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ON PARABOLIC QUASI-VARIATIONAL INEQUALITIES AND STATE-DEPENDENT SWEEPING PROCESSES

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1. Introduction

In this paper we consider the evolution problems

$$-u'(t) \in N_{C(t,u(t))}(u(t)) \quad \text{a.e. in } [0,T],\ u(0) = u_0 \in C(0,u_0), \tag{1.1}$$

in a Hilbert space H. We assume that

$$C(t,u) \subset H \text{ is nonempty, closed, and convex for } t \in [0,T],\ u \in H. \tag{1.2}$$

In (1.1), $N_{C(t,u)}(x)$ denotes the normal cone to $C(t,u)$ at $x \in C(t,u)$, cf. Section 2 below. We will treat the case of $(t,u) \mapsto C(t,u)$ being Lipschitz continuous w.r. to the Hausdorff distance d_H with constants $L_1, L_2 \geq 0$, i.e., we require

$$\mathrm{d}_H(C(t,u), C(s,v)) \leq L_1|t-s| + L_2|u-v|, \quad t,s \in [0,T],\ u,v \in H. \tag{1.3}$$

Note that a solution of (1.1) in particular has to satisfy $u(t) \in C(t,u(t))$ for $t \in [0,T]$.

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Problems of type (1.1) are generalizations of Moreau's classical sweeping process $-u'(t) \in N_{C(t)}(u(t))$ (cf. e.g. [8] and the references therein) with the moving convex set being additionally allowed to depend on the state u.

There is a special case of (1.1) which also deserves separate attention, namely parabolic quasi-variational inequalities of the form

$$\text{(1.4)} \quad \text{find } v(t) \in \Gamma(v(t)) : \langle v'(t) + f(t), w - v(t) \rangle \geq 0 \qquad \text{for all } w \in \Gamma(v(t)),\ v(0) = v_0 \in \Gamma(v_0),$$

where $v = v(t) : [0,T] \to H$, $f : [0,T] \to H$ is some inhomogeneity, and $\Gamma(v) \subset H$ is a set of constraints. Written somewhat differently, (1.4) means that

$$\text{(1.5)} \qquad -v'(t) \in N_{\Gamma(v(t))}(v(t)) + f(t) \quad \text{a.e. in } [0,T],\ v(0) = v_0 \in \Gamma(v_0).$$

Then, if v is a solution of (1.5), and if we define

$$\text{(1.6)} \quad u(t) = v(t) + \int_0^t f(s)\,ds \quad \text{and} \quad C(t,u) = \Gamma(u - \int_0^t f(s)\,ds) + \int_0^t f(s)\,ds,$$

it is found that u is a solution of (1.1), with initial value $u_0 = v_0 \in C(0, u_0)$. Thus indeed the quasi-variational inequalities (1.5) are particular cases of (1.1).

When dealing with (1.5), we shall always suppose that

$$\text{(1.7)} \qquad \Gamma(v) \subset H \text{ is nonempty, closed, and convex for } v \in H,$$

and that $v \mapsto C(v)$ is Lipschitz with constant $L \geq 0$, i.e.,

$$\text{(1.8)} \qquad \mathrm{d}_H(\Gamma(v), \Gamma(w)) \leq L|v - w|, \quad v, w \in H.$$

In case that $f \in L^\infty([0,T]; H)$ in (1.5) (which we will assume for simplicity, but $f \in L^1([0,T]; H)$ is sufficient), (1.8) implies that (1.3) holds for C defined by (1.6), with $L_1 = (L+1)|f|_{L^\infty([0,T];H)}$ and $L_2 = L$. This might be a bad estimate for L_1, but it will turn out that it is only the size of L_2 which determines the existence of a solution to (1.1).

There are several concrete examples where state-dependent sweeping processes as (1.1) or parabolic quasi-variational inequalities of type (1.5) yield the correct mathematical description of the underlying practical problem. State--dependent sweeping processes of type (1.1) occur, for instance, in the treatment of 2-D or 3-D quasistatical evolution problems with friction, as treated in [5, Chapter II, III] (see also the account given in [8, pp. 155–161]). In a differerent context, the state-dependent sweeping process is used in micromechanical damage models (the so-called Gurson-models) for iron materials with memory to describe the evolution of the plastic strain in presence of small damages; cf. [10], [3]. Examples of evolutionary quasi-variational inequalities may be found in [1] and the references therein, cf. in particular p. 242 f.

