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NON-COLLISION PERIODIC SOLUTIONS FOR A CLASS OF SYMMETRIC 3-BODY TYPE PROBLEMS

ANTONIO AMBROSETTI[1] — VITTORIO COTI ZELATI

Dedicated to Jean Leray

1. Introduction

In a recent paper [1] we have proved the existence of a periodic *weak* solution (see [2] for the definition; see also Section 2 below) with prescribed negative energy h for some Hamiltonian systems of N-body type, that is, solutions of

$$(1)\qquad m_i\ddot{x}_i + \nabla_{x_i}V(x_1,\dots,x_N) = 0, \qquad 1 \le i \le N,$$

such that

$$(2)\qquad \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{x}_i(t)|^2 + V(x_1(t),\dots,x_N(t)) = h$$

where

$$(3)\qquad V(x) = V(x_1,\dots,x_N) \simeq -\sum_{1\le i\ne j\le N}\frac{m_im_j}{|x_i-x_j|^\alpha}, \qquad 0<\alpha<2,$$

and V is even in x, i.e. $V(-x) = V(x)$.

Equation (1) describes the motion of N bodies of positions $x_i \in \mathbb{R}^k$ and masses $m_i > 0$ under the action of a potential of Keplerian type. The main

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purpose of this paper is to show that for some classes of such potentials system (1) possesses non-collision periodic solutions.

Critical point theory has been used to prove the existence of periodic solutions of Hamiltonian systems arising in Celestial Mechanics. See [2] and the extensive bibliography therein; see also the more recent papers [3] (dealing with 2-body problems), [10] and [4] (dealing with N-body type potentials like (3) with $\alpha > 2$ and $0 < \alpha < 2$, respectively).

The question of the existence of non-collision solutions has been addressed either by Morse theoretical arguments, or by comparison arguments. See Sections 13 and 14 of [2]. Here we will use the former to exclude double collisions and the latter to exclude triple collisions.

2. Existence result

In this section, for the reader's convenience, we will recall the existence result we refer to (Theorem B of [1]) and the variational procedure used to prove it. It is worth recalling that existence results are known in a much greater generality (see Remark 2 below).

Let $x = (x_1, \dots, x_N) \in \mathbb{R}^{kN}$ and

$$V(x) = \frac{1}{2} \sum_{1 \le i \ne j \le N} V_{ij}(x_i - x_j), \tag{4}$$

where $V_{ij} \in C^2(\mathbb{R}^k \setminus \{0\}, \mathbb{R})$ satisfies (for all $1 \le i \ne j \le N$)

(V1) $V_{ij}(\xi) = V_{ji}(\xi)\ \forall \xi \ne 0$;
(V2) $\exists \alpha \in [1, 2[$ such that $V'_{ij}(\xi) \cdot \xi \ge -\alpha V_{ij}(\xi)\ \forall \xi \ne 0$;
(V3) $\exists \delta \in]0, 2[$ and $r > 0$ such that $V'_{ij}(\xi) \cdot \xi \le -\beta V_{ij}(\xi)\ \forall 0 < |x| \le r$;
(V4) $V_{ij}(\xi) \to 0$ as $|\xi| \to \infty$;
(V5) $3V'_{ij}(\xi) \cdot \xi + V''_{ij}(\xi)\xi \cdot \xi > 0\ \forall \xi \ne 0$.

Setting $E = H^{1,2}(S^1; \mathbb{R}^{kN})$, $E_0 = \{u \in E \ :\ u(t + \frac{1}{2}) = -u(t)\}$ and

$$\Lambda_0 = \{u = (u_1, \dots, u_N) \in E_0 \ :\ u_i(t) \ne u_j(t),\ \forall t \in S^1,\ \forall 1 \le i \ne j \le N\},$$

we define $f \in C^2(\Lambda_0, \mathbb{R})$ by

$$f(u) = \frac{1}{2}\|u\|^2 \cdot \int_0^1 (h - V(u))\, dt, \qquad \text{where } \|u\|^2 = \int_0^1 \sum_{1 \le i \le N} m_i |\dot{u}_i|^2\, dt.$$

