Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 7, 1996, 205–217

TOPOLOGICAL CONTENT OF THE MAXWELL THEOREM ON MULTIPOLE REPRESENTATION OF SPHERICAL FUNCTIONS

V. Arnold

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

A spherical function of degree n on the unit sphere in $\mathbb{R}^3$ is the restriction to the sphere of a homogeneous harmonic polynomial of degree n.

In the present paper the topological consequences of the following classical fact are discussed.

Theorem 1. *The nth derivative of the function* $1/r$ *along* n *constant* (*translation-invariant*) *vector fields in* $\mathbb{R}^3$ *coincides on the sphere with a spherical function of degree* n. *Any nonzero spherical function of degree* n *can be obtained by this construction from some* n*-tuple of nonzero vector fields. These* n *fields are uniquely defined by the function* (*up to multiplication by nonzero constants and permutation of the* n *fields*).

The space of spherical functions of degree n is linear, of dimension $2n + 1$.

The set of functions representable by the multipole construction of the theorem is a priori highly nonlinear. The theorem implies that the image of the corresponding polylinear mapping is a *linear* space. The unicity statement can be reformulated in purely topological terms.

Theorem 2. *The configuration space of* n (*virtually coinciding*) *indistinguishable points on the real projective plane* (*i.e. the nth symmetric power*

1991 *Mathematics Subject Classification.* 33D55, 14D25, 14P25, 35C05, 55S15, 58C27, 78A30.

$\mathrm{Sym}^n(\mathbb{R}P^2)$) *is diffeomorphic to the real projective space of dimension* $2n$*:*

$$\mathrm{Sym}^n(\mathbb{R}P^2) \approx \mathbb{R}P^{2n}.$$

The symmetric powers of nonorientable surfaces have been computed independently of Maxwell by J. L. Dupont and G. Lusztig [4].

REMARK. Theorem 2 is a relative of the projective Viète's theorem

$$\mathrm{Sym}^n(\mathbb{C}P^1) \approx \mathbb{C}P^n$$

and is in a sense a quaternionic version of it.

Considering the Riemann sphere $\mathbb{C}P^1$ as a two-fold covering of the real projective space, we construct as a corollary an algebraic mapping $r : \mathbb{C}P^n \to \mathbb{R}P^{2n}$ of multiplicity 2^n and generalize the classical theorem $\mathbb{C}P^2/\mathrm{conj} \approx S^4$ to higher dimensions.

1. The main spaces and groups

Consider the n-dimensional arithmetical quaternionic space $\mathbb{H}^n = \bigoplus \mathbb{H}^1_p$ with its usual i-complex structure $(i(ae + bi + cj + dk) = ai - be + ck - dj)$. The left multiplication by j acts on $\mathbb{H}^1_p$, preserving complex lines. It sends each line to the hermitian orthogonal line and acts on $\mathbb{C}P^1_p = (\mathbb{H}^1_p \setminus 0)/\mathbb{C}^*$ as an antiholomorphic involution σ_p which has no fixed points.

Consider the Coxeter group $B(n)$, acting on the product $(\mathbb{C}P^1)^n$ as permutations of the factors, virtually accompanied by the σ_p's on some of the factors.

THEOREM 3. *The orbit space of the action of* $B(n)$ *on* $(\mathbb{C}P^1)^n$ *is the* $2n$*-dimensional real projective space*:

$$(\mathbb{C}P^1)^n/B(n) \approx \mathbb{R}P^{2n}.$$

The orbit space of the permutation group $S(n)$ is, by Viète's theorem, the complex projective space:

$$(\mathbb{C}P^1)^n/S(n) = \mathrm{Sym}^n(\mathbb{C}P^1) = \mathbb{C}P^n.$$

Thus we get the natural mapping $\varrho : \mathbb{C}P^n \to \mathbb{R}P^{2n}$ (which associates with an orbit ξ of the subgroup $S(n)$ the orbit $\varrho(\xi)$ of the group $B(n)$, containing ξ).

The group $B(n)$ also contains an interesting subgroup $\mathbb{Z}_2 \times S(n)$ (permutations and permutations accompanied by the antiholomorphic involutions σ_p in *every* factor). The product of the involutions σ_p acts on $(\mathbb{C}P^1)^n/S(n)$ as an involution $\sigma \in \mathbb{Z}_2$.

