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FIXED POINTS OF MULTIVALUED MAPPINGS IN CERTAIN CONVEX METRIC SPACES

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1. Introduction

Takahashi [10] introduced a notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. Let X be a metric space and $I = [0,1]$. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

X together with a convex structure W is called a *convex metric* space.

Recently, Shimizu and Takahashi [9] proved the following result:

Let X be a bounded convex metric space and let T be a multivalued nonexpansive mapping of X into itself such that $T(x)$ is a nonempty compact set for each $x \in X$. Then T has the almost fixed point property in X, i.e.,

$$\inf_{x \in X} d(x, Tx) = 0.$$

In 1974, Lim [5] showed a fixed point theorem for multivalued nonexpansive mappings in uniformly convex Banach spaces. After that, Goebel [2] gave a simpler proof of Lim's theorem using the notion of regular sequences. On the other hand, in 1980, Goebel, Sękowski and Stachura [4] studied hyperbolic

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metric spaces. They showed that a hyperbolic metric is, in some sense, uniformly convex, and showed fixed point theorems for single-valued nonexpansive mappings.

In this paper, we introduce a notion of uniform convexity in convex metric spaces and prove a fixed point theorem for multivalued nonexpansive mappings in such spaces by applying ultrafilters, without using the notion of regular sequences. This is a generalization of Lim's result [5] and the proof is simpler than that of [5].

2. Preliminaries

Let X be a nonempty set. A nonempty family $\mathcal{F}$ of subsets of X is called a *filter* on X if it has the following properties: (1) $\emptyset \notin \mathcal{F}$; (2) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$; (3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on X with $\mathcal{F}_1 \subset \mathcal{F}_2$, then we say that $\mathcal{F}_2$ is *finer* than $\mathcal{F}_1$. A filter $\mathcal{U}$ on X is called an *ultrafilter* if there is no filter on X which is strictly finer than $\mathcal{U}$. A nonempty class $\mathcal{B}$ of subsets of X is called a *filterbase* on X if it has the following properties: (1) $\emptyset \notin \mathcal{B}$; (2) for any A_1 and A_2 in $\mathcal{B}$, there exists A_3 in $\mathcal{B}$ such that $A_3 \subset A_1 \cap A_2$. If $\mathcal{B}$ is a filterbase on X, then

$$\mathcal{F} = \{A \subset X : B \subset A,\ B \in \mathcal{B}\}$$

is a filter on X. In this case, $\mathcal{B}$ is said to be a *base* of $\mathcal{F}$ or to *generate* $\mathcal{F}$. Let X be a topological space and let $\mathcal{B}$ be a filterbase on X. Then $\mathcal{B}$ is said to converge to a point x in X or to have a limit x in X if for any neighbourhood V of x, there is a set A in $\mathcal{B}$ such that $A \subset V$. If $\mathcal{U}$ is an ultrafilter on a compact set X, then $\mathcal{U}$ has a limit in X. Let $\mathcal{U}$ be an ultrafilter on a set X and P be a mapping of X into a set D. Then $P(\mathcal{U})$ is a filterbase on D and it generates an ultrafilter on D. In fact, it is obvious that since $\mathcal{U}$ is an ultrafilter on X, then $P(\mathcal{U})$ is a filterbase on D. Let

$$\mathcal{B} = \{B \subset D : P(A) \subset B \text{ for some } A \in \mathcal{U}\}$$

and let $\mathcal{K}$ be a filter on D with $\mathcal{K} \supset \mathcal{B}$. If $K \in \mathcal{K}$, then $P^{-1}K \in \mathcal{U}$ or $P^{-1}K^{\mathrm{c}} \in \mathcal{U}$, where K^{c} is the complement of K. Suppose $A = P^{-1}K^{\mathrm{c}} \in \mathcal{U}$. Then $P(A) = P(P^{-1}K^{\mathrm{c}}) \subset K^{\mathrm{c}}$ and hence $K^{\mathrm{c}} \in \mathcal{B}$. This is a contradiction. So, $P^{-1}K \in \mathcal{U}$. Since $P(P^{-1}K) \subset K$, we have $K \in \mathcal{B}$ and hence $\mathcal{K} = \mathcal{B}$. This implies that $\mathcal{B}$ is an ultrafilter on D; for details, see [1, 8].