Let $u \in \Lambda_0$ be a critical point of f at a positive level, that is, $f'(u) = 0$ and $f(u) = c > 0$; if we set

$$\omega^2 = \frac{\int_0^1 (h - V(u))\, dt}{\frac{1}{2}\|u\|^2}$$

then $x(t) = u(\omega t)$ is a collision-free periodic solution of (1) with energy h. Unfortunately, it is not known how to find directly critical points of f on Λ_0, because V_{ij} is singular at $\xi = 0$. Therefore one considers perturbed potentials

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{2} \sum_{1 \le i \ne j \le N} |x_i - x_j|^{-2} \tag{5}$$

and

$$f_\varepsilon(u) = \frac{1}{2}\|u\|^2 \cdot \int_0^1 (h - V_\varepsilon(x))\, dt.$$

If V satisfies (V1)–(V4), then f_ε has a Mountain Pass critical point. However, in order to have a variational characterization appropriate for the estimates we will discuss in Section 3, we set

$$\mathcal{M} = \left\{ u \in \Lambda_0 \ : \ \int_0^1 \left(V(u) + \frac{1}{2} V'(u) \cdot u \right) dt = h \right\}.$$

If $h < 0$ and (V5) holds, then for all $u \in \Lambda_0$ the half-line λu, $\lambda > 0$, meets $\mathcal{M}$ transversally. It follows that $\mathcal{M}$ is a smooth manifold and that critical points constrained on $\mathcal{M}$ are critical points of f_ε. Moreover, (PS) holds and f_ε has a minimum u_ε on $\mathcal{M}$ with $c_\varepsilon = f_\varepsilon(u_\varepsilon) > 0$. Finally, uniform estimates with respect to ε allow us to show that u_ε converges, uniformly on $[0,1]$, to $u \in E_0$, a weak solution of (1). It is worth recalling that, although $u_\varepsilon \in \Lambda_0$ and hence is collision-free, u might belong to $\partial\Lambda_0$. However, since u is a weak solution, the collision set $\Gamma = \{t \in S^1 \ : \ \exists i \ne j,\ u_i(t) = u_j(t)\}$ has zero measure, and u is a classical solution of (1) on $S^1 \setminus \Gamma$.

Summarizing, we have:

THEOREM 1. *Suppose that* $V_{ij} \in C^2(\mathbb{R}^k, \mathbb{R})$ *satisfies* (V1)–(V5) *and let* V *be of the form* (4). *Then, for all* $h < 0$, (1) *has a weak solution.*

REMARKS 2.

1. We recall that Theorem B of [1] holds for all $0 < \alpha < 2$. Moreover, in [4], the existence of a weak solution has been proved without assuming the symmetry condition (V1).

2. For future reference, we point out that the approximating critical points u_ε are minima of f_ε on $\mathcal{M}$. Moreover, in view of the specific features of $\mathcal{M}$, it follows that u_ε are *Mountain Pass* critical points of f_ε with Morse index $\mathrm{Morse}(u_\varepsilon) = 1$.

3. Estimates on triple collisions

In this section we want to prove that the weak solution found via Theorem 1 is free of triple collisions provided a pinching condition is satisfied. Let us recall that a weak solution $x(t)$ has a *triple collision* if there exists a $\bar{t} \in [0, T]$ such that $x_1(\bar{t}) = x_2(\bar{t}) = x_3(\bar{t})$, i.e. if the corresponding function $u \in E$ belongs to the set

$$\partial\Lambda_3 = \{u \in H^1([0,1];(\mathbb{R}^k)^3) \mid \exists \bar{t} \in [0,1] : u_1(\bar{t}) = u_2(\bar{t}) = u_3(\bar{t})\}. \tag{6}$$

In order to prove that u does not belong to $\partial\Lambda_3$, we will estimate $\mu_3 = \inf_{\mathcal{M}\cap\partial\Lambda_3} f$ and show that m_3 is larger than $\mu = \inf_{\mathcal{M}} f$. The idea of comparing those values in order to prove existence of collision-free solutions has been first used in [8] and [7] to find classical T-periodic solutions for the two-body problem. See also [5] and [14] for related results on the two body problem.