Our results are as follows. We will see in Example 3.1 below that there might be no solution of (1.5) in case that $L > 1$ in (1.8), and hence the same has to be said for (1.1), when $L_2 > 1$ in (1.3). In addition, Example 3.2 will show that also the solutions to both problems need not be unique (although we don't know this for (1.1) with $L_2 < 1$). This non-existence and non-uniqueness is in contrast to the situation in the classical state-independent sweeping process $-u'(t) \in N_{C(t)}(u(t))$, cf. [8], where unique solutions exist if $t \mapsto C(t)$ is Lipschitz continuous, no matter what is the Lipschitz constant.

Next, we prove in Theorem 3.3 that in fact (1.1) has a solution if $L_2 < 1$, and this transfers to (1.5) in case that $L < 1$ in (1.8), cf. Theorem 3.5. Then we will see in the one-dimensional Example 3.6 that one may have no solution to (1.1) for $L_2 = 1$ in (1.3), and the two-dimensional Example 3.7 shows that also (1.5) may fail to have solutions for $L = 1$ in (1.8). Afterwards we point out a difference between (1.1) and (1.5), because in Theorem 3.8 we obtain the existence of a solution for (1.5) with $L = 1$ in the one-dimensional setting. Thus we have clarified all cases of $L_2 \geq 0$ in (1.3) resp. $L \geq 0$ in (1.8).

Some results about a special problem of type (1.1) can be found in [5], where a fixed-point technique is used, together with a semi-implicit discretization for numerical purposes, the latter without proof. By semi-implicit discretization, we mean an iteration scheme $u_i^n = \mathrm{proj}(u_{i-1}^n, C(t_i^n, u_{i-1}^n))$ instead of (3.12) below. It appears that an approach like this still requires a compactness assumption in the infinite-dimensional case. Also in [10] the existence issue will be treated for $L_2 < 1$, with a different assumption on a certain retraction function of C, and the method used there relies on introducing an artificial time delay which is sent to zero to obtain finally a solution; cf. e.g. [4] for the method.

We start with some preliminaries before going on to the main subject in Section 3.

2. Preliminaries

We will always consider a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, norm $|\cdot|$, and closed balls $\overline{B}_r(x_0) = \{x \in H : |x - x_0| \leq r\}$ for $r > 0$ and $x_0 \in H$. For a closed convex $C \subset H$, the set $N_C(x) = \partial\delta_C(x) = \{y \in H : \langle y, c - x\rangle \leq 0 \text{ for all } c \in C\}$, $x \in C$, denotes the normal cone to C at x. Also, the Hausdorff distance between $C_1, C_2 \subset H$ is

$$\mathrm{d}_H(C_1, C_2) = \max\{\sup_{x \in C_2} \mathrm{dist}(x, C_1), \sup_{x \in C_1} \mathrm{dist}(x, C_2)\}$$

with $\mathrm{dist}(x, C_1) = \inf\{|x - y| : y \in C_1\}$. Finally, for a closed convex $C \subset H$ and $x \in H$, $y = \mathrm{proj}(x, C)$ will be the unique element of H such that $|y - x| = \mathrm{dist}(x, C)$.

Solutions of (1.1) (resp. (1.5)) are always understood to be Lipschitz continuous functions such that (1.1) (resp. (1.5)) holds.

The next lemma will be used below to establish the existence of an implicit discretization scheme for (1.1). There we will need the condition

$$\gamma(C(t,A)\cap \overline{B}_R(0)) < \gamma(A) \quad \text{for } t\in[0,T], \qquad A\subset H \text{ bounded with } \gamma(A)>0,\ R>0,$$

where $\gamma=\alpha$ or $\gamma=\beta$ is either the Kuratowski or the Hausdorff (ball-)measure of noncompactness, cf. [6] or [11]. Moreover, $C(t,A)=\bigcup_{u\in A}C(t,u)$. Thus (2.9) holds in particular in case that $C(t,A)\cap\overline{B}_R(0)\subset H$ is relatively compact for all bounded $A\subset H$ and $R>0$, and this means that condition (2.9) is satisfied in case that $\dim H<\infty$. Another typical situation where (2.9) holds is the case of $H=L^2(\mathbb{R}^n)$ and $C(t,A)\subset H^1(\mathbb{R}^n)$ bounded for bounded $A\subset L^2(\mathbb{R}^n)$.

