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ON PROVING EXISTENCE AND SMOOTHNESS OF INVARIANT MANIFOLDS IN SINGULAR PERTURBATION PROBLEM

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(*Submitted by A. Granas*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

Consider the following system of ordinary differential equations:

$$(\mathrm{S})_\varepsilon \qquad u' = f(u, v, \varepsilon), \quad \varepsilon v' = g(u, v, \varepsilon).$$

We make the following standing assumptions:

(H1) r is an integer with $r \geq 3$,

$$f : (u, v, \varepsilon) \in \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R} \mapsto f(u, v, \varepsilon) \in \mathbf{R}^m$$
$$g : (u, v, \varepsilon) \in \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R} \mapsto g(u, v, \varepsilon) \in \mathbf{R}^n$$

are C^r-bounded maps, $h : \mathbf{R}^m \to \mathbf{R}^n$ is a map of class C^r with all derivatives of order p with $1 \leq p \leq r$ globally bounded and such that $g(u, h(u), 0) \equiv 0$.

The author is indebted to Professor Pavol Brunovský for many stimulating conversations

(H2) For some integer k with $0 \leq k \leq n$ and some positive real μ the $n \times n$-matrix $A(u) := D_v g(u, h(u), 0)$ has, for every $u \in \mathbf{R}^m$, k eigenvalues with real part $< -2\mu$ and $n-k$ eigenvalues with real part $> 2\mu$.

The following result is known:

THEOREM 1. (i) *There is an* $\varepsilon_0 > 0$ *and a* C^{r-1}*-function* $h : \mathbf{R}^m \times [0, \varepsilon_0] \to \mathbf{R}^n$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *the set*

$$C_\varepsilon := \{(u, h(u, \varepsilon)) | u \in \mathbf{R}^m\}$$

is invariant with respect to $(\mathrm{S})_\varepsilon$ *and* $\sup_{u \in \mathbf{R}^m} |h(u, \varepsilon) - h(u)| \to 0$ *as* $\varepsilon \to 0$.

(ii) *There is a constant* $\delta > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0]$ *every solution* $t \mapsto (u(t), v(t))$ *of* $(\mathrm{S})_\varepsilon$ *which remains, for all* $t \in \mathbf{R}$, *in the* δ*-neighborhood of* $C_0 = \{(u, h(u)) | u \in \mathbf{R}^m\}$, *lies in* C_ε.

The center-like manifold C_ε is very important for the analysis of the singular perturbation problem $(\mathrm{S})_\varepsilon$ and we only refer the reader to the recent paper [16] for an interesting application of Theorem 1 to a model of electron transport in semiconductors.

A stronger and global version of Theorem 1 is given in [5, Theorem 9.1] with a proof based on the invariant manifold theory from [4]. The beatiful partly geometric and partly analytic technique (similar to the method presented in the lecture notes [9]) works for diffeomorphisms and flows on manifolds but does not appear to extend to the case of maps or semiflows (generated, for example, by semilinear parabolic equations).

An analytic proof which does generalize to maps and semiflows is presented in [14]. It is based on the method of functions of exponential growth developed in [3], [6], [7], [15], [19] and [18] and also used in [1], [10], [11], [17], [20] as well as in the excellent recent article [2]. Whereas in these papers the existence of the invariant manifolds in question can be obtained by a direct application of the contraction mapping principle to the appropiate fixed point equation this is no longer possible in the situation of Theorem 1 since the domain of definition of the corresponding contraction operator is not closed in the Banach space considered. To remedy this, Sakamoto first substitutes the solutions $t \mapsto u(t)$ (depending on initial values ξ and functions $t \mapsto v(t)$ as parameters) of the first equation in $(\mathrm{S})_\varepsilon$ into the second equation. This substitution leads to excessively complex expressions and estimates in the proof of higher order differentiability of the graph of C_r. This is probably the reason why Sakamoto only gives details of the existence and the C^1-smoothness part of the proof. (Cf. the remark made in the proof of [14, Lemma 2.5].)

In this note we give a simpler proof of Theorem 1 which is more in the spirit of the above quoted papers.

In fact we work directly in spaces of functions of exponential growth without any previous substitution of one equation into the other. We obtain existence of C_ε by a simple modification of the contraction mapping theorem. The approach we choose considerably simplifies the expressions for higher order derivatives of the operators involved so we are able to give a complete and reasonably short proof of the C^{r-1}-differentiability of C_ε. The proof is further shortened by the application of an abstract differentiability result for solutions of fixed point equations on scales of Banach spaces which was developed in [11]. In the course of the proof we obtain precise recursive formulas for the (higher order) Fréchet derivatives of the map generating the invariant manifold. Such formulas can be useful e.g. in applications of hard implicit function theorems (cf. [12] and [13]). The approach presented in this note can also be applied to maps or semiflows, e.g. to obtain smoothness of some invariant manifolds whose existence is established in [8]. However, this will not be treated here.

2. Existence

In the sequel, given $\rho \in \mathbf{R}$ and a normed space $(E, |\cdot|)$ we write, for an arbitrary function $y : \mathbf{R} \to E$

$$|y|_\rho := \sup_{t\in\mathbf{R}} \mathrm{e}^{-\rho|t|}|y(t)|.$$

By $\mathrm{BC}^\rho(E)$ we denote the vector space of all *continuous* functions $y : \mathbf{R} \to E$ such that $|y|_\rho < \infty$. Note that $|\cdot|_\rho$ is a norm on $\mathrm{BC}^\rho(E)$ which is complete if $|\cdot|$ is complete on E.

For the reader's convenience we first collect a few preliminary results which are explicitly or implicitly contained in [14].

