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HOMOCLINICS FOR AN ALMOST PERIODICALLY FORCED SINGULAR HAMILTONIAN SYSTEM

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

0. Introduction

Consider the Hamiltonian system

$$\text{(HS)} \qquad \ddot{q} + a(t)W'(q) = 0,$$

where a and W satisfy

(a_1) $a(t)$ is a continuous almost periodic function of t with $a(t) \geq a_0 > 0$ for all $t \in \mathbb{R}$.
(W_1) There is a $\xi \in \mathbb{R}^2 \backslash \{0\}$ such that $W \in C^2(\mathbb{R}^2 \backslash \{\xi\}, \mathbb{R})$.
(W_2) $\lim_{x \to \xi} W(x) = -\infty$.
(W_3) There is a neighborhood $\mathcal{N}$ of ξ and $U \in C^1(\mathcal{N} \backslash \{\xi\}, \mathbb{R})$ such that $|U(x)| \to \infty$ as $x \to \xi$ and

$$|U'(x)|^2 \leq -W(x) \qquad \text{for } x \in \mathcal{N} \backslash \{\xi\},$$

(W_4) $W(x) < W(0) = 0$ if $x \neq 0$ and $W''(0)$ is negative definite.
(W_5) There is a constant $W_0 < 0$ such that $\overline{\lim}_{x \to \infty} W(x) \leq W_0$.

When a is periodic in t and somewhat weaker conditions than (a_1) and (W_1)–(W_5) are satisfied, it was shown in [17] that (HS) possesses a pair of solutions that are homoclinic to 0 and wind around ξ in a positive and negative sense

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respectively. When $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$, $n > 2$, $a(t) \equiv 1$, and again weaker hypotheses than the above are satisfied, the existence of a single homoclinic solution of (HS) was proved by Tanaka [23]. In very recent work, Caldiroli and Nolasco [7] have shown that when $n = 2$, $a(t) \equiv 1$ and W satisfies an additional symmetry condition, (HS) possesses solutions which wind around ξ a prescribed number of times.

The goal of this paper is to obtain an analogue of the results of [17] when $a(t)$ is merely almost periodic. The proof of [17] was based on an elementary minimization argument. This argument no longer works in the current setting due to the loss of compactness in going from the case of periodic forcing to almost periodic $a(t)$. Recently Serra, Tarallo, and Terracini [21] established the existence of homoclinic solutions for a Hamiltonian system of another type which was subjected to almost periodic forcing. Using arguments motivated by and close to their work together with some ideas from [17], it will be shown here that as in [17], (HS) possesses a pair of homoclinic solutions Q^+, Q^- winding around ξ in opposite senses. This will be carried out in §1.

When $a(t)$ is periodic, e.g. with period 1, and $Q(t)$ is a homoclinic solution of (HS), then so is $Q(t-k)$ for all $k \in \mathbb{Z}$. This is no longer the case when a is almost periodic. However, under (a_1), there exists a sequence $\sigma_k \to \infty$ as $k \to \infty$ such that

$$\|a(\cdot) - a(\cdot - \sigma_k)\|_{L^\infty(\mathbb{R})} \to 0.$$

Exploiting this fact, it will be shown in §2 that (HS) possesses infinitely many homoclinic solutions. In fact, there are such solutions near $Q(t - \sigma_k)$ for large k whenever Q is an isolated local minimizer for a variational problem associated with (HS). In §3, some technical results used earlier will be treated.

In [18], it was shown that if $a(t)$ is periodic and Q^+ or Q^- are isolated minimizers of the variational problem that defines them, then there exist multibump homoclinic solutions of (HS). In a sequel to this paper, it will be shown that an analogous result obtains in the current setting. Such multibump solutions require a much more complicated construction than the simple arguments of §2 here.

There have been several other papers in recent years which use variational methods to find basic homoclinic or heteroclinic solutions of Hamiltonian systems and which also construct multibump solutions for periodically forced Hamiltonian systems. See e.g. Coti Zelati, Ekeland and Séré [8], Séré [19–20], Coti Zelati and Rabinowitz [9–10], Bessi [2], Bolotin [4], Bertotti and Bolotin [1], Caldiroli and Montecchiari [6], Montecchiari and Nolasco [14], and Strobel [22]. Moreover, in recent work, Buffoni and Séré [5] have obtained multibump for an autonomous Hamiltonian system.

1. A pair of homoclinics

In this section, a pair of solutions of (HS) which are homoclinic to 0 will be obtained as critical points of a corresponding functional. To formulate the variational problem, let $E = W^{1,2}(\mathbb{R}, \mathbb{R}^2)$ under its usual norn $\|\cdot\|$ and

$$\Lambda = \{q \in E \mid q(t) \neq \xi \text{ for all } t \in \mathbb{R}\}.$$

Set $V = aW$, $\mathcal{L}(q) = \frac{1}{2}|\dot{q}|^2 - V(t,q)$, and

$$I(q) = \int_{\mathbb{R}} \mathcal{L}(q)\,dt. \tag{1.1}$$

For what follows, even if not explicitly stated, it will always be assumed that (a_1) and (W_1)–(W_5) hold.

Proposition 1.2. *If (a_1) and (W_1)–(W_5) are satisfied, for any $M > 0$, there is a $\kappa(M) > 0$ such that if $q \in E$ and $I(q) \leq M$, then $|q(t) - \xi| \geq \kappa(M)$ for all $t \in \mathbb{R}$.*

Proof. This result is essentially due to Gordon [11] who obtained it in a different setting. It is here that hypotheses (W_2)–(W_3) play their role. The proof is essentially the same as the related argument given in Theorem 2.7 of [17] and will be omitted.

The next result gives the smoothness of I on Λ.

Proposition 1.3. *If (a_1) and (W_1)–(W_5) are satisfied, then $I \in C^1(\Lambda, \mathbb{R})$.*

Proof. This is proved as in Proposition 1.1 of [9] and will be omitted.

Remark. In fact, $I \in C^2(\Lambda, \mathbb{R})$ although the additional smoothness will not be employed.