Lemma 2.1. *Let $(t,u)\mapsto C(t,u)$ be a multifunction satisfying* (1.2), (1.3) *with* $0\le L_2<1$, *and* (2.9). *If $t\in[0,T]$ and $u\in C(s,u)$ for some $s\in[0,T]$, then there exists $v\in H$ such that $v=\operatorname{proj}(u,C(t,v))$ and $|v-u|\le L_1|t-s|/(1-L_2)$.*

Proof. Let $r=L_1|s-t|/(1-L_2)$, $D=\overline{B}_r(u)$, and $Fv=\operatorname{proj}(u,C(t,v))$ for $v\in D$. Then by (1.3) for $v\in D$

$$\begin{aligned}|Fv-u| &= |\operatorname{proj}(u,C(t,v))-u| = \operatorname{dist}(u,C(t,v)) \le \mathrm{d}_H(C(s,u),C(t,v))\\ &\le L_1|s-t|+L_2|u-v|\le L_1|s-t|+L_2r=r,\end{aligned}$$

and hence $F(D)\subset D$. In addition, $Fv\in C(t,v)$ implies $F(A)\subset C(t,A)\cap\overline{B}_R(0)$ with $R=|u|+r$, and thus $\gamma(F(A))\le\gamma(C(t,A)\cap\overline{B}_R(0))<\gamma(A)$ for $A\subset D$ with $\gamma(A)>0$ by (2.9). Moreover, F is continuous as may be seen from (1.3) and a geometrical inequality of Moreau for projections, cf. [8, Proposition 4.7, p. 26]. Therefore F is a condensing self-map of D, and consequently has a fixed point in D by Darbo's theorem, cf. [6] or [11]. □

Remark 2.2. To establish the discretization scheme, it is necessary to solve the equation $v=\operatorname{proj}(u,C(t,v))$ w.r.t. to v for known u. This fixed point problem cannot be solved by means of e.g. Banach's fixed point theorem, since only $|\operatorname{proj}(u,C(t,v))-\operatorname{proj}(u,C(t,\overline{v}))|\cong|v-\overline{v}|^{1/2}$ by (1.3) and the inequality for projections mentioned above. Thus we have to impose an additional compactness assumption like (2.9) and to use a more sophisticated fixed-point theorem.

The next lemmas will also be needed later.

Lemma 2.3. *Let $z\in\mathbb{R}$ and $I\subset\mathbb{R}$ a nonempty closed interval. Then*

$$|\operatorname{proj}(u+z,I)-u|\le\max\{|z|,\operatorname{dist}(u,I)\}. \tag{2.10}$$

PROOF. We only consider the case of a compact interval $I = [a, b]$. Let $v = \mathrm{proj}(u + z, I)$. If $u + z \in I$, then $v = u + z$ and (2.10) holds. If $u + z \notin I$ but $v = \mathrm{proj}(u, I)$, then $|v - u| = \mathrm{dist}(u, I)$. Finally, if $u + z \notin I$ and $\mathrm{proj}(u + z, I) \neq \mathrm{proj}(u, I)$, then w.l.o.g. $u+z < a < u$ (the case $u+z > b > u$ is treated similarly) and $v = a$, so that $|v - u| = u - a < |z|$. □

LEMMA 2.4. *If $C : [0, T] \to 2^{\mathbb{R}}$ has nonempty closed convex values and is d_H-Lipschitz with constant $K \geq 0$, then for every $f \in L^\infty([0, T])$ the unique solution of*

$$-u'(t) \in N_{C(t)}(u(t)) + f(t) \quad \text{a.e. in } [0, T], \; u(0) = u_0 \in C(0), \tag{2.11}$$

is Lipschitz with constant $\max\{|f|_{L^\infty([0,T])}, K\}$.

