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ELLIPTIC EQUATIONS WITH DISCONTINUOUS NONLINEARITIES[1]

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(*Submitted by L. Górniewicz*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

This paper deals with the existence or nonexistence of positive solutions for nonlinear elliptic equations with the nonlinear term discontinuous in the unknown function u. The prototype problem is illustrated by the equation:

$$\ell u = -\Delta u + \sum_{j=1}^{n} b_j(x) D_j u = \lambda f(x,u) \tag{1}$$

in a domain Ω of $\mathbb{R}^n$ with $n \geq 3$ and

$$f(x,u) = \begin{cases} g(x,u), & u > c, \\ 0, & u < c, \end{cases} \tag{2}$$

for some nonnegative smooth function g, monotone in u, and positive constant c. We always assume that $u = 0$ on $\partial\Omega$ (resp. u vanishes at ∞ for Ω unbounded). Observe that $f(x,c)$ is not specified and indeed it will be our purpose to obtain solutions

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u_λ such that the measure of the set $S_0(\lambda) = \{x \mid u_\lambda = c\}$ is zero. The motivation for the study of such equations arises from a variety of physical problems, [31]. In particular, if one considers the mathematical simulation of the electrical discharge in a gas, moving with a velocity $\vec{b}$ assumed unaffected by the discharge, then one obtains system (1), (2) in a subdomain of $\mathbb{R}^3$. Such situations arise, e.g., in arc welding problems, [21]. If the velocity $\vec{b} = (b_1, \dots, b_n) \equiv \vec{0}$ and Ω is bounded, then system (1), (2) includes the classical Elenbaas equation, [15]. Related discontinuous equations arise in vortex studies (see, e.g., [7], [8], [9], [13], [28] and the references therein).

Unlike the much better known case of continuous f, these problems have received relatively small attention. Apart from the above mentioned articles, we refer to the early results of Douchet, [17], Massabò-Stuart, [27], Nistri, [29], Stuart, [32]. In these papers the important concepts of solutions of type I and type II (precisely recalled below) were introduced. There is a close connection between discontinuous problems and multivalued problems obtained by "filling the discontinuity." For convenience, we refer to solutions of the multivalued problems as ones of type III. The theory of multivalued problems is well developed and we refer in particular to the paper of Chang, [14], where such topological concepts as the degree are discussed for multivalued maps. In conclusion we observe that if $\vec{b} \equiv \vec{0}$, then variational methods are applicable to (1), (2), and solutions may be found as critical points of the derivative of a functional. This procedure is the one followed in most of the above references, but it is clearly not applicable here.

The plan of this paper is as follows:

We consider equation (1) either in $\mathbb{R}^n$ or in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Since the proofs for the case of (1) defined on the whole of $\mathbb{R}^n$ usually also hold — with obvious changes — for the case of a smooth bounded Ω, our presentation will deal explicitly with the former case. The situations where this is not the case will be clearly indicated.

We first employ topological methods — specifically Degree Theory — to obtain the existence of solutions of type III, whose norm grows at a specified rate in λ, for (1) with $f \sim p(x)u^\gamma$ at $u = \infty$ and $0 \le \gamma < n/(n-2)$ for $\mathbb{R}^n$ — respectively $0 \le \gamma < (n+1)/(n-1)$ for Ω. Some results are also obtained for $\gamma < (n+2)/(n-2)$. The nonexistence of such solutions is also discussed. Next we use order arguments to show the existence of solutions of type I for $0 \le \gamma \le 1$. We emphasize that motivated by the arc-welding problem we wish to show not merely the existence of solutions but rather existence only for $\lambda > \lambda^* > 0$, for some $\lambda^* > 0$, and such that:

(i) $S(\lambda_1) \subseteq S(\lambda_2)$ if $\lambda_1 \le \lambda_2$, where $S(\lambda) \equiv \{x \,|\, u_\lambda > c\}$, and

(ii) for any compact $K \subset \mathbb{R}^n$ (resp. $\subset \Omega$), we have $K \subset S(\lambda)$ for suitably large λ.

We shall show that this occurs only for the sublinear problems ($\gamma < 1$) but not for the superlinear case ($\gamma > 1$), which, heuristically, has the opposite behaviour. Consequently, superlinear problems appear to be mainly of mathematical interest, with the linear problem ($\gamma = 1$) being a borderline case. The existence of type I solutions for the general superlinear case remains open, although it can be shown in specific cases using symmetry, e.g. for radial coefficient problems. However, we use moving plane arguments and the solution norm estimates in λ obtained earlier to show that for suitable conditions on Ω, λ and the coefficients, the surface $S_0(\lambda)$ is smooth. A brief discussion and comparison with other works concludes the paper. Our results are specifically illustrated by the following example theorems:

THEOREM A. *Let* $0 \le -\operatorname{div}(\vec{b})$, $\vec{b} \in L^{n/2}$ *and* $0 \le p(x)$ *be nontrivial smooth functions in* $\mathbb{R}^n$ *with* $p \in L^\infty \cap L^{2n/(n+2)}$, *and assume* $g(x,u) = p(x)u^\gamma$. *If* $0 < \gamma < 1$ *then there exists a* $\lambda^* > 0$ *such that problem* (1), (2) *in* $\mathbb{R}^n$ *has a positive type* I *solution* $u_\lambda \in C^{1+\alpha}_{\text{loc}}$ *for* $\lambda > \lambda^*$. *Furthermore, if* $\lambda_0 < \lambda_1 < \lambda_2$ *then there exist* $u_{\lambda_1}, u_{\lambda_2}$ *such that* $S(\lambda_1) \subset S(\lambda_2)$ *and for any compact set* K, $S(\lambda_1) \supset K$, *and* $\operatorname{meas}(S_0(\lambda_1)) = 0$ *if* λ_0 *is large enough. If* $\gamma = 1$ *the same result holds for* $0 < \mu - \lambda$ *small enough, with* μ *a simple eigenvalue of* $\ell w = \mu p(x) w$, *and* $w > 0$.

THEOREM B. *Let* Ω *be a locally strictly convex bounded domain of* $\mathbb{R}^n$, $0 \le -\operatorname{div}(\vec{b})$, $p \ge 0$ *and* $\vec{b}$, p *smooth. Assume* $\vec{\nabla}_x g \cdot \vec{n} < 0$ *on* $\partial\Omega$ *and* $\vec{b} = \vec{0}$ *near* $\partial\Omega$, *where* $\vec{n}$ = *outward normal. If* $1 < \gamma < (n+1)/(n-1)$, *and* g *is as in Theorem* A, *then problem* (1), (2) *has a positive solution* $u_\lambda \in C^{1+\alpha}$ *with* $S_0(\lambda)$ *a smooth surface for* λ *small enough. Furthermore, for any compact* $K \subset \Omega$ *we have* $K \subset S(\lambda) \subset S(\lambda_1)$ *for all small* λ, λ_1 *with* $\lambda/\lambda_1 \gg 0$.

