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THE DIRICHLET PROBLEM FOR THE PRESCRIBED CURVATURE QUOTIENT EQUATIONS

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Dedicated to Jean Leray

1. Introduction

In this paper we consider the classical Dirichlet problem for equations of prescribed curvature of the form

$$F[u] = f(\kappa) = \Psi(x, u) \tag{1.1}$$

in domains Ω in Euclidean n-space, $\mathbb{R}^n$, where $\kappa = (\kappa_1, \dots, \kappa_n)$ denotes the principal curvatures of the graph of u over Ω, Ψ is a prescribed positive function on $\Omega \times \mathbb{R}$ and f is a symmetric function of the form

$$f(\kappa) = S_{k,l} = \frac{S_k}{S_l}, \tag{1.2}$$

where $0 \leq l < k \leq n$ and S_k denotes the k-th order elementary symmetric function,

$$S_k = \sum \kappa_{i_1} \kappa_{i_2} \dots \kappa_{i_k}, \tag{1.3}$$

the sum being taken over all increasing k-tuples $i_1, i_2, \dots, i_k \subset \{1, \dots, n\}$. Taking $S_0 = 1$, we may write $S_{k,0} = S_k$. The mean, scalar, Gauss and harmonic

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curvatures correspond respectively to the special cases $l = 0$, $k = 1, 2, n$ and $l = n - 1$, $k = n$. The classical Dirichlet problem for the prescribed mean and Gauss curvature equations has been extensively studied (see for example [1], [4], [6] and [13]). For the mean curvatures of intermediate order, that is, the cases $f = S_k$, $1 < k < n$, the first breakthroughs were due to Caffarelli, Nirenberg and Spruck [3] and Ivochkina [7] and [8] for the case of convex domains and zero boundary values. Ivochkina [9] extended her approach to embrace general boundary values and domains subject to natural geometric restrictions. While her work dealt specifically with the cases $f = S_k$, that of Caffarelli, Nirenberg and Spruck treated more general curvatures but still excluded the quotients $S_{k,l}$ for $l \geq 1$. These cases were however included in the weak or viscosity solution approach of Trudinger [17] who established existence theorems for Lipschitz solutions. The solution and gradient bounds of Trudinger [15]–[17] lead to an improvement of the classical existence theorems of Ivochkina [9] for the higher order mean curvatures.

In this paper we extend the classical theory to the case of quotients of mean curvatures, that is, to the cases $f = S_{k,l}$, $l > 0$. As with [3], [8] and [9], the essence of our work lies in the derivation of second derivative estimates at the boundary for prospective solutions. For the mixed tangential-normal derivatives we follow the approach of Ivochkina [8], utilizing a fundamental inequality for quotients of elementary symmetric functions that we established in [11]. For the double normal derivatives we adapt a technique of Trudinger [18] from the case of Hessian equations.

In order to formulate our main existence theorem, we introduce some terminology from [17]. For a general continuous symmetric function f in (1.1), let us define its *admissible set* by

$$\mathcal{A}(f) = \{\kappa \in \mathbb{R}^n \mid f(\kappa + \eta) \geq f(\kappa) \ \forall \eta_i \geq 0, \ i = 1, \dots, n\} \tag{1.4}$$

and call a function $u \in C^2(\Omega)$ *admissible* for the operator F in Ω if the principal curvatures $\kappa = (\kappa_1, \dots, \kappa_n)$ of the graph of u belong to $\mathcal{A}(f)$ at every point $(x, u(x))$, $x \in \Omega$. The operator F will be degenerate elliptic with respect to an admissible function. When $f = S_{k,l}$, we have $\mathcal{A}(f) = \overline{\Gamma}_k$, where Γ_k is the open cone in $\mathbb{R}^n$ with vertex at the origin, given by

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \mid S_j(\kappa) > 0, \ j = 1, \dots, k\}. \tag{1.5}$$

Clearly $\Gamma_k \subset \Gamma_j$ for $j \leq k$ and Γ_n is the positive cone $\{\kappa \in \mathbb{R}^n \mid \kappa_i > 0, \ i = 1, \dots, n\}$. For $k = 1, \dots, n - 1$, we say that the domain Ω with boundary

$\partial\Omega \in C^2$ is *k-convex* (*uniformly k-convex*) if the principal curvatures of $\partial\Omega$, $\kappa' = (\kappa_1', \dots, \kappa_{n-1}')$, satisfy $S_j(\kappa') \geq 0$ (> 0) for $j = 1, \dots, k$, that is, $\kappa' \in \overline{\Gamma}_k(\Gamma_k)$. We then have the following existence theorem for the classical Dirichlet problem.

THEOREM 1.1. *Let* $0 \leq l < k < n$, $0 < \alpha < 1$. *Assume that*

(i) Ω *is a bounded* $(k-1)$*-convex domain in* $\mathbb{R}^n$, *with boundary* $\partial\Omega \in C^{4,\alpha}$;
(ii) $\Psi \in C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$, $\partial\Psi/\partial z \geq 0$, $\Psi > 0$ *on* $\overline{\Omega} \times \mathbb{R}$;
(iii) $\Psi(x, 0) \leq S_{k,l}(\kappa')$ *on* $\partial\Omega$.