PROOF. We assume $f \in C([0, T])$, the general case follows by approximation. The solution of (2.11) can be obtained through discretization, as a limit of (a subsequence) of the step functions

$$u_n(t) = u_i^n \quad \text{for } t \in [t_i^n, t_{i+1}^n],$$
$$\text{with } u_i^n = \mathrm{proj}(u_{i-1}^n - (t_i^n - t_{i-1}^n) f(t_i^n), C(t_i^n)) \in C(t_i^n),$$

cf. also the proof of Theorem 3.3 below. Hence, by Lemma 2.3,

$$\begin{aligned} |u_i^n - u_{i-1}^n| &= |\mathrm{proj}(u_{i-1}^n - (t_i^n - t_{i-1}^n) f(t_i^n), C(t_i^n)) - u_{i-1}^n| \\ &\leq \max\{(t_i^n - t_{i-1}^n)|f(t_i^n)|, \mathrm{dist}(u_{i-1}^n, C(t_i^n))\} \\ &\leq \max\{(t_i^n - t_{i-1}^n)|f|_{L^\infty([0,T])}, \mathrm{d}_H(C(t_{i-1}^n), C(t_i^n))\} \\ &\leq (t_i^n - t_{i-1}^n) \max\{|f|_{L^\infty([0,T])}, K\}. \end{aligned}$$

This estimate suffices to obtain in the limit that the solution u is Lipschitz with constant $\max\{|f|_{L^\infty([0,T])}, K\}$, cf. the proof of Theorem 3.3 which requires similar techniques. □

REMARK 2.5. It is a special feature of $H = \mathbb{R}$ that an estimate of the form $\max\{|f|_{L^\infty([0,T])}, K\}$ is possible for the solution to (2.11), mainly because there are "only two directions". Consider (2.11) e.g. in $H = \mathbb{R}^2$ with $C(t) = \{t\} \times [0, 1]$, $f(t) = (0, -1)$, and $u_0 = (0, 0)$. Then $K = 1$, so that $\max\{|f|_{L^\infty([0,T])}, K\} = 1$, but the unique solution $u(t) = (t, t)$ is Lipschitz with constant $\sqrt{2} > 1$. It is exactly this point that will allow a counterexample for $L = 1$ in $\dim H \geq 2$, whereas (1.5) with $L = 1$ in (1.8) has a solution in $H = \mathbb{R}$.

3. Existence and nonexistence of solutions

We start with a simple example showing that in general no solution of (1.5) can be expected for $L > 1$. Via (1.6), the same is true for (1.1).

Example 3.1. Let $L > 1$, $H = \mathbb{R}$, $\Gamma(v) = [Lv, 1]$ for $v \le 1/L$, and $\Gamma(v) = [1, Lv]$ for $v > 1/L$, $v \in \mathbb{R}$. Then $v \mapsto \Gamma(v)$ is d_H-Lipschitz with constant L. In addition, $v_0 := 0 \in \Gamma(0) = [0,1]$, but (1.5) with $f(t) = -1$, $t \in [0,1]$, has no solution. Indeed, assume $v : [0,1] \to H$ to be a solution. By continuity, $Lv(t) \le 1$ for $t \in [0,\delta]$ with a suitable $\delta > 0$, hence $v(t) \in \Gamma(v(t)) = [Lv(t), 1]$ and $L > 1$ imply $v(t) \le 0$ in $[0,\delta]$. On the other hand, for $x \le 0$,

$$N_{\Gamma(x)}(x) = \begin{cases}]-\infty, 0] & : x = 0, \\ \{0\} & : x < 0, \end{cases} \subset \,]-\infty, 0]\,,$$

so that (1.5) yields $v'(t) \ge -f(t) = 1$ in $[0,\delta]$, but this gives the contradiction $v(t) \ge t > 0 \ge v(t)$ in $]0,\delta]$. □

Our next example shows that if solutions to (1.5) exist, they need not be unique. Again this carries over to (1.1) by means of (1.6).