LEMMA 1. [14, Lemma 2.3]. *Under hypothesis* (H2) *for every positive real number N there exists an $\varepsilon_1 > 0$ and a $K \geq 1$ such that for every $u \in C^1(\mathbf{R}, \mathbf{R}^m)$ with $|u'| = \sup_{t\in\mathbf{R}} |u'(t)| \leq N$ and $\varepsilon \in (0, \varepsilon_1]$ there are projection operators $P^\varepsilon(s; u)$, $Q^\varepsilon(s; u) : \mathbf{R}^n \to \mathbf{R}^n$ with $P^\varepsilon(s; u) + Q^\varepsilon(s; u) = \mathrm{id}_{\mathbf{R}^n}$, $s \in \mathbf{R}$, such that the solution operator $T^\varepsilon(t, s; u)$ of the equation*

$$\varepsilon y' = A(u(t))v$$

satisfies the inequalities

$$|T^\varepsilon(t, s; u) \circ P^\varepsilon(s; u)| \leq K\mathrm{e}^{-\mu(t-s)/\varepsilon}, \qquad t \geq s,$$
$$|T^\varepsilon(t, s; u) \circ Q^\varepsilon(s; u)| \leq K\mathrm{e}^{\mu(t-s)/\varepsilon}, \qquad t \leq s.$$

Moreover, $\dim \operatorname{Range} P^\varepsilon(s; u) = k$ *and* $\dim \operatorname{Range} Q^\varepsilon(s; u) = n - k$.

DEFINITION 1. *For ε as in Lemma 1 define the Green's function $U^\varepsilon(t,s;u)$ as follows: $U^\varepsilon(t,s;u) := T^\varepsilon(t,s;u) \circ P^\varepsilon(s;u)$ for $t \geq s$, $U^\varepsilon(t,s;u) := -T^\varepsilon(t,s;u) \circ Q^\varepsilon(s;u)$ for $t < s$.*

LEMMA 2. (a) *For $\rho > 0$ and $u_1 \in \mathrm{BC}^\rho(\mathbf{R}^m)$ set*

$$u(t) := \int_0^t u_1(s)\,ds, \qquad t \in \mathbf{R}.$$

Then $u \in \mathrm{BC}^\rho(\mathbf{R}^m)$ and $|u|_\rho \leq (1/\rho)|u_1|_\rho$.

(b) *Let ε and u be as in Lemma 1, $\rho_1 < \mu/\varepsilon$ and $\rho_2 \in [\rho_1, \mu/\varepsilon)$. Then for every $\psi_1 \in \mathrm{BC}^{\rho_2}(\mathbf{R}^n)$ of*

$$\varepsilon\psi' = A(u(t))\psi + \psi_1(t).$$

The function ψ is given by the expression

$$\psi(t) = (1/\varepsilon)\int_{-\infty}^{\infty} U^\varepsilon(t,s;u)\psi_1(s)\,ds, \qquad t \in \mathbf{R}.$$

Moreover, $|\psi|_{\rho_2} \leq (2K/(\mu - \varepsilon\rho_1))|\psi_1|_{\rho_1}$.

PROOF. Lemma 1, Definition 1 and trivial integration. □

LEMMA 3. *The change of variables $u \to u$ and $v \to v + h(u)$ transforms system $(\mathrm{S})_\varepsilon$ into the equivalent system $(\mathrm{S'})_\varepsilon$:*

$$(\mathrm{S'})_\varepsilon \qquad u' = F(u,v,\varepsilon), \qquad \varepsilon v' = A(u(t))v + G(u,v,\varepsilon).$$

Here

$$\begin{aligned} F(u,v,\varepsilon) &:= f(u, v + h(u), \varepsilon), \\ g_1(u,v,\varepsilon) &:= g(u, v + h(u), \varepsilon) - \varepsilon \mathrm{D}_u h(u) f(u, v + h(u), \varepsilon), \\ G(u,v,\varepsilon) &:= g_1(u,v,\varepsilon) - g_1(u,0,0) - \mathrm{D}_v g_1(u,0,0)v. \end{aligned}$$

There is a constant C such that

$$\begin{aligned} |G(u,v,\varepsilon)| &\leq C(\varepsilon + |v|^2), \\ |\mathrm{D}_u G(u,v,\varepsilon)| &\leq C(\varepsilon + |v|^2), \\ |\mathrm{D}_v G(u,v,\varepsilon)| &\leq C(\varepsilon + |v|) \end{aligned}$$

for all (u,v,ε).

PROOF. A trivial calculation using the fact that $r \geq 3$ and the mean value theorem. (Note that $g_1(u,0,0) \equiv 0$.) □

LEMMA 4. *Suppose that* $0 < \varepsilon \leq \varepsilon_1$ *and that* $u : \mathbf{R} \to \mathbf{R}^m$ *and* $v : \mathbf{R} \to \mathbf{R}^n$ *are continuous functions with* v *bounded. Set* $N := \sup_{(u,v,\varepsilon)} |f(u,v,\varepsilon)|$. *Then the following properties are equivalent:*

(a) (u, v) *is a solution of* $(\mathrm{S'})_\varepsilon$.

(b) $u \in C^1(\mathbf{R}, \mathbf{R}^m)$, $|u'|_0 = \sup_{t\in\mathbf{R}} |u'(t)| \leq N$ *and there is a* $\xi \in \mathbf{R}^m$ *such that for all* $t \in \mathbf{R}$

$$u(t) = \xi + \int_0^t F(u(s), v(s), \varepsilon)\, ds,$$

$$v(t) = (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u) G(u(s), v(s), \varepsilon)\, ds.$$

PROOF. Use Lemmas 2 and 3. □

LEMMA 5. *If* $0 \leq \varepsilon \leq \varepsilon_1$, $w \in \mathrm{BC}^\rho(\mathbf{R}^n)$, $\rho < \mu/\varepsilon$, $u, u_0 \in C^1(\mathbf{R}, \mathbf{R}^m)$ *with* $|u'|$, $|u_0'| \leq N$ *and*

$$\widetilde{w}(t) := (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u) w(s)\, ds, \qquad t \in \mathbf{R}$$

then

$$\widetilde{w}(t) = (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u_0)[(A(u(s)) - A(u_0(s)))\widetilde{w}(s) + w(s)]\, ds, \quad t \in \mathbf{R}.$$