By a *locally strictly convex domain* $\Omega \subset \mathbb{R}^n$ we mean that for any $x \in \partial\Omega$ there exists a smooth strictly convex domain D such that $S \cap \partial\Omega \subset \partial D$ and $S \cap \Omega \subset D$ for some sphere S centered at x.

To the best of our knowledge, Theorem A represents the first result for discontinuous nonvariational problems in unbounded domains and Theorem B the first discontinuous nonvariational superlinear result in a bounded domain. Observe that in Theorem B, apart from global regularity, only some local assumptions are made on p and $\vec{b}$ near $\partial\Omega$, and furthermore p is allowed to vanish in subdomains of Ω.

In conclusion, we observe that the requirement that $u \to 0$ at ∞ leads to greater difficulties than if $u \to C > 0$ at ∞ (see, e.g., [5]).

2. Definitions and assumptions

Let ℓu be formally defined in $\mathbb{R}^n$, $n \geq 3$, by

$$\ell u = -\Delta u + \sum_{j=1}^{n} b_j(x) D_j u \tag{3}$$

with b_j bounded and smooth, and consider the equation

$$\ell u = \lambda f(x,u), \qquad \lambda \geq 0, \tag{4}$$

where

$$f(x,\xi) = \begin{cases} g(x,\xi), & \xi > c, \\ 0, & \xi < c, \end{cases}$$

with c a positive constant and $g(x,\xi) \geq 0$, smooth monotone increasing in $\xi \geq 0$. We denote by E the Hilbert space obtained by completing C_0^∞ in the energy norm, $\|\cdot\|_1$, where $\|\phi\|_1^2 = \int |\nabla\phi|^2 dx$, and recall that Hardy's inequality implies that $\|\phi\|_1^2 \simeq \int \{|\nabla\phi|^2 + [\varepsilon/(1+|x|^2)]\phi^2\}$ for some $\varepsilon > 0$. If B denotes the quadratic form associated with ℓ, we assume, for $\phi, \psi \in E$, that $|B(\phi,\psi)| \leq C\|\phi\|_1\|\psi\|_1$ and $\|\phi\|_\ell^2 \equiv B(\phi,\phi) \simeq \|\phi\|_1^2$. Examples of conditions on $\vec{b}$ for which this structure holds will be given at the end of the paper. For presentational convenience, we however always assume $\operatorname{div}(\vec{b}) \leq 0$. Following [14], [17], [27], [29], [32] we introduce the following concepts:

DEFINITION 1. A positive function $u \in E$ is called:

(i) a *type* I *solution* for (4) if (4) holds weakly a.e., i.e. if $\operatorname{meas}(S_0(\lambda)) = 0$;

(ii) a *type* II *solution* for (4) if $\ell u = \lambda \widetilde{f}(x,u)$ weakly a.e. where

$$\widetilde{f}(x,u) = \begin{cases} f(x,u), & u \neq c, \\ 0, & u = c; \end{cases}$$

(iii) a *type* III *solution* for (4) if $\ell u \in \lambda \widehat{f}(x,u)$ weakly a.e. where

$$\widehat{f}(x,u) = \begin{cases} f(x,u), & u \neq c, \\ [0, g(x,c)], & u = c. \end{cases}$$

This terminology does not always follow exactly the one given in the previous papers. We obviously have: (type I solution) $\to$ (type II solution) $\leftrightarrow$ (type III solution), but the other reverse implication does not hold in general, see [17], [32]. Since $\ell u \equiv 0 \quad$ a.e. in $S_0(\lambda)$, we observe that any a.e. solution of

$$\ell u = f^*(x,u) \tag{4'}$$

where

$$f^*(x,u) = \begin{cases} f(x,u), & u \neq c, \\ h(x), & u = c, \end{cases}$$

with $h(x) > 0$, is a type I solution regardless of the specific h chosen.

We conclude by observing that unless otherwise specified, the symbols C, K, C_i, K_i — with various subscripts i — denote constants whose values may vary within the same proof.

3. Solutions of type III

We now consider the existence or nonexistence of type III solutions, i.e. the solutions of multivalued problems, and treat superlinear, linear and sublinear nonlinearities separately.

Assume now that there exists a smooth function $0 \leq p \in L^\infty \cap L^{n/2}$ such that for any $\varepsilon > 0$ there exists a $K(\varepsilon)$ such that for any $\xi \geq 0$,

$$|g(x,\xi) - p(x)\xi^\gamma| \leq p(x)[\varepsilon\xi^\gamma + K(\varepsilon)]. \tag{5}$$

The same function $p(x)$ appears on both sides of (5) for simplicity. It could be replaced on the right hand side by another function $q(x)$ satisfying similar growth and regularity properties. Furthermore, if $K(\varepsilon)$ in (5) is replaced by $K(\varepsilon)\xi^\delta$, with $0 < \delta < \gamma$, then the conditions on p given below may be modified in obvious ways to obtain analogous results. We observe that (5) implies that $f(x,\xi) \leq C(\varepsilon,c)\xi^\gamma$ for all $\xi > 0$.

In this section we abuse somewhat notation and denote by f the Nemytskiĭ operator associated with the multivalued map $\widehat{f}$. Specifically, for any $u \in L^{2n/(n-2)}$, $f(u)$ is defined as follows:

$$f(u) = \{\eta \mid \eta \text{ measurable; } \eta(x) = f(x,u(x)) \text{ for a.a. } x \text{ if } u(x) \neq c;$$
$$0 \leq \eta(x) \leq g(x,c) \text{ for a.a. } x \text{ if } u(x) = c\}.$$

Suitable growth assumptions will be placed on p below, depending on the problem, in order to obtain that $f(u) \subset L^{2n/(n+2)}$ for any $u \in L^{2n/(n-2)}$. Therefore, by the properties of $\widehat{f}$ it is easy to show (see e.g. [25], Théorème 5.1) that $f(u)$ is a bounded, closed, convex subset of $L^{2n/(n+2)}$. Since E is continuously embedded in $L^{2n/(n-2)}$ with embedding i we can consider the Nemytskiĭ operator $fi : E \to L^{2n/(n+2)}$ which will be denoted simply by f in the sequel.