Let X be a convex metric space. A nonempty subset $K \subset X$ is *convex* if $W(x, y, \lambda) \in K$ whenever $(x, y, \lambda) \in K \times K \times I$. Takahashi [10] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(x, y) \le r\}$ are convex. Also, if $\{K_\alpha : \alpha \in A\}$ is a family of convex subsets of X, then $\bigcap\{K_\alpha : \alpha \in A\}$ is convex. For $A \subset X$, we denote by $\overline{\mathrm{co}}\, A$ the intersection of all closed convex sets containing A and by $\delta(A)$ the diameter of A. A convex metric space X is said to have the *property* (C) if every decreasing sequence of nonempty bounded closed convex subsets of X has nonempty intersection.

Let X be a convex metric space and let $\mathcal{B}$ be a filterbase on X which contains at least one bounded subset of X. Then we define

$$r(x,\mathcal{B}) = \inf_{A\in\mathcal{B}} \sup_{y\in A} d(x,y) = \lim_{A\in\mathcal{B}} \sup_{y\in A} d(x,y)$$

for every $x \in X$. Since for every $x, y \in X$, $|r(x,\mathcal{B}) - r(y,\mathcal{B})| \le d(x,y)$, the real-valued function $r(\cdot,\mathcal{B})$ on X is continuous. Further, for any real number α, the set

$$C = \{z \in X : r(z,\mathcal{B}) \le \alpha\}$$

is convex. In fact, let $z_1, z_2 \in C$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} r(W(z_1,z_2,\lambda),\mathcal{B}) &= \inf_{A\in\mathcal{B}} \sup_{y\in A} d(W(z_1,z_2,\lambda),y) \le \lambda r(z_1,\mathcal{B}) + (1-\lambda) r(z_2,\mathcal{B}) \\ &\le \lambda\alpha + (1-\lambda)\alpha = \alpha \end{aligned}$$

and hence $W(z_1,z_2,\lambda) \in C$.

3. Uniformly convex metric spaces

A convex metric space X is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that, for all $r > 0$ and $x, y, z \in X$ with $d(z,x) \le r$, $d(z,y) \le r$ and $d(x,y) \ge r\varepsilon$,

$$d(z, W(x,y,1/2)) \le r(1-\alpha) < r.$$

Example 1. Uniformly convex Banach spaces are uniformly convex metric spaces.

Example 2. Let H be a Hilbert space and let X be a nonempty closed subset of $\{x \in H : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, then $(\alpha x + \beta y)/\|\alpha x + \beta y\| \in X$ and $\delta(X) \le \sqrt{2}/2$; see [7]. Let $d(x,y) = \cos^{-1}\{(x,y)\}$ for every $x, y \in X$, where $(\cdot,\cdot)$ is the inner product of H. When we define a convex structure W for (X,d) properly, it is easily seen that (X,d) becomes a complete and uniformly convex metric space.

Remark. The module of convexity of Banach spaces and Goebel, Sękowski and Stachura's δ in Theorem 1 of [4] are continuous functions, but we only assume the existence of a positive number α such that α is a function of ε. Goebel, Sękowski and Stachura's δ depends on γ and ε, but our α only depends on ε.

Theorem 1. *Let X be a complete and uniformly convex metric space. Then X has the property (C).*