PROOF. Apply Lemma 2 to $\psi := \widetilde{w}$ and $\psi_1 := (A(u(s)) - A(u_0(s)))\widetilde{w}(s) + w(s)$, $s \in \mathbf{R}$. □

We can now prove existence of manifold C_ε:

LEMMA 6. *Set*

$$N := \sup_{(u,v,\varepsilon)} |f(u, v, \varepsilon)|,$$

$$N_1 := \sup_{(u,v,\varepsilon)} \max \{|\mathrm{D}_u f(u, v, \varepsilon)|, |\mathrm{D}_v f(u, v, \varepsilon)|\},$$

$$M_2 := \sup_u |\mathrm{D}_u A(u)|.$$

Let ρ, ε_0 *and* δ *be such that*

(1) $$0 < \varepsilon_0 \leq \varepsilon_1, \ \rho > N_1, \ \delta > 0, \ C(\varepsilon_0 + \delta^2) < (\mu/2K)\delta,$$

(2) $$0 < (2K/(\mu - \varepsilon_0\rho)) \max \{M_2\delta, C(\varepsilon_0 + \delta^2), C(\varepsilon_0 + \delta)\} < 1.$$

Then for every ε *with* $0 < \varepsilon \le \varepsilon_0$ *and every* $\xi \in \mathbf{R}^m$ *there exists a unique pair* $(u, v) = (u_\xi^\varepsilon, v_\xi^\varepsilon)$ *of functions* $u \in C^1(\mathbf{R}, \mathbf{R}^m)$, $v \in C(\mathbf{R}, \mathbf{R}^n)$ *with* $|u'|_0 \le N$ *and* $|v|_0 \le \delta$ *which satisfy the following system of equations for all* $t \in \mathbf{R}$

$$u(t) = \xi + \int_0^t F(u(s), v(s), \varepsilon)\, ds, \tag{3}$$

$$v(t) = (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u) G(u(s), v(s), \varepsilon)\, ds. \tag{4}$$

The assignment $\xi \mapsto (u_\xi^\varepsilon, v_\xi^\varepsilon)$ *defines a Lipschitzian map* $\phi = \phi^\varepsilon : \mathbf{R}^m \to \mathrm{BC}^\rho(\mathbf{R}^m) \times \mathrm{BC}^\rho(\mathbf{R}^n)$.

REMARK. For every ε with $0 < \varepsilon \le \varepsilon_0$ and every $\xi \in \mathbf{R}^m$ set

$$h(\xi, \varepsilon) := h(\xi) + \pi_v \phi^\varepsilon(\xi)(0)$$

where $\pi_v : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection operator. Then Lemmas 3, 4 and 6 easily imply all statements of Theorem 1 except for smoothness.

PROOF OF LEMMA 6. Let $\mathcal{A}$ be the set of all pairs (u, v) of functions $u \in C^1(\mathbf{R}, \mathbf{R}^m)$, $v \in C(\mathbf{R}, \mathbf{R}^n)$ with $|u'|_0 \le N$ and $|v|_0 \le \delta$. Since $\rho > 0$ it follows that $\mathcal{A} \subset \mathrm{BC}^\rho(\mathbf{R}^m) \times \mathrm{BC}^\rho(\mathbf{R}^n)$.

Fix ε with $0 < \varepsilon \le \varepsilon_0$ and define for $(u, v) \in \mathcal{A}$, $\xi \in X := \mathbf{R}^m$ and $t \in \mathbf{R}$

$$\mathcal{F}_1(u, v)(t) := \xi + \int_0^t F(u(s), v(s), \varepsilon)\, ds, \tag{5}$$

$$\mathcal{F}_2(u, v)(t) := (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u) G(u(s), v(s), \varepsilon)\, ds, \tag{6}$$

$$\mathcal{F}(u, v, \xi) := (\mathcal{F}_1(u, v, \xi), \mathcal{F}_2(u, v)). \tag{7}$$

From formula (1) and from Lemma 2 we conclude that $\mathcal{F}$ is well-defined map from $\mathcal{A} \times X$ to $\mathcal{A}$. By Lemma 2(a)

$$|\mathcal{F}_1(u, v, \xi) - \mathcal{F}_1(u_0, v_0, \xi)|_\rho \le (N_1/\rho)(|u - u_0|_\rho + |v - v_0|_\rho). \tag{8}$$

By Lemma 5

$$(\mathcal{F}_2(u, v) - \mathcal{F}_2(u_0, v_0))(t) = (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u) \psi_1(s)\, ds, \qquad t \in \mathbf{R} \tag{9}$$

where

$$\psi_1(s) = (A(u(s)) - A(u_0(s)))\mathcal{F}_2(u, v)(s) + G(u(s), v(s), \varepsilon) - G(u_0(s), v_0(s), \varepsilon). \tag{10}$$

The function $\psi_1 : \mathbf{R} \to \mathbf{R}^n$ is continuous and satisfies, by our assumptions, the estimate

$$|\psi_1(s)| \le \mathrm{e}^{\rho|s|} \max\{M_2\delta, C(\varepsilon_0 + \delta^2), C(\varepsilon_0 + \delta)\}(|u - u_0|_\rho - |v - v_0|_\rho).$$

Fix $\xi \in \mathbf{R}^m$. Choose any $(u_0, v_0) \in \mathcal{A}$ ($\neq \emptyset$!) and define the sequence (u_k, v_k) reqursively by

$$(u_{k-1}, v_{k+1}) = \mathcal{F}(u_k, v_k, \xi), \qquad k \geq 0.$$

It follows from Lemma 2 and formulas (1), (2), (8), (9) and (10) that $\mathcal{F}(\cdot,\cdot,\xi)$ is a contraction relative to the norm $|u|_\rho + |v|_\rho$. Thus (u_k, v_k) is a Cauchy sequence in $\mathrm{BC}^\rho(\mathbf{R}^m)\times\mathrm{BC}^\rho(\mathbf{R}^n)$ and so it has a limit $(u,v) = (u^\varepsilon_\xi, v^\varepsilon_\xi)$ in $\mathrm{BC}^\rho(\mathbf{R}^m)\times\mathrm{BC}^\rho(\mathbf{R}^n)$.

