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SEMILINEAR BOUNDARY VALUE PROBLEMS OF THE STRONG RESONANCE TYPE

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Dedicated to the memory of Juliusz Schauder

1. Introduction

In this paper we use saddle point techniques to solve resonance problems for semilinear equations. The resonance is permitted to be strong.

Let Ω be a domain in $\mathbb{R}^n$ and let A be a selfadjoint operator on $L^2(\Omega)$ having 0 as an isolated eigenvalue of finite multiplicity. If $f(x,t)$ is a Carathéodory function on $\Omega \times \mathbb{R}$, then the equation

$$Au = f(x,u) \tag{1.1}$$

is said to have *asymptotic resonance at infinity* if

$$f(x,t)/t \to 0 \qquad \text{as } |t| \to \infty. \tag{1.2}$$

Resonance problems for (1.1) have been studied by many authors; a partial list is included in the bibliography. Problem (1.1) is at strong resonance if

$$f(x,t) \to 0 \quad \text{as } |t| \to \infty, \qquad \int_0^t f(x,s)\,ds \quad \text{is bounded.} \tag{1.3}$$

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Comparatively few authors have studied the strong resonance case. In [23], Thews assumed that $f(x,t) \equiv g(t)$ is odd. In [3], Bartolo-Benci-Fortunato assumed that $f(x,t) \equiv g(t)$ satisfies

$$tg(t) \to 0 \qquad \text{as } |t| \to \infty,$$

$$f(t) \le b_0, \qquad t \in \mathbb{R},$$

$$F(t) \to b_0 \qquad \text{as } |t| \to \infty.$$

Ward [24] considered the following situation.

$$|f(x,t)| \le \gamma_0(x), \qquad |F(x,t) - tf(x)| \le \gamma_1(x), \qquad \gamma_j \in L^2(\Omega),$$

$$F_0(x) < F(x,t) - tf(x), \qquad x \in \Omega,\ t \in \mathbb{R},$$

(1.4) $$F(x,t) - tf(x) \to F_0(x) \qquad \text{uniformly in } x \text{ as } |t| \to \infty$$

and

(1.5) $$f(x,t) \to f(x) \qquad \text{as } |t| \to \infty,\ f(x) \in R(A).$$

In [20] we assumed (1.3),

$$\limsup_{|t|\to\infty} tf(x,t) \le W_1(x) \in L^1(\Omega),$$

(1.6) $$\liminf_{\|v\|\to\infty} 2\int_\Omega F(x,v)\,dx \ge b_0 > -\infty, \qquad v \in N(A),$$

and

$$\min(0, B_1) < 2c_1 + b_0$$

where

(1.7) $$B_1 = \int W_1(x)\,dx$$

and c_1 is the infimum of the energy functional corresponding to (1.1) on a subspace. In [21] we allowed

$$|f(x,t)| \le C(|t|^\gamma + 1), \qquad t \in \mathbb{R},$$

for some constant $\gamma < 1$ and assumed

$$\limsup_{|t|\to\infty} [2F(x,t) - tf(x,t)] \le W_1(x) \in L^1(\Omega)$$

and

$$B_1 < 2\int_\Omega F(x,v)\,dx, \qquad (Av,v) \le 0, \tag{1.8}$$

where B_1 is given by (1.7). In [22], Silva assumed (1.3), (1.6) and

$$2F(x,t) \le \overline{\lambda} t^2 + |\Omega|^{-1} b_0, \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{1.9}$$

where $\overline{\lambda}$ is the smallest positive point in the spectrum of A and $|\Omega|$ is the volume of Ω. In each of the cases mentioned the conditions are sufficient for the existence of a solution of (1.1).

In the present paper we wish to allow (1.5) but not require the other restrictions of [24]. In our first result we assume (1.6) and

$$2F(x,t) \le \overline{\lambda} t^2 + W_1(x), \qquad x \in \Omega, t \in \mathbb{R}. \tag{1.10}$$

We show that these two assumptions are sufficient for solutions of (1.1) to exist provided

$$B_1 \le b_0 + (f, u_1) \tag{1.11}$$

where B_1 is given by (1.7) and u_1 is the unique solution of

$$Au_1 = f, \qquad u_1 \in N(A)^\perp. \tag{1.12}$$

We then show that everything can be reversed. If we assume

$$\limsup_{\|v\|\to\infty} 2\int_\Omega F(x,v)\,dx \le b_1 < \infty, \qquad v \in N(A), \tag{1.13}$$

and

$$\underline{\lambda} t^2 - W_0(x) \le 2F(x,t), \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{1.14}$$

(where $\underline{\lambda}$ is the largest negative point of $\sigma(A)$), then a solution is assured if

$$B_0 = \int_\Omega W_0(x)\,dx \le -b_1 - (f, u_1). \tag{1.15}$$

The results of [3], [24], [22] and others are now corollaries. Our method is to apply a generalization of the saddle point theorem recently proved by the author [17, 18] (cf. also Silva [22]).

