therefore the L^1-norm of $f(\cdot, u, u', \lambda)$, satisfies

(7) $$|f(\cdot, u, u', \lambda)|_1 \leq c_1,$$

where $c_1 > 0$ is a constant independent on (u, λ). Then, from the equation in (P_λ) it follows that

$$|u''''|_1 \leq c_1,$$

and hence, using the fact that u''' has mean value zero in a period, we obtain

(8) $$|u'''|_\infty \leq c_1.$$

From (8) and (6), we now obtain

$$\|u\| \leq d + (1 + T + T^2)c_1.$$

Thus it remains only to check that the result is still true if we assume that $|u^+|_\infty$ (respectively, $|u^-|_\infty$) is bounded. To this end, we argue similarly as in [11]; some minor details will be omitted. Assume that

$$|u^+|_\infty \leq c_2$$

and consider the sets

$$\mathcal{A} := \{t \in [0,T] \,:\, u(t) < -d\}, \qquad \mathcal{B} := \{t \in [0,T] \,:\, -d \le u(t) \le c_2\}.$$

Then $\mathcal{A} \cup \mathcal{B} = [0,T]$; moreover, from (5) it follows that

$$|f(t,u(t),u'(t),\lambda)| = -f(t,u(t),u'(t),\lambda), \qquad \forall\, t \in \mathcal{A}.$$

Recall also that $\int_0^T f(t,u(t),u'(t),\lambda)\,dt = 0$.

Now we have

$$\begin{aligned}
|f(\cdot,u,u',\lambda)|_1 &= \int_0^T |f(t,u(t),u'(t),\lambda)|\,dt \\
&= \int_{t\in\mathcal{A}} -f(t,u(t),u'(t),\lambda)\,dt + \int_{t\in\mathcal{B}} |f(t,u(t),u'(t),\lambda)|\,dt \\
&= \int_{t\in\mathcal{A}} -f(t,u(t),u'(t),\lambda)\,dt + \int_{t\in\mathcal{B}} |f(t,u(t),u'(t),\lambda)|\,dt \\
&\quad + \int_0^T f(t,u(t),u'(t),\lambda)\,dt \\
&= \int_{t\in\mathcal{B}} f(t,u(t),u'(t),\lambda)\,dt + \int_{t\in\mathcal{B}} |f(t,u(t),u'(t),\lambda)|\,dt \\
&\le 2\int_{t\in\mathcal{B}} |f(t,u(t),u'(t),\lambda)|\,dt \\
&\le 2T(M + \max\{|g(x)| : -d \le x \le c_2\}) := c_3.
\end{aligned}$$

Thus we have proved an inequality like (7) (just with the constant c_3 in place of c_1). The rest of the proof follows the same steps as above and permits to conclude with

$$\|u\| \le d + (1 + T + T^2)c_3.$$

Clearly, the same conclusion holds if $|u^-|_\infty \le c_2$. The proof of Lemma 2 is complete. □

From Lemma 2 the following consequence is straightforward.

PROPOSITION 1. *There is a nondecreasing function* $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ := (0,+\infty)$, *with* $\eta(r) > r$ *for any* $r > 0$, *such that*

$$\|u\| \ge \eta(R), \qquad \textit{implies} \qquad \min\{|u^+|_\infty, |u^-|_\infty, |u'|_\infty, |u''|_\infty\} \ge R.$$

According to Proposition 1, we can now claim that any large solution u to (P_λ) must necessarily oscillate, i.e. it has at least two zeros in $[0,T)$. Our next goal is to show that for $\|u\|$ large the zeros of u are simple. With this respect, we recall that a basic feature of the second order differential equations without damping term, which allows to perform some useful estimates about the distribution of the zeros of the solutions, is the existence of an "energy relation" for the equation. In the next result, we find an energy-like relation for the equation

$$x'''' = g(u) + q(t), \tag{9}$$

where q is any continuous function defined in the real line. We also introduce the potential G of g defined by

$$G(x) := \int_0^x g(s)\, ds.$$

Observe that from (3) it follows that the function G is bounded below. Hence we can fix a constant $\gamma_1 > 0$ such that

$$2G(x) \geq -\gamma_1, \qquad \forall x \in \mathbb{R}.$$

LEMMA 3. *For any solution u of (9), defined on a interval $I \subset \mathbb{R}$, the following relation holds:*

$$\begin{aligned} \text{(E)}\quad & \frac{1}{2}u''(t)^2 + G(u(t)) - u'''(t)u'(t) \\ &= \frac{1}{2}u''(s)^2 + G(u(s)) - u'''(s)u'(s) - \int_s^t q(\xi)u'(\xi)\, d\xi, \qquad \forall t, s \in I. \end{aligned}$$

PROOF. It is sufficient to differentiate (E) with respect to t in order to obtain (9) multiplied by $u'(t)$. □

Our next result is a technical lemma which turns out to be useful in the proof of the simplicity of the zeros of the solutions. In what follows, u is an arbitrary solution of (P_λ), i.e. a T-periodic solution of (9) for $q(t) = p(t, u'(t), \lambda)$.