Proposition 1.4. *If (a_1) and (W_1)–(W_5) are satisfied, $q \in \Lambda$ and $I'(q) = 0$, i.e. q is a critical point of I on Λ, then q is a classical solution of* (HS) *with $|q(t)|, |\dot{q}(t)| \to 0$ as $|t| \to \infty$.*

Proof. If $q \in E$, standard embedding theorems imply $|q(t)| \to 0$ as $|t| \to \infty$. If q is a critical point of I, it is a weak solution of (HS) and then standard "elliptic" arguments show it is a classical solution of (HS). Finally, using (HS), (a_1), (W_1) and (W_4) as in [17] shows $\dot{q} \in E$ and therefore $|\dot{q}(t)| \to 0$ as $|t| \to \infty$.

By Propositions 1.2 and 1.4, to find solutions of (HS) homoclinic to 0, it suffices to find critical points of I on Λ. Towards that end, observe that as has already been noted above, if $q \in \Lambda$ and $I(q) < \infty$, then $q(t) \to 0$ as $|t| \to \infty$. Hence $q(\mathbb{R})$ is a closed curve in $\mathbb{R}^2 \backslash \{\xi\}$ and as such possesses a winding number with respect to ξ, WN(q), which equals its Brouwer degree with respect to ξ, $d(q)$. Let

$$\Gamma = \{q \in \Lambda \mid d(q) \neq 0\} = \Gamma^+ \cup \Gamma^-, \tag{1.5}$$

where $\Gamma^\pm = \{q \in \Gamma \mid \pm d(q) > 0\}$. Set

$$c^\pm = \inf_{q \in \Gamma^\pm} I(q). \tag{1.6}$$

PROPOSITION 1.7. *$c^\pm \geq c_0 > 0$.*

PROOF. Each $q \in \Gamma^\pm$ has a subloop starting in the sphere $\{|x| = |\xi|/2\}$, crossing the ray $\{s\xi \mid s \geq 1\}$ and returning to $\{|x| = |\xi|/2\}$. It is shown in [17] that this gives a lower bound c_0 for $I(q)$.

In [17], it was shown that when V is periodic in t, there exist functions $Q^\pm \in \Gamma^\pm$ such that $I(Q^\pm) = c^\pm$ and $Q^\pm$ are critical points of I. The proof relies heavily on the fact that if V is e.g. 1-periodic in t, $j \in \mathbb{Z}$, and $q \in \Lambda$, and if we set

$$\tau_j q(t) \equiv q(t-j), \tag{1.8}$$

then

$$I(\tau_j q) = I(q). \tag{1.9}$$

Unfortunately, (1.9) is no longer valid under hypothesis (a_1). However, the definition of almost periodicity (see e.g. [21] or [3]) implies there is a sequence $(\sigma_m) \subset \mathbb{R}$ such that $\sigma_m \to \infty$ and

$$\|\tau_{\sigma_m} a - a\|_{L^\infty(\mathbb{R})} \to 0 \tag{1.10}$$

as $m \to \infty$. This fact will play an important role here.

To continue, let $M > 0$ and

$$I^M = \{w \in \Lambda \mid I(w) \leq M\}.$$

Bounds needed for I^M are provided by the next result.

PROPOSITION 1.11. *If $M > 0$ and $q \in I^M$, then there is an $\omega(M) > 0$ such that $\|q\| \leq \omega(M)$.*

PROOF. If $q \in M$, by (1.1) and (a_1), (W_4),

$$\|\dot{q}\|_{L^2}^2 \leq 2M \tag{1.12}$$

so all that is needed is a bound for q in L^2. By (W_4), there is a $\beta > 0$ such that

$$W''(0)(y, y) \geq 2\beta |y|^2 \tag{1.13}$$

for $y \in \mathbb{R}^2$. Hence there is a $\delta > 0$ such that for $0 \le |y| \le \delta$,

$$-W(y) \ge \beta |y|^2. \tag{1.14}$$

Consequently, if $q \in I^M$ is fixed and

$$S(\delta) = \{t \in \mathbb{R} \mid |q(t)| > \delta\} \quad \text{and} \quad \widehat{S}(\delta) = \{t \in \mathbb{R} \mid |q(t)| \le \delta\},$$

then

$$M \ge a_0\beta \int_{\widehat{S}(\delta)} |q|^2\,dt - \int_{S(\delta)} aW(q)\,dt. \tag{1.15}$$

Let

$$\gamma = \gamma(\delta) = \inf_{|x| \ge \delta} -W(x). \tag{1.16}$$

Then

$$M \ge a_0\beta \int_{\widehat{S}(\delta)} |q|^2\,dt + a_0\gamma(\delta) \operatorname{meas} S(\delta). \tag{1.17}$$

For $t \in S(\delta)$, $\delta \le |q(t)| \le \|q\|_{L^\infty}$. Therefore by a slight variant of Lemma 3.6 of [16],

$$M \ge I(q) \ge \sqrt{2a_0\gamma(\delta)}\,|q(s) - q(\overline{s})|, \tag{1.18}$$

where $|q(s)| = \delta$ and $|q(\overline{s})| = \|q\|_{L^\infty}$. Now (1.18) provides an L^∞ bound for q:

$$\|q\|_{L^\infty} \le \frac{M + \sqrt{2a_0\gamma(\delta)}\,\delta}{\sqrt{2a_0\gamma(\delta)}} \equiv M_1. \tag{1.19}$$

Therefore by (1.17) and (1.19),

$$\begin{aligned} \int_{\mathbb{R}} |q|^2\,dt &\le \int_{\widehat{S}(\delta)} |q|^2\,dt + \int_{S(\delta)} |q|^2\,dt \\ &\le M a_0^{-1}(\beta^{-1} + M_1^2\gamma^{-1}) \end{aligned} \tag{1.20}$$

and Proposition 1.11 is proved.

The first existence result for (HS) can now be formulated. It provides homoclinic solutions of (HS) which may not belong to Γ. A slight variant of an argument from [21] is used to obtain the theorem. In particular, the following technical result whose proof is postponed until §3 is required. Its statement is essentially the same as the analogous result in [21].