THEOREM 1.1. *Let N be a closed subspace of a Hilbert space H and let $M = N^\perp$. Assume that at least one of the subspaces M, N is finite dimensional. Let G be a C^1 functional on H such that*

$$m_0 := \sup_{v \in N} \inf_{w \in M} G(v+w) > -\infty \tag{1.16}$$

and

$$m_1 := \inf_{w \in M} \sup_{v \in N} G(v+w) < \infty. \tag{1.17}$$

Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset H$ such that

$$m_0 \leq c \leq m_1, \qquad G(u_k) \to c, \qquad G'(u_k) \to 0. \tag{1.18}$$

Our results are stated in Section 2 and proved in Section 3.

2. Semilinear boundary value problems

Let Ω be a domain in $\mathbb{R}^n$ and let A be a selfadjoint operator on $L^2(\Omega)$ such that:

(A) the essential spectrum $\sigma_e(A)$ of A is contained in $(0, \infty)$,
(B) there is a function $V_0(x) > 0$ such that multiplication by V_0 is a compact operator from $D := D(|A|^{1/2})$ to $L^2(\Omega)$,
(C) if $u \in N(A) \setminus \{0\}$, then $u \neq 0$ a.e. in Ω.

Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ such that:

(D) $|f(x,t)| \leq V(x) \in L^2(\Omega)$, $x \in \Omega$, $t \in \mathbb{R}$,
(E) $f(x,t) \to f(x)$ a.e. as $|t| \to \infty$.

We wish to solve

$$Au = f(x,u), \qquad u \in D(A). \tag{2.1}$$

We have

THEOREM 2.1. *In addition to* (A)–(E) *assume that*

$$b_0 := \liminf_{\substack{\|v\|\to\infty \\ v\in N(A)}} 2\int F(x,v)\,dx > -\infty \tag{2.2}$$

where

$$F(x,t) := \int_0^t f(x,s)\,ds. \tag{2.3}$$

If $b_0 < \infty$, *assume further that*

$$2F(x,t) \le \overline{\lambda} t^2 + W_1(x), \qquad x\in\Omega,\ t\in\mathbb{R}, \tag{2.4}$$

and

$$B_1 := \int_\Omega W_1(x)\,dx \le b_0 + (f,u_1) \tag{2.5}$$

where $\overline{\lambda}$ *is the smallest positive point in the spectrum* $\sigma(A)$ *of* A *and* u_1 *is the unique solution in* $N(A)^\perp$ *of* $Au_1 = f$. *Then* (2.1) *has a solution.*

THEOREM 2.2. *In addition to* (A)–(E) *assume that*

$$b_1 := \limsup_{\substack{\|v\|\to\infty \\ v\in N(A)}} 2\int F(x,v)\,dx < \infty. \tag{2.6}$$

If $b_1 > -\infty$, *assume further that*

$$\underline{\lambda} t^2 - W_0(x) \le 2F(x,t), \qquad x\in\Omega,\ t\in\mathbb{R}, \tag{2.7}$$

and

$$B_0 := \int_\Omega W_0(x)\,dx \le -b_1 - (f,u_1) \tag{2.8}$$

where $\underline{\lambda}$ *is the largest negative point in* $\sigma(A)$. *Then* (2.1) *has a solution.*

REMARK 2.3. The hypotheses of Theorems 2.1 and 2.2 imply that $f(x)$ is orthogonal to $N(A)$.

COROLLARY 2.4. *Assume hypotheses* (A)–(E) *with* $f(x) \equiv 0$. *Assume also*

$$-W_0(x) < 2F(x,t) \le \overline{\lambda} t^2 + W_1(x), \qquad x\in\Omega,\ t\in\mathbb{R}, \tag{2.9}$$

$$2F(x,t) \to F_0(x) \qquad a.e. \ \ as \ |t| \to \infty \tag{2.10}$$

and

$$\int_\Omega W_1(x)\,dx \le \int_\Omega F_0(x)\,dx. \tag{2.11}$$

Then (2.1) *has a solution.*

COROLLARY 2.5. *Assume hypotheses* (A)–(E) *with* $f(x) \equiv 0$. *Assume also*

$$\underline{\lambda} t^2 - W_0(x) \le 2F(x,t) \le W_1(x), \qquad x \in \Omega, \ t \in \mathbb{R}, \tag{2.12}$$