Proof. Let $\{K_n\}$ be a decreasing sequence of nonempty bounded closed convex subsets of X. If $\delta(K_n) > 0$ for every positive integer n, then there exist $x, y \in K_n$ such that $d(x,y) \geq \delta(K_n)/2$. Since $d(z,x) \leq \delta(K_n)$, $d(z,y) \leq \delta(K_n)$ for all $z \in K_n$ and the space is uniformly convex, there exists $\alpha > 0$ such that

$$d(z, W(x,y,1/2)) \leq \delta(K_n)(1-\alpha) < \delta(K_n)$$

for all $z \in K_n$ and hence we obtain $u_n^1 \in K_n$ such that

$$d(z, u_n^1) \leq \delta(K_n)(1-\alpha)$$

for all $z \in K_n$. Let

$$K_n^1 = \{u_n^1, u_{n+1}^1, u_{n+2}^1, \dots\}.$$

Then it is obvious that $K_n^1 \neq \emptyset$ and $K_n^1 \supset K_{n+1}^1$ for every n. Suppose $\delta(K_n^1) > 0$ for every n. Then there exist $x, y \in K_n^1$ such that $d(x,y) \geq \delta(K_n^1)/2$. Put

$$B_n^1 = \bigcap_{k=0}^{\infty} B[u_{n+k}^1, \delta(K_n^1)].$$

Then $B_n^1 \supset \overline{\text{co}}(K_n^1)$ and $d(z,x) \leq \delta(K_n^1)$, $d(z,y) \leq \delta(K_n^1)$ for every $z \in \overline{\text{co}}\, K_n^1$. Since X is uniformly convex, there exists $u_n^2 \in \overline{\text{co}}\, K_n^1 \subset K_n$ such that

$$d(z, u_n^2) \leq \delta(K_n)(1-\alpha)^2$$

for all $z \in \overline{\text{co}}\, K_n^1$. By the same method, we obtain $\overline{\text{co}}\, K_n^2, \overline{\text{co}}\, K_n^3, \dots$ and $u_n^3, u_n^4, \dots$ It is obvious that

$$K_n \supset \overline{\text{co}}\, K_n^1 \supset \overline{\text{co}}\, K_n^2 \supset \dots \quad \text{and} \quad \delta(\overline{\text{co}}\, K_n^m) \to 0$$

as $m \to \infty$. Since X is complete, there exists $u_n \in X$ such that

$$\bigcap_{m=1}^{\infty} \overline{\text{co}}\, K_n^m = \{u_n\}$$

for every n. From

$$\bigcap_{m=1}^{\infty} \overline{\text{co}}\, K_n^m \supset \bigcap_{m=1}^{\infty} \overline{\text{co}}\, K_{n+1}^m,$$

we obtain $u_1 = u_2 = u_3 = \dots$ Therefore, there exists u with $u \in K_n$ for all n and hence $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

LEMMA. *Let X be a complete and uniformly convex metric space. Let K be a nonempty closed convex subset of X. If $\mathcal{F}$ is a filter on X which contains at least a bounded subset of X, then there exists a unique point $u_0 \in K$ such that*

$$r(u_0, \mathcal{F}) = \inf_{x \in K} r(x, \mathcal{F}).$$

PROOF. Let $r = \inf_{x \in K} r(x, \mathcal{F})$ and define

$$K_n = \{z \in K : r(z, \mathcal{F}) \leq r + 1/n\}$$

for every positive integer n. Then it is obvious that K_n is nonempty, closed and convex. Further, K_n is bounded. In fact, let $u, v \in K_n$. Then there exists $A \in \mathcal{F}$ such that

$$\sup_{y \in A} d(u, y) < r + 2/n \quad \text{and} \quad \sup_{y \in A} d(v, y) < r + 2/n.$$

So, we have

$$d(u, v) \leq \sup_{y \in A} d(u, y) + \sup_{y \in A} d(v, y) < 2(r + 2/n).$$

Since $\{K_n\}$ is a bounded decreasing sequence of nonempty closed convex subsets of K, we have

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Further, we prove that $\bigcap_{n=1}^{\infty} K_n$ consists of one point. Let $x, y \in \bigcap_{n=1}^{\infty} K_n$. If $r = 0$, then $d(x, y) < 4/n$ for every positive integer n. Hence $x = y$. In the case of $r > 0$, suppose $x \neq y$. Then, for a fixed positive number b, there exists a positive number ε such that

$$d(x, y) \geq (r + a)\varepsilon$$

for every $a \in [0, b]$. We can also choose $a_0 \in (0, b)$ such that

$$(r + a_0)(1 - \alpha(\varepsilon)) < r.$$

Then there exists $A \in \mathcal{F}$ such that

$$\sup_{z \in A} d(x, z) < r + a_0 \quad \text{and} \quad \sup_{z \in A} d(y, z) < r + a_0.$$