PROPOSITION 1.21. *Suppose* $(p_m) \subset \Lambda$, $I(p_m) \to b > 0$ *and* $I'(p_m) \to 0$ *as* $m \to \infty$ *(i.e.* (p_m) *is a Palais–Smale sequence for* I*). If in addition*

$$\|p_m - p_{m-1}\| \to 0 \tag{1.22}$$

as $m \to \infty$, *then there is a sequence* $(\theta_m) \subset \mathbb{R}$ *and an* $r > 0$ *such that*

$$\varliminf_{m\to\infty} |\tau_{\theta_m} p_m(0)| \geq r \tag{1.23}$$

and

$$|\theta_m - \theta_{m-1}| \to 0 \tag{1.24}$$

as $m \to \infty$.

Now we have

THEOREM 1.25. *Suppose* (a_1) *and* (W_1)–(W_5) *are satisfied. Let* $q \in \Gamma$. *Then there exists a homoclinic solution* $Q \in \Lambda$ *of* (HS) *with* $I(Q) \in (0, I(q)]$.

PROOF. If $I'(q) = 0$, the result obtains with $Q = q$. Thus suppose $I'(q) \neq 0$. Let $\mathcal{V}(x)$ be a locally Lipschitz continuous pseudogradient vector field for I, i.e. $\mathcal{V}$ is locally Lipschitz continuous on $\widehat{E} = \{y \in E \mid I'(y) \neq 0\}$ and satisfies

$$\begin{aligned} &\text{(i)} \quad \|\mathcal{V}(x)\| \leq 2\|I'(x)\|, \\ &\text{(ii)} \quad I'(x)\mathcal{V}(x) \geq \|I'(x)\|^2. \end{aligned} \tag{1.26}$$

For the existence of such a $\mathcal{V}$, see e.g. Lemma A.2 of [15].

Consider the ordinary differential equation in E:

$$\frac{d\eta}{ds} = -\frac{\mathcal{V}(\eta)}{1 + \|\mathcal{V}(\eta)\|} \equiv -\mathcal{W}(\eta) \tag{1.27}$$

with $\eta(0) = q$. Then $\mathcal{W}$ is locally Lipschitz continuous on $\widehat{E}$ and $\|\mathcal{W}(x)\| \leq 1$ for all $x \in \widehat{E}$. Therefore the solution of (1.27) exists for all $s > 0$ (see e.g. [15]) and by (1.26)(ii),

$$\frac{dI(\eta(s))}{ds} = -I'(\eta(s))\mathcal{W}(\eta(s)) < 0. \tag{1.28}$$

Since $\eta(0) \in \Gamma$, Proposition 1.2 and (1.28) show $\eta(s) \in \Gamma$ for all $s > 0$. Hence by (1.28), (1.6) and Proposition 1.7,

$$\inf_{s\geq 0} I(\eta(s)) = \lim_{s\to\infty} I(\eta(s)) \geq c^{\pm} > 0 \tag{1.29}$$

depending on whether $q \in \Gamma^+$ or Γ^-. Let $s_m \to \infty$ as $m \to \infty$ and further satisfy

$$|s_m - s_{m-1}| \to 0 \tag{1.30}$$

as $m \to \infty$. By a corollary to Ekeland's Theorem (see Mawhin–Willem [13], Corollary 4.1) there is a sequence $\varphi_m \to \infty$ as $m \to \infty$ such that

$$|\varphi_m - s_m| \to 0, \tag{1.31}$$

$$I(\eta(\varphi_m)) \leq I(\eta(s_m)) \tag{1.32}$$

and

$$\frac{d}{ds} I(\eta(\varphi_m)) \to 0 \tag{1.33}$$

as $m \to \infty$. Set $q_m = \eta(\varphi_m)$. Then by (1.33) and (1.26)(ii),

$$I'(q_m) \to 0 \tag{1.34}$$

as $m \to \infty$. Hence by (1.27) and (1.30)–(1.31),

$$\|q_m - q_{m-1}\| \leq \left\| \int_{\varphi_{m-1}}^{\varphi_m} \frac{d\eta}{ds}\, ds \right\| \leq |\varphi_m - \varphi_{m-1}| \to 0 \tag{1.35}$$

as $m \to \infty$. Consequently, (φ_m) satisfies the hypotheses of Proposition 1.21. Therefore there is a sequence $(\theta_m) \subset \mathbb{R}$ and an $r > 0$ such that

$$\varliminf_{m \to \infty} |\tau_{\theta_m} q_m(0)| \geq r \tag{1.36}$$

and

$$|\theta_m - \theta_{m-1}| \to 0 \tag{1.37}$$

as $m \to \infty$.

Suppose θ_m has a bounded subsequence. Then along a further subsequence, $\theta_m \to \overline{\theta}$. Moreover, $I(q_m) < I(q)$ by (1.28). Therefore by Proposition 1.11, (q_m) is bounded. Hence there is a $Q \in \Lambda$ such that q_m converges to Q weakly in E and strongly in L^∞_{loc} along a subsequence. Thus

$$I'(q_m)\varphi \to I'(Q)\varphi \tag{1.38}$$

for all $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ and by (1.34),

$$I'(Q)\varphi = 0. \tag{1.39}$$

Moreover, by (1.36), $\tau_{\overline{\theta}} Q(0) \neq 0$. Consequently, Q is a nontrivial homoclinic solution of (HS) and Theorem 1.25 is proved for this case.