$$2F(x,t) \to F_1(x) \qquad a.e. \ \ as \ |t| \to \infty \tag{2.13}$$

and

$$\int_\Omega W_0(x)\,dx \le -\int_\Omega F_1(x)\,dx. \tag{2.14}$$

Then (2.1) *has a solution.*

COROLLARY 2.6. *Assume hypotheses* (A)–(E) *with* $f(x) \equiv 0$. *Assume also*

$$F_0(x) \le F(x,t) \le F_1(x), \qquad x \in \Omega, \ t \in \mathbb{R}, \tag{2.15}$$

and either

$$F(x,t) \to F_0(x) \qquad a.e. \ \ as \ |t| \to \infty \tag{2.16}$$

or

$$F(x,t) \to F_1(x) \qquad a.e. \ \ as \ |t| \to \infty. \tag{2.17}$$

Then (2.1) *has a solution.*

3. The method

In this section we shall prove the theorems and corollaries of Section 2 using Theorem 1.1.

PROOF OF THEOREM 2.1. Let

$$N' = \bigoplus_{\lambda<0} N(A-\lambda), \quad N = N' \oplus N(A), \quad M' = N^{\perp} \cap D, \quad M = M' \oplus N(A). \tag{3.1}$$

By hypothesis (A), N', $N(A)$ and N are finite dimensional and

$$D = M' \oplus N = M \oplus N'. \tag{3.2}$$

In view of hypothesis (D), it is easily verified that the functional

$$G(u) := (Au, u) - 2\int_{\Omega} F(x, u)\, dx \tag{3.3}$$

is continuously differentiable on D and that

$$(G'(u), v) = 2(Au, v) - 2(f(x, u), v), \qquad u, v \in D. \tag{3.4}$$

By hypothesis (A) there is a constant K such that $A + K \geq 1$. We take

$$\|u\|_D^2 := (Au, u) + K\|u\|^2 \geq \|u\|^2 \tag{3.5}$$

as the norm squared in D. By (3.4),

$$G'(u) = 0 \tag{3.6}$$

is equivalent to (2.1). Note that

$$(Av, v) \leq \underline{\lambda}\|v\|^2, \qquad v \in N', \tag{3.7}$$

$$(Aw, w) \geq \overline{\lambda}\|w\|^2, \qquad w \in M'. \tag{3.8}$$

Let

$$a(u, v) := (Au, v), \qquad a(u) := a(u, u), \qquad u, v \in D. \tag{3.9}$$

We use the first decomposition in (3.2). For $v \in N$ we write $v = v' + v_0$, where $v' \in N'$ and $v_0 \in N(A)$. By (D) and (2.3),

$$\int_{\Omega} F(x, v_0)\, dx \leq \int_{\Omega} F(x, v)\, dx + \|V\|\,\|v'\|.$$

Hence

$$G(v) \leq \underline{\lambda}\|v'\|^2 + 2\|V\|\,\|v'\| - 2\int_{\Omega} F(x, v_0)\, dx, \qquad v \in N.$$

Consequently,

$$\limsup_{\substack{\|v\|\to\infty \\ v\in N}} G(v) \leq -b_0 < \infty. \tag{3.10}$$

On the other hand,

$$G(w) \geq \overline{\lambda}\|w\|^2 - 2\|V\|\,\|w\|, \qquad w \in M'. \tag{3.11}$$

Consequently,

$$m_0 := \inf_{M'} G > -\infty, \qquad m_1 := \sup_N G < \infty. \tag{3.12}$$

It now follows from Theorem 1.1 that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \qquad m_0 \leq c \leq m_1, \qquad G'(u_k) \to 0. \tag{3.13}$$

We write

$$u_k = v_k + w_k + \rho_k y_k, \quad v_k \in N',\ w_k \in M',\ y_k \in N(A),\ \|y_k\| = 1,\ \rho_k \geq 0. \tag{3.14}$$

By (3.13)

$$a(u_k, h) - (f(x, u_k), h) = o(\|h\|_D), \qquad h \in D. \tag{3.15}$$

In view of (D) this implies

$$a(v_k) = O(\|v_k\|), \qquad a(w_k) = O(\|w_k\|_D). \tag{3.16}$$

Thus

$$\|v_k\|_D \leq C, \qquad \|w_k\|_D \leq C. \tag{3.17}$$

Hence there is a renamed subsequence such that

$$v_k \to v_1 \quad \text{in } N', \qquad w_k \to w_1 \quad \text{weakly in } M'. \tag{3.18}$$