Then, provided there exists any bounded admissible subsolution of equation (1.1) *in* Ω, *there exists a unique admissible solution* $u \in C^{4,\alpha}(\overline{\Omega})$ *satisfying* $u = 0$ *on* $\partial\Omega$.

The existence of a bounded subsolution (which can be taken in the viscosity sense of [17]) can be replaced by conditions guaranteeing *a priori* solution bounds, for example (from [17]),

$$\sup_{x\in\Omega} \Psi(x,0)\left(\frac{\operatorname{diam}\Omega}{2}\right)^{k-l} < \binom{n}{k}\Big/\binom{n}{l}. \tag{1.6}$$

More general conditions involving quermassintegrals are provided in [15], [16] and [19].

For convenience we have expressed the smoothness conditions in Theorem 1.1 in terms of Hölder spaces (see [4] for definitions and notation). A minimal assumption for our proof would be $\partial\Omega \in C^{3,1}$ and $\Psi \in C^{1,1}(\overline{\Omega}\times\mathbb{R})$ with resulting solution $u \in C^{2,\alpha}(\overline{\Omega})$ for all $\alpha < 1$.

The uniqueness of the solution in Theorem 1.1 follows from a comparison principle (see [3], [8] and [17]). Namely, if $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $u \leq v$ on $\partial\Omega$, $F[u] > F[v]$ in Ω and u is admissible with respect to F, then it follows that $u \leq v$ in Ω. If in addition F is elliptic with respect to u (which is implied by $F[u] > 0$ in our case $f(\kappa) = S_{k,l}(\kappa)$), then we can relax $F[u] \geq F[v]$.

The hypotheses of Theorem 1.1 imply that Ω is uniformly k-convex. The case $k = n$ is omitted from Theorem 1.1, as condition (iii) implies $\Psi = 0$ on $\partial\Omega$. We may embrace this case by assuming the existence of a subsolution satisfying the boundary condition (as done by Caffarelli, Nirenberg and Spruck [3]).

THEOREM 1.2. *Let* $0 \leq l < k \leq n$, $0 < \alpha < 1$. *Assume that hypothesis* (ii) *of Theorem* 1.1 *holds with hypotheses* (i) *and* (iii) *replaced by:*

(i)′ Ω *is uniformly k-convex,* $k < n$, *and uniformly convex for* $k = n$, *with* $\partial\Omega \in C^{4,\alpha}$.

Then, provided there exists an admissible subsolution $u_0 \in C^{0,1}(\overline{\Omega})$, *with* $u_0 = 0$ *on* $\partial\Omega$, *there exists a unique admissible solution* $u \in C^{4,\alpha}(\overline{\Omega})$ *of equation* (1.1), *satisfying* $u = 0$ *on* $\partial\Omega$.

The proofs of the above theorems utilize the method of continuity which reduces the problem of existence to that of *a priori* estimates for a related family of problems in the Hölder space $C^{2,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ (see [4]). For the curvature quotients, the new estimates needed in this paper are those of second derivatives of prospective solutions on the boundary of the domain Ω. Our assumption of the existence of a bounded subsolution u_0 automatically ensures, by the comparison principle, a solution bound,

$$u_0 - \sup_{\partial\Omega} u_0 \leq u \leq 0, \tag{1.7}$$

which also implies a boundary gradient estimate in the case of Theorem 1.2. Estimates for the first and second derivatives of solutions in terms of their boundary values are provided by Caffarelli, Nirenberg and Spruck [3] for more general equations, while the boundary gradient estimate for Theorem 1.1, under condition (iii), is given in Trudinger [17]. Note that condition (iii) is the natural extension of the Serrin condition for the mean curvature case [13]. The Hölder bounds for second derivatives follow from the Krylov theory (see [10], [4] and [14]).

In the following section, we list some of the basic properties of elementary symmetric functions and their quotients, to be used in this paper. In Section 3, we derive the mixed tangential-normal boundary estimates for second derivatives, while the double normal derivative estimation is treated in Section 4, thereby completing the estimation of the second derivatives on the boundary. These estimations are carried out for the more general degenerate case, $\Psi \geq 0$ (although we need to assume some smoothness of $\Psi^{1/m}$). Finally, in Section 5, we complete the proofs of Theorems 1.1, 1.2 and derive related results. In further investigations, we consider the extension of our existence theorems to the degenerate case, as well as to the Dirichlet problem for general boundary values.