Next suppose (θ_m) does not have a bounded subsequence. Set $v_m = \tau_{\theta_m} q_m$. Since a is almost periodic, there is an unbounded sequence σ_m (in the same direction as θ_m) such that

$$\|\tau_{-\sigma_m} a - a\|_{L^\infty} \to 0. \tag{1.40}$$

Choose a subsequence (θ_{m_k}) of (θ_m) satisfying

$$|\theta_{m_k} - \sigma_k| \to 0 \tag{1.41}$$

as $k \to \infty$. This is possible via (1.37). The functions (q_m) and therefore (v_m) are bounded in Λ as in the previous case. Consequently, there is a $Q \in \Lambda$ such that along a subsequence, v_{m_k} converges to Q weakly in E and strongly in L^∞_{loc}. By (1.36), $Q(0) \neq 0$. Moreover,

$$\|\tau_{-\theta_{m_k}} a - a\|_{L^\infty} \le \|\tau_{-\theta_{m_k}} a - \tau_{-\sigma_k} a\|_{L^\infty} + \|\tau_{-\sigma_k} a - a\|_{L^\infty} \to 0 \tag{1.42}$$

as $k \to \infty$ via (1.40)–(1.41). Note also that if $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$,

$$\begin{aligned} I'(Q)\varphi &= \int_{\mathbb{R}} (\dot Q \cdot \dot\varphi - W'(Q)\cdot\varphi)\,dt \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} (\dot v_{m_k} \cdot \dot\varphi - aW'(v_{m_k})\cdot\varphi)\,dt \\ &= \lim_{k\to\infty} \int_{\mathbb{R}} (\dot q_{m_k} \cdot \tau_{-\theta_{m_k}}\dot\varphi - (\tau_{-\theta_{m_k}} a) W'(q_{m_k})\cdot \tau_{-\theta_{m_k}}\varphi)\,dt = 0 \end{aligned} \tag{1.43}$$

by (1.42), (1.30), the L^∞ bounds on q_{m_k} provided by Proposition 1.11 and the lower bounds for $|q_{m_k}(t) - \xi|$ given by Proposition 1.2. This completes the proof of Theorem 1.25.

It is natural to ask whether the solution Q obtained above lies in Γ. We do not know whether this is the case in the generality of Theorem 1.25. However, the next theorem shows that $Q \in \Gamma^\pm$ if $I(q)$ is close enough to $c^\pm$. A preliminary result is needed first.

LEMMA 1.44. *There is a $\varrho > 0$ such that if $w \in \Lambda\backslash\{0\}$ is a solution of* (HS)*, then $\|w\|_{L^\infty} > \varrho$.*

PROOF. By (W_4), there are constants $\varrho, \beta > 0$ such that if $|x| \le \varrho$, then

$$-W'(x)\cdot x \ge \beta|x|^2. \tag{1.45}$$

Suppose $w \in \Lambda\backslash\{0\}$ is a solution of (HS) with $\|w\|_{L^\infty} \le \varrho$. Then

$$0 = I'(w)w = \int_{\mathbb{R}} (|\dot w|^2 - aW'(w)\cdot w)\,dt \ge \int_{\mathbb{R}} (|\dot w|^2 + a\beta|w|^2)\,dt > 0, \tag{1.46}$$

a contradiction.

THEOREM 1.47. *Suppose (a_1) and (W_1)–(W_5) are satisfied. Then there is an $\varepsilon_0 > 0$ such that whenever $q \in \Gamma^\pm$ with*

$$I(q) < c^\pm + \varepsilon_0, \tag{1.48}$$

then the solution, Q, of (HS) *given by Theorem* 1.25 *lies in $\Gamma^\pm$ and $I(Q) \in [c^\pm, I(q)]$.*

PROOF. The + case will be proved using an argument from [17]. If $Q \notin \Gamma^+$, then $d(Q) \le 0$. Let $\delta \in (0, \varrho/2)$, with ϱ given by Lemma 1.44. Since $Q \in E$, there is a $T = T(\delta) > 0$ such that for $|t| \geq T$,

$$|Q(t)| \leq \delta. \tag{1.49}$$

Let $Q_k(t) = q_k(t)$ if Q was obtained via Case 1 of the proof of Theorem 1.25; set $Q_k(t) = v_{m_k}(t)$ if Case 2 obtains. It can be assumed that Q_k converges to Q uniformly for $|t| \leq T+1$ as $k \to \infty$. Note that

$$0 < d(Q_k) = \mathrm{WN}(Q_k|_{-\infty}^{-T-1}) + \mathrm{WN}(Q_k|_{-T-1}^{T+1}) + \mathrm{WN}(Q_k|_{T+1}^{\infty}) \tag{1.50}$$

and

$$\mathrm{WN}(Q_k|_{-T-1}^{T+1}) \to \mathrm{WN}(Q|_{-T-1}^{T+1}) \tag{1.51}$$

as $k \to \infty$. Moreover, since $d(Q) \leq 0$, by (1.49), the right hand side of (1.51) is near a nonpositive integer. Consequently, the first or third term on the right hand side of (1.50) is near a positive integer since $Q_k(\pm\infty) = 0$ and $Q_k(\pm(T+1))$ is within δ of 0. The argument for either case is similar so suppose $\mathrm{WN}(Q_k|_{-\infty}^{-T-1})$ is near a positive integer. Define a new function

$$\widehat{q}_k(t) = \begin{cases} Q_k(t), & t \leq -T-1, \\ 0, & t \geq -T, \\ -(t+T)Q_k(t), & -T-1 < t < -T. \end{cases} \tag{1.52}$$

Then $d(\widehat{q}_k) \geq 1$ so $\widehat{q}_k \in \Gamma^+$. For the case of $Q_k = q_k$,

$$\begin{aligned} I(\widehat{q}_k) &= I(Q_k) + \int_{-T-1}^{-T} \mathcal{L}(\widehat{q}_k)\,dt - \int_{-T-1}^{\infty} \mathcal{L}(Q_k)\,dt \\ &< c^+ + \varepsilon_0 - \int_{-T-1}^{T+1} \mathcal{L}(Q_k)\,dt + O(\delta^2) \end{aligned} \tag{1.53}$$

as $\delta \to 0$. Since Q_k converges to $Q \in \Lambda$ uniformly for $t \in [-T-1, T+1]$, by Lemma 1.44, in this interval the curve Q_k passes from $\partial D_{2\delta}(0)$, the sphere of radius 2δ about 0 in $\mathbb{R}^2$, to $\partial D_{\varrho}(0)$ and ultimately back to $\partial D_{2\delta}(0)$. Therefore as in (1.18), by Lemma 3.6 of [16],

$$c^+ + \varepsilon_0 > I(Q_k) \geq \int_{-T-1}^{T+1} \mathcal{L}(Q_k)\,dt \geq 2\sqrt{2a_0\gamma\left(\frac{\varrho}{2}\right)}\frac{\varrho}{2} \equiv \varepsilon_1, \tag{1.54}$$

where γ is as in (1.16). Combining (1.53)–(1.54) gives

$$I(\widehat{q}_k) < c^+ + \varepsilon_0 + O(\delta^2) - \varepsilon_1 \tag{1.55}$$

as $\delta \to 0$. Hence choosing $\varepsilon_0 = \varepsilon_1/2$ and δ sufficiently small shows

$$I(\widehat{q}_k) < c^+, \tag{1.56}$$

contrary to $\widehat{q}_k \in \Gamma^+$. Thus $Q \in \Gamma^+$, $I(Q) \geq c^+$, and this case is proved.