The group inclusions $S(n) \to \mathbb{Z}_2 \times S(n) \to B(n)$ provide the orbit space mappings

$$(\mathbb{C}P^1)^n/S(n) \xrightarrow{\alpha} (\mathbb{C}P^1)^n/S(n) \times \mathbb{Z}_2 \xrightarrow{\beta} (\mathbb{C}P^1)^n/B(n)$$

of multiplicities 2 and 2^{n-1} respectively.

The involution $\sigma : (\mathbb{C}P^1)^n/S(n) \to (\mathbb{C}P^1)^n/S(n)$ permutes both preimages $\alpha^{-1}(\,\cdot\,)$.

Theorem 4. *For even n, the involution σ acts on $(\mathbb{C}P^1)^n/S(n) \approx \mathbb{C}P^n$ as the complex conjugation* conj.

Thus for even n we get the real algebraic maps

$$\mathbb{C}P^n \xrightarrow{\alpha} \mathbb{C}P^n/\mathrm{conj} \xrightarrow{\beta} \mathbb{R}P^{2n}$$

of multiplicities 2 and 2^{n-1} respectively.

Remark. For $n = 2$ the space in the middle is smooth (see, e.g., [1] and [2]; perhaps this was known even before [2]),

$$\mathbb{C}P^2/\mathrm{conj} \approx S^4.$$

In this case the multiplicity of β is equal to 2.

The involution, interchanging the 2 preimages, acts on S^4 as the antipodal involution.

I am grateful to S. Donaldson for this remark, which shows that the (strange) Maxwell Theorem is in a sense a higher-dimensional extension of the (equally strange) theorem $\mathbb{C}P^2/\mathrm{conj} \approx S^4$.

As we shall show later in this paper, the Maxwell theorem provides an explicit formula for the diffeomorphism $\mathbb{C}P^2/\mathrm{conj} \to S^4$.

Remark. In most cases, when stating that two manifolds "coincide" we shall only provide explicitly a real algebraic homeomorphism between the corresponding manifolds. The pedantical checking that these homeomorphisms can be smoothened is in some cases left to the reader (see, however, Section 4).

2. Some theorems of real algebraic geometry

Consider a real homogeneous polynomial f of degree n in three variables (x, y, z). The strange algebraic byproduct of the Maxwell theorem is the following

Theorem 5. *Every real homogeneous polynomial f of degree n can be represented in one and only one way as the sum of two such polynomials, one of which is the product of n linear real factors, the other being divisible by $x^2 + y^2 + z^2$.*

In particular, every real algebraic curve of degree n in the metric projective plane defines n real "main axes", intrinsically associated with it.

Proof. The real equation $x^2 + y^2 + z^2 = 0$ defines in $\mathbb{C}P^2$ a real curve S with no real points. It is called the imaginary circle, is rational and is topologically a sphere. The complex conjugation sends S to itself and acts on S as an antiholomorphic involution having no fixed points (it is the usual antipodal involution of S^2).

The equation $f = 0$ defines in $\mathbb{C}P^2$ a real algebraic curve K of degree n. The complex conjugation $\mathrm{conj} : \mathbb{C}P^2 \to \mathbb{C}P^2$ sends K to itself and permutes the $2n$ points of the intersection of K with S.

Each pair of conjugate intersection points defines a line connecting them (the points are different, since conj has no fixed points on S^2). This line is real (since conj permutes two points on it) and may be defined by an equation $ax + by + cz = 0$ with real coefficients.

Consider the product g of the n real linear functions that we have constructed. We shall prove that f is proportional to g along S.

The homogeneous polynomial f vanishes at the $2n$ common points of the curves K and S. The curve S is rational. Choose a rational parameter t (say, $x = 2t$, $y = t^2 - 1$, $z = i(t^2 + 1)$). By the choice of coordinates we may eliminate the case where $t = \infty$ is one of the intersection points of S and K. Now the polynomials $f(x(t), y(t), z(t))$ and $g(x(t), y(t), z(t))$ of degree $2n$ have $2n$ common roots, and $g \neq 0$. Hence along S we have everywhere $f = cg$, $c = \mathrm{const}$.

The homogeneous polynomial $f - cg$ is thus vanishing on S. Hence it is of the form $(x^2 + y^2 + z^2)h(x, y, z)$, where h is a real homogeneous polynomial of degree $n - 2$. The theorem is proved.