In particular $(u_k(s), v_k(s)) \to (u(s), v(s))$ uniformly for s in compact subsets of $\mathbf{R}$. Hence, by (5),

$$u = \mathcal{F}_1(u, v, \xi)$$

and so, in particular, $u \in C^1(\mathbf{R}, \mathbf{R}^m)$ and $|u'|_0 \leq N$. By the same token $|v|_0 \leq \delta$. It follows that $(u,v) \in \mathcal{A}$. (The point here is that $\mathcal{A}$ is not closed in $\mathrm{BC}^\rho(\mathbf{R}^m) \times \mathrm{BC}^\rho(\mathbf{R}^n)$, so the contraction principle cannot be used directly).

Thus (u,v) satisfies (3) and (4) for all $t \in \mathbf{R}$. The contractivity property of $\mathcal{F}$ implies the uniqueness statement of the lemma. Finally, as the map $\mathcal{F}(u_0, v_0, \cdot) : \mathbf{R}^m \to \mathrm{BC}^\rho(\mathbf{R}^m) \times \mathrm{BC}^\rho(\mathbf{R}^n)$ is Lipschitzian, the assignment $\xi \mapsto (u^\varepsilon_\xi, v^\varepsilon_\xi)$ defines a Lipschitzian map $\phi^\varepsilon : \mathbf{R}^m \to \mathrm{BC}^\rho(\mathbf{R}^m) \times \mathrm{BC}^\rho(\mathbf{R}^n)$. The lemma is proved. □

3. Smoothness

To facilitate a direct comparison with the arguments in [14] we shall show in this section that the map $\xi \mapsto h(\xi, \varepsilon)$ is C^{r-1}-smooth for fixed ε sufficiently small. Once the details are understood for this case, the proof of joint smoothness in (ξ, ε) can safely be left to the reader (similarly as it is done in [14]).

The main ingredient in the proof is the following

LEMMA 7. *Let N, N_1, M_2 be as in Lemma 6 and let b, ρ, ε_0 and δ be such that*

(1) $$b > 1,\ 0 < \varepsilon_0 \leq \varepsilon_1,\ \rho > N_1,\ \delta > 0,\ C(\varepsilon_0 + \delta^2) < (\mu/2K)\delta$$

(2) $$0 < \kappa_1 := (2K/(\mu - \varepsilon_0\rho(r-1)b)) \max\{M_2\delta, C(\varepsilon_0 + \delta^2), C(\varepsilon_0 + \delta)\} < 1$$

Then for every ε with $0 < \varepsilon \leq \varepsilon_0$ the map $\phi = \phi^\varepsilon$ defined in Lemma 6 is of class C^{r-1} from $\mathbf{R}^m$ to $\mathrm{BC}^\zeta(\mathbf{R}^m) \times \mathrm{BC}^\zeta(\mathbf{R}^n)$ for every $\zeta \geq (r-1)\rho b$.

We shall prove Lemma 7 below. In the following we fix ε with $0 < \varepsilon \leq \varepsilon_0$ and write ϕ, u_ξ, v_ξ, $\mathcal{F}$, $F(u,v)$, $G(u,v)$ instead of ϕ^ε, u^ε_ξ, v^ε_ξ, $\mathcal{F}^\varepsilon$, $F(u,v,\varepsilon)$, $G(u,v,\varepsilon)$. Note that ϕ satisfies the fixed-point equation

$$\phi(\xi) = \mathcal{F}(\xi, \phi(\xi)).$$

Write $\psi(\xi) := (\xi, \phi(\xi))$.

Let us first proceed heuristically. If the higher-order chain rule were applicable to the composite map $\mathcal{F} \circ \psi$ then we could express the Fréchet derivatives $\mathrm{D}^k\psi(\xi)$ of $\phi = \mathcal{F} \circ \psi$ as finite sums involving derivatives $\mathrm{D}^j\mathcal{F}(\xi, \phi(\xi))$. Let us see what these latter derivatives should look like. Let us abbreviate the notation writing $y, y_1, \ldots$ for $(u, v), (u_1, v_1), \ldots$ etc. Abusing slightly the notation we also write, e.g. $y(t)$ instead of $(u(t), v(t))$ so that y becomes a function from $\mathbf{R}$ to $\mathbf{R}^m \times \mathbf{R}^n$. Now formula (5) in the proof of Lemma 6 suggests the following definitions:

$$\mathrm{D}^1_{\text{formal}}\mathcal{F}_1(\xi, y)(\xi_1, y_1)(t) := \xi_1 + \int_0^t \mathrm{D}F(y(s))y_1(s)\,ds \tag{3}$$

and

$$\begin{aligned}(4) \qquad & \mathrm{D}^j_{\text{formal}}\mathcal{F}_1(\xi, y)(\xi_1, y_1)\ldots(\xi_j, y_j)(t) \\ & \qquad := (1/\varepsilon)\int_0^t \mathrm{D}^j_{\text{formal}}F(y(s))y_1(s)\ldots y_j(s)\,ds,\end{aligned}$$

for $2 \leq j \leq r-1$. Here $t \in \mathbf{R}$, $\xi, \xi_1, \ldots, \xi_j \in \mathbf{R}^m$ and $y, y_1, \ldots, y_j$ are appropriate functions from $\mathbf{R}$ to $\mathbf{R}^m \times \mathbf{R}^n$.