LEMMA 4. *For any constant $R \geq 0$, there is a constant $L(R) > 0$ such that, if t_0 is a point of local minimum of $u(\cdot)$ with $u(t_0) \geq -R$, then $0 \leq u''(t_0) \leq L(R)$ and, respectively, if t_0 is a point of local maximum of $u(\cdot)$ with $u(t_0) \leq R$, then $-L(R) \leq u''(t_0) \leq 0$.*

PROOF. We prove only the first inference as the argument for the second one is completely similar.

Suppose that t_0 is a point of local minimum for $u(\cdot)$ with $u(t_0) \geq -R$, and assume that $u''(t_0) > 0$ (if $u''(t_0) = 0$, the result holds trivially). Set also $z(t) := u''(t)$.
Since z is a continuous T-periodic function with mean value zero and with $z(t_0) > 0$, we can find an open interval $J = (t_1, t_2)$ containing t_0 such that $z(t) > 0$ for all

$t \in J$ and $z(t_1) = z(t_2) = 0$. Then the function u is convex in J with $u(t) \geq -R$, for all $t \in J$ and hence from (3) and (4) we obtain

$$z''(t) = u''''(t) \geq -K, \qquad \text{for all } t \in J,$$

with

$$K = K(R) := M + \max\{|g(x)| : -R \leq x \leq d\}.$$

From the Taylor formula and using the fact that $z'' \geq -K$ in J, we have

$$0 = z(t_1) \geq z(t_0) + z'(t_0)(t_1 - t_0) - \frac{K}{2}(t_1 - t_0)^2 \geq z(t_0) + z'(t_0)(t_1 - t_0) - \frac{KT^2}{2},$$

$$0 = z(t_2) \geq z(t_0) + z'(t_0)(t_2 - t_0) - \frac{K}{2}(t_2 - t_0)^2 \geq z(t_0) + z'(t_0)(t_2 - t_0) - \frac{KT^2}{2}.$$

Observe now that $t_1 - t_0$ and $t_2 - t_0$ have opposite sign and, therefore, at least one of the two numbers $z'(t_0)(t_1 - t_0)$, $z'(t_0)(t_2 - t_0)$ is nonnegative. Hence we have

$$z(t_0) \leq L(R) := \frac{KT^2}{2},$$

which proves the result. □

Now we are in position to state the next result.

PROPOSITION 2. *There is $R_1 > 0$ such that if $\|u\| \geq R_1$, then $u'(t) \neq 0$ for each t such that $u(t) = 0$.*

PROOF. Let t_0 be such that

$$u(t_0) = u'(t_0) = 0.$$

We consider two possibilities:

Either $u''(t_0) \neq 0$, or $u''(t_0) = 0$.

In the former case, t_0 is a point of local minimum or local maximum for u and we can apply Lemma 4 with $R = 0$ and obtain that

$$|u''(t_0)| \leq L := L(0).$$

On the other hand, if $u''(t_0) = 0$, then such inequality holds as well.

Now from the energy relation (E), computed at the points $s = t_0$ and $t = t^*$, with t^* such that

$$|u''(t^*)| = |u''|_\infty := r, \qquad u'''(t^*) = 0,$$

we obtain

$$\begin{aligned} r^2 &= -2G(u(t^*)) + L^2 - 2\int_{t_0}^{t^*} p(\xi, u'(\xi), \lambda)u'(\xi)\, d\xi \\ &\le L^2 + \gamma_1 + 2M\int_0^T |u'(\xi)|\, d\xi \\ &\le L^2 + \gamma_1 + 2MT^2 r, \end{aligned}$$

where we recall that $-2G(x) \le \gamma_1$, for all x. Thus we obtain

$$|u''|_\infty < R_0 := 1 + \max\{(2MT^2+1), L^2 + \gamma_1\}.$$

Now we can apply Proposition 1, which implies that $\|u\| < \eta(R_0)$. Then the result is proved for any choice of $R_1 \ge \eta(R_0)$. □

Proposition 2 ensures that the zeros of a solution u with $\|u\|$ sufficiently large are isolated and therefore only a finite number of zeros can belong to the interval $[0, T)$. Moreover, the solution changes sign at any point at which it vanishes. On the other hand, note that Proposition 1 guarantees that if $\|u\| \ge R_1$, then $\min u < 0 < \max u$, so that we know that zeros for u do exist and they have all the desired properties. Hence we can now introduce the following notation: Let u be a solution of (P_λ) with $\|u\| \ge R_1$. Then, it follows from Proposition 2 that there is an *even* number

$$n := \mathbf{n}(u)$$

and there are $n+1$ points

$$t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_n = t_0 + T,$$

such that $t_0 \in [0, T)$ and

$$\begin{aligned} u(t_i) &= 0, & u'(t_i) &\ne 0, \quad \forall\, i : 0 \le i \le n-1, \\ u(t) &> 0, & \forall\, t &\in I_i^+ := (t_i, t_{i+1}), \quad i \text{ even}, \\ u(t) &< 0, & \forall\, t &\in I_i^- := (t_i, t_{i+1}), \quad i \text{ odd}. \end{aligned}$$

The points t_i are uniquely determined by the function u if we choose as t_0 the first point in the interval $[0, T)$ where $u = 0$ and $u' > 0$.