2. Preliminaries

In this section we list some properties of elementary symmetric functions and the associated curvature operators, which we shall use. Defining for any fixed s-tuple $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$, $\lambda \in \mathbb{R}^n$

$$S_{k;i_1 \ldots i_s}(\lambda) = S_k|_{\lambda_{i_1} = \cdots = \lambda_{i_s} = 0}, \tag{2.1}$$

we have for $\lambda \in \Gamma_k$,

$$S_{l;i_1\dots i_s}(\lambda) > 0 \tag{2.2}$$

for all $\{i_1, \dots, i_s\} \subset \{1, \dots, n\}$, $l + s \leq k$. In particular,

$$\frac{\partial S_k}{\partial \lambda_i} = S_{k-1;i} > 0 \tag{2.3}$$

for $\lambda \in \Gamma_k$. Introducing the normalized functions

$$\sigma_k = \frac{1}{\binom{n}{k}} S_k, \qquad \sigma_{k,l} = \frac{\sigma_k}{\sigma_l}, \qquad k \geq l \geq 0, \tag{2.4}$$

we also have on Γ_k the Newton-Maclaurin inequalities

$$(\sigma_{k,l})^{1/(k-l)} \leq (\sigma_{r,s})^{1/(r-s)} \tag{2.5}$$

provided $k \geq r, l \geq s$. In particular, it follows that

$$\begin{aligned} \frac{\partial S_{k,l}}{\partial \lambda_i} &= \frac{S_l S_{k-1;i} - S_k S_{l-1;i}}{S_l^2} \\ &= \frac{S_{l;i} S_{k-1;i} - S_{k;i} S_{l-1;i}}{S_l^2} \\ &\geq \frac{n(k-l)}{k(n-l)} \frac{S_{l;i} S_{k-1;i}}{S_l^2} > 0 \end{aligned} \tag{2.6}$$

on Γ_k. Inequality (2.6) implies that the operator (1.1) will be elliptic whenever the principal curvatures $\kappa \in \Gamma_k$. From (2.6), we also have the formula

$$\sum_{i=1}^{n} \frac{\partial S_{k,l}}{\partial \lambda_i} = (n-k+1)\frac{S_{k-1}}{S_l} - (n-l+1)\frac{S_k S_{l-1}}{S_l^2} \tag{2.7}$$

so that by (2.5),

$$\left(1 - \frac{l}{k}\right) S_{k-1,l} \leq \frac{1}{n-k+1} \sum_{i=1}^{n} \frac{\partial S_{k,l}}{\partial \lambda_i} \leq S_{k-1,l}. \tag{2.8}$$

We also need the concavity of the functions $(S_{k,l})^{1/(k-l)}$ on Γ_k. Taking account of homogeneity, we can express this by the inequality

$$m[S_{k,l}(\lambda)]^{1-1/m}[S_{k,l}(\mu)]^{1/m} \leq \sum_{i=1}^{n} \frac{\partial S_{k,l}(\lambda)}{\partial \lambda_i} \mu_i, \qquad m = k - l, \tag{2.9}$$

for all $\lambda, \mu \in \Gamma_k$. Inequalities (2.5) and (2.9) are proved, for example, in [5] and [12]. The essential new inequality we need for our work is established by us in [11]. For any fixed $r = 1, \dots, n$, we have

$$\frac{\partial S_{k,l}(\lambda)}{\partial \lambda_r}\lambda_r^2 \leq C(k,l,n)\sum_{i\neq r}\frac{\partial S_{k,l}(\lambda)}{\partial \lambda_i}\lambda_i^2 \tag{2.10}$$

for $\lambda \in \Gamma_k$, $l > 0$, where C is a constant depending on k, l, n. Note that (2.10) is not valid for $l = 0$. In this case, we obtain

$$\frac{\partial S_k(\lambda)}{\partial \lambda_r}\lambda_r^2 \leq \lambda_r S_k(\lambda) + C(k,n)\sum_{i\neq r}\frac{\partial S_k(\lambda)}{\partial \lambda_i}\lambda_i^2 \tag{2.11}$$

provided $k > 1$; inequality (2.11) improves a key inequality of Ivochkina [8]. Following Ivochkina [8], we extend the cones Γ_k to symmetric matrices. Namely, for $p \in \mathbb{R}^n$, let us define

$$\Gamma_k(p) = \{r \in \mathbf{S}^n \mid \lambda(p,r) \in \Gamma_k\}, \tag{2.12}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of the matrix

$$\left(I - \frac{p \otimes p}{1+|p|^2}\right)r. \tag{2.13}$$

Writing

$$S_k(p,r) = S_k(\lambda), \qquad S_{k,l} = \frac{S_k}{S_l}, \tag{2.14}$$

it then follows that the matrix

$$\frac{\partial S_{k,l}}{\partial r} = \left[\frac{\partial S_{k,l}}{\partial r_{ij}}\right] > 0 \tag{2.15}$$

on $\Gamma_k(p)$ and moreover, from (2.9), $S_{k,l}^{1/m}$ is concave with respect to r, and

$$m[S_{k,l}(p,r)]^{1-1/m}[S_{k,l}(p,s)]^{1/m} \leq \sum \frac{\partial S_{k,l}}{\partial r_{ij}}s_{ij}, \qquad m = k-l, \tag{2.16}$$

for all $r, s \in \Gamma_k(p)$. Furthermore, by [9], we see that $\Gamma_{k+1}(0) \subset \Gamma_k(p)$ and

$$S_{k,l}(p,r) \geq \frac{1}{1+|p|^2}S_{k,l}(0,r) \tag{2.17}$$

for all $r \in \Gamma_{k+1}(0)$. Note that the operator F is given by

$$F[u] = S_{k,l}(\kappa) = (1+|Du|^2)^{-m/2}S_{k,l}(Du, D^2u)$$

in this notation.