Let $h \in E^*$ and consider the map $h \to u \in E$ given by $B(u, \phi) = h(\phi)$ for all $\phi \in E$, where we recall that B denotes the quadratic form associated with ℓ. In particular, if $\psi \in E$ then we set $h(\phi) = \int p\psi\phi$; we thus have a map from E to E given by $u = \ell^{-1}(p\psi)$. The maps $\psi \to (\ell + \tau p)^{-1}(p\psi)$, $\psi \to \ell^{-1}(G(\psi))$, and $\psi \to \ell^{-1}(F(\psi))$ are then defined in the obvious way for any constant $\tau \geq 0$. Here G and F denote the Nemytskiĭ operators associated with the functions g and f respectively.

LEMMA 1. *Let $p \in L^{p_0}$, where $p_0 = 2n/[2n - (\gamma + 1)(n - 2)]$. Then $\ell^{-1}f$: $E \to E^* \simeq E$ is upper semicontinuous and compact.*

PROOF. We only consider the case of $\mathbb{R}^n$ since otherwise the result is well known, in fact $E = H^{1,2}(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is an open, bounded set. Given $u \in E$, we observe that

$$E \underset{f}{\to} L^{2n/(n+2)} \hookrightarrow E^* \underset{\ell^{-1}}{\to} E.$$

Since the embeddings and ℓ^{-1} are continuous, we need only show that $f : L^{2n/(n-2)} \to L^{2n/(n+2)}$ and $\ell^{-1}f : E \to E$ are respectively upper semicontinuous and compact. The upper semicontinuity may be shown by direct calculation or by appealing to the general results of ([25], Chap. V) which continue to hold even for $\mathbb{R}^n$. As for the compactness, note first that the procedure of [5] fails, since it depends on the continuity of the Nemytskiĭ operator. Instead, let $\{u_m\}$ be bounded in E and $w_m \in \ell^{-1}[f(u_m)]$. We conclude that $B(w_m, \varphi) = (z_m, \varphi)$ for any $\varphi \in C_0^\infty$ with $0 \leq z_m \leq Cp(x)u_m^\gamma$.

Choose a sequence of nested balls $\{B_i\}$ which exhausts $\mathbb{R}^n$ and note that for any given B_i we have $p \in L^\infty(B_i)$, $\gamma < (n+2)/(n-2)$ and $u_m \in L^{2n/(n-2)}(B_i)$, whence $pu_m^\gamma \in L^{(2n+\varepsilon)/(n+2)}(B_i)$ for some $\varepsilon > 0$. We apply [1, Theorem 6.1] to B_j with $j > i$ and conclude $\{w_m\}$ is bounded in $H^{2,\frac{2n+\varepsilon}{n+2}}(B_i)$ and thus compact in $H^{1,2}(B_i)$. By the standard diagonal method we construct a subsequence — also denoted by $\{w_m\}$ — which is Cauchy in $H^{1,2}(B_j)$ for any j. We now show that

$\{w_m\}$ is Cauchy in E. Let $\varepsilon > 0$ be given and $0 \le \varphi \in C_0^\infty(\mathbb{R}^n)$, and note

$$\begin{aligned}\|w_m - w_r\|_\ell^2 &= \|(w_m - w_r)\varphi + (w_m - w_r)(1-\varphi)\|_\ell^2 \\ &\le C\left[\|(w_m - w_r)\varphi\|_\ell^2 + \|(w_m - w_r)(1-\varphi)\|_\ell^2\right].\end{aligned}$$

Direct calculations then show that

$$\|(w_m - w_r)\varphi\|_\ell^2 \le K\|w_m - w_r\|_{H^{1,2}(\mathrm{supp}\varphi)}^2$$

and

$$\begin{aligned}\|(w_m - w_r)(1-\varphi)\|_\ell^2 &= B\big((w_m - w_r)(1-\varphi), (w_m - w_r)(1-\varphi)\big) \\ &\le K\|w_m - w_r\|_{H^{1,2}(\mathrm{supp}|\nabla\varphi|)}^2 + B((w_m - w_r), (1-\varphi)^2(w_m - w_r)),\end{aligned}$$

i.e.,

$$\begin{aligned}\|(w_m - w_r)(1-\varphi)\|_\ell^2 \le K\{&\|w_m - w_r\|_{H^{1,2}(\mathrm{supp}|\nabla\varphi|)}^2 \\ &+ \|p\|_{L^{p_0}(\mathrm{supp}(1-\varphi))} \cdot \big(\|u_m\|_\ell^\gamma + \|u_r\|_\ell^\gamma\big)\|(w_m - w_r)(1-\varphi)\|_\ell\}.\end{aligned}$$

Choosing R large and $\varphi \equiv 1$ if $|x| < R$ and applying Hardy's inequality then shows $\|w_m - w_r\|_\ell < \varepsilon$ if m, r are large enough, and the result follows. □

We continue by recalling the following results and indicate briefly the proof.

LEMMA 2. (a) *Any nontrivial type* III *solution of* (4) *is positive.*

(b) *Let* $0 \le u \in E$ *be a nontrivial type* III *solution of* (4). *Then* $u \in C_{\mathrm{loc}}^{1+\alpha}$, *for some* $\alpha > 0$, *and* $u \to 0$ *at* ∞.

(c) *The linear problem* $\ell^* J = \mu p J$ *has a real positive eigenvalue* μ *with corresponding normalized positive eigenvector* $J \in E$, *where* ℓ^* *denotes the formal adjoint of* ℓ. *The same result holds for* ℓ, *i.e.* $\ell v = \mu p v$ *for some* $v > 0$.

PROOF. (a) Let $\ell u = w \in \widehat{f}(x,u)$ a.e., and observe that $w(x) \ge 0$. We then apply the maximum principle using the coercivity of ℓ to conclude $u \ge 0$. Part (b) will show that $u \in C^{1+\alpha}$ and we apply [20, Theorem 8.18] to conclude $u > 0$.

(b) If $\gamma \le 1$ in (5) then $f(x,u) \le Cp(x)(u+1)$ for some constant C. We recall the estimate ([20, p. 194])

$$u(x) \le C[\|u\|_{L^{2n/(n-2)}(B_1(x))} + \|p\|_{L^q(B_1(x))}]$$

for some $q > n/2$ where $B_1(x)$ denotes the ball of radius 1 centered at x. It follows that $u \in L^\infty$ and hence $u \in C_{\mathrm{loc}}^{1+\alpha}$ for some α (see [20, p. 211]). Since $u \in E$, and

$p \in L^q$ the same estimate shows $u \to 0$ at ∞. If $\gamma > 1$, then $f(x,u) \leq Cp(x)u^\gamma$ implies that $u \in L^t$ for any large t (see, e.g., [5]). We then repeat the arguments used for $\gamma \leq 1$.