Finally, for any i even, we denote by s_i the first maximum point of u in the interval I_i^+ and for i odd, we denote by s_i the first minimum point of u in the interval I_i^-, so that

$$u(s_i) = u_i^+ := \max\{u(t) : t \in I_i^+\}, \qquad i \text{ even},$$

$$u(s_i) = u_i^- := \min\{u(t) : t \in I_i^-\}, \qquad i \text{ odd}.$$

Clearly, we have that $u'(s_i) = 0$, for each index i.

REMARK 1. Assume $\|u\| \geq R_1$ so that $(u(t), u'(t)) \neq (0,0)$ for all $t \in \mathbb{R}$ and we can pass to polar coordinates and write $u(t) = \rho(t)\cos\theta(t)$, $u'(t) = \rho(t)\sin\theta(t)$. Hence, as

$$\pi = \theta(t_i) - \theta(t_{i+1}) = -\int_{t_i}^{t_{i+1}} \frac{d}{dt}(\tan^{-1}[u'(t)/u(t)])\,dt$$
$$= -\int_{t_i}^{t_{i+1}} \frac{u'(t)^2 - u(t)u''(t)}{u(t)^2 + u'(t)^2}\,dt,$$

for each i, we obtain the following formula:

$$-\mathbf{n}(u) = \frac{1}{\pi}\int_0^T \frac{u'(t)^2 - u(t)u''(t)}{u(t)^2 + u'(t)^2}\,dt.$$

Now we can state a more refined version of Proposition 1 as follows.

PROPOSITION 3. *There is a nondecreasing function* $\beta : [R_1, +\infty) \to \mathbb{R}_0^+$, *with* $\beta(r) > r$ *for any* $r \geq R_1$, *such that*

$$\|u\| \geq \beta(R), \quad \textit{implies} \quad \min\{|u_i^{\pm}| : 0 \leq i \leq \mathbf{n}(u) - 1\} \geq R.$$

PROOF. For ease in the notation during the proof, we set $u_i := |u_i^{\pm}|$, $z_i := |u''(s_i)|$.

From Lemma 3 we know that

$$z_i \leq L(u_i),$$

thus, a bound for u_i implies a bound for z_i.

Now, arguing as in the proof of Proposition 2, from the energy relation (E), computed at the points $s = s_i$ and $t = t^*$, with t^* such that $|u''(t^*)| = |u''|_\infty := r$, $u'''(t^*) = 0$, we obtain

$$r^2 \leq 2\max\{|G(x)| : |x| \leq |u_i|\} - 2G(u(t^*)) + z_i^2 - 2\int_{s_i}^{t^*} p(\xi, u'(\xi), \lambda)u'(\xi)\,d\xi$$
$$\leq \gamma(u_i) + z_i^2 + \gamma_1 + 2M\int_0^T |u'(\xi)|\,d\xi \leq \gamma(u_i) + z_i^2 + \gamma_1 + 2MT^2 r,$$

where we have set $\gamma(s) := 2\max\{|G(x)| : |x| \leq s\}$ and γ_1 was already defined above. Thus we obtain

$$|u''|_\infty < \rho = \rho(u_i, z_i) := 1 + \max\{(2MT^2 + 1), \gamma(u_i) + z_i^2 + \gamma_1\}.$$

Then we can apply Proposition 1, which implies that $\|u\| < \eta(\rho)$. Since $\rho(u_i, z_i)$ is bounded if u_i is bounded, the result follows immediately. □

A more precise picture about the behaviour of $u(t)$ and $u'(t)$ comes from the next results.

LEMMA 5. *There is $R_2 \geq R_1$ such that if $\|u\| \geq R_2$, then $u'(t) \neq 0$, for any $t \neq s_i$, $(i = 0, \dots, \mathrm{n}(u) - 1)$, $t \in [t_0, t_0 + T]$.*

PROOF. In order to avoid distinctions among various sub-cases, we develop the proof only for the first interval $I_0 = I_0^+ = (t_0, t_1)$ where $u > 0$, $u'(t_0) > 0 > u'(t_1)$ and $\max u = u_0 = u_0^+ = u(s_0) > 0$. The same argument will work (just changing the indexes) for any interval $I_i^\pm$.

For the interval I_0 we have to prove that

$$u'(t) > 0, \ \forall\, t \in (t_0, s_0), \quad u'(t) < 0, \ \forall\, t \in (s_0, t_1).$$

To this end, we observe that it is sufficient to show that if s^* is any point in the interval where $u'(s^*) = 0$, then, necessarily $u''(s^*) < 0$ and therefore any critical point of u in I_0 is a strict local maximum. Clearly, this in turn immediately will imply that there is only one critical point of u in I_0 and such a point is s_0.