Let us recall that, given $u \in \Lambda_0$, there exists a unique $\lambda(u) \in \mathbb{R}$ such that $\lambda(u)u \in \mathcal{M}$ and

$$f(\lambda(u)u) = \max_{\lambda>0} f(\lambda u).$$

We define, for all $u \in \overline{\Lambda}_0$,

$$I(u) = \max_{\lambda>0} f(\lambda u).$$

Then the following holds:

PROPOSITION 3. *Assume*

(V6) $\quad -V(x) \geq -\frac{a}{2} \sum_{1\leq i\neq j\leq 3} m_i m_j |x_i - x_j|^{-\alpha}.$

Then

$$\inf_{u\in\mathcal{M}\cap\partial\Lambda_3} I(u) = \inf_{u\in\partial\Lambda_3} I(u) \geq 4\pi^2 \frac{K}{M}, \tag{7}$$

where $M = m_1 + m_2 + m_3$ *and*

$$\begin{aligned} K &= K(m_1, m_2, m_3, \alpha, |h|) \\ &= \alpha \left(\frac{a}{2}\right)^{2/\alpha} \left(\frac{2-\alpha}{2}\right)^{(2-\alpha)/2} |h|^{(\alpha-2)/2} \frac{(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/\alpha}}{M}. \end{aligned} \tag{8}$$

PROOF. Consider $u \in \partial\Lambda_3$, $u \not\equiv$ const. If $-\int_0^1 V(u)\,du = +\infty$, there is nothing to prove. So assume $-\int_0^1 V(u)\,du < +\infty$. This can easily be shown to imply that $I(u) < +\infty$.

From (V6) we deduce that

$$f(\lambda u) \geq f_a(\lambda u) := \frac{\lambda^2}{2}\|u\|^2 \int_0^1 \left[h - \frac{a}{2\lambda^\alpha} \sum_{i\neq j} \frac{m_i m_j}{|u_i - u_j|^\alpha}\right] dt \tag{9}$$

and hence

$$I(u) \geq \max_{\lambda>0} f_a(\lambda u). \tag{10}$$

Recalling that [6, Lemma 2.1]

$$\frac{1}{2}\sum_{i\neq j}\frac{m_i m_j}{|x_i - x_j|^\alpha} \geq \frac{1}{2^{(\alpha+2)/2}}\frac{(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/2}}{M^{\alpha/2}}\frac{1}{(\sum_{i=1}^3 m_i |x_i|^2)^{\alpha/2}}, \tag{11}$$

we also have, for all $\xi \in \mathbb{R}^n$,

$$\frac{1}{2}\sum_{i\neq j}\frac{m_i m_j}{|x_i - x_j|^\alpha} \geq \frac{1}{2^{(\alpha+2)/2}}\frac{(\sum_{i\neq j} m_i m_j)^{(2+\alpha)/2}}{M^{\alpha/2}}\frac{1}{(\sum_{i=1}^3 m_i |x_i - \xi|^2)^{\alpha/2}}. \tag{12}$$

Since $u \in \partial\Lambda_3$, there exists a $\bar{t} \in [0,1]$ such that $u_1(\bar{t}) = u_2(\bar{t}) = u_3(\bar{t}) = \xi$. Set

$$R_1(t)^2 = \begin{cases} \frac{1}{M}\sum_{i=1}^3 m_i|u_i(t+\bar{t}) - \xi|^2, & 0 \leq t \leq 1/4, \\ \frac{1}{M}\sum_{i=1}^3 m_i|u_i(1-t+\bar{t}) + \xi|^2, & 1/4 \leq t \leq 1/2. \end{cases} \tag{13}$$

Then

1. $R_1(0) = 0$;
2. $R_1(\frac{1}{2})^2 = \frac{1}{M}\sum_{i=1}^3 m_i|u_i(\frac{1}{2}+\bar{t}) + \xi|^2 = \frac{1}{M}\sum_{i=1}^3 m_i|-u_i(\bar{t}) + \xi|^2 = 0$;
3. $R_1(\frac{1}{4}-) = \frac{1}{M}\sum_{i=1}^3 m_i|u_i(\frac{1}{4}+\bar{t}) - \xi|^2$;
4. $R_1(\frac{1}{4}+) = \frac{1}{M}\sum_{i=1}^3 m_i|u_i(\frac{1}{4}+\bar{t}+\frac{1}{2}) + \xi|^2 = R_1(\frac{1}{4}-)$;