Remark. The unicity of the decomposition $f = cg + (x^2 + y^2 + z^2)h$ can be proved independently of the existence. Suppose we have a second decomposition involving $c'g'$ and h'. Then $cg - c'g' = (x^2 + y^2 + z^2)h''$, where g and g' are the products of n linear factors. Suppose $g = 0$ on a line l. On this line the product polynomial $c'g'$ has n real roots, and it is equal to the right hand side product which has at most $n - 2$ real roots. Thus $c'g' \equiv 0$, $h'' = 0$ on l. Hence we can divide all the three terms by the equation of l and thus prove the unicity by induction on n. (For $n = 1$ everything is evident, since $h'' = 0$ and hence $cg = c'g'$.)

Consider the projective space $\mathbb{R}P^N$ of real algebraic curves of degree n ($N = n(n + 3)/2$).

The real algebraic curves consisting of n real lines form in this projective space a real algebraic closed variety T of dimension $2n$ (the image of $(\mathbb{R}P^2)^n$ under a polylinear mapping). The curves containing S form a projective subspace $P = \mathbb{R}P^M$, $M = (n - 2)(n + 1)/2$. In these terms our Theorem 5 takes the following form.

Theorem 6. *The varieties T and P are linked in $\mathbb{R}P^N$ with coefficient* 1 *in such a way that through every point of $\mathbb{R}P^N$ which belongs neither to T nor to P, there passes a unique real projective line connecting T to P. This line intersects T (as well as P) in one point.*

Now it is easy to prove Theorem 2.

Proof of Theorem 2. Consider a plane H of dimension $M+1$, containing the M-dimensional projective space P. *Every such plane intersects* T. Indeed, choose a point O on H which belongs neither to P nor to T. The line connecting O to T and to P belongs to H, and intersects T.

The intersection point is unique. Otherwise we would find on H a point O and two lines connecting T to P and passing through O.

Thus we have constructed a homeomorphism between $T \approx \mathrm{Sym}^n(\mathbb{R}P^2)$ and the manifold of planes H of dimension $M+1$ containing P. The last manifold is $\mathbb{R}P^{N-M-1}$, and since $N-M-1 = 2n$, Theorem 2 is proved (at least at the topological level).

3. From algebraic geometry to spherical functions

The derivatives of a harmonic function along translation-invariant vector fields are evidently harmonic functions. Hence all the multiple derivatives of $1/r$ are harmonic outside 0. The following lemmas are also well known.

Lemma 1. *The nth derivative of* $1/r$ *along n translation-invariant fields has the form* P/r^{2n+1}*, where P is a homogeneous polynomial of degree n.*

Proof. Let A be a homogeneous polynomial of degree a. Then

$$\frac{\partial}{\partial x}\frac{A}{r^b} = \frac{r^2(\partial A/\partial x) - bAx}{r^{b+2}}.$$

The numerator is a homogeneous polynomial of degree $a+1$. Thus differentiation increases its degree by 1 and that of the denominator by 2, as required.

Lemma 2. *The homogeneous polynomial P of Lemma* 1 *is a harmonic function.*

Proof. This follows from the classical inversion theorem, which I briefly recall here. The harmonic function P/r^{2n+1} is homogeneous, of degree $-(n+1)$. On the unit sphere it coincides with P. This implies that the continuation of this function from the unit sphere to the whole space as a homogeneous function of degree n is also harmonic. The continuation is just P.

To prove the inversion theorem denote by $\widetilde{\Delta}$ the Laplace operator div grad on the unit sphere. We extend it to the homogeneous functions of degree k in $\mathbb{R}^m \setminus 0$ in such a way that every homogeneous function F of degree k is sent to a homogeneous function $\widetilde{\Delta}F$ of the same degree.

An easy computation (based on scaling of divergence and gradient only) leads to the spherical Laplacian formula

$$\widetilde{\Delta}F = r^2\Delta F - \Lambda F, \quad \Lambda = k^2 + k(m-2).$$

This formula shows that:

(i) The restriction of a harmonic homogeneous function of degree k to the unit sphere in $\mathbb{R}^m$ is an eigenfunction of the spherical Laplacian with eigenvalue $-\Lambda$.

(ii) An eigenfunction of the spherical Laplacian with eigenvalue $-\Lambda$ becomes a harmonic function when continued from the sphere to $\mathbb{R}^m \setminus 0$ as a homogeneous function of degree k.