Next suppose $Q_k = v_{m_k} = \tau_{\theta_{m_k}} q_{m_k}$. Then

$$I(Q_k) = I(\tau_{\theta_{m_k}} q_{m_k}) + \int_{\mathbb{R}} (a - \tau_{-\theta_{m_k}} a) W(q_{m_k})\, dt < c^+ + \varepsilon_0 \tag{1.57}$$

for large k due to the uniform L^∞ bounds on (q_{m_k}) given by Proposition 1.11 and the estimate (1.42). Consequently, this case follows in a similar fashion to the previous one and Theorem 1.47 is proved.

REMARK 1.58. If in Theorem 1.47, $q \in \Gamma^\pm$, $c^\pm < I(q) < c^\pm + \varepsilon$ and $I'(q) \neq 0$, then $I(Q) \in [c^\pm, I(q))$.

2. Some multiplicity results

At this point, the existence of homoclinic solutions of (HS) in $\Gamma^\pm$ has been obtained. The multiplicity of such solutions will be studied in this section. It will be shown that each of Γ^+, Γ^- contains infinitely many homoclinic solutions of (HS). In fact, stronger results hold. To make a more precise statement, let

$$\mathcal{K} = \{q \in \Lambda \backslash \{0\} \mid I'(q) = 0\},$$

i.e. $\mathcal{K}$ is the set of nontrivial solutions of (HS) that are homoclinic to 0. Set

$$\mathcal{Q}^\pm = \mathcal{K} \cap \Gamma^\pm \cap I^{c^\pm + \varepsilon_0}.$$

Then we have:

THEOREM 2.1. *If* (a_1) *and* (W_1)–(W_5) *hold, then* $\mathcal{Q}^+$, $\mathcal{Q}^-$ *are each infinite sets.*

PROOF. It will be proved that $\mathcal{Q}^+$ is an infinite set. Suppose the result is false. By Theorem 1.47, for each $q \in I^{c^+ + \varepsilon_0}$, there is a corresponding solution Q of (HS) in $\mathcal{Q}^+$ with $c^+ \le I(Q) \le I(q)$. Therefore the set of such functions Q is finite: Suppose there are j distinct Q's: $Q_1, \dots, Q_j$. Each Q_i must be an isolated critical point of I. It can be assumed that $I(Q_1) \le \dots \le I(Q_j)$. Then $I(Q_1) = c^+$ for otherwise there would exist another solution Q in $\mathcal{Q}^+$ with $c^+ \le I(Q) < I(Q_1)$. The function Q_1 is a global minimizer of I (in Γ^+) and also an isolated critical point of I.

These properties of Q_1 yield infinitely many critical points of I in $\mathcal{Q}^+$ by means of the following result which is of independent interest. For $x \in E$, let $B_\varrho(x)$ denote the open ball of radius ϱ about x.

THEOREM 2.2. *If* P *is a local minimizer and an isolated critical point of* I, *and* (σ_k) *is as in* (1.40), *then there is an* $r_0 > 0$ *and* $k_0 = k_0(r) \in \mathbb{N}$ *defined for* $0 < r \le r_0$ *such that* I *has a local minimum in* $B_r(\tau_{\sigma_k} P)$ *for all* $k \ge k_0$.

REMARK. Note that if (HS) is autonomous, P cannot be an isolated critical point of I since $\tau_\theta P$ is also a critical point for all $\theta \in \mathbb{R}$. Hence dependence on t of $a(t)$ is a necessary condition for P to be isolated.

Assume Theorem 2.2 for the moment.

Completion of proof of Theorem 2.1. Set $P = Q_1$ in Theorem 2.2. Since $d(\tau_{\sigma_k} Q_1) = d(Q_1)$ we have $\tau_{\sigma_k} Q_1 \in \Gamma^+$. Moreover, for r_0 small (independently of k), if $q \in B_r(\tau_{\sigma_k} Q_1)$, then $d(q) = d(\tau_{\sigma_k} Q_1)$. Therefore $B_r(\tau_{\sigma_k} Q_1) \subset \Gamma^+$. Consequently, the local minimum, Q_k, of I in $B_r(\tau_{\sigma_k} Q_1)$ given by Theorem 2.2 lies in Γ^+ for all $k \geq k_0(r)$. Moreover, if $q \in B_r(\tau_{\sigma_k} Q_1)$, writing $q = \tau_{\sigma_k} v$ where $v \in B_r(Q_1)$ and arguing as in (1.57) shows

$$I(q) = I(v) + \int_{\mathbb{R}} (a - \tau_{-\sigma_k} a) W(q)\, dt < c^+ + \varepsilon_0 \tag{2.3}$$

for r_0 sufficiently small and $k \geq k_0(r)$. Therefore $Q_k \in I^{c^+ + \varepsilon_0}$ and Theorem 2.1 is proved.