Example 3.2. Let $H = \mathbb{R}$, $\Gamma(v) = [v, 1]$ for $v \le 1$ resp. $\Gamma(v) = [1, v]$ for $v > 1$, $v \in \mathbb{R}$. Define $v_0 = 0$ and $f(t) = 0$, $t \in [0,1]$. Since $N_{[x,1]}(x) =]-\infty, 0]$ for $x < 1$ and $N_{\{x\}}(x) = \mathbb{R}$, all sufficiently regular functions $v : [0,1] \to \mathbb{R}$ satisfying $v(0) = 0$, $v(t) \le 1$, and $v'(t) \ge 0$ for $t \in [0,1]$ are solutions to (1.5). □

Next we give a positive result for (1.1) with $L_2 < 1$.

Theorem 3.3. *Let* (1.2), (1.3) *with some* $0 \le L_2 < 1$, *and* (2.9) *hold for* C. *If* $u_0 \in C(0, u_0)$, *then* (1.1) *has a solution on* $[0,T]$.

Proof. We discretize the problem as follows. For $n \in \mathbb{N}$ fix partitions $0 = t_0^n < t_1^n < \ldots < t_{I_n}^n = T$ of $[0,T]$ such that $\varepsilon_n = \max\{t_{i+1}^n - t_i^n : 0 \le i \le I_n - 1\} \to 0$ as $n \to \infty$. Let $u_0^n = u_0$. Since $u_0 \in C(0, u_0)$, by Lemma 2.1 we find u_1^n with $u_1^n = \mathrm{proj}(u_0^n, C(t_1^n, u_1^n))$ and additionally also $|u_1^n - u_0^n| \le L_1 t_1^n/(1-L_2)$. Thus in particular $u_1^n \in C(t_1^n, u_1^n)$, and hence Lemma 2.1 applies again to yield u_2^n such that $u_2^n = \mathrm{proj}(u_1^n, C(t_2^n, u_2^n))$ and $|u_2^n - u_1^n| \le L_1(t_2^n - t_1^n)/(1 - L_2)$. Iterating this procedure, for $i = 1, \ldots, I_n$ we find u_i^n with

$$u_i^n = \mathrm{proj}(u_{i-1}^n, C(t_i^n, u_i^n)) \quad \text{and} \quad |u_i^n - u_{i-1}^n| \le L_1(t_i^n - t_{i-1}^n)/(1 - L_2). \tag{3.12}$$

Thus inductively

$$|u_i^n| \le |u_0| + L_1 t_i^n/(1 - L_2) \le |u_0| + L_1 T/(1 - L_2), \quad n \in \mathbb{N},\ 0 \le i \le I_n. \tag{3.13}$$

Next define for $n \in \mathbb{N}$ the right-continuous step approximations $u_n : [0,T] \to H$ through

$$u_n(t) = u_n^i \quad \text{for } t \in \left[t_i^n, t_{i+1}^n\right[, \tag{3.14}$$

and $u_n(T) = u^n_{I_n}$. Then

$$\sup_{n\in\mathbb{N}} |u_n|_{L^\infty([0,T];H)} \le |u_0| + L_1T/(1-L_2) =: R \tag{3.15}$$

by (3.13). In addition,

$$\operatorname{var}(u_n) = \sum_{i=1}^{I_n} |u^n_i - u^n_{i-1}| \le L_1T/(1-L_2)$$

by (3.12), and thus also

$$\sup_{n\in\mathbb{N}} \operatorname{var}(u_n) \le L_1T/(1-L_2).$$

Thus we may extract a subsequence (for simplicity not relabeled) such that for some function $u : [0,T] \to H$ of bounded variation we have $u_n(t) \rightharpoonup u(t)$ in H for $t \in [0,T]$, cf. [8, Theorem. 2.1, p. 10], $\rightharpoonup$ denoting weak convergence. By (3.12) we also obtain

$$|u_n(t) - u_n(s)| \le \frac{L_1}{1-L_2}(|t-s| + \varepsilon_n), \quad t,s \in [0,T]. \tag{3.16}$$