To "calculate" $\mathrm{D}^j_{\text{formal}}\mathcal{F}_2$ note that $\mathcal{F}_2$ satisfies the equation

$$\varepsilon(\mathcal{F}_2(y))'(t) = A(u(t))\mathcal{F}_2(y)(t) + G(y(t)).$$

Differentiating the last equation formally, assuming that the Fréchet derivatives and the time derivative commute and using the Leibniz rule we obtain

$$\begin{aligned}(5) \qquad & \varepsilon(\mathrm{D}^j_{\text{formal}}\mathcal{F}_2(y)y_1\ldots y_j)'(t) \\ & \qquad = \sum_{(N,M)\in\mathcal{S}} D_u^{\#N}A(u(t))[u_i(t)]_{i\in N}\mathrm{D}^{\#M}_{\text{formal}}\mathcal{F}_2(y)[y_i]_{i\in M}(t) \\ & \qquad\quad + \mathrm{D}^jG(y(t))y_1(t)\ldots y_j(t).\end{aligned}$$

Here $\mathcal{S}$ is defined as the set of all pairs (N, M) with $N, M \subset \{1, \ldots, j\}$, $N \cup M = \{1, \ldots, j\}$ and $N \cap M = \emptyset$. Note that N or M may be empty so we set

$$\mathrm{D}^0_{\text{formal}}\mathcal{F}_2(y)[y_i]_{i\in\emptyset} := \mathcal{F}(y)$$

and similarly for D^0_uA. We also use the following self-explanatory notation: if $(x_1, \ldots, x_n)$ is an n-tuple and $S \subset \{1, \ldots, n\}$ then $[x_i]_{i\in S} := (x_{i_1}, \ldots, x_{i_m})$ where $i_1 < i_2 < \ldots < i_m$ are all the elements of S put in ascending order. We can write (5) in the form

$$\varepsilon(\mathrm{D}^j_{\text{formal}}\mathcal{F}_2(y)y_1\ldots y_j)'(t) = A(u(t))\mathrm{D}^j_{\text{formal}}\mathcal{F}_2(y)y_1\ldots y_j(t) + \psi_1(t).$$

Here, $\mathcal{S}'$ is the set of all $(N, M) \in \mathcal{S}$ with N nonepmty and

$$(6) \qquad \psi_1(t) := \sum_{(N,M)\in\mathcal{S}'} \mathrm{D}_u^{\#N} A(u(t))[u_i(t)]_{i\in N} \mathrm{D}_{\mathrm{formal}}^{\#M} \mathcal{F}_2(y)[y_i]_{i\in M}(t) + \mathrm{D}^j G(y(t)) y_1(t) \dots y_j(t).$$

Thus Lemma 2(b) with

$$\psi := \mathrm{D}_{\mathrm{formal}}^j \mathcal{F}_2(y) y_1 \dots y_j$$

suggests that

$$(7) \qquad \mathrm{D}_{\mathrm{formal}}^j \mathcal{F}_2(y) y_1 \dots y_j(t) := (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u) \psi_1(s)\, ds, \qquad t \in \mathbf{R}$$

Write

$$(8) \qquad f_{0,k}(\xi) := \mathcal{F}_k(\xi, \phi(\xi)), \qquad k = 1, 2,$$

and

$$(9) \qquad f_{j,1}(\xi)(\xi_1, y_1) \dots (\xi_j, y_j) := \mathrm{D}_{\mathrm{formal}}^j \mathcal{F}_1(\xi, \phi(\xi))(\xi_1, y_1) \dots (\xi_j, y_j), \qquad j = 1, \dots, r-1,$$

$$(10) \qquad f_{j,2}(\xi)(\xi_1, y_1) \dots (\xi_j, y_j) := \mathrm{D}_{\mathrm{formal}}^j \mathcal{F}_2(\phi(\xi)) y_1 \dots y_j, \qquad j = 1, \dots, r-1,$$

the right-hand sides of (9) and (10) being defined by (3), (4), (6) and (7) whenever these right-hand sides make sense.

The relevant properties of the maps $f_{j,k}$ are started in the following

LEMMA 8. *Set*

$$\kappa := \max\{\kappa_1, N_1/\rho\} < 1$$

where κ_1 is defined in the statement of Lemma 7.

Let a be a number with $0 < a < \mu/\varepsilon$. Then the following statements hold:

(1) *There is a constant $C' = C'(a)$ such that for every $j \in \{1, \dots, r-1\}$, every tuple $(\eta_1, \dots, \eta_j)$ of positive real numbers with $\eta := \eta_1 + \dots + \eta_j \le a$, all $\xi, \xi_1, \dots, \xi_j \in X := \mathbf{R}^m$ and all $y_1, \dots, y_j$ with $y_i \in \mathrm{BC}^{\eta_i}(\mathbf{R}^n)$ for $i = 1, \dots, j$, the function $f_{j,k}(\xi)(\xi_1, y_1) \dots (\xi_j, y_j)$, $k = 1, 2$, is well-defined, it lies in $\mathrm{BC}^\eta(\mathbf{R}^m)$ for $k = 1$, it lies in $\mathrm{BC}^\eta(\mathbf{R}^n)$ for $k = 2$ and*

$$|f_{j,k}(\xi)(\xi_1, y_1) \dots (\xi_j, y_j)|_\eta \le C' \prod_{i=1}^{j} (|\xi_i| + |y_i|_{\eta_i}).$$

(2) *Let $j \in \{0, \dots, r-1\}$, $(\eta_1, \dots, \eta_j)$ be a tuple of positive real numbers and let $\eta := \eta_1 + \dots + \eta_j$ if $j \ne 0$, $\eta := \rho$, if $j = 0$. Suppose that $\eta \le a$. Then for every $\zeta < \eta$ and $k = 1, 2$*

$$|(f_{j,k}(\xi + h) - f_{j,k}(\xi))(\xi_1, y_1) \dots (\xi_j, y_j)|_\zeta \to 0 \quad \text{as} \quad h \to 0$$

uniformly for $\xi_i \in X$, $y_i \in \mathrm{BC}^{\eta_i}(\mathbf{R}^m \times \mathbf{R}^n)$ *with* $|\xi_i| \le 1$ *and* $|y_i|_{\eta_i} \le 1$ *for* $i = 1, \dots, j$.