(c) This follows from positive operator arguments and eigencurve considerations. The procedures given, e.g., in [4] for the bounded domain case still hold since for $\tau \geq 0$ the map $Tw = (\ell + \tau p)^{-1}[pw]$ defines a continuous compact map in E which leaves invariant the cone of nonnegative functions. Note that if $0 \leq \phi \in C_0^\infty$, with $p\phi$ not identically zero, then $T\phi \geq \alpha\phi$ for some α, by the maximum principle. We then apply [24, p. 67]. □

LEMMA 3. *Assume that* (5) *holds and* $1 < \gamma < n/(n-2)$ (*resp.* $1 < \gamma < (n+1)/(n-1)$) *if* $\Omega = \mathbb{R}^n$ (*resp.* $\Omega =$ *bounded domain*). *Furthermore, if* $\Omega = \mathbb{R}^n$ *then at infinity* $|\vec{b}(x)| \sim |x|^{-\alpha}$, $\alpha > 1$, $p \in L^{2n/(n+2)}$ *and* $P \in L^{2/(1-\theta)}$ *where* $P(x) = p(x)|x|^2$ *and* $\theta = \gamma(n-2)/n$. *We then have:*

(i) *If* u *solves* $\ell u \in \tau\lambda\widehat{f}(x,u)$ *for some* $0 \leq \tau \leq 1$ *then* $\|u\|_\ell \geq C_0(\lambda, c)$;

(ii) *If* u *solves* $\ell u \in \lambda\widehat{f}(x,u) + tp(x)J$ *for some* $t \geq 0$ *then* $t + \|u\|_\ell \leq C_1(\lambda, c)$.

PROOF. (i) If $\ell u \in \tau\lambda\widehat{f}(x,u)$ we have

$$\|u\|_\ell^2 \leq \tau\lambda\Big[\int_{\{u\geq c\}} g(x,u)u\Big].$$

Inequality (5) and the Sobolev embedding theorem yield

$$\|u\|_\ell^2 \leq \widehat{C}_0(\lambda, c)\|p\|_{L^{p_0}}\|u\|_\ell^{\gamma+1}$$

with $p_0 = 2n/[2n - (\gamma+1)(n-2)]$ and the first estimate holds.

(ii) If Ω is a bounded domain, this follows immediately from the procedures of Brezis-Turner, [11], taking advantage of the fact that $f(x,\xi) = 0$ if $\xi < c$. If $\Omega = \mathbb{R}^n$, we may still proceed along the lines of [11]. Specifically, from (5) we have

$$\mu\int Jpu = \int J\ell u \geq \lambda\int_{\{u>c\}} Jp[(1-\varepsilon)u^\gamma - K(\varepsilon)] + \int tpJ^2.$$

We conclude that for some constant K,

$$\int_{\{u\geq c\}} JpK + \mu\int_{\{u\leq c\}} Jpu \geq \mu\int_{\{u>c\}} Jpu + t\int pJ^2.$$

Observe that $\|pJ\|_{L^1} \leq \|p\|_{L^{2n/(n+2)}}\|J\|_{L^{2n/(n-2)}}$, whence $pJ \in L^1$ and it follows that t, $\int pJu$, $\int pJu^\gamma$ are bounded. If we now proceed analogously to [30, Chapt. 3], reproduced briefly here for convenience, we find

$$\|u\|_\ell^2 \leq C\int pu^{\gamma+1} + t\int upJ.$$

Set $a = 2/n$ and choose a ball $B \subset \mathbb{R}^n$. We have

$$(6)\qquad \|u\|_\ell^2 \le C_0\Big[\Big[\int Jpu^\gamma\Big]^\alpha \cdot \Big\{\Big[\int_B \frac{u^{\gamma+1/(1-\alpha)}p}{J^{\alpha/(1-\alpha)}}\Big]^{1-\alpha} + \Big[\int_{\mathbb{R}^n - B} \frac{u^{\gamma+1/(1-\alpha)}p}{J^{\alpha/(1-\alpha)}}\Big]^{1-\alpha}\Big\} + 1\Big].$$

Note that $pJ^{-\alpha/(1-\alpha)} \in L^\infty(B)$ and $\gamma + (1-\alpha)^{-1} < 2n/(n-2)$. The first two integrals on the right hand side of (6) are thus bounded. To estimate the third integral we observe that $J(x) \ge C_1|x|^{2-n}$ in $\mathbb{R}^n - B$ by [19, Lemma H′] and applying Hölder's inequality and Sobolev's embedding theorem, we obtain

$$\|u\|_\ell^2 \le C_2[\|P\|_{L^{2/(1-\theta)}}^{(n-2)/n}\|u\|_\ell^{1+\theta} + 1] + C_3.$$

Since $\theta < 1$, the result follows. □

THEOREM 1. *Under the conditions of Lemma* 3 *the following results hold.*

(a) *Problem* (4) *has a positive solution of type* III *for any* $\lambda > 0$.

(b) *If* $0 < u$ *is any type* III *solution of* (4) *in* E *then* $\|u\|_\ell \sim \lambda^{-1/(\gamma-1)}$ *as* $\lambda \to 0$.

(c) *For any compact set* K, $S(\lambda) \supset K$ *as* $\lambda \to 0$.

PROOF. (a) This is immediate since the first part of Lemma 3 shows $\deg(I - T, B_r, 0) = 1$ for some small r by homotopy to the identity map, where $T(u) = \lambda\ell^{-1}(f(u))$, deg( ) stands for the topological degree for compact, convex-valued, upper semicontinuous vector fields (see [12], [16], [22], [25], [26]), and B_r is the ball in E of radius r centered at the origin. The second part of Lemma 3 yields $\deg(I - T, B_R, 0) = 0$ for some large R by homotopy to $I - T - t\ell^{-1}[pJ]$ for t large.

(b) If $0 < u$ is any positive solution of type III then $\|u\|_\ell^{\gamma-1} \ge C/\lambda$ for some C, by the proof of the first part of Lemma 3. Setting $u = \alpha v$ in (4) with $\alpha = \lambda^{-1/(\gamma-1)}$ and applying an argument analogous to the one in the second part of Lemma 3 (for $t = 0$), replacing c by a suitably large constant shows $\|v\|_\ell \le C_1$ and the result.