To prove Theorem 2.2, some additional preliminaries are needed. Let $\mathcal{H}(a)$ denote the hull of the almost periodic function a, i.e. the closure (under $\|\cdot\|_{L^\infty}$) of the set of all uniform limits of translates of a. (We recall that by a theorem of Bochner, a is almost periodic if and only if $\mathcal{H}(a)$ is compact. See e.g. [12].) Note that for each $h \in \mathcal{H}(a)$, $h(t) \geq a_0$ for all $t \in \mathbb{R}$. Corresponding to each such h is a Hamiltonian system of the form

$$\ddot{q} + hW'(q) = 0 \tag*{$(\mathrm{HS})_h$}$$

with associated functional

$$I_h(q) = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{q}|^2 - hW(q) \right) dt. \tag{2.4}$$

Let

$$\mathcal{K}_h = \{q \in \Lambda \backslash \{0\} \mid I_h'(q) = 0\},$$

the set of nontrivial critical points of I_h in Λ or equivalently the set of nontrivial solutions of $(\mathrm{HS})_h$ which are homoclinic to 0. Finally, set

$$\mathcal{K}^* = \bigcup_{h \in \mathcal{H}(a)} \mathcal{K}_h\,. \tag{2.5}$$

Remark 2.6. The argument of Lemma 1.44 shows that the ϱ obtained there can be chosen so that $\|q\|_{L^\infty} > \varrho$ for all $q \in \mathcal{K}^*$.

To continue, it is essential to understand the behavior of Palais–Smale sequences for I. The next proposition provides this information.

Proposition 2.7. *Let $(x_m) \subset \Lambda$ satisfy $I(x_m) \to b > 0$ and $I'(x_m) \to 0$. Then there is a $j = j(b) \in \mathbb{N}$, $v_1, \ldots, v_j \in \mathcal{K}^*$, and sequences $(k_m^1), \ldots, (k_m^j) \subset$*

$\mathbb{R}$ *such that along a subsequence, as* $m \to \infty$,

$$\left\| x_m - \sum_{i=1}^{j} \tau_{k_m^i} v_i \right\| \to 0, \tag{2.8}$$

$$|k_m^i - k_m^p| \to 0 \qquad \text{if } i \neq p, \tag{2.9}$$

and

$$\sum_{i=1}^{j} I_{h_i}(v_i) = b, \qquad \text{where } v_i \in \mathcal{K}_{h_i}. \tag{2.10}$$

The proof of Proposition 2.7 is similar to related results in e.g. [18], [20], [21] and will be discussed in §3.

An important consequence of Proposition 2.7 is:

PROPOSITION 2.11. *If* P *is a local minimum and an isolated critical point of* I*, there is an* $r_1 > 0$ *and* $\delta = \delta(r, \underline{r}) > 0$ *such that if* $0 < \underline{r} < r \leq r_1$ *and* $x \in \overline{B_r(P) \backslash B_{\underline{r}}(P)}$*, then* $\|I'(x)\| \geq 2\delta$.

PROOF. Choose r_1 so that

$$\mathcal{K} \cap \overline{B}_{r_1}(P) = \{P\}. \tag{2.12}$$

If Proposition 2.11 is false, there is a sequence $(x_m) \subset \overline{B_r(P) \backslash B_{\underline{r}}(P)}$ such that $I'(x_m) \to 0$. The form of I shows it is bounded on bounded subsets of E which avoid ξ and this is the case here for r_1 sufficiently small. Therefore (x_m) can be assumed to be a Palais–Smale sequence. Applying Proposition 2.7, if $j > 1$, by (2.8)–(2.9),

$$\left\| P - \sum_{i=1}^{j} \tau_{k_m^i} v_i \right\| \leq 2r \leq 2r_1 \tag{2.13}$$

while

$$\varliminf_{m \to \infty} \left\| P - \sum_{i=1}^{j} \tau_{k_m^i} v_i \right\| \geq \min_{1 \leq i \leq j} \|v_i\| \geq \inf_{v \in \mathcal{K}^*} \|v\| \geq \frac{1}{2} \inf_{v \in \mathcal{K}^*} \|v\|_{L^\infty} \geq \frac{\varrho}{2}. \tag{2.14}$$

Hence if $4r_1 < \varrho$, then $j > 1$ is impossible. Therefore $j = 1$. If (k_m^1) is unbounded, (2.8)–(2.9) again show

$$2r_0 \geq \frac{\varrho}{2}. \tag{2.15}$$

Consequently, (k_m^1) is bounded so without loss of generality it can be assumed that $k_m^1 \to k$. Hence by (2.8), $x_m \to \tau_k v_1$ as $m \to \infty$ and $I'(\tau_k v_1) = 0$. Then $\tau_k v_1 \in \mathcal{K}$, contrary to (2.12) unless $\tau_k v_1 = P$. But $(x_m) \subset \overline{B_r(P) \backslash B_{\underline{r}}(P)}$ so $x_m \to P$ as $m \to \infty$ cannot occur. Therefore there exists a $\delta = \delta(r, \underline{r})$ as claimed.

COROLLARY 2.16. *Let* (σ_k) *be as in* (1.40). *Then there is a* $k_1 = k_1(r, \underline{r})$ *such that for all* $k \geq k_1$,

$$\|I'(x)\| \geq \delta(r, \underline{r}), \qquad x \in \overline{B_r(\tau_{\sigma_k} P)\backslash B_{\underline{r}}(\tau_{\sigma_k} P)}. \tag{2.17}$$

PROOF. Let $\varphi \in E$ with $\|\varphi\| = 1$ and let $y \in \overline{B_r(P)\backslash B_{\underline{r}}(P)}$. Then if $x = \tau_{\sigma_k} y$,

$$\begin{aligned} I'(x)\varphi &= \int_{\mathbb{R}} (\dot{x} \cdot \dot{\varphi} - aW'(x) \cdot \varphi)\, dt \\ &= I'(y)\tau_{-\sigma_k}\varphi + \int_{\mathbb{R}} (a - \tau_{-\sigma_k} a) W'(y) \cdot \tau_{-\sigma_k}\varphi\, dt \\ &\geq I'(y)\tau_{-\sigma_k}\varphi - \|a - \tau_{-\sigma_k} a\|_{L^\infty} \left(\int_{\mathbb{R}} |W'(y)|^2\, dt \right)^{1/2}. \end{aligned} \tag{2.18}$$

Hence (2.17) follows from (2.18), Proposition 2.11, (1.40), and the boundedness of $\|W'(y)\|_{L^2}$ on $B_{r_0}(P)$.