and $R_1 \in H_0^1([0,1/2];\mathbb{R}^+)$. Similarly, setting

$$R_2(t)^2 = \begin{cases} \frac{1}{M}\sum_{i=1}^3 m_i|u_i(\frac{1}{2}-t+\bar{t}) + \xi|^2, & 0 \leq t \leq 1/4, \\ \frac{1}{M}\sum_{i=1}^3 m_i|u_i(t+\bar{t}+\frac{1}{2}) + \xi|^2, & 1/4 \leq t \leq 1/2, \end{cases} \tag{14}$$

we see that $R_2 \in H_0^1([0,1/2];\mathbb{R}^+)$.

Moreover, for $0 \leq t \leq 1/4$, we have

$$\begin{aligned} R_1(t)\dot{R}_1(t) &= \frac{1}{M}\sum_{i=1}^3 m_i \dot{u}_i(t+\bar{t})(u_i(t+\bar{t}) - \xi) \\ &\leq \left(\frac{1}{M}\sum_{i=1}^3 m_i|\dot{u}_i(t+\bar{t})|^2\right)^{1/2}\left(\frac{1}{M}\sum_{i=1}^3 m_i|u_i(t+\bar{t}) - \xi|^2\right)^{1/2} \\ &= \left(\frac{1}{M}\sum_{i=1}^3 m_i|\dot{u}_i(t+\bar{t})|^2\right)^{1/2} R_1(t), \end{aligned}$$

so that, defining

$$\phi(t)^2 = \frac{1}{M}\sum_{i=1}^3 m_i|\dot{u}_i(t)|^2,$$

we have

$$\dot{R}_1(t) \le \phi(t+\bar{t}), \qquad 0 \le t \le 1/4,$$

and, similarly,

$$\begin{aligned} \dot{R}_1(t) &\le \phi(1-t+\bar{t}), && 1/4 \le t \le 1/2, \\ \dot{R}_2(t) &\le \phi(1/2-t+\bar{t}), && 0 \le t \le 1/4, \\ \dot{R}_2(t) &\le \phi(1/2+t+\bar{t}), && 1/4 \le t \le 1/2. \end{aligned}$$

Finally,

$$\int_0^{1/2} \dot{R}_1^2(t)\,dt + \int_0^{1/2} \dot{R}_2^2(t)\,dt \le \frac{1}{M}\int_0^1 \Big(\sum_{i=1}^3 m_i|\dot{u}_i(t)|^2\Big)\,dt = \frac{1}{M}\|u\|^2.$$

Using (12), we also deduce that, for $0 \le t \le 1/4$,

$$\frac{1}{2}\sum_{i\neq j} \frac{m_i m_j}{|u_i(t+\bar{t}) - u_j(t+\bar{t})|^\alpha} \ge \frac{C}{M^\alpha}\frac{1}{R_1(t)^\alpha} \tag{15}$$

where

$$C = C(m_1, m_2, m_3, \alpha) = \frac{1}{2^{(\alpha+2)/2}}\Big(\sum_{i\neq j} m_i m_j\Big)^{(2+\alpha)/2}, \tag{16}$$

with similar relations holding for $1/4 \le t \le 1/2$ and for R_2. This implies that

$$\frac{1}{2}\sum_{i\neq j}\int_0^1 \frac{m_i m_j}{|u_i - u_j|^\alpha} \ge CM^{-\alpha}\Big[\int_0^{1/2}\frac{dt}{R_1(t)^\alpha} + \int_0^{1/2}\frac{dt}{R_2(t)^\alpha}\Big].$$

In order to estimate

$$I(u) \ge \max_{\lambda>0} \frac{1}{2}\lambda^2\|u\|^2 \int_0^1 \Big(h + \frac{a}{2\lambda^\alpha}\sum_{i\neq j}\frac{m_i m_j}{|\xi_i - u_j|^\alpha}\Big)$$

we can always assume that

$$g(\lambda u) = \int_0^1 \Big(h + \frac{a}{2\lambda^\alpha}\sum_{i\neq j}\frac{m_i m_j}{|\xi_i - u_j|^\alpha}\Big)dt > 0.$$