3. Mixed second derivative boundary estimates

Our approach to the estimation of the mixed tangential-normal second derivatives, on the boundary, follows that of Ivochkina [8], which employs the same auxiliary function as in the uniformly elliptic case, Trudinger [14]. It will be convenient for our calculations to write equation (1.1) in the form

$$F[u] = F(Du, D^2u) = f(\lambda) = \Psi(x, u, Du), \tag{3.1}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues (in decreasing order) of the matrix

$$\mathcal{C} = (I - \nu \otimes \nu) D^2 u, \qquad \nu = \frac{Du}{\sqrt{1 + |Du|^2}}, \tag{3.2}$$

and f and Ψ are given by

$$f(\lambda) = S_{k,l}(\lambda), \qquad \Psi(x, z, p) = \Psi(x, z)(1 + |p|^2)^{(k-l)/2}. \tag{3.3}$$

Fixing a point y on the boundary $\partial\Omega$ of the domain Ω, we choose the coordinate axes so that the x_n axis is directed along the inner normal at y. Let ξ be a C^2 vector field in some neighbourhood $\mathcal{N}$ of y which is tangential on $\mathcal{N} \cap \partial\Omega$, and consider the function

$$w = D_\xi u - \frac{1}{2} \sum_{s=1}^{n-1} (D_s u - D_s u(y))^2. \tag{3.4}$$

If $u \in C^3(\mathcal{N} \cap \Omega)$ is an admissible solution of equation (3.1), then we show that w satisfies an elliptic differential inequality of the form

$$F^{ij} D_{ij} w - \Psi^i D_i w \leq C(|\widetilde{D}\Psi| + F^{ii} + F^{ij} D_i w D_j w), \tag{3.5}$$

with coefficients F^{ij}, Ψ^i given by

$$F^{ij} = \frac{\partial F}{\partial r_{ij}}(Du, D^2u), \qquad \Psi^i = \frac{\partial \Psi}{\partial p_i}(x, u, Du) \tag{3.6}$$

and constant C depending on n, $|\xi|_2$ and $|Du|_0$, where $\widetilde{D}$ denotes the gradient in $\mathbb{R}^{2n+1}(x, z, p)$. To derive (3.5) we use the same coordinate system as [8], which corresponds to the projection of principal curvature directions of the graph of u onto $\mathbb{R}^n$. Fixing a point $y \in \mathcal{N} \cap \Omega$, we choose a basis of eigenvectors $\tau_1, \dots, \tau_n$ of the matrix (3.2) at y, corresponding to the eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ and orthonormal with respect to the inner product given by the matrix

$$A = I + Du \otimes Du. \tag{3.7}$$

Using a subscript α to denote differentiation with respect to $\tau_\alpha, \alpha = 1, \dots, n$, so that

$$u_\alpha = \tau_\alpha^i D_i u, \qquad u_{\alpha\alpha} = \lambda_\alpha = \tau_\alpha^i \tau_\alpha^j D_{ij} u,$$

we then obtain, by differentiation of equation (3.1),

$$\begin{aligned} D_k F[u] &= F^{ij} D_{ijk} u + F^i D_{ik} u \\ &= f_\alpha \tau_\alpha^i \tau_\alpha^j D_{ijk} u - 2 f_\alpha \lambda_\alpha \tau_\alpha^i \tau_\alpha^j D_i u D_{jk} u \\ &= f_\alpha (D_k u)_{\alpha\alpha} - 2 f_\alpha \lambda_\alpha u_\alpha (D_k u)_\alpha \\ &= D_k \Psi \end{aligned} \tag{3.8}$$

where

$$f_\alpha = \frac{\partial f}{\partial \lambda_\alpha}, \qquad \tau_\alpha = (\tau_\alpha^1, \dots, \tau_\alpha^n), \ \alpha = 1, \dots, n, \ F^i = D_{p_i} F.$$

Formula (3.8) agrees with formula (4.7) in [8] (for the case $f(\lambda) = S_k(\lambda)$) but the proof in [8] appears incomplete (see also [21]). Observe that the right hand side of (3.8) is independent of our choice of τ_α when the eigenvalue λ_α is not simple. Using (3.8) we now calculate

$$\begin{aligned} F^{ij} D_{ij} w = {} & F^{ij} \xi_k D_{ijk} u + 2 F^{ij} (D_i \xi_k) D_{jk} u + D_k u F^{ij} D_{ij} \xi_k \\ & - (D_s u - D_s u(y)) F^{ij} D_{ijs} u - F^{ij} D_{is} u D_{js} u \\ = {} & 2 f_\alpha \lambda_\alpha u_\alpha W_\alpha - 2 f_\alpha \lambda_\alpha u_\alpha D_k u (\xi_k)_\alpha \\ & + 2 f_\alpha (\xi_k)_\alpha (D_k u)_\alpha + D_k u f_\alpha (\xi_k)_{\alpha\alpha} - f_\alpha (D_s u)_\alpha^2 \\ & + \xi_k D_k \Psi - (D_s u - D_s u(y)) D_s \Psi \end{aligned} \tag{3.9}$$

