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CONTINUATION METHOD FOR CONTRACTIVE MAPS

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Dedicated to Jean Leray

1. Introduction

We recall that a map $T : A \to X$ between metric spaces is contractive provided $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in A$, where $0 \leq \alpha < 1$. The well known Contraction Principle of S. Banach states that any contractive operator T acting on a complete metric space has a unique fixed point $y_0 = Ty_0$.

It is the aim of this note to establish the continuation method for contractive maps. Using suitably defined notions of "essential" maps and "homotopy" for contractive operators, we show that the property of a map to be essential is invariant under homotopy. This result (which we call "the topological transversality theorem") provides a convenient tool for deducing in a simple manner several fixed point theorems of the Leray-Schauder type, which we shall prove. Our approach uses only some simple facts of metric topology; although entirely elementary, it has points in common with the more advanced homotopy extension approach developed for compact operators in [2].

2. The notion of homotopy. Essential maps

In what follows we fix arbitrarily a complete metric space (X, d). For a domain U in X, the set of all contractive maps $T : \overline{U} \to X$ will be denoted by $\mathcal{K}(U)$. Given a map T in $\mathcal{K}(U)$ we let $\operatorname{Fix} T = \{x \in \overline{U} : x = Tx\}$; clearly, because T is contractive, this set is either empty or consists of exactly one point. We shall also consider families $\{H_t : \overline{U} \to X\}$ in $\mathcal{K}(U)$, depending on a parameter

$t \in [0,1]$. Such a family will be called *α-contractive* provided

$$d(H_t(x_1), H_t(x_2)) \le \alpha d(x_1, x_2) \qquad \text{for all } t \in [0,1] \text{ and } x_1, x_2 \in X,$$

and

$$d(H_{t_1}(x), H_{t_2}(x)) \le M|t_1 - t_2| \qquad \text{for some } M > 0 \text{ and all } x \in X \text{ and } t_1, t_2 \in [0,1].$$

The number M will be called the *Lipschitz constant* of the family $\{H_t\}$. Clearly, if $\{H_t\}$ is α-contractive, then the map $H : [0,1] \times \overline{U} \to X$ given by $H(t,x) = H_t(x)$ is continuous.

To formulate the main result we consider the set

$$\mathcal{K}_0(U) = \{T \in \mathcal{K}(U) : \operatorname{Fix} T \cap \partial U = \emptyset\},$$

in which we introduce a suitable notion of "homotopy".

DEFINITION 2.1. By a *homotopy in* $\mathcal{K}_0(U)$ is meant an α-contractive family $\{H_t : \overline{U} \to X\}$ such that $\alpha < 1$ and all $H_t \in \mathcal{K}_0(U)$. Two maps $S, T \in \mathcal{K}_0(U)$ are called *homotopic*, $S \sim T$, provided there is a homotopy $\{H_t\}$ in $\mathcal{K}_0(U)$ such that $H_0 = S$ and $H_1 = T$. Clearly "$\sim$" is an equivalence relation in $\mathcal{K}_0(U)$; under this relation, $\mathcal{K}_0(U)$ decomposes into disjoint homotopy classes of contractive maps.

Next, for members of $\mathcal{K}_0(U)$, we introduce the notion of an essential map.

DEFINITION 2.2. A map $T \in \mathcal{K}_0(U)$ is said to be *essential* provided T has a fixed point. Otherwise T is called *inessential*.

3. The main theorem

We now establish our main result which says that the property of a map to be essential depends only on the homotopy class of the map.

THEOREM 3.1 (Topological Transversality). *Let $\{H_t\}$ be a homotopy in $\mathcal{K}_0(U)$. If H_0 is essential, then so is H_t for every t in $[0,1]$.*

PROOF. Consider the set

$$\Lambda = \{\lambda \in [0,1] : x = H_\lambda(x) \text{ for some } x \in U\}.$$

Because, by assumption, H_0 is essential, i.e. $0 \in \Lambda$, the set Λ is non-empty.