(iii) For every degree of homogeneity k in $\mathbb{R}^m$ there exists a dual degree $k' = 2 - m - k$ such that a homogeneous harmonic function of degree k remains harmonic if we restrict it to the sphere and then continue it as a homogeneous function of the dual degree. For $m = 3$ we get the duality condition $k + k' = -1$.

In particular, P/r^{2n+1} of Lemma 1 has degree $k = -(1+n)$, $m = 3$, hence the dual degree is $k' = n$, and Lemma 2 follows.

Lemma 3. *Every spherical function of degree n on S^2 can be represented in the form $f(X,Y,Z)(1/r)$, where f is a homogeneous polynomial of degree n, and where $X = \partial/\partial x$, $Y = \partial/\partial y$, $Z = \partial/\partial z$, $r^2 = x^2 + y^2 + z^2$.*

Proof. The space of harmonic homogeneous polynomials of degree n is linear. It contains the subspace of harmonic polynomials whose restrictions are representable as in Lemma 3 (by Lemmas 1 and 2). This subspace is (evidently) rotation invariant. But the representation of SO(3) in the space of spherical functions of degree n is irreducible (everything can be obtained from the Legendre polynomial of degree n by rotations and taking linear combinations of the rotated functions).

Thus the subspace of Lemma 3 coincides with the whole space of spherical functions of degree n.

Lemma 4. *Every spherical function of degree n can be represented in the form $f_T(X,Y,Z)(1/r)$, where $f_T = \prod_{i=1}^n(\alpha_i X + \beta_i Y + \gamma_i Z)$ is the product of n real linear factors.*

Proof. By Theorem 5 there exists a decomposition $f(x,y,z) = f_T(x,y,z) + g(x,y,z)(x^2+y^2+z^2)$. Applying this to the representation of Lemma 3 we get

$$f(X,Y,Z)(1/r) = f_T(X,Y,Z)(1/r) + 0,$$

since $X^2 + Y^2 + Z^2 = \Delta$ and $\Delta(1/r) = 0$. Thus every spherical function has a multipole representation of Theorem 1.

Lemma 5. *The multipole representation is unique (i.e. the polynomial f_T is unambiguously defined by the spherical function).*

Proof. Let f_T and f'_T be two completely decomposable polynomials, and let

$$f_T(X,Y,Z)(1/r) = f'_T(X,Y,Z)(1/r).$$

According to Lemma 1 (used n times),

$$f_T(X,Y,Z)(1/r) = (cf_T(x,y,z) + r^2 g)r^{-(2n+1)}, \qquad c \neq 0,$$
$$f'_T(X,Y,Z)(1/r) = (c'f'_T(x,y,z) + r^2 g')r^{-(2n+1)}, \qquad c' \neq 0.$$

Hence $f_T(x,y,z) - f'_T(x,y,z) = r^2 h(x,y,z)$. But this is only possible if $f_T = f'_T$ (Theorem 5). Theorem 1 is thus proved.

4. Explicit formulas

The quaternionic left multiplication by j sends the vector $(z,w) = zj + we$ of $\mathbb{C}^2 \approx \mathbb{H}^1$ to $(\overline{w}, -\overline{z})$. This vector is hermitian-orthogonal to the original vector.

We thus get the explicit formula $t \mapsto -1/\overline{t}$ for the antiholomorphic involution of $\mathbb{C}P^1$ which has no fixed points.

We can use the pairs of hermitian-orthogonal lines in $\mathbb{C}^2$ to parametrize the points of $\mathbb{R}P^2$. We thus deduce Theorem 3 from Theorem 2. We shall now write explicit formulas for the diffeomorphism

$$\mathrm{Sym}^n(\mathbb{R}P^2) \approx \mathbb{R}P^{2n}$$

of Theorem 2, using as coordinates on $\mathrm{Sym}^n(\mathbb{R}P^2)$ n-tuples of pairs of hermitian-orthogonal lines in $\mathbb{C}^2$.

Consider first the case $n = 1$.