(c) Let $W = \{x \mid u(x) > \varepsilon\lambda^{-1/(\gamma-1)}\}$. It follows that

$$\begin{aligned} K\lambda^{-2/(\gamma-1)} \le \|u\|_\ell^2 &\le \lambda \int_{\{u \ge c\}} p[u^\gamma \cdot 2 + K_1]u \\ &\le C(c)\lambda\Big[\int_W pu^{\gamma+1} + \int_{-W} pu^{\gamma+1}\Big] \\ &\le C(c)\lambda\Big[\|p\|_{L^{p_0}(W)}\|u\|_\ell^{\gamma+1} + |u|_{L^\infty(-W)}^\delta \int_{-W} pu^{\gamma+1-\delta}\Big] \end{aligned}$$

for some $\delta > 0$, i.e.

$$K\lambda^{-2/(\gamma-1)} \leq C(c)\lambda[\|p\|_{L^{p_0}(W)}\lambda^{-(\gamma+1)/(\gamma-1)} + \varepsilon^\delta \lambda^{-\delta/(\gamma-1)}\|p\|_{L^{p_1}} \cdot \|u\|_\ell^{\gamma+1-\delta}]$$
$$\leq C(c)[\|p\|_{L^{p_0}(W)} + \varepsilon^\delta\|p\|_{L^{p_1}}]\lambda^{-2/(\gamma-1)}$$

with $p_0 = 2n/[2n-(\gamma+1)(n-2)]$ and $p_1 = 2n/[2n-(\gamma+1-\delta)(n-2)]$. We choose ε small enough, and conclude $\|p\|_{L^{p_0}(W)} > C_2$, i.e. $\text{meas}(W) > C_3$ with C_3 independent of λ. Finally, note that $p \in L^{p_0}$ and it follows that $\text{meas}(W \cap B) > C_3/2$ for some ball B dependent on p. Hence

$$\ell u \geq C\lambda p\varepsilon^\gamma \lambda^{-\gamma/(\gamma-1)}\chi(W\cap B) = Cp\varepsilon^\gamma\lambda^{-\frac{1}{\gamma-1}}\chi(W \cap B)$$

where χ denotes the characteristic function. We conclude that for any ball $B_1 \supset B$, $u \geq w$ where

$$\ell w = Cp\varepsilon^\gamma\lambda^{-1/(\gamma-1)}\chi(W\cap B),$$
$$w = 0 \quad \text{on} \quad \partial B_1$$

and thus $u \to \infty$ in B as $\lambda \to 0$. □

Observe that in form, Theorem 1(a) is identical to the classical superlinear result for the continuous problem $\ell u = \lambda g(x, u)$. We mention briefly that for $\gamma < 1$, i.e. for the sublinear case, and Ω bounded, the continuous problem also has a solution for all λ whose existence may be shown in the same way by demonstrating $\deg(I - T,\ B_R - \overline{B}_r, 0) = 1$ (see [4]). This approach fails for f discontinuous since it is now not clear that $\deg(I - T, B_r, 0) = 0$ if r is small enough and indeed we shall show that the sublinear discontinuous problem has no solutions for λ small. Instead, we have

THEOREM 2. *Let $0 < \gamma < 1$ in (5), $\vec{b} \in L^n$, and $p \in L^{p_0}$ with $p_0 = 2n/(n+2)$ (resp. $p_0 < 2n/[2n-(\gamma+1)(n-2)]$ if $K(\varepsilon) = 0$ in (5)). There exists a $\lambda^* > 0$ such that*

- (a) *equation (4) has a positive solution u of type III for $\lambda > \lambda^*$ and with f replaced by $f(x,u) + \phi(x)$, where $0 \leq \phi \in C_0^\infty$;*
- (b) *u satisfies $\|u\|_\ell \sim \lambda^{1/(1-\gamma)}$ as $\lambda \to \infty$;*
- (c) *for any compact set K, $S(\lambda) \supset K$ as $\lambda \to \infty$.*

PROOF. (a) For $0 < \varepsilon$, α and $0 \leq u \in E$ set $h(\varepsilon, \alpha, x, u) = \alpha^\gamma f(x, u/\alpha) + \alpha^\gamma\phi - p(u^+)^\gamma$. For any given $\varepsilon' > 0$ and u in some ball B_R it follows that

$$\|\ell^{-1}(h(\varepsilon, \alpha, u))\|_\ell < \varepsilon'$$

by choosing ε and then α small enough and applying (5). Here, abusing notation, h denotes the Nemytskiĭ operator associated with the function h. To see this, we observe:

$$|\alpha^\gamma f(x,u/\alpha)+\alpha^\gamma\phi-p(u^+)^\gamma|\le\begin{cases}\alpha^\gamma\phi+p[\varepsilon u^\gamma+K(\varepsilon)\alpha^\gamma], & u>c\alpha,\\ \alpha^\gamma\phi+Cp(u^+)^\gamma, & u\le c\alpha,\end{cases}$$

for some constant C independent of u. We then have

$$C_0\|\ell^{-1}h\|_\ell^2\le B(\ell^{-1}h,\ell^{-1}h)\le K[\alpha^\gamma\|\phi\|_{L^\infty}+\|p(u^+)^\gamma\|_{L^{\frac{2n}{n+2}}(u\le c\alpha)}$$
$$+\alpha^\gamma K(\varepsilon)\|p\|_{L^{p_0}}+\varepsilon\|p\|_{L^{p_1}}\|u\|_\ell^\gamma]\|\ell^{-1}h\|_\ell$$

where $p_1=2n/[2n-(\gamma+1)(n-2)]$. We estimate the second term on the right hand side:

$$\|p(u^+)^\gamma\|_{L^{\frac{2n}{n+2}}(u\le c\alpha)}\le K\|u\|^\delta_{L^\infty(u\le c\alpha)}\|p\|_{L^{p_2}}\|u\|_\ell^{\gamma-\delta}$$
$$\le K\alpha^\delta\|p\|_{L^{p_2}}\|u\|_\ell^{\gamma-\delta}$$

where $p_2=2n/[2n-(\gamma+1-\delta)(n-2)]$, for α small. The continuous problem

$$\ell u=p(u^+)^\gamma \tag{7}$$

has the property that $\deg(I-G,B_R-\overline{B}_r,0)=1$, for R,r respectively large and small enough, where $G(u)=\ell^{-1}(p(u^+)^\gamma)$. As mentioned earlier, a proof of this fact may be found in [4] for Ω bounded. For the case of $\Omega=\mathbb{R}^n$ we proceed as follows: If u is a solution of $\ell u=\tau p(u^+)^\gamma$ with $0\le\tau\le1$ then

$$\|u\|_\ell^2\le K\|p\|_{L^{p_0}}\|u\|_\ell^{\gamma+1}$$

whence $\|u\|_\ell$ is bounded independently of τ. On the other hand, if $\ell u=p(u^+)^\gamma+tJp$ for some $t\ge0$, we apply an extension of Picone's identity, [3], and obtain