Indeed, fix $s \in \left[t^n_i, t^n_{i+1}\right[$ and $t \in \left[t^n_j, t^n_{j+1}\right[$ with $j > i$. Then

$$\begin{aligned}
|u_n(t) - u_n(s)| = |u^n_j - u^n_i| &\le \sum_{k=0}^{j-i-1} |u^n_{i+k+1} - u^n_{i+k}| \\
&\le \frac{L_1}{1-L_2} \sum_{k=0}^{j-i-1} (t^n_{i+k+1} - t^n_{i+k}) \\
&= \frac{L_1}{1-L_2}(t^n_j - t^n_i) \le \frac{L_1}{1-L_2}(|t-s| + \varepsilon_n),
\end{aligned}$$

thus showing (3.16). From (3.16) and $u_n(t) \rightharpoonup u(t)$ we therefore find

$$|u(t) - u(s)| \le \liminf_{n\to\infty} |u_n(t) - u_n(s)| \le \frac{L_1}{1-L_2}|t-s|, \quad t,s \in [0,T],$$

i.e., u is Lipschitz continuous, and hence differentiable a.e.

We are going to show that u is a solution of (1.1). For this, we first note that $u(0) = u_0$, because $u_n(0) = u_0$ for $n \in \mathbb{N}$ and $u_n(0) \rightharpoonup u(0)$. Next we will verify that

$$u(t) \in C(t, u(t)), \quad t \in [0,T]. \tag{3.17}$$

By (3.12) and the definition (3.14) of u_n,

$$u_n(t) = u^n_i = \operatorname{proj}(u^n_{i-1}, C(t^n_i, u^n_i)) \in C(t^n_i, u^n_i) = C(t^n_i, u_n(t)), \quad t \in \left[t^n_i, t^n_{i+1}\right[,$$

and as a consequence of (1.3) we thus deduce

$$u_n(t) \in C(t, u_n(t)) + \overline{B}_{L_1\varepsilon_n}(0) \subset C(t, u(t)) + \overline{B}_{L_1\varepsilon_n + L_2|u_n(t)-u(t)|}(0), \tag{3.18}$$

for $t \in [0,T]$. We claim that $A(t) = \{u_n(t) : n \in \mathbb{N}\} \subset H$ is relatively compact for every $t \in [0,T]$. Suppose on the contrary that $\gamma(A(t)) > 0$. Then $\gamma(A(t)) - \gamma(C(t, A(t)) \cap \overline{B}_{R+1}(0)) \geq 2\delta > 0$ for some $\delta \in \,]0,1]$, according to (2.9), with R chosen from (3.15). Fix $n_0 \in \mathbb{N}$ such that $2L_1\varepsilon_n \leq \delta$ for $n \geq n_0$. Then the first inclusion in (3.18), the properties of γ, and (3.15) imply

$$\begin{aligned}\gamma(A(t)) = \gamma(\{u_n(t) : n \geq n_0\}) &\leq \gamma(C(t, A(t)) \cap \overline{B}_{R+1}(0)) + 2L_1\varepsilon_n \\ &\leq \gamma(A(t)) - 2\delta + \delta = \gamma(A(t)) - \delta,\end{aligned}$$

a contradiction. Therefore every $A(t)$ is relatively compact in H, and together with the weak convergence $u_n(t) \rightharpoonup u(t)$ this yields

$$u_n(t) \to u(t) \quad \text{strongly in } H \text{ for } t \in [0,T]. \tag{3.19}$$

As a consequence of (3.19) and the second inclusion in (3.18),

$$\operatorname{dist}(u_n(t), C(t, u(t))) \to 0 \quad \text{as } n \to \infty,$$

and thus the closedness of $C(t, u(t))$ and (3.19) yield (3.17).