Since X is uniformly convex, we have

$$d(z, W(x, y, 1/2)) \leq (r + a_0)(1 - \alpha(\varepsilon)) < r$$

for every $z \in A$. This implies

$$\sup_{z \in A} d(z, W(x, y, 1/2)) \leq (r + a_0)(1 - \alpha(\varepsilon)) < r$$

and hence $r(W(x, y, 1/2), \mathcal{F}) < r$. This is a contradiction, because $W(x, y, 1/2) \in K$. Therefore we have $x = y$.

4. Fixed point theorem

Let X be a metric space. Then, for $x \in X$ and $A \subset X$, we define $d(x,A) = \inf\{d(x,y) : y \in A\}$. Let $BC(X)$ be the family of all nonempty bounded closed subsets of X. Then a mapping T of X into $BC(X)$ is said to be *nonexpansive* if

$$H(Tx,Ty) \le d(x,y) \qquad \text{for every } x,y \in X,$$

where H is the Hausdorff metric with respect to d, i.e.,

$$H(A,B) = \max\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\}$$

for every $A,B \in BC(X)$. Now, we can prove a fixed point theorem for multivalued nonexpansive mappings in uniformly convex metric spaces.

Theorem 2. *Let X be a bounded, complete and uniformly convex metric space. If T is a multivalued nonexpansive mapping which assigns to each point of X a nonempty compact subset of X, then T has a fixed point in X.*

Proof. By Theorem 1 of [9], there exists a sequence $\{x_n\}$ in X such that $d(x_n,Tx_n) \to 0$ as $n \to \infty$. For every positive integer n, define

$$A_n = \{x_n, x_{n+1}, \dots\}.$$

Then $\{A_n\}$ is a filterbase on X and generates a filter $\mathcal{F}$ on X. From [1, 8], we know that there is an ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$. Clearly we have

$$\inf_{A\in\mathcal{U}} \sup_{x\in A} d(x,Tx) = 0.$$

By the Lemma, there exists a unique element $u_0 \in X$ such that

$$r(u_0,\mathcal{U}) = \inf_{x\in X} r(x,\mathcal{U}).$$

Since for each $x \in X$, Tx is nonempty and compact, we obtain elements $Sx \in Tx$ and $Px \in Tu_0$ such that

$$d(x,Sx) = d(x,Tx) \quad \text{and} \quad d(Sx,Px) = d(Sx,Tu_0).$$

Thus, we have got a mapping $P : X \to Tu_0$. We know that $P(\mathcal{U})$ is a filterbase on Tu_0 and the filter generated by $P(\mathcal{U})$ is an ultrafilter on Tu_0. Since

Tu_0 is compact, $P(\mathcal{U})$ has a limit p_0 in Tu_0. So, we have

$$\begin{aligned} r(p_0,\mathcal{U}) &= \inf_{A\in\mathcal{U}} \sup_{x\in A} d(p_0,x) \le \inf_{A\in\mathcal{U}} \sup_{x\in A}\{d(p_0,Px)+d(Px,Sx)+d(Sx,x)\} \\ &= \inf_{A\in\mathcal{U}} \sup_{x\in A}\{d(p_0,Px)+d(Sx,Tu_0)+d(x,Tx)\} \\ &\le \inf_{A\in\mathcal{U}} \sup_{x\in A}\{d(p_0,Px)+H(Tx,Tu_0)+d(x,Tx)\} \\ &\le \inf_{A\in\mathcal{U}} \sup_{x\in A}\{d(p_0,Px)+d(x,u_0)+d(x,Tx)\} \\ &= \inf_{A\in\mathcal{U}} \sup_{x\in A} d(x,u_0) = r(u_0,\mathcal{U}). \end{aligned}$$

By the Lemma, we have $u_0 = p_0 \in Tu_0$. This completes the proof.

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