$$0\le\int\Big(\frac{\ell+\ell^*}{2}+H\Big)(\varphi)\varphi-\int\frac{\varphi^2}{u}\,(\ell u)$$

for any $\varphi\in C_0^\infty$, with $H=\sum b_j^2/4$. Since $H\in L^{n/2}$, we observe that there exists an eigenvalue θ with positive eigenvector z in E such that $\big(\frac{\ell+\ell^*}{2}+H\big)(z)=\theta pz$. Letting φ approach z yields

$$0\le\theta\int pz^2-\Big(\int\frac{z^2}{u}\,[pu^\gamma+tpJ]\Big).$$

It follows that $\int z^2 p u^{\gamma-1} \leq K$. Set $\beta = [2n-(n-2)(1-\gamma)]/8$ and observe that $2\beta > 1$ and thus $0 < \int p^{2\beta} z^2 < \|p\|_{L^\infty}^{2\beta-1} \int p z^2 < \infty$. We conclude

$$\begin{aligned}\int p^{2\beta} z^2 &= \int p^\beta z^2 \frac{p^\beta u^{(1-\gamma)/2}}{u^{(1-\gamma)/2}} \\ &\leq \left(\int \frac{p^{2\beta} z^4}{u^{1-\gamma}}\right)^{1/2} \left(\int p^{2\beta} u^{1-\gamma}\right)^{1/2} \\ &\leq \|p\|_{L^\infty}^{(2\beta-1)/2} \|z\|_{L^\infty} K \left[\|p\|_{L^{n/2}}^{2\beta} \|u\|_{L^{2n/(n-2)}}^{1-\gamma}\right]^{1/2}\end{aligned}$$

and we obtain $\|u\|_\ell > C$ for some C independent of u, and thus the result.

It follows that $\deg(I - G - \ell^{-1}h, B_R - \overline{B}_r, 0) \neq 0$ for ε, α small enough, and thus we obtain the existence of a solution $w \in B_R - \overline{B}_r$ of $\ell w \in \alpha^\gamma f(x, w/\alpha) + \alpha^\gamma \phi(x)$.

(b) Setting $u = w/\alpha$ then gives a solution of $\ell u \in \lambda[f(x,u) + \phi(x)]$ with $\lambda = \alpha^{\gamma-1}$. Since $r \leq \|w\|_\ell \leq R$ $\|u\|_\ell \sim \lambda^{\frac{1}{1-\gamma}}$ as $\lambda \to \infty$.

(c) The proof is identical to that given for Theorem 1(c). □

We next consider the case of $\gamma = 1$ in (5). We recall that $\ell v = \mu p v$, $\ell^* J = \mu p J$, with $J, v > 0$. Our result in this situation will follow from

THEOREM 3. *Let $\phi \in L^\infty$ be smooth and μ be simple. Suppose $J_1 : \mathbb{R}_1 \to \mathbb{R}_1$ given by*

$$J_1(e) = \int_\Omega J[g_1(x, e\mu v) + p\phi],$$

has a nondegenerate zero at $e_1 > 0$ (i.e. $J_1'(e_1) \neq 0$). Then for all $\varepsilon > 0$ sufficiently small there exists a solution $0 < u \in E$ of

$$\ell u \in \mu \begin{cases} (1-\varepsilon)[g_1(x, \frac{\varepsilon u}{1-\varepsilon}) + p\phi] + pu, & u > c, \\ (1-\varepsilon)p\phi, & u < c, \end{cases}$$

with the interval previously defined at $u = c$ such that $\|u\|_\ell \sim 1/\varepsilon$ as $\varepsilon \to 0$. We assume here g_1 is smooth and $|g_1|$ satisfies (5) with $1 \leq \gamma < (n+2)/(n-2)$, and $p \in L^{p_0}$ with $p_0 = 2n/(n+2)$.

PROOF. Let

$$T\begin{pmatrix} u_2 \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ e - \operatorname{sign}(J_1'(e_1)) \int J z(\varepsilon, u_2 + (e+e_1)v) \end{pmatrix},$$

$$Q_\varepsilon \begin{pmatrix} u_2 \\ e \end{pmatrix} = \mu \begin{pmatrix} L^{-1}[z(\varepsilon, u_2 + (e+e_1)v) - Jp \int Jz(\varepsilon, u_2 + (e+e_1)v)] \\ 0 \end{pmatrix},$$

where $L^{-1} = (\ell - \mu p)^{-1}$ is defined on $\mathrm{Im}(\ell - \mu p) = (\mathrm{Ker}(\ell^* - \mu p))^{\perp}$ in E, where $E \simeq E^*$, and

$$\varepsilon z(\varepsilon, u) = \begin{cases} \frac{\varepsilon(1-\varepsilon)}{\mu-\varepsilon}\left[g_1\left(x, \frac{(\mu-\varepsilon)}{1-\varepsilon}u\right) + p\phi\right], & u > \frac{\varepsilon c}{\mu-\varepsilon}, \\ -pu + \frac{\varepsilon p\phi(1-\varepsilon)}{\mu-\varepsilon}, & u < \frac{\varepsilon c}{\mu-\varepsilon}. \end{cases}$$

Degree theory arguments (see [4]) then imply that

$$\deg(I - T',\ B_{\varepsilon'}(0) \times (-\varepsilon', \varepsilon'), 0) = 1$$

for small ε', where T' is the same as T with $\frac{1}{\mu}[g_1(x, \mu u_2 + \mu(e + e_1)u_1) + p\phi]$ in place of z, and thus, using $0 \leq \sigma \leq 1$ as a homotopy parameter,

$$\deg(I - \sigma(T' + (T - T') + \varepsilon Q_\varepsilon),\ B_{\varepsilon'}(0) \times (-\varepsilon', \varepsilon'), 0) = 1$$

for small ε since

$$\mathrm{meas}([x \mid u_2 + (e + e_1)v \leq \varepsilon c/(\mu - \varepsilon)] \cap B) \to 0$$

as $\varepsilon \to 0$ for any ball $B \subset \mathbb{R}^n$ and fixed ε' by the positivity of v. We thus have the existence of a nontrivial solution $w = u_2 + (e + e_1)v$ of

$$\ell w - \mu p(x) w \in \mu \begin{cases} \frac{\varepsilon(1-\varepsilon)}{\mu-\varepsilon}\left[g_1\left(x, \frac{(\mu-\varepsilon)w}{1-\varepsilon}\right) + p\phi\right], & w > \frac{\varepsilon c}{\mu-\varepsilon}, \\ -pw + \frac{\varepsilon p\phi(1-\varepsilon)}{\mu-\varepsilon}, & w < \frac{\varepsilon c}{\mu-\varepsilon}. \end{cases}$$