where summation over s is only taken from 1 to $n-1$ (all other repeated indices are summed from 1 to n). As with the uniformly elliptic case the term

$$F^{ij} D_{is} u D_{js} u = f_\alpha (D_s u)_\alpha^2$$

is the key element to control other terms on the right hand side of (3.9). Letting $[\eta_i^\alpha]$ denote the inverse matrix to $[\tau_\alpha^i]$, we write

$$(D_s u)_\alpha = \tau_\alpha^i D_{is} u = \eta_s^\alpha \lambda_\alpha \tag{3.10}$$

so that

$$\sum_{s=1}^{n-1} f_\alpha (D_s u)_\alpha^2 = f_\alpha \lambda_\alpha^2 \sum_{s=1}^{n-1} (\eta_s^\alpha)^2. \tag{3.11}$$

Now we reason similarly to [8]. Suppose for all $\alpha = 1, \dots, n$ we have

$$\sum_{s=1}^{n-1} (\eta_s^\alpha)^2 \geq \varepsilon^2, \tag{3.12}$$

where ε is any positive number satisfying

$$\varepsilon \leq [2n!(1 + M_1^2)^{(n-1)/2}]^{-1}, \qquad M_1 = |Du|_0. \tag{3.13}$$

Then we clearly have the estimate

$$\sum f_\alpha \lambda_\alpha^2 \leq \frac{1}{\varepsilon^2} \sum_{s=1}^{n-1} f_\alpha (D_s u)_\alpha^2. \tag{3.14}$$

On the other hand, if (3.12) is not true, then

$$\sum_{s=1}^{n-1} (\eta_s^\gamma)^2 < \varepsilon^2 \tag{3.15}$$

for some γ, which implies (see [8])

$$\sum_{s=1}^{n-1} (\eta_s^\alpha)^2 \geq \delta^2$$

for all $\alpha \neq \gamma$ where

$$\delta = [2(n-1)!(1 + M_1^2)^{(n-1)/2}]^{-1}$$

and hence

$$\sum_{\alpha \neq \gamma} f_\alpha \lambda_\alpha^2 \leq \frac{1}{\delta^2} \sum_{s=1, \alpha \neq \gamma}^{n-1} f_\alpha (D_s u)_\alpha^2. \tag{3.16}$$

At this point we use, for the first time in this argument, our particular form of f so that by (2.10) we conclude, in both cases,

$$\sum f_\alpha \lambda_\alpha^2 \leq C \sum_{s=1}^{n-1} f_\alpha (D_s u)_\alpha^2, \tag{3.17}$$

where C is a constant depending on n and M_1. Substituting (3.17) into (3.9) and using Cauchy's inequality, we thus obtain

$$\begin{aligned} F^{ij} D_{ij} w - \Psi^i D_i w &\leq C\left(|\widetilde{D}\Psi| + \sum f_\alpha + f_\alpha w_\alpha^2 \right) \\ &\leq C(|\widetilde{D}\Psi| + F^{ii} + F^{ij} D_i w D_j w) \end{aligned} \tag{3.18}$$

as asserted. The inequality (3.18) can be simplified further for

$$\sum f_\alpha \geq \frac{(n-k+1)(k-l)}{k} S_{k-1,l}$$
$$\geq c\Psi^{1-1/m}, \qquad m = k-l,\ c = c(k,l),$$

by (2.8), (2.5) and hence

$$F^{ij}D_{ij}w - \Psi^i D_i w \leq C(F^{ii} + F^{ij}D_i w D_j w), \tag{3.19}$$

where C depends also on $|\widetilde{D}\Psi^{1/m}|_0$. But then setting

$$\widetilde{w} = 1 - e^{-aw} - b|x-y|^2 \tag{3.20}$$

we obtain

$$F^{ij}D_{ij}\widetilde{w} \leq \Psi^i D_i \widetilde{w} \tag{3.21}$$

for constants a, b depending on n, $|\xi|_2$, $|\widetilde{D}\Psi^{1/m}|_0$ and $|Du|_0$ provided $\operatorname{diam}\mathcal{N}$ is sufficiently small in terms of the same quantities.

REMARK. The precise form of the first derivative term in (3.21) is crucial for our subsequent barrier argument. If we replace the quotient $S_{k,l}$ by S_k then a similar argument, using (2.11), yields

$$F^{ij}D_{ij}\widetilde{w} \leq C|D\widetilde{w}| \tag{3.22}$$

for some constant C, (as in [8]).