With the above preliminaries in hand, we are ready for the

PROOF OF THEOREM 2.2. Note first that the infimum of I over $\overline{B}_{r/2}(\tau_{\sigma_k} P)$ is achieved for all $k \in \mathbb{Z}$. Indeed, any minimizing sequence (x_m) for

$$\inf_{B_{r/2}(\tau_{\sigma_k} P)} I$$

is bounded via Proposition 1.11. Therefore it converges weakly in E and strongly in L^∞_{loc} to $z \in \overline{B}_r(\tau_{\sigma_k} P)$. Writing I as

$$I(x) = \frac{1}{2}\|x\|^2 - \int_{\mathbb{R}} \left(\frac{1}{2}|x|^2 + a(t)W(x) \right) dt$$

shows, for any $l > 0$,

$$I(x_m) \geq \frac{1}{2}\|x_m\|^2_{W^{1,2}[-l,l]} - \int_{-l}^{l} \left(\frac{1}{2}|x_m|^2 + a(t)W(x_m) \right) dt$$

via (W_4). Therefore

$$\varliminf_{m\to\infty} I(x_m) \geq \frac{1}{2}\|z\|^2_{W^{1,2}[-l,l]} - \int_{-l}^{l} \left(\frac{1}{2}|z|^2 + a(t)W(z) \right) dt. \tag{2.19}$$

Since (2.19) holds for all $l > 0$,

$$\varliminf_{m\to\infty} I(x_m) \geq I(z) \tag{2.20}$$

and z minimizes I in $\overline{B}_{r/2}(\tau_{\sigma_k} P)$.

Suppose that I does not possess an interior minimum in $B_{r/2}(\tau_{\sigma_k} P)$. Then $z \in \partial B_{r/2}(\tau_{\sigma_k} P)$. Consider the ordinary differential equation

$$\frac{d\eta}{ds} = -\mathcal{V}(\eta), \qquad \eta(0) = z, \tag{2.21}$$

where $\mathcal{V}$ is a locally Lipschitz continuous pseudogradient vector field for I on $\widehat{E}$. For any $s > 0$ for which $\eta(s)$ is defined,

$$\frac{d}{ds} I(\eta(s)) = -I'(\eta(s))\mathcal{V}(\eta(s)) < 0 \tag{2.22}$$

via (1.26)(ii). Hence $\eta(s) \notin \overline{B}_{r/2}(\tau_{\sigma_k} P)$ since $I(\eta(s)) < I(z) \le I(x)$ for all $x \in \overline{B}_{r/2}(\tau_{\sigma_k} P)$. Let $\delta = \delta(r, r/2)$ as given by Corollary 2.16. Then for all s for which $\eta(s) \in \overline{B}_r(\tau_{\sigma_k} P) \backslash B_{r/2}(\tau_{\sigma_k} P)$, by (1.26)(ii),

$$\frac{d}{ds} I(\eta(s)) \le -\delta^2. \tag{2.23}$$

Since $I(x) \ge 0$ for all $x \in E$, (2.23) implies that after a finite time T, $\eta(s)$ reaches $\partial B_r(\tau_{\sigma_k} P)$ and

$$\begin{aligned} \frac{r}{2} &\le \|\eta(T) - \eta(0)\| = \left\| \int_0^T \frac{d\eta}{ds}\, ds \right\| \\ &\le \int_0^T \|\mathcal{V}(\eta(s))\|\, ds \le 2 \int_0^T \|I'(\eta(s))\|\, ds \end{aligned} \tag{2.24}$$

via (1.26)(i). On the other hand, by (1.26)(ii) again,

$$\begin{aligned} I(\eta(0)) - I(\eta(T)) &= \int_T^0 \frac{dI}{ds}(\eta(s))\, ds = \int_0^T I'(\eta(s))\mathcal{V}(\eta(s))\, ds \\ &\ge \int_0^T \|I'(\eta(s))\|^2\, ds \ge \delta \int_0^T \|I'(\eta(s))\|\, ds. \end{aligned} \tag{2.25}$$

Combining (2.24)–(2.25) yields

$$I(\tau_{\sigma_k} P) - I(\eta(T)) \ge I(z) - I(\eta(T)) \ge \delta r/4. \tag{2.26}$$

Now $\eta(T) = \tau_{\sigma_k} x_k$ for some $x_k \in \partial B_r(P)$. Hence

$$I(\tau_{\sigma_k} x_k) \le I(\tau_{\sigma_k} P) - \delta r/4 \tag{2.27}$$

and as in (1.57), for $k \ge k_0(r)$,

$$I(\tau_{\sigma_k} P) \le I(P) + \delta r/16. \tag{2.28}$$

Therefore

$$I(\tau_{\sigma_k} x_k) \le I(P) - 3\delta r/16. \tag{2.29}$$

As in (1.57) again,

$$I(\tau_{\sigma_k} x) - I(x) \le \|a - \tau_{-\sigma_k} a\|_{L^\infty} \int_{\mathbb{R}} |W(x)|\, dt. \tag{2.30}$$

Since $B_r(P)$ is a bounded set in E all members of which avoid ξ, as earlier it can be assumed that if $k \ge k_0(r)$ and $x \in \overline{B}_r(P)$,

$$|I(\tau_{\sigma_k} x) - I(x)| \le \delta r/8. \tag{2.31}$$

Therefore by (2.31) and (2.29),

$$I(x_k) \le I(P) - \delta r/16. \tag{2.32}$$

But P is a strict local minimum for I in $\overline{B}_r(P)$. Therefore (2.32) is impossible and $B_{r/2}(\tau_{\sigma_k}P)$ must contain a local minimum of I for all large k. The proof of Theorem 2.2 is complete.