Accordingly, suppose, by contradiction, that

$$u'(s^*) = 0 \leq u''(s^*).$$

We consider now two poss]ibilities:

Either $u(s^*) \leq d$, or $u(s^*) > d$.

In the former case, if $u''(s^*) > 0$, then s^* is a local minimum and Lemma 4 implies that

$$0 \leq u''(s^*) \leq L := L(0).$$

On the other hand, this inequality is clearly true even if $u''(s^*) = 0$. Then, from the energy relation (E), computed at the points $s = s^*$ and $t = t^*$, with t^* such that $|u''(t^*)| = |u''|_\infty := r$, (as in Propositions 2, 3) we obtain

$$\begin{aligned} r^2 &\leq 2\max\{|G(x)| : 0 \leq x \leq d\} - 2G(u(t^*)) + L^2 - 2\int_{s_i}^{t^*} p(\xi, u'(\xi), \lambda) u'(\xi)\, d\xi \\ &\leq \gamma(d) + L^2 + \gamma_1 + 2MT^2 r, \end{aligned}$$

with $\gamma(s)$ and γ_1 defined in the proof of Proposition 3. Thus we have

$$|u''|_\infty < \rho_1 := \rho(d, L) = 1 + \max\{(2MT^2 + 1), \gamma(d) + L^2 + \gamma_1\}$$

and Proposition 1 yields $\|u\| < \eta(\rho_1)$. Hence, this first case can be excluded if we take $\|u\| \geq \eta(\rho_1)$.

Suppose now that we are in the latter situation, that is, $u(s^*) > d$. Here we can use (5) and obtain

$$(u'')''(s^*) \geq \varepsilon > 0, \quad \text{and} \quad (u'')''(t) \geq \varepsilon, \quad \forall\, t \in \mathcal{D} := \{t \in I_0 : u(t) \geq d\}.$$

By the Taylor formula applied to u'' we have that

$$u''(t) \geq u''(s^*) + u'''(s^*)(t - s^*),$$

for any t belonging to a subinterval of $\mathcal{D}$. Hence, choosing t at the left or at the right of s^* so that $u'''(s^*)(t - s^*) \geq 0$, we have that $u''(t) \geq u''(s^*) \geq 0$, for all t in a one-sided (right or left) connected neighbourhood J of s^* contained in $\mathcal{D}$.

Now taking J maximal with respect to the inclusion in $\mathcal{D}$, we can assume that either $J = [t_-, s^*)$ or $J = (s^*, t_+]$, with, respectively $u(t_-) = d$, or $u(t_+) = d$. Then, applying the Taylor formula for u, we find

$$u(t) \geq u(s^*), \qquad \forall\, t \in J,$$

which implies, respectively $u(t_-) \geq u(s^*)$, or $u(t_+) \geq u(s^*)$. Thus, in any case, we obtain $u(s^*) \leq d$, which contradicts our starting assumption.

Thus, we have seen that the second case in our discussion can never occur and therefore, from the first case, we have proved the result by choosing any $R_2 \geq \max\{R_1, \eta(\rho_1)\}$. □

From Lemma 5 we have that if $\|u\| \geq R_2$, then

$$u'(t) > 0, \qquad \forall t \in J_i^+ := (s_{i-1}, s_i), \qquad i \text{ even}$$

and

$$u'(t) < 0, \qquad \forall t \in J_i^- := (s_{i-1}, s_i), \qquad i \text{ odd},$$

for $0 \leq i \leq \mathbf{n}(u) - 1$, where, with obvious "cyclic" convention, we read $s_{-1} = -T + s_{n-1}$.

We note also that, as $u'(s_i) = 0$, for each i and $u'(t) \neq 0$, for all $t \neq s_i$, then for any i even, we can choose the first maximum point σ_i of u' in the interval J_i^+ and for i odd, we denote by σ_i the first minimum point of u' in the interval J_i^-, so that

$$u'(\sigma_i) = \max\{u'(t) : t \in J_i^+\}, \qquad i \text{ even},$$
$$u'(\sigma_i) = \min\{u'(t) : t \in J_i^-\}, \qquad i \text{ odd}.$$

Clearly, we have that $u''(\sigma_i) = 0$, for each index i.

Now we can state an analog of Lemma 5 for u' as follows:

LEMMA 6. *If $\|u\| \geq R_2$, then $u''(t) \neq 0$, for any $t \neq \sigma_i$, $(i = 0, \dots, \mathbf{n}(u) - 1)$, $t \in [s_{-1}, s_{-1} + T]$.*

PROOF. First of all, we claim that $u''(s_i) \neq 0$, for each i.