We establish the following two facts:

(i) Λ is closed in $[0,1]$: Letting $\lambda_n \to \lambda_0$ with $\lambda_n \in \Lambda$, we show that λ_0 also belongs to Λ. For any n, let x_n be a fixed point of H_{λ_n} in U. Then

$$\begin{aligned} d(x_n, x_m) &\le d(H_{\lambda_n}(x_n), H_{\lambda_m}(x_n)) + d(H_{\lambda_m}(x_n), H_{\lambda_m}(x_m)) \\ &\le M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m) \end{aligned}$$

and therefore $d(x_n, x_m) \le \frac{M}{1-\alpha}|\lambda_n - \lambda_m|$.

Since $\{\lambda_n\}$ is a Cauchy sequence, so is $\{x_n\}$. Hence for some $x_0 \in \overline{U}$ we have $x_n \to x_0$ and therefore, by continuity of H, we get $x_n = H_{\lambda_n}(x_n) \to H_{\lambda_0}(x_0)$ and thus $x_0 = H_{\lambda_0}(x_0)$; because $\{H_t\}$ is a homotopy in $\mathcal{K}_0(U)$, we conclude that x_0 must be in U. Thus $\lambda_0 \in \Lambda$ and the proof of (i) is complete.

(ii) Λ is open in $[0,1]$: Letting $\lambda_0 \in \Lambda$ and $x_0 = H_{\lambda_0}(x_0)$, we fix $\varepsilon > 0$ such that $\varepsilon \leq (1-\alpha)r/M$, where $r < \operatorname{dist}(x_0, \partial U)$ and M is the Lipschitz constant of the homotopy $\{H_t\}$. Let now $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. From the estimate

$$\begin{aligned} d(H_\lambda(x), x_0) &\leq d(H_\lambda(x), H_\lambda(x_0)) + d(H_\lambda(x_0), H_{\lambda_0}(x_0)) \\ &\leq \alpha d(x, x_0) + (1-\alpha)r \end{aligned}$$

we conclude that, if $d(x, x_0) \leq r$, then $d(H_\lambda(x), x_0) \leq r$; consequently, H_λ maps the closed ball $K(x_0, r) \subset U$ into itself, and so, by the Banach Theorem, H_λ has a fixed point in U. Thus $\lambda \in \Lambda$ for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and the proof of (ii) is complete.

Because $[0,1]$ is connected and Λ is non-empty we conclude that $\Lambda = [0,1]$. This implies that every H_t is essential and the proof of the theorem is complete.

4. Nonlinear Alternative for contractive maps

From now on we assume that $X = E$ is a Banach space. This assumption permits us to get several consequences of the main theorem. We observe first that the property of a map $T \in \mathcal{K}_0(U)$ to be essential depends only on the behaviour of T on the boundary ∂U of U. Precisely: if $S, T \in \mathcal{K}_0(U)$ are such that $S|\partial U = T|\partial U$, then T is essential if and only if so is S.

THEOREM 4.1 (Nonlinear Alternative). *Let U be an open bounded subset of E with $0 \in U$. Then any $T \in \mathcal{K}(U)$ has at least one of the following properties:*

(i) *T has a fixed point,*

(ii) *there exist $y_0 \in \partial U$ and $\lambda \in (0,1)$ such that $y_0 = \lambda T y_0$.*

PROOF. Examine the α-contractive family $\{H_t\}$ in $\mathcal{K}(U)$ given by $H_t(x) = tT(x)$ for $(t,x) \in [0,1] \times \overline{U}$. Assume that $\{H_t\}$ is a homotopy in $\mathcal{K}_0(U)$. By Theorem 3.1, since H_0 is essential, so is $H_1 = T$ and thus T has a fixed point. If $\{H_t\}$ is not a homotopy in $\mathcal{K}_0(U)$, then tT must have a fixed point on the boundary for some t; clearly $t \neq 0$ (because $0 \in U$) and therefore, either T has a fixed point on ∂U, or property (ii) holds. The proof is complete.