Barrier construction. Now let us assume that $\partial\Omega$ is uniformly k-convex for $k < n$ (uniformly convex for $k = n$), with $u = 0$ on $\partial\Omega$. Fixing ξ and $\mathcal{N}$ as above we select a function $g \in C^2(\mathcal{N} \cap \Omega)$ satisfying

$$g \leq \widetilde{w} \quad \text{on } \partial(\mathcal{N} \cap \Omega), \qquad g(y) = w(y), \tag{3.23}$$

$$F^{ij}D_{ij}g \geq 0 \qquad \text{in } \mathcal{N} \cap \Omega. \tag{3.24}$$

By virtue of (2.16), (3.24) follows if $D^2 g \in \Gamma_k(Du)$ and hence by (2.17), if $D^2 g \in \Gamma_{k+1}(0)$ if $k < n$. Accordingly we can construct g, similarly to Ivochkina [8], of the form

$$g(x) = -a_0|x-y|^2 + (e^{-b_0 d(x)} - 1)c_0 \tag{3.25}$$

for appropriate constants a_0, b_0 and c_0, where $d(x) = \text{dist}\,(x, \partial\Omega)$. Having fixed g, we then consider as a barrier

$$w^* = Au + g \tag{3.26}$$

for a further constant A to be chosen. With Ψ now given by (3.3), we then have

$$F^{ij}D_{ij}w^* - \Psi^i D_i w^* \geq m\Psi(x,u)(1+|Du|^2)^{m/2-1}(A - Du \cdot Dg) \geq 0 \tag{3.27}$$

for $A \geq \max_{\mathcal{N}\cap\Omega} |Du||Dg|$. By the maximum principle we then conclude $w^* \leq \widetilde{w}$ in $\mathcal{N} \cap \Omega$, and hence

$$D_n\widetilde{w}(y) \leq D_n w^*(y), \tag{3.28}$$

yielding an estimate for $D_{in}u(y)$, $i = 1, \ldots, n-1$, as required.

THEOREM 3.1. *Let Ω be a uniformly k-convex domain in $\mathbb{R}^n$ for $k < n$ (uniformly convex if $k = n$), with boundary $\partial\Omega \in C^{2,1}$. Let Ψ be a non-negative function in $\Omega \times \mathbb{R}$, with $\Psi^{1/m} \in C^{0,1}(\overline{\Omega} \times \mathbb{R})$. Then if $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is an admissible solution of equation (3.1) satisfying $u = 0$ on $\partial\Omega$, we have, at any point $y \in \partial\Omega$, the estimate*

$$|D_{in}u(y)| \leq C, \qquad i = 1, \ldots, n-1, \tag{3.29}$$

with constant C depending on n, $\partial\Omega$, $|D\Psi^{1/m}|_0$ and $|Du|_0$.

The above argument establishes the case $l > 0$ in Theorem 3.1. The case $l = 0$ is covered by Ivochkina [9] and is simpler in that w^* already furnishes a barrier for inequality (3.22) when $A = 0$. Moreover, it is then possible to allow u to take on arbitrary boundary values $\phi \in C^{2,1}(\partial\Omega)$.

4. Normal second derivative boundary estimates

To complete the estimation of second derivatives of solutions of equation (1.1) on the boundary $\partial\Omega$, we need to estimate the double normal derivative $D_{nn}u$ in terms of the other second derivatives. Our argument below, which is adapted from the case of Hessian equations [18], is new even for the case $l = 0$ [9]. We will utilize the same barrier construction as for the estimation of the mixed derivatives in the previous section and this will have the effect of restricting our argument to the case of zero boundary values when $l > 0$. Suppose thus that $u \in C^3(\overline{\Omega})$ is an admissible solution of equation (1.1) in Ω, satisfying $u = 0$ on $\partial\Omega$, with $\partial\Omega \in C^4$, uniformly k-convex for $k < n$ (uniformly convex if $k = n$).

Letting γ denote the unit, outer normal vector field on $\partial\Omega$, we have, with respect to a principal coordinate system at any point $y \in \partial\Omega$ ([4], §14.6),

$$D_{ij}u = (D_\gamma u)\kappa_i' \delta_{ij}, \qquad i,j < n,\ D_\gamma u = -D_n u, \tag{4.1}$$

where $\kappa' = (\kappa_1', \dots, \kappa_{n-1}')$ denotes the principal curvatures of $\partial\Omega$ at y. Consequently, the matrix $\mathcal{C}$ in (3.2) is given by

$$\begin{cases} \mathcal{C}_{ij} = D_{ij}u = (D_\gamma u)\kappa_i' \delta_{ij}, & i,j < n, \\ \mathcal{C}_{in} = \mathcal{C}_{ni} = \frac{1}{v} D_{in}u, & i < n, \\ \mathcal{C}_{nn} = \frac{1}{v^2} D_{nn}u, \end{cases} \tag{4.2}$$

where

$$v = \sqrt{1 + |Du|^2} = \sqrt{1 + (D_\gamma u)^2}.$$

Note that by the maximum principle $D_\gamma u \geq 0$ on $\partial\Omega$. Furthermore, if u_0 is the solution of the mean curvature equation