Actually, this assertion has been already checked in the proof of Lemma 5 for $s_i = s^*$. For completeness, we give here another short and direct proof of the claim, assuming $R_2 \geq \beta(d+1)$ (this is not restrictive: just take a larger R_2 in Lemma 5, if necessary).

Indeed, if, by contradiction, $u''(s_i) = 0$ for some i, then, using the Taylor formula and recalling (5) and $|u(s_i)| > d$ (which follows from Proposition 3), we obtain

$$|u(t)| > |u(s_i)| + \frac{1}{6}u'''(s_i)(t-s_i)^3 \text{sign}(u(s_i))$$

for $t \neq s_i$ in an open interval containing s_i where $|u(t)| > d$. Then, taking t such that $u'''(s_i)(t-s_i)^3\text{sign }(u(s_i)) \geq 0$, we contradict the fact that $|u(s_i)|$ is a local maximum of $|u(\cdot)|$. Thus our claim is proved.

The proof now follows a similar argument to that of Lemma 5; this time u' plays the role of u.

We consider only the first interval $J_0 = J_0^+ = (s_{-1}, s_0)$ where $u' > 0$ and $\max u' = u'(\sigma_0) > 0$. The same argument will work (just changing the indexes) for any interval $J_i^{\pm}$.

For the interval J_0 we have to prove that

$$u''(t) > 0,\ \forall\, t \in (s_{-1}, \sigma_0), \qquad u''(t) < 0,\ \forall\, t \in (\sigma_0, s_0).$$

To this end, we observe that it is sufficient to show that if σ^* is any point in the interval where $u''(\sigma^*) = 0$, then, necessarily $u'''(\sigma^*) < 0$ and therefore any critical point of u' in J_0 is a strict local maximum. Clearly, this in turn will immediately imply that there is only one critical point of u' in J_0 and such a point is σ_0.

Accordingly, suppose, by contradiction, that

$$u''(\sigma^*) = 0 \leq u'''(\sigma^*).$$

We consider two possibilities:

Either $|u(\sigma^*)| \leq d$, or $|u(\sigma^*)| > d$.

In the former case, from the energy relation (E), computed at the points $s = \sigma^*$ and $t = t^*$, with t^* such that $|u''(t^*)| = |u''|_\infty := r$ and observing that $-u'''(\sigma^*)u'(\sigma^*) \leq 0$, we obtain

$$\begin{aligned} r^2 &\leq 2\max\{|G(x)| : |x| \leq d\} - 2G(u(t^*)) - 2\int_{s_i}^{t^*} p(\xi, u'(\xi), \lambda)u'(\xi)\, d\xi \\ &\leq \gamma(d) + \gamma_1 + 2MT^2 r. \end{aligned}$$

Thus we have

$$|u''|_\infty < \rho_2 := \rho(d,0) = 1 + \max\{(2MT^2+1), \gamma(d) + \gamma_1\}$$

and Proposition 1 yields $\|u\| < \eta(\rho_2) \leq \eta(\rho_1)$, where ρ_1 was defined in Lemma 5. Hence, this first case can be excluded if we take $\|u\| \geq R_2 \geq \eta(\rho_1)$.

Suppose now that we are in the latter situation, that is, $|u(\sigma^*)| > d$. Just to fix the next discussion, say that $u(\sigma^*) > d$. Hence, as $u' > 0$ in J_0, we have that $u(t) > d$, for all $t \geq \sigma^*$. Now we can use (5) and obtain for the function $y(t) := u'(t)$ that

$$y(\sigma^*) > 0, \quad y'(\sigma^*) = 0, \qquad y''(\sigma^*) \geq 0, \qquad y'''(t) \geq \varepsilon, \qquad \forall\, t \in [\sigma^*, s_0).$$

From the Taylor formula applied to y we find

$$y(t) \geq y(\sigma^*) > 0, \qquad \forall\, t \in [\sigma^*, s_0).$$

Hence, passing to the limit as $t \to s_0$, we obtain $0 = y(s_0) \geq y(\sigma^*)$, contradicting the fact that $y = u' > 0$ in (s_{-1}, s_0). Clearly, the same argument works if we assume $u(\sigma^*) < -d$ (the only difference is that this time we would find a contradiction letting $t \to s_{-1}$).

Thus we have seen that the second case in our discussion can never occur and therefore the result is proved. □

From Lemma 6 we have that if $\|u\| \geq R_2$, then

$$u''(t) < 0, \qquad \forall\, t \in (\sigma_i, \sigma_{i+1}), \qquad i \text{ even}$$

and

$$u''(t) > 0, \qquad \forall\, t \in (\sigma_i, \sigma_{i+1}), \qquad i \text{ odd},$$

for $0 \leq i \leq \mathbf{n}(u) - 1$.

We introduce now a further definition. Let $D > 0$ be a given constant and consider the set

$$\mathcal{W}_D(u) := \{t \in \mathbb{R} : \ |u(t)| < D\}.$$

It is clear that $\mathcal{W}_D(u)$ is a T-periodic set and we already know that if $\|u\| \geq \max\{\beta(D), R_2\}$, then $u'(t) \neq 0$ for all $t \in \mathcal{W}_D(u)$ (see Proposition 3 and Lemma 5). More details are given in the following result.