(3) *Let* $j \in \{0, \dots, r-2\}$, $(\eta_1, \dots, \eta_j)$ *be a tuple of positive real numbers and let* $\eta := \rho + \eta_1 + \dots + \eta_j$ *if* $j \ne 0$, $\eta := \rho$, *if* $j = 0$. *Suppose that* $\eta \le a$. *Then for every* $\zeta > \eta$ *and* $k = 1, 2$

$$|(f_{j,k}(\xi + h) - f_{j,k}(\xi))(\xi_1, y_1) \dots (\xi_j, y_j) \\ - f_{j+1,k}(\xi)(h, \phi(\xi + h) - \phi(\xi))(\xi_1, y_1) \dots (\xi_j, y_j)|_\zeta = o(|h|) \quad \text{as } h \to 0$$

uniformly for $\xi_i \in X$, $y_i \in \mathrm{BC}^{\eta_i}(\mathbf{R}^m \times \mathbf{R}^n)$ *with* $|\xi_i| \le 1$ *and* $|y_i|_{\eta_i} \le 1$ *for* $i = 1, \dots, j$.

(4) $|f_{1,1}(\xi)(0, y_1)|_\zeta + |f_{1,2}(\xi)(0, y_1)|_\zeta \le \kappa |y_1|_\zeta$ *for all* ζ *with* $\rho \le \zeta \le (r-1)\rho b$, $\xi \in X$ *and* $y_1 \in \mathrm{BC}^\zeta(\mathbf{R}^m \times \mathbf{R}^n)$.

In the proof of Lemma 8 we shall use the following simple Lemma 9, whose proof is left to the reader.

LEMMA 9. *Let* X, Y *and* Z *be normed spaces,* $A \subset Y$, $\psi : A \to Z$ *be continuous and bounded and* $\phi : X \to C(\mathbf{R}, A)$ *be a map satisfying the following assumption:*

$$(*) \qquad \begin{cases} \text{For every compact interval } I \subset \mathbf{R} \text{ the map} \\ X \ni \xi \mapsto \phi(\xi)(t) \in A \subset Y \text{ is continuous} \\ \text{uniformly for } t \in I. \end{cases}$$

Then for all $\xi \in X$ *and every* $\alpha > 0$

(a) $\lim_{h \to 0} \sup_{t \in \mathbf{R}} e^{-\alpha|t|} |\psi(\phi(\xi + h)(t)) - \psi(\phi(\xi)(t))| = 0$ *and*

(b) *if* A *is convex, then*

$$\lim_{h \to 0} \sup_{t \in \mathbf{R}} e^{-\alpha|t|} \sup_{\tau \in [0,1]} |\psi(\tau\phi(\xi + h)(t) + (1 - \tau)\phi(\xi + h)(t)) - \psi(\phi(\xi)(t)| = 0.$$

Assumption $(*)$ *is satisfied if* ϕ *is continuous from* X *to* $\mathrm{BC}^\rho(Y)$ *for some* $\rho > 0$. □

PROOF OF LEMMA 8. (1) For $k = 1$ part (1) of lemma is an obvious consequence of Lemma 2(a) and the definition of $f_{j,1}$. For $k = 2$ part (1) of the lemma is easily proved by induction on $i = 1, \dots, r-1$ using the fact that $|\mathcal{F}_2(\phi(\xi))|_0 \le \delta$ together with Lemma 2(b) and the definition of $f_{j,2}$.

(2) Case $k = 1$: By Lemma 2(a) and the definition of $f_{j,1}$

$$|(f_{j,1}(\xi + h) - f_{j,1}(\xi)(\xi_1, y_1) \dots (\xi_j, y_j)|_\zeta \\ \le (1/\zeta) \sup_{t \in \mathbf{R}} e^{-\zeta|t|} |\mathrm{D}^j F(\phi(\xi + h)(t)) - \mathrm{D}^j F(\phi(\xi)(t)| \prod_{i=1}^{j} e^{\eta_i |t|} |y_i|_{\eta_i} \\ \le (1/\zeta) \sup_{t \in \mathbf{R}} e^{-\alpha|t|} |\mathrm{D}^j F(\phi(\xi + h)(t)) - \mathrm{D}^j F(\phi(\xi)(t)| \to 0 \quad \text{as } h \to 0.$$

Here we used Lemma 9(a) with $\psi = \mathrm{D}^j F$ and $\alpha := \zeta - \eta$.

Case $k = 2$: induction on $i = 0, \dots, r-1$. For $i = 0$ part (2) of the lemma follows since ϕ is Lipschitzian into $\mathrm{BC}^\rho(\mathbf{R}^m) \times \mathrm{BC}^\rho(\mathbf{R}^n) \cong \mathrm{BC}^\rho(\mathbf{R}^m \times \mathbf{R}^n)$. Now let $1 \le j \le r-1$ and suppose that part (2) of the lemma has been proved for all $j' < j$. By Lemmas 3, 5, 6 and the definition of $f_{j,2}$

$$
\begin{aligned}
&f_{j,2}(\xi + h)(\xi_1, y_1) \dots (\xi_j, y_j)(t) \\
&= (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u_\xi)[(A(u_{\xi+h}(s)) - A(u_\xi(s)))(\xi_1, y_1) \dots (\xi_j, y_j)(t) \\
&\quad + \sum_{(N,M)\in S'} D_u^{\#N} A(u_{xi+h}(s)[[u_i(s)]_{i\in N} f_{\#M,2}(\xi + h)[(\xi_i, y_i)]_{i\in M}(s) \\
&\quad + \mathrm{D}^j G(\phi(\xi + h)(s)) y_1(s) \dots y_j(s)]\, ds.
\end{aligned}
\tag{11}
$$