$$S_1(\kappa) = n \left(\frac{\binom{n}{l}\Psi}{\binom{n}{k}} \right)^{1/(k-l)} \tag{4.3}$$

satisfying $u_0 = 0$ on $\partial\Omega$, we have by (2.5) and the comparison principle, $u \leq u_0$ in Ω, whence

$$D_\gamma u \geq D_\gamma u_0 \geq \chi > 0 \tag{4.4}$$

for some positive constant χ depending on n, Ω, and Ψ (provided $\Psi \not\equiv 0$). Setting

$$\nu_\gamma = \frac{D_\gamma u}{v} = \frac{D_\gamma u}{\sqrt{1 + (D_\gamma u)^2}}, \tag{4.5}$$

we then have the formulae

$$\begin{aligned} S_k(\kappa) =& \frac{1}{v^3} \nu_\gamma^{k-1} S_{k-1}(\kappa') D_{nn}u + \nu_\gamma^k S_k(\kappa') \\ & - \frac{1}{v^4} \nu_\gamma^{k-2} \sum_{i=1}^{n-1} S_{k-2;i}(\kappa')(D_{in}u)^2 \\ \equiv & A_k D_{nn}u + B_k, \end{aligned} \tag{4.6}$$

so that equation (1.1) may be written as

$$F[u] = \frac{A_k D_{nn}u + B_k}{A_l D_{nn}u + B_l} = \Psi. \tag{4.7}$$

Since u is admissible, we must have

$$A_k - A_l\Psi \geq 0 \tag{4.8}$$

(with strict inequality if $\Psi > 0$), and a bound for $D_{nn}u$ will follow if the quantity,

$$\frac{A_k}{A_l} - \Psi = \nu_\gamma^{k-l} S_{k-1,l-1}(\kappa') - \Psi \tag{4.9}$$

is bounded away from zero. Note that (4.8) provides a complementary estimate to (4.4) when $\Psi > 0$, namely

$$\nu_\gamma \geq \left(\frac{\Psi}{S_{k-1;l-1}(\kappa')}\right)^{1/(k-l)} \equiv \widetilde{\Psi}, \tag{4.10}$$

Following the idea in [18], we let $y \in \partial\Omega$ be a point in $\partial\Omega$ where the quantity $\nu_\gamma - \widetilde{\Psi}$ is minimized. It follows that the function

$$g = D_\gamma u - v^3(y)\widetilde{\Psi} \tag{4.11}$$

is also minimized at y, since, by the concavity of ν_γ with respect to $D_\gamma u$,

$$\begin{aligned} D_\gamma u(x) - D_\gamma u(y) &\geq v^3(y)(\nu_\gamma(x) - \nu_\gamma(y)) \\ &\geq v^3(y)(\widetilde{\Psi}(x) - \widetilde{\Psi}(y)). \end{aligned} \tag{4.12}$$

Now observe that in (3.4) we can replace that vector field ξ by any extension of γ in $C^2(\mathcal{N})$, with the differential inequalities (3.5), (3.21) continuing to hold. Accordingly, by the same barrier construction, we obtain the *one-sided* estimate

$$D_{nn}u(y) \leq C \tag{4.13}$$

with positive constant C depending on $n, \partial\Omega, |D\Psi^{1/m}|_1$ and $|Du|_0$. Note that unlike estimate (3.9), C depends additionally on second derivatives of $\Psi^{1/m}$ and fourth derivatives of local representations of $\partial\Omega$. To estimate $D_{nn}u$ on all of $\partial\Omega$, we must pass from (4.13) to an estimate from below in (4.8). We consider first the case $\Psi \geq \Psi_0 > 0$, which implies the ellipticity of F at y and hence from (4.7),

$$\frac{\partial F}{\partial r_{nn}} = \frac{A_k B_l - A_l B_k}{[S_l(\kappa)]^2} \geq \delta_0 \tag{4.14}$$

for some positive constant δ_0, depending on n, Ψ_0, $\partial\Omega$, $|Du|_0$, $|D\Psi|_0$. Consequently,

$$A_k B_l - A_l B_k \geq \overline{\delta}_0 \tag{4.15}$$

for a further such positive constant $\overline{\delta}_0$, and hence, multiplying by Ψ,

$$A_k(B_l\Psi - B_k) + B_k(A_k - A_l\Psi) \geq \overline{\delta}_0\Psi_0.$$

Therefore if

$$B_k(A_k - A_l\Psi) \leq \frac{1}{2}\overline{\delta}_0\Psi_0, \tag{4.16}$$

we have

$$A_k(B_l\Psi - B_k) \geq \frac{1}{2}\overline{\delta}_0\Psi_0,$$

so that by (4.7),

$$A_k - A_l\Psi \geq \frac{1}{C}(B_l\Psi - B_k) \geq \frac{1}{2CA_k}\overline{\delta}_0\Psi_0, \tag{4.17}$$

where C is the constant in (4.13). Combining (4.16), (4.17), we thus have a lower bound for the quantity $A_k - A_l\Psi$ when Ψ is positive on $\partial\Omega$. For the degenerate case, $\Psi \geq 0$, we use (4.4), which provides a lower bound for

$$\Psi \leq \Psi_0 = \frac{1}{2}\left(\frac{\chi}{\sqrt{1+\chi^2}}\right)^{k-l} S_{k-1,l-1}(\kappa'). \tag{4.18}$$

From a lower bound for $A_k - A_l\Psi$ at its minimum y, we thus infer from equation (4.4) an upper bound for $D_{\gamma\gamma}u$ on $\partial\Omega$. From (4.1) and Theorem 3.1, we then conclude an estimate for D^2u on the boundary $\partial\Omega$. Note that by considering a conical function in place of Ψ in (4.3) the constant τ can be shown to depend only on Ω and $|\Psi^{1/m}|_1$.