With every pair of hermitian-orthogonal vectors in $\mathbb{C}^2$, (z,w) and $(\overline{w}, -\overline{z})$, we associate a quadratic form on the dual plane, which is the product of the linear forms represented by the vectors:

$$f = (zx + wy)(\overline{w}x - \overline{z}y) = f_0x^2 + f_1xy + f_2y^2.$$

Here (x,y) are the coordinates on the dual plane $\mathbb{C}^2$. Thus the coefficients of the quadratic form f are

$$f_0 = z\overline{w}, \quad f_1 = w\overline{w} - z\overline{z}, \quad f_2 = -\overline{z}w.$$

Note that f_1 is real, while $f_2 = -\overline{f}_0$. We shall consider f_0 and f_1 as coordinates in the space $\mathbb{R}^3$ of forms f.

The multiplication of the initial vector (z,w) by a complex number multiplies the resulting vector by the squared absolute value of this number. Hence we get a mapping

$$F : \mathbb{C}P^1 \to S^2 = (\mathbb{R}^3 \setminus 0)/\mathbb{R}^+,$$

sending complex lines in $\mathbb{C}^2$ to real rays in $\mathbb{R}^3$.

Chosen a point (z, w) on the complex line, (f_0, f_1) sends it to a definite point on the ray. For instance, choosing $z = t$, $w = 1$ we get $f_0 = t$, $f_1 = 1 - t\bar{t}$.

If the initial point on the line is normalized by the condition $|z|^2 + |w|^2 = 1$, the image lies on the ellipsoid $|2f_0|^2 + |f_1|^2 = 1$ (which we may also call a sphere, considering $2f_0$ and f_1 as coordinates).

Hence the Riemann sphere $\mathbb{C}P^1 = S^3/S^1$ is sent by $(2f_0, f_1)$ diffeomorphically onto the unit sphere in $\mathbb{R}^3$ with coordinates $2f_0$ and f_1. Considering $t = z/w$ as the coordinate in $\mathbb{C}P^1$, we obtain a mapping from the t-plane to the unit sphere in $\mathbb{R}^3$, which is just the stereographical projection. The formulas that we shall write below are in this sense higher-dimensional generalizations of the stereographical projection.

Replacing the initial line in $\mathbb{C}P^2$ by the hermitian-orthogonal line one changes the sign of F. Indeed, replacing z by $\overline{w}$ and w by $-\overline{z}$ one changes the signs of both f_0 and f_1. Thus F transforms the involution $j : \mathbb{C}P^1 \to \mathbb{C}P^1$ (sending each line to its hermitian orthocomplement) into the antipodal involution of S^2 in $\mathbb{R}^3$.

Now we shall apply a similar construction to the nth symmetric power of $\mathbb{R}P^2$. Start from the nth symmetric power of $\mathbb{C}P^1$. A point of this complex manifold

$$\mathrm{Sym}^n(\mathbb{C}P^1) \approx \mathbb{C}P^n$$

is defined as an (unordered) set of n lines in $\mathbb{C}^2$.

Choose n nonzero vectors (z_k, w_k) and multiply the corresponding linear forms on the dual plane to obtain the binary n-form

$$H(x, y) = \prod_{k=1}^{n} (z_k x + w_k y) = h_0 x^n + \ldots + h_n y^n.$$

The coefficients of this form are the (homogeneous) coordinates in $\mathbb{C}P^n = \mathrm{Sym}^n(\mathbb{C}P^1)$ (defining its smooth holomorphic structure).

If $w_k \neq 0$, we may choose $w_k = 1$. In this case we get as affine coordinates the basic symmetric functions of $\{z_k\}$,

$$h_0 = \sigma_n(z), \ldots, h_{n-1} = \sigma_1(z) \quad (h_n = 1).$$

Now we shall use the same σ_k as *local* coordinates on $\mathrm{Sym}^n(\mathbb{R}P^2)$.

We start from n pairs of hermitian-orthogonal lines in $\mathbb{C}^2$. We choose a representative of each pair in such a way that no line chosen coincides with a nonchosen one (at a given point and hence at some neighbourhood of it, where our local coordinate system will work). It suffices to choose always the same line whenever a pair is repeated several times to fulfill the above nondegeneracy condition.

Let $(z_k, 1)$ be the vectors defining the chosen lines ($k = 1, \ldots, n$). We shall use as local coordinates on the real variety $\mathrm{Sym}^n(\mathbb{R}P^2)$ the (real and imaginary parts of the) n complex numbers

$$\sigma_1(z_1, \ldots, z_n), \ldots, \sigma_n(z_1, \ldots, z_n).$$

The orthogonal lines are defined by the vectors $(1, -\overline{z}_k)$. We construct the symmetrizing $2n$-form

$$f = \prod_{k=1}^{n}(z_k x + y)\prod_{k=1}^{n}(x - \overline{z}_k y) = f_0 x^{2n} + \ldots + f_{2n} y^{2n}.$$

We shall see that the coefficients f_k are polynomials in σ and $\overline{\sigma}$.