Thus adding and subtracting terms we obtain

$$
(f_{j,2}(\xi + h) - f_{j,2}(\xi))(\xi_1, y_1) \dots (\xi_j, y_j)(t) = (1/\varepsilon) \int_{-\infty}^{\infty} U^\varepsilon(t, s; u_\xi) \psi(s)\, ds
$$

where

$$
\begin{aligned}
\psi(s) =& (A(u_{\xi+h}(s)) - A(u_\xi(s))) f_{j,2}(\xi + h)(\xi_1, y_1) \dots (\xi_j, y_j)(s) \\
&+ \sum_{(n,M)\in S'} \Big\{ (D_u^{\#N} A(u_{\xi+h}(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi + h)[(\xi_i, y_i)]_{i\in M}(s) \\
&- (D_u^{\#N} A(u_\xi(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi + h)[(\xi_i, y_i)]_{i\in M}(s) \\
&+ (D_u^{\#N} A(u_\xi(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi + h)[(\xi_i, y_i)]_{i\in M}(s) \\
&- (D_u^{\#N} A(u_\xi(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi)[(\xi_i, y_i)]_{i\in M}(s)) \Big\} \\
&+ (\mathrm{D}^j G(\phi(\xi + h)(s)) y_1(s) \dots y_j(s) - \mathrm{D}^j G(\phi(\xi)(s)) y_1(s) \dots y_j(s))
\end{aligned}
\tag{12}
$$

Estimating the summands in (12) using Lemma 2(b), the induction hypothesis, the first part of this lemma and Lemma 9(a) (similarly as we did in the case $k = 1$) we complete the proof of part 2.

(3) Case $k = 1$: By Lemma 2(a), the definition of $f_{j,1}$ and the mean value theorem

$$
\begin{aligned}
|(&f_{j,1}(\xi + h) - f_{j,1}(\xi))(\xi_1, y_1) \dots (\xi_j, y_j) \\
&\quad - f_{j+1,1}(\xi)(h, \phi(\xi + h) - \phi(\xi))(\xi_1, y_1) \dots (\xi_j, y_j)|_\zeta \\
&\le (1/\zeta) \sup_{t\in\mathbf{R}} \mathrm{e}^{-\zeta|t|} |\mathrm{D}^j F(\phi(\xi + h)(t)) - \mathrm{D}^j F(\phi(\xi)(t)) \\
&\quad - \mathrm{D}^{j+1} F(\phi(\xi)(t))(\phi(\xi + h)(t) - \phi(\xi)(t))| \prod_{i=1}^{j} \mathrm{e}^{\eta_i|t|} |y_i|_{\eta_i}
\end{aligned}
$$

$$
\begin{aligned}
&\le (1/\zeta) \sup_{t\in\mathbf{R}} \mathrm{e}^{-\zeta|t|} |\mathrm{D}^{j+1}F(\tau\phi(\xi+h)(t)) + (1-\tau)\phi(\xi)(t)) \\
&\quad - \mathrm{D}^{j+1}F(\phi(\xi)(t))| \prod_{i=1}^{j} \mathrm{e}^{\eta_i|t|}|y_i|_{\eta_i} \cdot \mathrm{e}^{\rho|t|}|\phi(\xi+h)-\phi(\xi)|_\rho \\
&\le (L/\zeta)|h|\beta(h)
\end{aligned}
$$

with

$$\beta(h) := \sup_{t\in\mathbf{R}} \mathrm{e}^{-\alpha|t|} \sup_{\tau\in[0,1]} |\mathrm{D}^{j+1}F(\tau\phi(\xi+h)(t) + (1-\tau)\phi(\xi)(t)) - \mathrm{D}^{j+1}F(\phi(\xi)(t))|.$$

Here L is a Lipschitz constant of ϕ and $\alpha := \zeta - \eta$. Lemma 9(b) implies that $\beta(h) \to 0$ as $h \to 0$. This proves part (3) of the lemma for $k = 1$.

Case $k = 2$: induction on $j \in \{0, \dots, r-2\}$. Suppose that $j \in \{0, \dots, r-2\}$ and that the statement has been proved for all $j' < j$. Then by the definition of $f_{j,2}$, $f_{j+1,2}$ and by formula (11), which is also valid for $j = 0$ (with the usual convention that the sum over the empty set of indices is zero), we obtain after a simple combinatorial argument

$$
\begin{aligned}
&((f_{j,2}(\xi+h) - f_{j,2}(\xi))(\xi_1,y_1)\dots(\xi_j,y_j) \\
&\qquad - f_{j_1,2}(\xi)(h,\phi(\xi+h)-\phi(\xi))(\xi_1,y_1)\dots(\xi_j,y_j))(t) \\
&= (1/\varepsilon)\int_{-\infty}^{\infty} U^\varepsilon(t,s;u_\xi)\psi_1(s)\,ds
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_1(s) &= (A(u_{\xi+h}(s)) - A(u_\xi(s)))f_{j,2}(\xi+h)(\xi_1,y_1)\dots(\xi_j,y_j)(s) \\
&- \mathrm{D}A(u_\xi(s))(u_{\xi+h}(s)-u_\xi(s))f_{j,2}(\xi+h)(\xi_1,y_1)\dots(\xi_j,y_j)(s) \\
&+ \sum_{(n,M)\in S'} \{\mathrm{D}_u^{\#N}A(u_{\xi+h}(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi+h)[(\xi_i,y_i)]_{i\in M}(s) \\
&- \mathrm{D}_u^{\#N}A(u_\xi(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi)[(\xi_i,y_i)]_{i\in M}(s) \\
&- \mathrm{D}_u^{\#N+1}A(u_\xi(s))(u_{\xi+h}(s)-u_\xi(s))[u_i(s)]_{i\in N} f_{\#M,2}(\xi)[(\xi_i,y_i)]_{i\in M}(s) \\
&- \mathrm{D}_u^{\#N}A(u_\xi(s))[u_i(s)]_{i\in N} f_{\#M+1,2}(\xi)(h,\phi(\xi+h)-\phi(\xi))[(\xi_i,y_i)]_{i\in M}(s)\} \\
&+ (\mathrm{D}^jG(\phi(\xi+h)(s))y_1(s)\dots y_j(s) - \mathrm{D}^jG(\phi(\xi)(s))y_1(s)\dots y_j(s) \\
&- \mathrm{D}^{j+1}G(\phi(\xi))(s))(\phi(\xi+h)(s)-\phi(\xi)(s))y_1(s)\dots y_j(s)).
\end{aligned}
\tag{13}
$$

Adding and subtracting terms in (13), using the induction hypothesis and Lemma 9(b) as in the preceding part of the proof we obtain the assertion of part (3), case $k = 2$.