Indeed, $g(\lambda u) \to +\infty$ as $\lambda \to 0$. Recalling that $f(\lambda(u)u) = \max_{\lambda>0} f(\lambda u)$, we deduce

$$\begin{aligned} f(\lambda(u)u) &\ge \frac{\lambda^2 M}{2}\Big(\int_0^{1/2}\dot{R}_1(t)^2\,dt + \int_0^{1/2}\dot{R}_2(t)^2\,dt\Big)g(\lambda u) \\ &\ge \frac{\lambda^2 M}{2}\Big(\int_0^{1/2}\dot{R}_1(t)^2\,dt + \int_0^{1/2}\dot{R}_2(t)^2\,dt\Big) \\ &\quad \times \Big(h + \frac{aC}{M^\alpha\lambda^\alpha}\int_0^{1/2}\frac{dt}{R_1(t)^\alpha} + \frac{aC}{M^\alpha\lambda^\alpha}\int_0^{1/2}\frac{dt}{R_2(t)^\alpha}\Big) \\ &\equiv \psi(\lambda R_1, \lambda R_2). \end{aligned}$$

Collecting the above facts and setting $X_0 = H_0^1([0,1/2];\mathbb{R}^+)$ we deduce that

$$\begin{aligned}\inf_{u\in\partial\Lambda_3} I(u) &= \inf_{u\in\partial\Lambda_3}\max_{\lambda>0} f(\lambda u)\\ &\geq \inf_{R_1,R_2\in X_0}\max_{\lambda>0}\psi(\lambda R_1,\lambda R_2)\\ &= \inf_{R\in X_0}\max_{\lambda>0}\lambda^2 M\Big(\int_0^{1/2}\dot R^2\,dt\Big)\Big(h+\frac{2aC}{M^\alpha\lambda^\alpha}\int_0^{1/2}\frac{dt}{R^\alpha}\Big).\end{aligned}$$

Using Jensen's inequality, we finally find

$$\inf_{u\in\partial\Lambda_3} I(u) \geq \inf_{R\in E_0}\max_{\lambda>0}\lambda^2 M\Big(\int_0^{1/2}\dot R^2\,dt\Big)\Big[h+\frac{2^\alpha aC}{M^\alpha\lambda^\alpha}\Big(\int_0^{1/2}\frac{dt}{R}\Big)^\alpha\Big].$$

Easy computations show that

$$\begin{aligned}\Phi(R) &:= \max_{\lambda>0}\lambda^2 M\int_0^{1/2}\dot R^2\,dt\Big[h+\frac{2^\alpha aC}{M^\alpha\lambda^\alpha}\Big(\int_0^{1/2}\frac{dt}{R}\Big)^\alpha\Big]\\ &= 2\alpha a^{2/\alpha}\Big(\frac{2-\alpha}{2}\Big)^{(2-\alpha)/\alpha}\frac{C^{2/\alpha}}{M|h|^{(2-\alpha)/\alpha}}\Big(\int_0^{1/2}\dot R^2\,dt\Big)\Big(\int_0^{1/2}\frac{dt}{R}\Big)^2\\ &= K\Big(\int_0^{1/2}\dot R^2\,dt\Big)\Big(\int_0^{1/2}\frac{dt}{R}\Big)^2.\end{aligned}$$

In order to evaluate $\inf_{R\in E_0}\Phi(R)$, let us remark that such an infimum is attained at some $R_0\in E_0$ and that such a R_0 satisfies, for all $t\in\,]0,1/2[$,

$$\Big(\int_0^{1/2}\frac{dt}{R}\Big)^2\ddot R+\Big(\int_0^{1/2}\dot R^2\,dt\Big)\Big(\int_0^{1/2}\frac{dt}{R}\Big)\frac{1}{R^2}=0,$$

or, taking also into account the boundary conditions,

$$\begin{cases}\ddot R+T^2R^{-2}=0 & \text{for all } t\in\,]0,1/2[,\\ R(0)=R(1/2)=0,\end{cases}$$

where $T^2=(\int_0^{1/2}\dot R^2)/(\int_0^{1/2}\frac{1}{R})$. Then (see [9])

$$\frac12\int_0^{1/2}\dot R^2\,dt=\frac12(2\pi T^2)^{2/3}\qquad\text{and}\qquad T^2\int_0^{1/2}\frac{dt}{R}=(2\pi T^2)^{2/3},$$

which implies

$$\Big(\int_0^{1/2}\dot R^2\,dt\Big)\Big(\int_0^{1/2}\frac{dt}{R}\Big)^2=(2\pi T^2)^{2/3}\frac{(2\pi T^2)^{4/3}}{T^4}=4\pi^2.$$

We deduce from such an estimate that

$$\inf_{u\in\partial\Lambda} I(u)\geq K(m,\alpha,|h|)4\pi^2$$

and the proposition follows. □

We now estimate the infimum of I over Λ_0.