We conclude this section with a final observation. Let (y_m) be a minimizing sequence for (1.6), e.g. for the + case. Then by a simple variant of Lemma 1.44 (using (W_4)), $\|y_m\|_{L^\infty} \ge \varrho_1 > 0$. Since $|y_m(t)| \to 0$ as $|t| \to \infty$, there exists a $T_m \in \mathbb{R}$ such that $|y_m(T_m)| \ge \varrho_1$. If (T_m) possesses a bounded subsequence, earlier arguments show there is a subsequence of y_m which converges weakly in E to $Q \in E\backslash\{0\}$ satisfying $I(Q) \le c^+$. A further argument as in the proof of Theorem 1.47 or [17] implies $Q \in \Gamma^+$ and therefore $I(Q) = c^+$. If on the other hand, (T_m) is unbounded, by setting $v_m = \tau_{-T_m} y_m$, it follows as above that v_m converges along a subsequence to a function $Q \in E\backslash\{0\}$ with $I_h(Q) \le c^+$ and $h = \lim \tau_{-T_m} a$, the limit being taken along a subsequence. It is not difficult to see that

$$\inf_{\Gamma^+} I_\varphi = \inf_{\Gamma^+} I$$

for all $\varphi \in \mathcal{H}(a)$. Therefore if $Q \in \Gamma^+$, then $I_h(Q) = c^+$. To prove that $Q \in \Gamma^+$, note that $d(v_m) = d(y_m) > 0$ so $v_m \in \Gamma^+$. Then arguing once again as in Theorem 1.47 or in [17] yields $Q \in \Gamma^+$.

These remarks show there always is an $h^\pm \in \mathcal{H}(a)$ and $Q^\pm \in \Gamma^\pm$ such that $I_{h^\pm}(Q^\pm) = c^\pm$. If $Q^\pm$ is an isolated critical point of I_h, Theorem 2.2 applies to it. However, a priori $Q^\pm$ may not be an isolated point of I_h.

3. Technical results

This section deals with Proposition 1.21 and Proposition 2.7. The statement of Proposition 1.21 is the same as that of Proposition 3.13 of [21] although the technical frameworks of the two results are different. In [21], Serra, Tarallo, and Terracini consider a Hamiltonian system of the form

$$\ddot{q} - q + a(t)G'(q) = 0, \tag{3.1}$$

where $q \in \mathbb{R}^n$, $a(t)$ satisfies (a_1), and $G \in C^2$ is a superquadratic potential, i.e.

(G1) There is a $\theta > 2$ such that

$$0 < \theta G(x) \le x \cdot G'(x)$$

for all $x \in \mathbb{R}^n \backslash\{0\}$.

Thus G is a rather different nonlinearity than W. Nevertheless (3.1) and (HS) have several common features that make the proof of Proposition 1.21 nearly identical with that of Proposition 3.13 of [21]. The proof of the latter requires several pages of work. Therefore rather than repeat this argument here, those properties for (HS) that combined with [21] allow for the same proof will be indicated.

One of the important ingredients in obtaining Proposition 1.21 is Proposition 2.7. For the proof of this latter result note first that if $h \in \mathcal{H}(a)$, then $a(t) \geq a_0 > 0$. Therefore for small $x \in E$,

$$
\begin{aligned}
I_h(x) &= \int_{\mathbb{R}} \Big(\frac{1}{2}|\dot{x}|^2 - hW(x)\Big)\,dt \\
&= \int_{\mathbb{R}} \Big(\frac{1}{2}|\dot{x}|^2 + \frac{h}{2}W''(0)(x,x)\Big)\,dt + o(\|x\|^2) \\
&\geq \frac{1}{2}\min(1,a_0)\|x\|^2.
\end{aligned} \tag{3.2}
$$

Combining (3.2) with Remark 2.6 shows for any $h \in \mathcal{H}(a)$,

$$I_h(\mathcal{K}_h) \geq c_0 > 0, \tag{3.3}$$

where c_0 is independent of h. Lastly, observe that by Proposition 1.11, a Palais–Smale sequence for I (or I_h) is bounded in E. With these observations, the proof of Proposition 2.7 proceeds as in [21], Proposition 2.16, or in [18] or [20].

REMARK 3.4. Actually, [21] states the result for sequences (x_m) which converge weakly to 0. If (x_m) converges weakly to $v \neq 0$, the proof of Proposition 2.7 shows that $v_1 \in \mathcal{K}_a = \mathcal{K}$ and k_m^1 can be taken to be 0 for all m.

Next let

$$\mathcal{M} = \bigcup_{n\in\mathbb{N}} \Big\{ \sum_{i=1}^{n} |\tau_{\theta_i} v_i(t)|^2 \;\Big|\; v_i \in \mathcal{K}^* \text{ and } \theta_i \in \mathbb{R} \Big\}.$$

Suppose $\varphi = \sum_{i=1}^{n} |\tau_{\theta_i} v_i(t)|^2 \in \mathcal{M}$. Then $v_i \in \mathcal{K}_{a_i}$, and

$$
\begin{aligned}
\varphi''(t) &= 2\sum_{i=1}^{n} |\tau_{\theta_i}\dot{v}_i(t)|^2 + 2\sum_{i=1}^{m} \tau_{\theta_i} v_i(t)\cdot \tau_{\theta_i} v_i''(t) \\
&= 2\sum_{i=1}^{n} |\tau_{\theta_i}\dot{v}_i(t)|^2 - 2\sum_{i=1}^{n} \tau_{\theta_i} v_i(t)\cdot \tau_{\theta_i}\varphi_i(t) W'(\tau_{\theta_i} v_i(t)).
\end{aligned} \tag{3.5}
$$

Let ε_1 be such that

$$a_0\varepsilon_1 < 1. \tag{3.6}$$

By (W_4), there is a $\varrho_2 > 0$ such that if $|x| < \sqrt{2\varrho_2}$, then $|W'(x)| \leq \varepsilon_1 |x|$. Thus if

$$0 < \varphi(t) < 2\varrho_2, \tag{3.7}$$

by the definition of φ, $|\tau_{\theta_i} v_i(t)| \le \sqrt{2\varrho_2}$ and $|W'(\tau_{\theta_i} v_i(t))| \le \varepsilon_1 |\tau_{\theta_i} v_i(t)|$. Therefore when (3.7) holds,

$$\varphi''(t) \ge 2(1 - a_0\varepsilon_1)\varphi(t). \tag{3.8}$$

Consequently, φ cannot have a local maximum when (3.7) holds.

With this observation, we have all of the ingredients needed to complete the proof of Proposition 1.21 as in [21].

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