(4) This part is obvious from the definition of f_1 and κ.

The lemma is proved. □

PROOF OF LEMMA 7. We apply Theorem 2.1 in [11]. To this end we just have to verify hypotheses (H1) and (H2) of that theorem. But a look at those hypotheses shows that this is obvious in view of Lemma 8. We just have to set $U = X := \mathbf{R}^m$, $m := r-1$, $E_s := \mathrm{BC}^{\rho s}(\mathbf{R}^m) \times \mathrm{BC}^{\rho s}(\mathbf{R}^n) \cong \mathrm{BC}^{\rho s}(\mathbf{R}^m \times \mathbf{R}^n)$ (with the corresponding norm), $s \in S := \{1, 2, \dots, r-1\} \cup \{b, 2b, \dots, (r-1)b\}$,

$$f_j(\xi)(\xi_1, y_1)\dots(\xi_j, y_j) := (f_{j,1}(\xi)(\xi_1, y_1)\dots(\xi_j, y_j), f_{j,2}(\xi)(\xi_1, y_1)\dots(\xi_j, y_j))$$

$j = 1, \dots, r-1$ and $M := C'(a)$ with $a := (r-1)\rho b$.

Now Theorem 2.1 in [11] together with definition of $h(\xi, \varepsilon)$, given in the remark following the statement of Lemma 6, implies, that, for fixed ε with $0 < \varepsilon \leq \varepsilon_0$, the map $\xi \mapsto h(\xi, \varepsilon)$ is of class C^{r-1} with all derivatives of order p with $1 \leq p \leq r-1$ globally bounded. □

To complete the proof of Theorem 1 we only have to prove joint smoothness of $h(\xi, \varepsilon)$ in (ξ, ε). We proceed as it is explained in [14, p. 51], namely we pass to the fast variable $\tau = t/\varepsilon$ and the fast system:

$$(\mathrm{F})_\varepsilon \qquad u' = \varepsilon f(u, v, \varepsilon), \qquad v' = g(u, v, \varepsilon).$$

This is a regular perturbation problem, equivalent, for $\varepsilon \neq 0$ to the slow system $(\mathrm{S})_\varepsilon$. Proceeding as in the smoothness proof above but calculating the formal derivatives with respect to (ξ, y, ε) (which only slightly complicates the resulting formulas) and establishing an analogue of Lemma 8 we obtain the joint smoothness of $h(\xi, \varepsilon)$ in (ξ, ε).

This completes the proof of Theorem 1. □

REMARK. If the system $(\mathrm{S})_\varepsilon$ depends C^r-smoothly on some additional parameters then the above proof also shows C^{r-1}-smoothness of h in those parameters.

REFERENCES

[1] S. N. CHOW AND K. LU, *Invariant manifolds for flows in Banach spaces*, J. Differential Equations **74** (1988), 285–317.

[2] S. N. CHOW, X-B. LIN AND K. LU, *Smooth invariant foliations in infinite dimensional spaces*, J. Differential Equations **94** (1991), 266–291.

[3] O. DIEKMANN AND S. A. VAN GILS, *Invariant manifolds for Volterra integral equations of convolution type*, J. Differential Equations **54** (1984), 139–180.

[4] N. FENICHEL, *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. Math. J. **21** (1971), 193–226.

[5] ———, *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations **31** (1979), 53–98.

[6] G. FISCHER, *Zentrumsmannigfaltigkeiten bei elliptischen Differentialgleichungen*, Math. Nachr. **115** (1984), 137–157.

[7] S. A. VAN GILS, *Some studies in dynamical systems theory*, Ph. D. Thesis, TH Delft, 1984.

[8] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. **840** (1981), Springer.

[9] M. W. HIRSCH, C. C. PUGH AND M. SHUB, *Invariant Manifolds*, Lecture Notes in Math. **583** (1977), Springer.

[10] A. MIELKE, *A reduction principle for nonautonomous systems in infinite dimensional spaces*, J. Differential Equations **65** (1986), 68–88.

[11] K. P. RYBAKOWSKI, *An abstract approach to smoothness of invariant manifolds*, Appl. Analysis (to appear).

[12] ———, *Realization of arbitrary vector fields on center manifolds of parabolic Dirichlet BVPs*, J. Differential Equations (to appear).

[13] ———, *Realization of arbitrary vector fields on invariant manifolds of delay equations*, J. Diff. Equations (to appear).

[14] K. SAKAMOTO, *Invariant manifolds in singular perturbation problems for ordinary differential equations*, Proc. Roy. Soc. Edinburgh **116A** (1990), 45–78.

[15] K. R. SCHNEIDER, *On the application of integral manifolds to Hopf bifurcation*, Math. Nachr. **97** (1980), 313–323.

[16] P. SZMOLYAN, *Analysis of a singularly perturbed traveling wave problem*, SIAM J. Appl. Math. **52** (1992), 485–493.

[17] A. VANDERBAUWHEDE, *Invariant manifolds in infinite dimensions*, in: Dynamics of Infinite Dimensional Systems (S. N. Chow and J. Hale, eds.), NATO Adv. Sci. Inst. Ser., Ser. Dynam. Syst. Appl., **97** (1987), 409–420.

[18] ———, *Centre manifolds, normal forms and elementary bifurcations*, in: Dynam. Report. **2**, 1989, pp. 89–170.

[19] A. VANDERBAUWHEDE AND S. A. VAN GILS, *Center manifolds and contractions on a scale of Banach spaces*, J. Funct. Anal. **72** (1987), 209–224.

[20] A. VANDERBAUWHEDE AND G. IOOS, *Center manifold theory in infinite dimensions*, Dynam. Report., (N.S.) **1**, Springer, 1992, pp. 125–162.

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