PROPOSITION 4. *Assume*

(V7) $-V(x) \le -\frac{b}{2}\sum_{1\le i\neq j\le 3} m_i m_j |x_i - x_j|^{-\alpha}$.

Then

$$\inf_{u\in\mathcal{M}} f(u) = \inf_{u\in\Lambda_0} I(u) \le \frac{\alpha\pi^2}{2}\left[\frac{2-\alpha}{2}\right]^{(2-\alpha)/\alpha} \frac{b^{2/\alpha}(\sum_{i\neq j} m_i m_j)^{(\alpha+2)/\alpha}}{M|h|^{(2-\alpha)/\alpha}}.$$

PROOF. We observe that

$$f(\lambda u) \le f_b(\lambda u)$$

so that

$$I(u) = \max_{\lambda>0} f(\lambda u) \le I_b(u) = \max_{\lambda>0} f_b(\lambda u).$$

We will evaluate

$$\max_{\lambda>0} f_b(\lambda u)$$

for a particular $u \in \Lambda_0$.

Let $\xi = (1,0,0)$, $\eta = (0,1,0)$ and define

$$\begin{aligned}\overline{u}_i(t) = \xi\left[\cos\left(2\pi t + \frac{2\pi i}{3}\right) - \frac{1}{M}\sum_{\ell=1}^{3} m_\ell \cos\left(2\pi t + \frac{2\pi\ell}{3}\right)\right] \\ + \eta\left[\sin\left(2\pi t + \frac{2\pi i}{3}\right) - \frac{1}{M}\sum_{\ell=1}^{3} m_\ell \sin\left(2\pi t + \frac{2\pi\ell}{3}\right)\right].\end{aligned}$$

Then, as in [6], one obtains

$$\sum_{i=1}^{3}\frac{m_i}{2}\int_0^1 |\dot{\overline{u}}_i(t)|^2\,dt = \frac{4\pi^2}{M}\sum_{i,j} m_i m_j \sin^2\frac{\pi(i-j)}{3} = \frac{3\pi^2}{M}\sum_{i\neq j} m_i m_j$$

and

$$|\overline{u}_i(t) - \overline{u}_j(t)|^2 = 4\sin^2\frac{\pi(i-j)}{3} = 3$$

and

$$-V(\overline{u}_1(t), \overline{u}_2(t), \overline{u}_3(t)) \le \frac{b}{3^{\alpha/2}}\sum_{i\neq j} m_i m_j.$$

We deduce that

$$f(\lambda\overline{u}) \le \frac{\lambda^2 3\pi^2}{M}\left(\sum_{i\neq j} m_i m_j\right)\left[h + \frac{b}{2\cdot 3^{\alpha/2}\lambda^\alpha}\left(\sum_{i\neq j} m_i m_j\right)\right]$$

and hence

$$\max_{\lambda>0} f(\lambda\overline{u}) \le \frac{\alpha\pi^2}{2}\left[\frac{2-\alpha}{2}\right]^{(2-\alpha)/\alpha} b^{2/\alpha}|h|^{(2-\alpha)/\alpha}\frac{(\sum_{i\neq j} m_i m_j)^{(\alpha+2)/\alpha}}{M}.$$

□

4. Existence of non-collision solutions

In this section we will use the results of Sections 2 and 3 to prove existence of a non-collision solution.