Theorem 7. *The mapping $F : \mathbb{C}^n \to \mathbb{R}^{2n+1}$ sending $(\sigma_1, \ldots, \sigma_{2n})$ to $(f_0, \ldots, f_n)$ defines a* (*local*) *diffeomorphism of the manifold* $\mathrm{Sym}^n(\mathbb{R}P^2)$ *to the space $\mathbb{R}P^{2n}$ of rays in $\mathbb{R}^{2n+1}$. In coordinates the mapping F is given by the following explicit formulas:*

$$\begin{aligned}
&f_0 = \sigma_n,\\
&f_1 = \sigma_{n-1} - \sigma_n\overline{\sigma}_1,\\
&f_2 = \sigma_{n-2} - \sigma_{n-1}\overline{\sigma}_1 + \sigma_{n-2}\overline{\sigma}_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&f_n = 1 - \sigma_1\overline{\sigma}_1 + \sigma_2\overline{\sigma}_2 - \ldots + (-1)^n\sigma_n\overline{\sigma}_n.
\end{aligned}$$

Proof. Evidently, we find

$$\begin{aligned}
&\prod_{k=1}^{n}(z_k x + y) = \sigma_n x^n + \sigma_{n-1}x^{n-1}y + \ldots + y^n,\\
&\prod_{k=1}^{n}(x - \overline{z}_k y) = x^n - \overline{\sigma}_1 x^{n-1} y + \overline{\sigma}_2 x^{n-2} y^2 + \ldots + (-1)^n \overline{\sigma}_n y^n.
\end{aligned}$$

Multiplying these two polynomials, we obtain (following F. Aicardi) the above formulas for the coefficients of the product.

We also get $f_{2n-k} = (-1)^{n-k}\overline{f}_k$, in particular, the middle coefficient f_n is real.

It remains to prove that the Jacobian nowhere vanishes in our domain. This can be seen with no computations. The Jacobian that we need is of order $2n+1$. One of the columns is the vector Φ whose components are

$$(f_0, \overline{f}_0, f_1, \overline{f}_1, \ldots, f_{n-1}, \overline{f}_{n-1}, f_n).$$

The other $2n$ columns are its derivatives

$$(\partial\Phi/\partial\sigma_1, \partial\Phi/\partial\overline{\sigma}_1, \ldots, \partial\Phi/\partial\sigma_n, \partial\Phi/\partial\overline{\sigma}_n).$$

We represent this nonholomorphic Jacobian as the value at the point $\tau = \overline{\sigma}$ of a holomorphic Jacobian $\mathcal{T}(\sigma, \tau)$, defined by the following construction.

Consider the product

$$\prod_{k=1}^{n}(z_k x + y)\prod_{k=1}^{n}(x - w_k y) = F_0 x^n + \ldots + F_{2n} y^{2n}.$$

The coefficients F_k are some polynomials in $\sigma_1(z), \ldots, \sigma_n(z)$ and $\tau_1 = \sigma_1(w), \ldots,$ $\tau_n = \sigma_n(w)$. Denote by $\Psi = (F_0, \ldots, F_{2n})$ the vector-function depending on σ and τ, and construct the determinant with the columns

$$(\Psi, \partial\Psi/\partial\sigma_1, \partial\Psi/\partial\tau_1, \ldots, \partial\Psi/\partial\sigma_n, \partial\Psi/\partial\tau_n).$$

This determinant $\mathcal{T}(\sigma, \tau)$ does not vanish at the point $\sigma(z)$, $\tau = \sigma(\overline{z})$. Indeed, our noncoincidence condition implies that no z_k coincides with $-1/\overline{z}_l$. Hence, $(\sigma_1, \ldots, \sigma_n)$ and $(\tau_1, \ldots, \tau_n)$ form a local holomorphic coordinate system in $\mathbb{C}P^{2n} = \mathrm{Sym}^{2n}(\mathbb{C}P^1)$ at our point $\sigma, \tau = \overline{\sigma}$ and in its neighbourhood. Thus, this large determinant $\mathcal{T}(\sigma, \tau)$ does not vanish at our point. But the determinant we need to evaluate coincides with the value of $\mathcal{T}(\sigma, \overline{\sigma})$, since $f_{2n-k} = (-1)^{n-k}\overline{f}_k$. We have thus proved that our mapping $\mathrm{Sym}^n(\mathbb{R}P^2) \to \mathbb{R}P^{2n}$ is a local diffeomorphism.