Putting $u = \frac{(\mu-\varepsilon)w}{\varepsilon}$ then gives

$$\ell u \in \mu \begin{cases} (1-\varepsilon)\left[g_1\left(x, \frac{u\varepsilon}{1-\varepsilon}\right) + p\phi\right] + pu, & u > c, \\ (1-\varepsilon)p\phi, & u < c, \end{cases} \tag{8}$$

with $\|u\|_\ell \sim 1/\varepsilon$. □

Finally, if μ is not simple, we may repeat the procedure projecting z over the entire (finite dimensional) kernel of L^*. The scalars e, e_1 are now replaced by the vectors $\vec{e}, \vec{e}_1$ with $\dim(\vec{e}, \vec{e}_1) = \dim(\ker(L^*)), \vec{e}_1 = (e_1, 0, \dots, 0)$ and the matrix $J'(\vec{e}_1)$ assumed definite. We recall that explicit conditions for μ to be simple follow from [24]. We observe that $0 \leq u$ by coercivity (see the proof of Lemma 2(a)) and that Theorem 3 did not require monotonicity nor positivity of $g_1(x, \xi)$. Motivated by the Elenbaas equation we then have

COROLLARY 1. *Let $g_1(x, \xi) = p(x)(d - \xi)$. Then for $0 < \mu - \lambda$ small enough,* (4) *with $g = p(u + d)$ has a type* III *solution u such that $\|u\|_\ell \sim (\mu - \lambda)^{-1}$. Furthermore, given any compact set K then $S(\lambda) \supset K$ if $\mu - \lambda$ is small enough.*

PROOF. In this case, by direct calculation (8) becomes

$$\ell u \in \mu p(x)\begin{cases} (1-\varepsilon)(d+u+\phi), & u>c,\\ (1-\varepsilon)\phi, & u<c, \end{cases} \tag{9}$$

$$= \mu(1-\varepsilon)p(x)\begin{cases} u+d+\phi, & u>c,\\ \phi, & u<c. \end{cases}$$

The first result follows by letting $\lambda = \mu(1-\varepsilon)$.

Next, let $W = \{x \mid u(x) > \varepsilon(\mu-\lambda)^{-1}\}$ with ε to be determined below. We have

$$\begin{aligned} K(\lambda-\mu)^{-2} \le \|u\|_\ell^2 &\le \lambda\Big[\int_{W\cap\{u\ge c\}} p[u+d]u + \int_{(-W)\cap\{u\ge c\}} p[u+d]u + \int pu\phi\Big]\\ &\le \lambda C(c)[\|p\|_{L^{n/2}(W)}\|u\|_\ell^2 + |u|^\delta_{L^\infty(-W)}\|p\|_{L^{n/2-\delta}}\|u\|_\ell^{2-\delta}\\ &\quad + \|p\|_{L^{n/2}}\|u\|_\ell\|\phi\|_\infty]\\ &\le \lambda C(c)[\|p\|_{L^{n/2}(W)}(\lambda-\mu)^{-2} + \varepsilon^\delta\|p\|_{L^{n/2-\delta}}(\lambda-\mu)^{-2}\\ &\quad + \|p\|_{L^{n/2}}\|\phi\|_\infty(\mu-\lambda)^{-1}]. \end{aligned}$$

Choosing ε and $\mu-\lambda$ small enough shows $\|p\|_{L^{n/2}(W)} > C_1 > 0$ whence $\text{meas}(W) > C_2 > 0$ independent of λ. The rest of the proof is analogous to the one given for Theorem 1(c). Observe that the classical Elenbaas equation corresponds to $\phi \equiv 0$, and that $u > 0$ by Lemma 2(a). □

It is interesting to note that here g_1 is not the same function g which appears on the right hand side of (4). Also, the nonlinearity in Theorem 3 is allowed to be in the full subcritical range: $\gamma < (n+2)/(n-2)$, and one may thus obtain some existence results for $\ell u - \lambda p(x)u \in f(x,u)$ in this case.

We remark that, unlike the superlinear case, it is not true that all solutions grow at the rates given in Theorem 2 and Corollary 1 for sublinear and linear problems respectively. Explicit examples of this statement may be found in [14], [29]. We do, however, have the following results

THEOREM 4. *Let $g(x,u)/u \ge p(x)u^\delta$ with $\delta \ge 0$ in (5) and $H = \sum b_j^2/4 \in L^{n/2}$. Then for any ball $B \subset \mathbb{R}^n$ and $\varepsilon > 0$, and for λ large enough we have* $\text{meas}(S(\lambda)\cap B) < K + \varepsilon$ *if* $\text{meas}([p=0]\cap B) = K$.

PROOF. Let τ denote the least eigenvalue of $\left(\frac{\ell+\ell^*}{2} + H\right)(z) = \tau p z$ and let z be the associated eigenvector. We again apply an extension of Picone's identity, [3], to $\varphi \in C_0^\infty$ and obtain

$$0 \le \int \varphi\Big(\frac{\ell+\ell^*}{2} + H\Big)(\varphi) - \int \frac{\varphi^2}{u}\,\ell u.$$

Let $\varphi \to z$ in E and observe that

$$\lambda \int_{\{u>c\}} \frac{z^2}{u} pu^{\delta+1} \leq \tau \int pz^2.$$

Consequently,

$$c^\delta \int_{\{u>c\}} pz^2 \leq (\tau/\lambda) \int pz^2$$

and the result follows. □

The above result shows that both linear and superlinear problems do not behave in accordance with our requirements for λ large. As for nonexistence results, we have:

THEOREM 5. *If* $\gamma \leq 1$ *in* (5) *and* u *is a nontrivial solution of* (4), *then*

$$\lambda \geq [C(c)\|p\|_{L^{n/2}}]^{-1}$$

for some positive constant $C(c)$.

PROOF. Observe that $\ell u \in \lambda f(x,u)$ implies in this case $\ell u \leq \lambda C(c)pu$, whence

$$\|u\|_\ell^2 \leq \lambda C(c)\|p\|_{L^{n/2}}\|u\|_\ell^2$$

and $\lambda \geq [C(c)\|p\|_{L^{n/2}}]^{-1}$. □

We observe that Theorem 6 thus yields the estimate $\lambda^* \geq C(c,\gamma)\mu$.