As an immediate consequence we obtain:

COROLLARY 4.2. *Let U be an open bounded subset of E with $0 \in U$ and let $T \in \mathcal{K}_0(U)$. Then:*

(i) *if $Tx \neq \mu x$ for all $x \in \partial U$ and $\mu > 1$, then T is essential,*

(ii) *if there is a point $c \in E$, $c \neq 0$, such that $Tx + \mu c \neq x$ for all $x \in \partial U$ and all $\mu > 0$, then T is inessential.*

PROOF. (i) This is clearly a consequence of Theorem 4.1.

(ii) Assume to the contrary that T is essential. For a fixed $\mu > 0$, consider the family $\{H_t\}$ given by $H_t(x) = Tx + t\mu c$. Because of (ii), $\{H_t\}$ is a homotopy joining $H_0 = T$ and $H_1 = T + \mu c$. We conclude from the above that $T + \mu c$ has a fixed point. Since μ was arbitrary, we can get (by taking $\mu = 1, 2, \dots$) a sequence $\{x_n\}$ of points x_n in U such that $Tx_n + nc = x_n$. From this, because the set U is bounded, we obtain a contradiction. Thus T is inessential.

5. Other consequences. Leray-Schauder Alternative

We now establish a few other consequences of Theorem 4.1.

THEOREM 5.1. *Let U be an open bounded subset of E with $0 \in U$ and $T \in \mathcal{K}_0(U)$ be a map such that for all $x \in \partial U$ one of the following conditions is satisfied:*

(i) $\|T(x)\| \le \|x\|$ *(E. Rothe);*
(ii) $\|T(x)\| \le \|T(x) - x\|$;
(iii) $\|T(x)\| \le (\|x\|^2 + \|T(x) - x\|^2)^{1/2}$ *(M. Altman);*
(iv) $\|T(x)\| \le \max\{\|x\|, \|T(x) - x\|\}$;
(v) $\langle Tx, x\rangle \le \langle x, x\rangle$, *where $\langle\cdot,\cdot\rangle$ is a scalar product in E.*

Then $\operatorname{Fix} T \ne \emptyset$.

PROOF. The routine verification that property (ii) in Theorem 4.1 cannot occur is left to the reader.

THEOREM 5.2 (Antipodal Theorem). *Let U be an open bounded subset of E symmetric with respect to the origin, $0 \in U$ and let $T \in \mathcal{K}_0(U)$ be such that $T(x) = -T(-x)$ for all $x \in \partial U$. Then* $\operatorname{Fix} T \ne \emptyset$.

PROOF. Because U is symmetric with respect to 0 and T is contractive and odd on ∂U, it follows at once that $\|T(x)\| \le \|x\|$ for any $x \in \partial U$. Thus our assertion follows from Theorem 5.1(i).

THEOREM 5.3 (Leray-Schauder Alternative). *Let $T : E \to E$ be α-contractive on every ball $K(0,r)$ in E, and let $\mathcal{E}_T = \{x \in E : x = \lambda T(x)$ for some $\lambda \in (0,1)\}$. Then either $\mathcal{E}_T$ is unbounded or T has a fixed point.*

PROOF. Assuming $\mathcal{E}_T$ is bounded, let $K(0,r)$ be a ball containing $\mathcal{E}_T$ in its interior. Since no $x \in \partial K(0,r)$ can satisfy the second property in Theorem 4.1, the map T has a fixed point and the proof is complete.

6. Remarks

1. It is easily seen that Theorem 4.1 remains valid in the case when $X = C$ is a closed cone in a Banach space E and U is a bounded open set in C with

$0 \in U$. Consequently, Corollary 4.2 and also Theorems 5.1 and 5.2 remain valid in this setting.

2. Theorem 4.1 for the cones in E is a special case of the Non-linear Alternative for k-set contractions (cf. [1], p. 70); for Banach spaces it follows from the theory of topological degree for k-set contractions as developed by R. Nussbaum in [6]; in the special case when U is a ball $K(0,r)$ in E, this theorem was established in a direct manner in [3].

3. Topological transversality theorem can be extended to multivalued contractive maps in the sense of S. Nadler [5]; this and related matters are considered in the article of M. Frigon and the author (cf. [2]).

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