Since $\|y_k\| = 1$, there is a renamed subsequence such that $y_k \to y$ in $N(A)$. Since $y \not\equiv 0$, we know that $y \neq 0$ a.e. by hypothesis (C). Assume that

$$\rho_k \to \infty. \tag{3.19}$$

Then

$$|u_k(x)| = |v_k(x) + w_k(x) + \rho_k y_k(x)| \to \infty \qquad \text{a.e.}$$

If we put $u'_k = v_k + w_k = u_k - \rho_k y_k$, we have by (3.15),

$$a(u'_k, h) - (f(x, u_k), h) \to 0, \qquad h \in D.$$

Consequently,

$$a(u_1, h) - (f(x), h) = 0, \qquad h \in D, \tag{3.20}$$

where $u_1 = v_1 + w_1$. Then u_1 is the unique solution in $N(A)^\perp$ of $Au_1 = f$. By (3.15) and (3.20) we have

$$a(u'_k - u_1, h) - (f(x, u_k) - f(x), h) = o(\|h\|_D).$$

Taking $h = w_k - w_1$, we see that in view of (3.18),

$$u'_k \to u_k \qquad \text{in } D. \tag{3.21}$$

Since

$$\int_\Omega [F(x, u_k) - F(x, \rho_k y_k)]\, dx = \int_\Omega \int_0^1 f(x, \rho_k y_k + \theta u'_k) u'_k \, d\theta\, dx \to (f, u_1) \tag{3.22}$$

we have

$$G(u_k) = a(u_k) - 2\int_\Omega F(x, \rho_k y_k)\, dx - 2(f, u_1) + o(1).$$

Thus

$$\limsup_{k\to\infty} G(u_k) \le -(f, u_1) - b_0. \tag{3.23}$$

Consequently, by (3.13),

$$m_0 \le -(f, u_1) - b_0. \tag{3.24}$$

If $b_0 = \infty$, this contradicts (3.12). Hence assumption (3.19) must by false, i.e., for a renamed subsequence

$$\rho_k \le C. \tag{3.25}$$

But then we have a renamed subsequence such that $u_k \to u$ in D. It then follows from (3.15) that

$$a(u, h) = (f(x, u), h), \qquad h \in D, \tag{3.26}$$

showing that indeed (2.1) has a solution.

Let us now assume that $b_0 < \infty$ and that (2.4), (2.5) and (3.19) hold. By the former we have

$$G(w) \geq a(w) - B_1, \qquad w \in M'. \tag{3.27}$$

Thus $m_0 \geq -B_1$. Assume first that $m_0 > -B_1$. Then (2.5) and (3.24) imply

$$-B_1 < m_0 \leq -(f, u_1) - b_0 \leq -B_1,$$

again providing a contradiction to (3.19). Again (3.25) provides a solution to (2.1) via (3.26).

Finally, assume that $m_0 = -B_1$. From the definition of m_0, there is a minimizing sequence $\{w_k\} \subset M'$ such that $G(w_k) \to m_0$. Thus there is a renamed subsequence such that $w_k \to w_0$ weakly in M'. By hypothesis (B) there is another renamed subsequence such that $V_0 w_k \to V_0 w_0$ in $L^2(\Omega)$ and a.e. in Ω. By (D)

$$\int_\Omega [F(x, w_k) - F(x, w_0)]\, dx = \int_\Omega \int_0^1 f(x, w_0 + \Theta(w_k - w_0))(w_k - w_0)\, d\Theta\, dx \to 0.$$

Thus $G(w)$ is weakly lower semi-continuous on M' and

$$G(w_0) \leq \lim G(w_k) = m_0 = -B_1.$$

Thus

$$\overline{\lambda}\|w_0\|^2 \leq 2\int F(x, w_0) - B_1 \leq \overline{\lambda}\|w_0\|^2. \tag{3.28}$$

Consequently,

$$a(w_0) = \overline{\lambda}\|w_0\|^2,$$

showing that

$$Aw_0 = \overline{\lambda} w_0. \tag{3.29}$$

Moreover, we also see from (3.28) that

$$\int_\Omega [2F(x, w_0) - \overline{\lambda} w_0^2 - W_1(x)]\, dx = 0. \tag{3.30}$$