PROPOSITION 4. *For any $D > 0$ and $A > 0$, there is a constant $R(D, A) \geq R_2$ such that if u is any solution of* (P_λ), *then*

$$\|u\| \geq R(D, A), \qquad \text{implies} \qquad \min\{|u'(t)| \ : \ t \in \mathcal{W}_D(u)\} \geq A.$$

PROOF. As remarked above, from Proposition 3 and Lemma 5, we have that if $\|u\| \geq \max\{\beta(D), R_2\}$, then $\mathcal{W}_D$ is made by the union of disjoint open intervals and it can be easily seen that $\mathcal{W}_D(u) \cap [s_0, s_0 + T]$ consists of the union of exactly

$\mathbf{n}(u)$ open intervals. Now, let (τ_0, τ_1) be one of such intervals and, for definiteness, suppose that

$$u'(t) < 0, \qquad \forall\, t \in (\tau_0, \tau_1) \subset (s_0, s_1) := J_1^-.$$

By Lemma 6, we know that $u' < 0$ in J_1^- and $u''(t) \neq 0$ for any $t \in J_1^- \setminus \{\sigma_1\}$. We now examine possibilities. First,

$$(\mathrm{a}_1) \qquad \sigma_1 \in (s_0, \tau_0].$$

In this case, $(\sigma_1, s_1) \supset [\tau_0, s_1)$, and therefore

$$u'(t) < 0, \qquad \text{and} \;\; u''(t) > 0, \qquad \forall\, t \in [\tau_0, s_1).$$

Hence,

$$(10) \qquad \min\{|u'(t)| : \; t \in [\tau_0, \tau_1]\} = |u'(\tau_1)| = \max\{|u'(t)| : \; t \in [\tau_1, s_1]\}.$$

Therefore, from the second equality in (10) and using the fact that $u(\tau_1) = -D$ and $u(s_1) = \min\{u(t) \;:\; t \in [\tau_1, s_1]\} = \min\{u(t) \;:\; t \in (t_1, t_2)\} = u_1^-$, we easily obtain

$$(11) \qquad |u_1^-| \leq D + (s_1 - \tau_1)\max|u'| \leq D + T|u'(\tau_1)|.$$

Then, from (11), the first equality in (10) and Proposition 3, we have that

$$(12) \qquad \min\{|u'(t)| \;:\; t \in [\tau_0, \tau_1]\} \geq A, \qquad \text{for } \;\|u\| \geq \beta(D + AT).$$

The second possibility to consider is

$$(\mathrm{a}_2) \qquad \sigma_1 \in (\tau_0, \tau_1).$$

In this case, a moment of reflection (arguing like in (a_1)), shows that

$$\begin{aligned} \min\{|u'(t)| \;:\; t \in [\tau_0, \tau_1]\} &= \min\{|u'(\tau_0)|, |u'(\tau_1)|\} \\ &= \min\left\{\max\{|u'(t)| \;:\; t \in [s_0, \tau_0]\}, \max\{|u'(t)| \;:\; t \in [\tau_1, s_1]\}\right\}. \end{aligned}$$

From this, we easily obtain

$$\min\{u_0^+, |u_1^-|\} \leq D + T\min\{|u'(\tau_0)|, |u'(\tau_1)|\},$$

where we recall that $u_0^+ = \max\{u(t) \;:\; t \in (t_0, t_1)\} = \max\{u(t) \;:\; t \in [s_0, \tau_0]\} = u(s_0)$. Hence Proposition 3 yields (12) as well.

The third and last possibility is

$$(\mathrm{a}_3) \qquad \sigma_1 \in [\tau_1, s_1).$$

Clearly, this case is completely symmetric to (a_1) and (12) can be proved in the same manner.

Since the estimates we have found are independent of the particular subinterval of $\mathcal{W}_D(u)$ that we have chosen, we can conclude that the result is proved, just taking

$$R(D,A) := \max\{\beta(D+AT), R_2\}.$$

□

REMARK 2. Take $D = A = 1$ and apply Proposition 3 and Proposition 4 to obtain that $(u(t),u'(t)) \notin (-1,1)^2$, for all $t \in \mathbb{R}$. Hence, $u(t)^2 + u'(t)^2 \geq 1$ for all t and therefore Remark 1 and the definition of the functional ϕ yield

$$\phi(u,\lambda) = \frac{\mathbf{n}(u)}{2}. \tag{13}$$

Then, the first condition of Lemma 1 (i.e. (k_1)) is fulfilled with $R^* = R(1,1)$.