We already know from Section 2 that it is a homeomorphism. Theorem 2 is now proved completely.

5. Maxwell theorem and $\mathbb{C}P^2/\mathrm{conj} = S^4$

The explicit formulas of Section 4 provide also a diffeomorphism to S^{2n} of the following orbit space.

Start from the complex manifold $(\mathbb{C}P^1)^n$ of ordered sets of n lines in $\mathbb{C}^2$. Consider the following smooth (nonholomorphic) action of the Coxeter group $D(n)$ on this manifold. An element of $D(n)$ is a permutation of the factors accompanied by the replacement of an *even* number of lines by their hermitian orthocomplements.

Theorem 8. *$(\mathbb{C}P^1)^n/D(n) \approx S^{2n}$, and the diffeomorphism is locally defined by the formulas of Theorem* 7.

Proof. The permutations do not change the binary $2n$-form f. The replacement of a line by the complementary line changes the sign of the corresponding factor $(z_k x + y)(x - \overline{z}_k y)$. Hence an even number of replacements does not change f (while an odd number of replacements changes f to $-f$).

The relation $(\mathbb{C}P^1)^n/D(n) \approx S^{2n}$ that we have thus proved seems interesting as an informal extension of the Chevalley theorem: the orbit space of the action of a real $2n-1$-dimensional group in $\mathbb{C}^{2n}$ (which should be thus considered as a generalization of a Coxeter group) is the smooth real space $\mathbb{R}^{2n+1}$.

Example. For $n = 2$ we get

$$(\mathbb{C}P^1)^2/D(2) \approx S^4,$$

where $D(2)$ is the group of 4 elements, acting on the couples of lines in $\mathbb{C}^2$, permuting them and (virtually) replacing both lines by their hermitian complements.

But $(\mathbb{C}P^1)^2/S(2) \approx \mathrm{Sym}^2(\mathbb{C}P^1) = \mathbb{C}P^2$. Hence $(\mathbb{C}P^1)^2/D(2) = \mathbb{C}P^2/(j)$, where (j) is the replacement of both lines by their complements.

Consider the complex manifold $\mathrm{Sym}^n(\mathbb{C}P^1) = \mathbb{C}P^n$ of unordered sets of n lines in $\mathbb{C}^2$. The operation j of replacement of each line by its hermitian orthocomplementary line acts on $\mathbb{C}P^n$ as an (antiholomorphic) involution.

Theorem 9. *For even n, the involution $j : \mathbb{C}P^n \to \mathbb{C}P^n$ coincides with the complex conjugation* (*in some coordinates).*

Proof. The natural (homogeneous) coordinates in $\mathbb{C}P^n = \mathrm{Sym}^n(\mathbb{C}P^1)$ are the coefficients of the form

$$H_{z,w}(x,y) = \prod_{k=1}^{n}(z_k x + w_k y) = \sum_{k=0}^{n} h_k x^{n-k} y^k.$$

We now compute the action of j on the coefficients h_k.

We transport the action of j to the dual plane:

$$\overline{w}_k x - \overline{z}_k y = \overline{z_k(-\overline{y}) + w_k(\overline{x})}.$$

Hence the transformed form is

$$H_{\overline{w},-\overline{z}}(x,y) = \overline{H_{z,w}(-\overline{y},\overline{x})}.$$

In terms of the coefficients we get the following expression for the transformed form:

$$\overline{\sum_{k=0}^{n} h_k(-\overline{y})^{n-k}(\overline{x})^k} = \sum_{k=0}^{n}(-1)^{n-k}\overline{h}_k x^k y^{n-k}.$$

Hence j acts on the coefficients of H as $(jh)_k = (-1)^k \overline{h}_{n-k}$. Since n is even, we also find $(jh)_{n-k} = (-1)^k \overline{h}_k$. Now the required coordinates are $h_k + h_{n-k}$ and $i(h_k - h_{n-k})$ for even k, $(h_k + h_{n-k})$ and $h_k - h_{n-k}$ for odd k (we never take $h_{n/2} - h_{n/2}$, of course).