By (2.4) we see that

$$2F(x, w_0) \equiv \overline{\lambda} w_0^2 + W_1(x). \tag{3.31}$$

Let

$$\Phi(u) = \int_\Omega [2F(x,u) - \overline{\lambda} u^2]\,dx. \tag{3.32}$$

Then (3.30) implies

$$\Phi(u) \le \Phi(w_0), \qquad u \in D. \tag{3.33}$$

Since

$$(\Phi'(u), h) = 2(f(x,u), h) - 2\overline{\lambda}(u,h) \tag{3.34}$$

and (3.33) implies $\Phi'(w_0) = 0$, we must have

$$f(x, w_0) = \overline{\lambda} w_0.$$

Thus by (3.29) we must have

$$Aw_0 = \overline{\lambda} w_0 = f(x, w_0)$$

and we see that w_0 is a solution of (2.1). On the other hand, if (3.25) holds, we obtain a solution as before.

PROOF OF THEOREM 2.2. In this case we use the second decomposition in (3.2). In this case we have

$$G(v) \le \underline{\lambda}\|v\|^2 + 2\|V\|\,\|v\|, \qquad v \in N', \tag{3.35}$$

and

$$G(w) \ge \overline{\lambda}\|w'\|^2 - 2\int_\Omega F(x, w_0)\,dx - 2\|V\|\|w'\|, \qquad w \in M.$$

where $w = w' + w_0$, $w' \in M'$, $w_0 \in N(A)$. Thus we have

$$m_0 := \inf_M G > -\infty, \qquad m_1 := \sup_{N'} G < \infty. \tag{3.36}$$

Again we apply Theorem 1.1 to conclude that there is a sequence in D satisfying (3.3)–(3.18). Assume that (3.19) holds. Again we find that $u_1 = v_1 + w_1 \in N(A)^\perp$ satisfies (3.20), (3.21) and $Au_1 = f$. From (3.22) we see that

$$\liminf_{k\to\infty} G(u_k) \ge -(f, u_1) - b_1. \tag{3.37}$$

and consequently,

$$m_1 \geq -(f, u_1) - b_1. \tag{3.38}$$

If $b_1 = -\infty$, this contradicts (3.13), showing that (3.19) cannot hold. Once we have (3.25) we proceed as before to obtain a solution of (2.1). Assume now that $b_1 > -\infty$ and that (3.19), (2.7) and (2.8) hold. By (2.7),

$$G(v) \leq a(v) - \underline{\lambda}\|v\|^2 + B_0, \qquad v \in N'.$$

By (3.7) we see that $m_1 \leq B_0$. Assume first that $m_1 < B_0$. Then (2.8) and (3.38) imply

$$B_0 \leq -(f, u_1) - b_1 \leq m_1 < B_0$$

again providing a contradiction to (3.19). We can now use (3.5) to proceed as before. Finally, assume that $m_1 = B_0$. Let v_k be a maximizing sequence in N' such that $G(v_k) \to m_1$. By (3.35), $\|v_k\|_D \leq C$ and there is a renamed subsequence such that $v_k \to v_0$ in N'. By continuity $G(v_k) \to G(v_0)$. Hence

$$G(v_0) = m_1 = B_0.$$

Thus

$$\underline{\lambda}\|v_0\|^2 \leq 2\int F(x, v_0) + B_0 = a(v_0) \leq \underline{\lambda}\|v\|^2,$$

and consequently,

$$a(v_0) = \underline{\lambda}\|v_0\|^2.$$

Thus

$$Av_0 = \underline{\lambda}v_0.$$

We also have

$$\int_\Omega [2F(x, v_0) - \underline{\lambda}v_0^2 + W_0(x)]\, dx = 0$$

showing that

$$2F(x, v_0) \equiv \underline{\lambda}v_0^2 - W_0(x).$$

Let

$$\Phi(u) = \int_\Omega [2F(x, u) - \underline{\lambda}u^2]\, dx.$$

Then

$$(\Phi'(u), h) = 2(f(x, u), h) - 2\underline{\lambda}(u, h)$$

and

$$\Phi(u) \geq \Phi(v_0), \qquad u \in D.$$

Thus

$$\Phi'(v_0) = 2f(x, v_0) - 2\underline{\lambda} v_0 = 0.$$

Consequently,

$$Av_0 = \underline{\lambda} v_0 = f(x, v_0)$$

and v_0 is a solution of (2.1).

PROOF OF COROLLARY 2.4. We apply Theorem 2.1. In this case

$$b_0 = \lim_{|t|\to\infty} 2 \int_\Omega F(x,t)\, dx = \int_\Omega F_0(x)\, dx.$$

PROOF OF COROLLARY 2.5. In this case

$$b_1 = \int_\Omega F_1(x)\, dx$$

and we apply Theorem 2.2.

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