We notice, that until now, only the sign condition (i) has been used. To proceed further, we need to find some estimates for the distance of two consecutive zeros of u. Hence we set

$$\Delta(u) := \max_{0 \leq i \leq \mathbf{n}(u)-1} \{t_{i+1} - t_i\}$$

and observe that

$$\mathbf{n}(u) \geq \frac{T}{\Delta(u)}$$

so that, we can prove that $\mathbf{n}(u) \to +\infty$ as $\|u\| \to +\infty$ if we show, that at the same moment $\Delta(u) \to 0$. To do this, we need an upper bound for $\Delta(u)$. This comes from our last lemma below.

LEMMA 7. *Let $D_0 \geq d$ and $K > 0$ be such that*

$$\frac{g(x)}{x} \geq H_0^4 > K^4, \qquad \text{for all } |x| \geq D_0.$$

Then there is a constant $r_K \geq R_2$ such that

$$\|u\| \geq r_K, \qquad \text{implies } \Delta(u) \leq \frac{2\pi}{K}.$$

PROOF. First of all, we take $D \geq D_0$ and $H > 0$ with

$$H_0^4 > H^4 > K^4,$$

such that

$$f(t,x,y,\lambda)\mathrm{sign}(x) \geq H^4|x|, \qquad \forall\, |x| \geq D,\ (t,y) \in \mathbb{R}^2,\ \lambda \in [0,1].$$

Hence, we have immediately that for any solution u of (P_λ),

$$u''''(t)\mathrm{sign}(u(t)) \geq K^4|u(t)|, \qquad \forall\, t \in \mathcal{M} := \{t \in \mathbb{R} : |u(t)| \geq D\} = \mathbb{R} \setminus \mathcal{W}_D(u).$$

Taking $\|u\|$ sufficiently large (e.g. $\|u\| \geq \max\{\beta(D), R_2\}$), we have that the points s_i, which are the points of local maximum (for i even) and minimum (for i odd) for u, are such that $|u(s_i)| > D$, with $u'(t) \neq 0$ for all $t \neq s_i$ (see Proposition 3 and Lemma 5). Then, the set $\mathcal{M}$ is made by the disjoint union of closed intervals. In particular, $\mathcal{M} \cap [t_0, t_0 + T]$ is made by the union of $\mathbf{n}(u)$ disjoint intervals where $u(t)$ alternates its sign.

Let $[\alpha_0, \alpha_1] \subset (t_0, t_1)$ be one of the intervals of $\mathcal{M}$, so that we can assume;

$$u(t) > D, \qquad \forall\, t \in (\alpha_0, \alpha_1),$$

$$0 < u(t) < D, \qquad \forall\, t \in (t_0, \alpha_0) \cup (\alpha_1, t_1)$$

and $s_0 \in (\alpha_0, \alpha_1)$, with s_0 the point of maximum of u in $[t_0, t_1]$.

We claim that

$$\alpha_1 - \alpha_0 \leq \frac{2\pi}{H}. \tag{14}$$

Indeed, assume, by contradiction, that $\alpha_1 - \alpha_0 > \frac{2\pi}{H}$ then we choose a number α such that

$$s_0 \in (\alpha - \frac{\pi}{H}, \alpha + \frac{\pi}{H}) \subset (\alpha_0, \alpha_1)$$

and consider the function

$$v(t) := 1 + \cos H(t - \alpha) \qquad \text{for } \ a := \alpha - \frac{\pi}{H} \leq t \leq \alpha + \frac{\pi}{H} := b.$$

It is easy to check that all the following properties are satisfied.

$$v(a) = v(b) = v'(a) = v'(b) = v'''(a) = v'''(b) = 0,$$

$$v''(a) = v''(b) = H^2, \quad v''''(t) = H^4(v(t) - 1), \qquad \forall\, t \in [a, b],$$
$$u'(a) > 0 > u'(b), \qquad u''''(t) \geq H^4 u(t), \qquad \forall\, t \in [a, b].$$

Then we have

$$\begin{aligned} 0 > H^2(u'(b) - u'(a)) &= \int_a^b \frac{d}{dt}(u'''v - v'''u - u''v' + u'v'')\, dt \\ &= \int_a^b (u''''(t)v(t) - v''''(t)u(t))\, dt \geq \int_a^b H^4(u(t)v(t) - (v(t) - 1)u(t))\, dt \\ &= H^4 \int_a^b u(t)\, dt > 0 \end{aligned}$$

and a contradiction is obtained. Thus (14) holds true.

Now we choose a constant

$$A > \frac{1}{\pi}\frac{DHK}{H - K}$$

and apply Proposition 4 from which we know that if $\|u\| \geq R(D, A)$, then $|u'(t)| \geq A$ for all $t \in (t_0, \alpha_0) \cup (\alpha_1, t_1)$. Consequently, we obtain

$$(\alpha_0 - t_0) + (t_1 - \alpha_1) \leq 2D/A = 2\pi \frac{H-K}{HK}. \tag{15}$$

From (14) and (15) we have that

$$t_1 - t_0 = (\alpha_0 - t_0) + (\alpha_1 - \alpha_0) + (t_1 - \alpha_1) \leq \frac{2\pi}{H} + 2\pi \frac{H-K}{HK} = \frac{2\pi}{K}.$$

Clearly, the same upper bound can be obtained for any of the intervals $[t_i, t_{i+1}]$ and hence we have proved our result, for

$$r_K := R(D, A),$$

with D and A defined in the course of the proof. □

Now we are in a position to conclude.