4. Solutions of type I

We now show that for the linear or sublinear case the earlier procedures yield the existence of type I solutions with the desired properties. Specifically:

THEOREM 6. (a) *Under the conditions of Corollary* 1, *with* $\phi \equiv 0$, *there exist solutions of type* I *for* $0 < \lambda^* < \lambda < \mu$ *such that* $\|u\|_\ell \to \infty$, $S(\lambda) \supset K$ *for any given compact* K *as* $\lambda \to \mu$, *and for any given* $\lambda_2 > \lambda^*$ *and* u_{λ_2} *there exists for any* $\lambda_1 \geq \lambda_2$ *a* u_{λ_1} *such that* $S(\lambda_1) \supset S(\lambda_2)$.

(b) *Under the conditions of Theorem* 2 *with* $K(\varepsilon) = M$, $\phi \equiv 0$, *for* $0 < \lambda^* < \lambda$ *there exist type* I *solutions with the same properties as in part* (a) *for* $\lambda \to \infty$.

PROOF. (a) Let $\lambda^* < \lambda < \mu$. If the solution u found in Corollary 1 is not of type I then it is a subsolution to

$$\ell u = \lambda p(x) \begin{cases} u + d, & u > c, \\ \varepsilon, & u = c, \\ 0, & u < c. \end{cases} \tag{10}$$

Choose a solution of

$$\ell w \geq \lambda_1 p(x) \begin{cases} w + M, & u \geq c, \\ M, & u < c, \end{cases}$$

for M and $1/(\mu - \lambda_1)$ large enough. This is clearly a supersolution to (10) and $w > u$ for M large. If we express (10) as $u = \lambda T(u)$ with $T : L^{2n/(n-2)} \to L^{2n/(n-2)}$, and T monotone, and observe that the cone of nonnegative functions in $L^{2n/(n-2)}$ is strongly minihedral then by [6], [23, Chapt. 6], there exists a solution v to (10) which is a solution of type I for (4) in this case. Finally, observe that $v \geq u$ by construction, whence

$$\|v\|_\ell^2 = \lambda \int_{\{v \geq c\}} pv(v + d) \geq \lambda \int_{\{u \geq c\}} pu(u + d) \geq \|u\|_\ell^2$$

and we conclude that $\|v\|_\ell \uparrow \infty$ as $\lambda \to \mu^-$. We thus have the existence of type I solutions u_λ for all λ such that $\lambda > \mu - \varepsilon$. Select a λ_0, u_{λ_0} in this set and put

$$P = \{t \mid \text{for any } \lambda \in [t, \mu) \text{ there exists a type I solution } u_\lambda \text{ with } u_\lambda \geq u_{\lambda_0}\}.$$

P is not empty, since we may choose λ_1 near μ and use u_{λ_0} as a subsolution. Let $t_0 = \inf \{t \mid t \in T\}$. We claim $t_0 \leq \lambda_0$, for if $t_0 > \lambda_0$ then choose $\lambda_0 < t_1 < t_0$ and again use u_{λ_0} as a subsolution. We observe that $u_{t_1} \geq u_{\lambda_0}$ and thus have a contradiction by using u_{t_1} as a subsolution for the cases $t > t_1$. We conclude that given any λ_1, λ_2 near μ with $\lambda_1 > \lambda_2$, there exist type I solutions $u_{\lambda_1}, u_{\lambda_2}$ with $u_{\lambda_1} \geq u_{\lambda_2}$ and hence $S(\lambda_1) \supset S(\lambda_2)$. Given any compact set K, we observe that $S(\lambda) \supset K$ by the second part of Corollary 1 and monotonicity.

(b) The proof for the sublinear result is identical. □

We remark that the procedures of [17], [32] of constructing super and subsolutions for the bounded nonlinearity case also hold for $\mathbb{R}^n$. One can thus obtain analogous results for this case directly by the order procedures.

It is interesting to note that the earlier growth conditions yield criteria under which $S_0(\lambda)$ is actually a smooth surface. Specifically:

THEOREM 7. *Let Ω be a locally strictly convex bounded domain, $\vec{\nabla}_x g(x,\xi)\cdot\vec{n} < 0$ and $\vec{b} = 0$ near $\partial\Omega$. Then there exists a solution u with $S_0(\lambda)$ a smooth surface if λ is near ∞ (sublinear), near 0 (superlinear) or near μ (linear).*

PROOF. In all the above cases we have shown that for any compact set $K \subset \Omega$ we have $K \subset S(\lambda)$ in the above situation. We recall that the extension of the moving plane results of Gidas, Ni and Nirenberg [18] given by Amick and Fraenkel, [9], shows that under the conditions of Theorem 7 there exists a neighbourhood B_ε of $\partial\Omega$, independent of u, in which $\nabla u \neq 0$. We choose $K = \Omega - B_\varepsilon$ and the result follows. □

5. Conclusions

We first observe that Theorems A and B are merely collections of results from the previous section. Note that in these cases $(\phi, \ell\phi) \sim \|\phi\|_1^2$. Indeed, in Theorem A,

$$(\phi, \ell\phi) = \int_\Omega |\nabla\phi|^2 - \frac{\operatorname{div}(\vec{b})}{2}\,\phi^2 \geq \|\phi\|_1^2$$

while

$$(\phi, \ell\phi) \leq \|\phi\|_1^2 + C\|\vec{b}\|_{L^{n/2}}\|\phi\|_1\|\phi\|_{L^{2n/(n-2)}} \leq [1 + C\|\,\vec{b}\,\|_{L^{n/2}}]\|\phi\|_1^2.$$

The situation in Theorem B is identical. We remark that if, e.g., $-\operatorname{div}(\vec{b}) > C > 0$ then it becomes possible to work with the $H^{1,2}$ norm. The conditions on $p(x)$ may then be expressed in terms of the M_α spaces of Berger and Schechter, [10].

In comparing in detail our results with earlier work, we observe that while Chang, [14], employed degree theory, he considered only the bounded domain case and did not distinguish between the various types of solution nor analyzed the properties of $S(\lambda)$. The studies dealing with vortex rings and [8], [9], [28], and/or related questions, [7], [13], [15] considered variational situations and thus such methods as Lagrange multipliers and mountain pass arguments were directly applicable. Some analysis of the behaviour of $S(\lambda)$ was given for this case in [7]. The early results, [17], [29], [32], were primarily based on sub-super solution arguments, considered only the bounded domain case and did not investigate the properties of $S(\lambda)$. Our results have been restricted to $\gamma < n/(n-2)$ in the $\mathbb{R}^n$ case — unlike some of the references above. Some results can also be obtained here for $\gamma < (n+2)/(n-2)$ by employing Theorem 3 as mentioned earlier. The general case remains open, however, as does the existence of solutions of type I for the general superlinear case.

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