Our main result is the following:

THEOREM 5. *Suppose that V satisfies* (V1)–(V7) *and, moreover, that*

$$\frac{b}{a} < 2^{(3\alpha-2)/2}. \tag{17}$$

Then, for all $h < 0$, system (1) *has a weak periodic solution satisfying* (2) *without triple collision. If, in addition, V satisfies*

(V8) $V_{ij}(\xi) = -\frac{1}{|\xi|^\alpha} + U_{ij}(\xi)$ *where*

$$\begin{aligned} &|U_{ij}(\xi)||\xi|^\alpha \to 0 && \text{as } |\xi| \to 0, \\ &|U'_{ij}(\xi)||\xi|^{\alpha+1} \to 0 && \text{as } |\xi| \to 0, \\ &|U''_{ij}(\xi)||\xi|^{\alpha+2} \to 0 && \text{as } |\xi| \to 0, \end{aligned}$$

then such a solution is a classical one.

PROOF. The proof will be carried out in two steps.

STEP 1. *The solution found via Theorem 1 has no triple collisions.*

We recall that the weak solution x found via Theorem 1 is obtained as the limit as $\varepsilon \to 0$ of functions x_ε. The latter are classical solutions of the problem (1) and (2), with V_ε replacing V (V_ε is defined in (5)). They are obtained from the minima u_ε of the perturbed functional f_ε on the manifold $\mathcal{M}$. It is then easy to see that $f(u) \le \liminf_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon)$, u being the point in E_0 corresponding to x.

If x has a triple collision, then using Propositions 3 and 4 one deduces that

$$\begin{aligned} 4\pi^2 K M^{-1} &\le \inf_{w\in\mathcal{M}\cap\partial\Lambda_3} \le f(w) \le \inf_{w\in\mathcal{M}} f(w) \\ &\le \frac{\alpha\pi^2}{2}\left[\frac{2-\alpha}{2}\right]^{(2-\alpha)/\alpha} |h|^{(2-\alpha)/\alpha} \frac{b^{2/\alpha}(\sum_{i\neq j} m_i m_j)^{(\alpha+2)/\alpha}}{M}, \end{aligned}$$

that is,

$$a^{2/\alpha} 2^{(3\alpha-2)/\alpha} \le b^{2/\alpha},$$

a contradiction which proves Step 1.

STEP 2. *The solution found via Theorem* 1 *has no double collisions.*

Suppose that u has, possibly, a certain number ν of double collisions. Then, close to any such collision, the problem can be regarded as a perturbed 2-body problem. This remark enables us to use a result due to Tanaka [13] (see also [12] for similar regularity results dealing with solutions of fixed period of some 2-body problems), that we are going to recall briefly, for the reader's convenience.

Consider a 2-body Keplerian problem

$$\tag{18} \begin{aligned} &\ddot{x} + V'(x) = 0, \\ &\frac{1}{2}|\dot{x}|^2 + V(x) = h, \end{aligned}$$

where $V : \mathbb{R}^k \setminus \{0\} \to \mathbb{R}$ satisfies (V1)–(V8). Periodic solutions of (18) can be found with the same procedure sketched in Section 2. Using the same notation, let v correspond to a weak solution of (18) and let $v_\varepsilon \in \Lambda_0$ be the corresponding critical point of f_ε such that $v_\varepsilon \to v$. It is shown in [13] that

$$\tag{19} (k-2)\kappa \le \liminf_{\varepsilon \to 0} \operatorname{Morse}(v_\varepsilon),$$

where κ denotes the number of collisions of v. Let us point out explicitly that this result makes only use of the local properties of V near the singularity $x = 0$. As a consequence, we can repeat in our situation the arguments of [13], yielding, as in (19), that

$$(k-2)\nu \le \liminf_{\varepsilon \to 0} \operatorname{Morse}(u_\varepsilon).$$

Since now $\operatorname{Morse}(u_\varepsilon) = 1$ (see Remark 2.2), it follows that the number ν of double collisions of u is zero. This completes the proof of the theorem. □

REMARK 6. The existence of non-collision periodic solutions with prescribed period T for the 3-body problem has been proved without any pinching condition in [11], using Morse theoretic arguments. We point out that the arguments therein make use of the fact that T-periodic solutions of symmetric N-body problems can be found as (limits of) minima of the Lagrangian Action.

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ANTONIO AMBROSETTI
Scuola Normale Superiore
56100 Pisa, ITALY

VITTORIO COTI ZELATI
Istituto di Matematica, Facoltà di Architettura
Università degli Studi di Napoli
Napoli, ITALY