4. Proof of Theorem 1

As announced before, we use Lemma 1.

In Remark 2, we have already seen that (k_1) is satisfied and, via formula (13), $\phi(u, \lambda) = \frac{\mathrm{n}(u)}{2}$, for each $(u, \lambda) \in \Sigma$ with $\|u\| \geq R^*$, so that $\phi(u, \lambda) \geq 1$.

In order to prove (k_2), we fix any natural number $k \geq 1$ and consider the set $\mathcal{U}_k$ of the solutions u of (P_λ) such that $\mathrm{n}(u) = 2k$. Using (j), we find $D_k \geq d$ and $K = K_k = (4k\pi/T) + 1$ satisfying

$$\frac{g(x)}{x} \geq (K+1)^4, \qquad \text{for all } |x| \geq D_k.$$

Then, there is a constant $R_k^* := r_K$ which makes the inference of Lemma 7 fulfilled. Now we claim that

$$\|u\| < R_k^*, \quad \forall u \in \mathcal{U}_k.$$

Indeed, if, by contradiction, there is a solution u with $\|u\| \geq R_k^*$, then, by the choice of K, $\Delta(u) < T/2k$ and therefore, $\mathrm{n}(u) \geq T/\Delta(u) > 2k$, which implies $u \notin \mathcal{U}_k$.

Then also, (k_2) is satisfied and the theorem is proved. □

5. Final remarks

The result can be extended to the slightly more general problem

$$(\mathrm{P}') \qquad \begin{cases} u'''' = g(u) + e(t, u, u', u'', u'''), \\ u(T) - u(0) = u'(T) - u'(0) = u''(T) - u''(0) = u'''(T) - u'''(0) = 0, \end{cases}$$

where $e : [0,T]\times\mathbb{R}^4 \to \mathbb{R}$ is a Caratheodory function such that $|e(t,x,y,x,w)| \leq q(t)$ for almost every $t \in [0,T]$ and all $(x,y,z,w) \in \mathbb{R}^4$, with $q \in L^1([0,T],\mathbb{R}^+)$.

It is also possible to obtain a variant of Theorem 1 to the case of "singularities", assuming that g is defined on a open interval $(A,B) \subset \mathbb{R}$ and replacing the condition of superlinear growth at $\pm\infty$ with a similar assumption at the singularities, like in [6]; that is, for some $c \in (A,B)$,

$$\lim_{x\to A^+} \frac{g(x)}{x-c} = \lim_{x\to B^-} \frac{g(x)}{x-c} = +\infty,$$

and

$$\lim_{x\to A^+} \int_c^x g(s)\,ds = \lim_{x\to B^-} \int_c^x g(s)\,ds = +\infty.$$

The fast oscillatory behaviour of large solutions of (P_λ) suggests the possibility of investigating the existence of infinitely many periodic solutions in the case of problem (P) with $e = e(t)$. This problem is completely solved in the affirmative for the second order equation $-u'' = g(u)+e(t)$, with g satisfying (j), via the Poincaré-Birkhoff fixed point theorem. Here it seems not obvious how to apply such a method which provides fixed points for an area – preserving homeomorphism of the plane. Other tools which have been used to obtain multiplicity of solutions in the superlinear case, for periodic problems having variational structure, are based on critical point theory and replace condition (j) with the more restrictive assumption

$$(\ell) \qquad xg(x) \geq kG(x) > 0, \qquad \text{for } |x| \geq d > 0, \qquad (k > 2),$$

(see [2, 9]). It should be interesting to see whether it is possible to apply these variational methods for problem (1) under condition (j) or (ℓ).

The proof we follow makes strong use, in several steps, of the fact that we are analyzing the solutions of a fourth order equation, and hence some parts of our argument would not apply to higher order boundary value problems like

$$(\mathrm{P}_n) \qquad \begin{cases} (-1)^n u^{(2n)} = g(u) + e(t), \\ u(\cdot) \quad T\text{ - periodic.} \end{cases}$$

The problem to extend our result to (P_n) with $n \geq 3$ remains open. Finally, we recall that for problems of the form

$$\begin{cases} \pm u^{(2k+1)} = g(u) + e(t), \\ u(\cdot) \quad T\text{ - periodic,} \end{cases}$$

the existence of solutions is ensured by a simple sign condition on g which is implied by (i) (cf. [10]).

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Jean Mawhin
Université Catholique de Louvain
Institut de Mathématique Pure et Appliquée
Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, BELGIUM

Fabio Zanolin
Università di Udine
Dipartimento di Matematica e Informatica
via Zanon 6, I-33100 Udine, ITALY