THEOREM 4.1. *Let Ω be a uniformly k-convex domain in $\mathbb{R}^n$ (uniformly convex if $k = n$), with boundary $\partial\Omega \in C^{3,1}$. Let Ψ be a non-negative function in $\Omega \times \mathbb{R}$, with $\Psi^{1/m} \in C^{1,1}(\overline{\Omega} \times \mathbb{R})$. Then if $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is an admissible solution of equation* (3.1) *satisfying $u = 0$ on $\partial\Omega$, we have the estimate*

$$\max_{\partial\Omega} |D^2u| \leq C, \tag{4.19}$$

where the constant C depends on n, $\partial\Omega$, $|\Psi^{1/m}|_2$ and $|Du|_0$.

REMARKS. (i) We can still carry out the above proof with the sum in (3.4) taken from $s = 1$ to n. This method can be extended to more general functions f than the quotient (1.2) (see [18]).

(ii) As for the mixed tangential-normal derivatives, the case $l = 0$ is covered by Ivochkina [9]. But our approach yields an alternative proof, which also extends to embrace non-zero boundary values $\phi \in C^{3,1}(\partial\Omega)$ when Ψ is positive on $\partial\Omega \times \mathbb{R}$. In this case, the quantity to be minimized on $\partial\Omega$ is

$$A_k = \frac{1 + |\partial\phi|^2}{v^{k+2}} S_{k-1}(\partial\phi, \partial^2\phi + D_\gamma u \partial\gamma), \tag{4.20}$$

where ∂ is the tangential gradient in $\partial\Omega$. An estimate for $D_{nn}u$ then follows by further replacement of $D_\xi u$ in (3.4). Indeed, our calculations in Section 3 show that a differential inequality of the form (3.5) will continue to hold for any function w given by

$$w = g(x, Du) - \frac{K}{2} \sum_{s=1}^{n-1} (D_s u - D_s u(y))^2, \tag{4.21}$$

where $g \in C^2(\mathcal{N} \times \mathbb{R}^n)$, provided K is a sufficiently large constant. If g is concave in Du, then we can take $K = 1$ as before.

(iii) It is possible to weaken the geometric conditions in Theorem 4.1 to Ω being uniformly $(k-1)$-convex, by more elaborate barrier considerations.

5. Completion of existence proofs

To apply the method of continuity to the proofs of Theorems 1.1 and 1.2, we need to designate a suitable family of problems. For Theorem 1.1, we may select any admissible function $g \in C^{4,\alpha}(\overline{\Omega})$, with the same boundary values, such that

$$0 < F[g] \leq \Psi. \tag{5.1}$$

Such a function is readily constructed as a multiple of a function $g_0 \in \Gamma_{k+1}(0)$ if $k < n$. Then we consider the problems

$$\begin{cases} F[u] = t\Psi + (1-t)F[g] & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

for $0 \leq t \leq 1$. The same family may also serve for Theorem 1.2 or we may replace g by u_0. Clearly all our preceding estimates are independent of the parameter t. The same is true of the remaining first and second derivative estimates ([3], [10], [17]), so that under the hypotheses of Theorem 1.1 or 1.2, we conclude an *a priori* estimate of the form

$$|u|_{2,\alpha} \leq C \tag{5.3}$$

with constant C depending on n, Ω, u_0, and Ψ, and hence the unique solvability of the Dirichlet problems (5.2) in the class of admissible functions $u \in C^{4,\alpha}(\overline{\Omega})$.

As mentioned in the introduction, the condition (1.6) may be replaced by conditions involving quermassintegrals, namely, for $l \geq 1$,

$$\frac{1}{l} \sup_{x \in \Omega} \Psi(x,0) \int_{\partial E} H_{l-1}(\partial E) \leq \frac{1-\chi}{k} \int_{\partial E} H_{k-1}(\partial E) \tag{5.4}$$

for every $(k-1)$-convex set $E \subset \Omega$ and some positive constant $\chi > 0$, [19]. Here the boundary mean curvatures $H_m(\partial E)$ are given by

$$H_m(\partial E) = S_m(\mu_1, \dots, \mu_{n-1}), \qquad m = 0, \dots, n-1, \tag{5.5}$$

where $\mu_1, \dots, \mu_{n-1}$ are the principal curvatures of ∂E. The case $l = 0$ is treated in [16].

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