Finally, for $n = 2$, our results reduce to an explicit formula for the classical diffeomorphism $\mathbb{C}P^2/\mathrm{conj} \approx S^4$. The "Maxwell theorem"

$$(\mathbb{C}P^1)^n/D(n) \approx S^{2n}$$

extends this diffeomorphism to higher dimensions.

6. The history of the Maxwell theorem

Maxwell's own version can be found in his main book *Electricity and Magnetism*, vol. I, chapter IX, n. 129–133 (pages 222–233 in [5]).

Sylvester criticized his reasoning, saying in [7]: "I am a little surprised that this distinguished writer should not have noticed that there is always one, and only one, *real* system of poles appertaining to any given harmonic...

With all possible respect for Professor Maxwell's great ability, I must own that to deduce purely analytical properties of spherical harmonics, as he has done, from "Green's theorem" and the "principle of potential energy"... seems to me a proceeding at variance with sound method, and of the same kind and as reasonable as if one should... make the rule for the extraction of the square root flow as a consequence from Archimedes' law of floating bodies."

Sylvester proposed his own approach, perhaps equivalent to that of Theorem 5 above:

"The method of poles for representing spherico-harmonics, devised or developed by Professor Maxwell, really amounts to neither more nor less than the choice of an apt canonical form for a ternary quantic, subject to the condition that the sum of the squares of its variables (here differential operators) is zero."

However, he had not taken the trouble to work out the subject in detail ("being very much pressed for time and within twenty-four hours of steaming back to Baltimore"). The details appeared in [6], and later in [3].

Sylvester mentioned the relation of his theory of integrals of products of spherical harmonics to the Ivory theorem on the ellipsoid attraction and proposed several generalizations of these ideas.

It seems that both the algebraic and the philosophical ideas of that *Note* of Sylvester were never understood or developed by the mathematical community. The pages containing the note were not cut in the copy of his *Collected Papers* in the library of the Paris Ecole Normale Supérieure.

The note contains the following (anti-bourbakist) paragraph:

"It is by no means uncommon in mathematical investigation... for the part to be in a sense greater than the whole—the groundwork of this wonder-striking intellectual phenomenon being that, for mathematical purposes, all quantities and relations ought to be considered (so experience teaches) as in a state of flux."

This general philosophy leads him to the conclusion that "... the general proposition should be more easily demonstrable than any special case of it".

We can deduce from the way Sylvester arrived at this conclusion that the truth of this very important principle of Sylvester (borrowed almost hundred years later by Bourbaki) does not imply the necessity of the disastrous petrification of the "flux" of mathematics.

References

[1] V. Arnold, *Ramified covering $\mathbb{C}P^2 \to S^4$, hyperbolicity and projective topology*, Sibirsk. Mat. Zh. **29** (1988), no. 5, 36–47 (Russian); English transl., Siberian Math. J. **29** (1988), 717–726.

[2] ———, *On the disposition of ovals of real plane algebraic curves, involutions of four-dimensional smooth manifolds and arithmetic of integer quadratic forms*, Funktsional. Anal. i Prilozhen. **5** (1971), no. 3, 1–9 (Russian); English transl., Funct. Anal. Appl. **5** (1971), 169–176.

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Interscience, New York, 1953, ch. 7, §5, pp. 514–521.

[4] J. L. Dupont and G. Lusztig, *On manifolds satisfying $w_1^2 = 0$*, Topology **10** (1971), 81–92.

[5] J. C. Maxwell, *Traité d'Electricité et de Magnétisme*, T. 1, Gauthier-Villars, Paris, 1885.

[6] A. Ostrowski, *Die Maxwellsche Erzeugung der Kugelfunktionen*, Jahresber. Deutsch. Math.-Verein. **33** (1925), 245–251.

[7] J. J. Sylvester, *Note on spherical harmonics*, Philosophical Magazine, II (1876), 291–307, The Collected Mathematical Papers of J. J. Sylvester, vol. III, Cambridge, 1909, pp. 37–51.

Manuscript received April 24, 1996

CEREMADE
Université Paris-Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16, FRANCE
and
Steklov Mathematical Institute
42, Vavilova Street
117966 Moscow, GSP-1, RUSSIA

E-mail address: arnold@paris9.dauphine.fr