Finally we have to show that the inclusion in (1.1) is satisfied. To see this, first note that

$$\langle u_i^n - u_{i-1}^n, u_i^n - x\rangle \leq 0 \quad \text{for } x \in C(t_i^n, u_i^n),\ n \in \mathbb{N},\ 1 \leq i \leq I_n \tag{3.20}$$

by (3.12) and the properties of a projection. Define the continuous approximations $v_n : [0,T] \to H$ through

$$v_n(t) = \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}\,(u_i^n - u_{i-1}^n) + u_{i-1}^n, \quad t \in [t_{i-1}^n, t_i^n].$$

Then $v_n(0) = u_0$ and

$$|v_n(t) - u_n(t)| \leq \frac{L_1}{1 - L_2}\varepsilon_n, \quad t \in [0,T],$$

due to (3.12); whence we have

$$v_n(t) \to u(t) \quad \text{strongly in } H \text{ for } t \in [0,T]$$

by (3.19). Moreover, v_n is differentiable a.e. with derivative

$$v_n'(t) = \frac{u_i^n - u_{i-1}^n}{t_i^n - t_{i-1}^n}, \quad t \in \,]t_{i-1}^n, t_i^n\,[\,,$$

and therefore

$$\sup_{n\in\mathbb{N}} |v_n'|_{L^\infty([0,T];H)} \leq \frac{L_1}{1 - L_2}$$

by (3.12). Hence in particular (w.l.o.g) $v_n' \rightharpoonup w$ in $L^2([0,T];H)$ for some $w \in L^2([0,T];H)$, and this implies $u' = w$ a.e., thus

$$v_n' \rightharpoonup u' \quad \text{in } L^2([0,T];H). \tag{3.21}$$

We claim that for $t \in [0,T] \setminus P$ where $P = \{t_i^n : n \in \mathbb{N}, 0 \le i \le I_n\}$, $n \in \mathbb{N}$, and $x \in C(t, u(t))$ we have

$$\langle v_n'(t), u_n(t) - x\rangle \le \frac{L_1}{1-L_2} r_n(t) \tag{3.22}$$

with

$$r_n(t) = \left(1 + \frac{2L_2}{1-L_2}\right) L_1 \varepsilon_n + L_2 |u_n(t) - u(t)|.$$

Indeed, choose i with $t \in \left]t_{i-1}^n, t_i^n\right[$. Then by (1.3) there exists $\widetilde{x} \in C(t_i^n, u_i^n)$ with

$$|\widetilde{x} - x| \le L_1\varepsilon_n + L_2|u_i^n - u(t)| = L_1\varepsilon_n + L_2|u_n(t_i^n) - u(t)| \le r_n(t),$$

the latter by the triangle inequality and (3.16). Hence as a consequence of (3.20) and (3.12),

$$\begin{aligned}\langle v_n'(t), u_n(t) - x\rangle &= \frac{1}{t_i^n - t_{i-1}^n}\langle u_i^n - u_{i-1}^n, u_{i-1}^n - x\rangle \\ &= \frac{1}{t_i^n - t_{i-1}^n}\langle u_i^n - u_{i-1}^n, u_i^n - \widetilde{x}\rangle + \frac{1}{t_i^n - t_{i-1}^n}\langle u_i^n - u_{i-1}^n, \widetilde{x} - x\rangle \\ &\quad - \frac{1}{t_i^n - t_{i-1}^n}\left|u_i^n - u_{i-1}^n\right|^2 \le \frac{L_1}{1-L_2} r_n(t),\end{aligned}$$

as claimed in (3.22). The estimate (3.22) can be used as follows. Choose $t_0 \in [0,T] \setminus P$ and $x_0 \in C(t_0, u(t_0))$. Fix a continuous selection $x : [0,T] \to H$ of $C(\,\cdot\,, u(\cdot))$ with $x(t_0) = x_0$. Then for $h > 0$ small by (3.22)

$$\int_{t_0-h}^{t_0+h} \langle v_n'(t), u_n(t) - x(t)\rangle\, dt \le \frac{L_1}{1-L_2}\int_0^T r_n(t)\, dt. \tag{3.23}$$

Because (3.19), (3.15), and Lebesgue's convergence theorem imply $u_n \to u$ in $L^1([0,T]; H)$, we have $r_n \to 0$ in $L^1([0,T])$. Since also $u_n \to u$ in $L^2([0,T]; H)$, (3.23) and (3.21) yield as $n \to \infty$

$$\int_{t_0-h}^{t_0+h} \langle u'(t), u(t) - x(t)\rangle\, dt \le 0$$

for h small. Dividing by $2h$ and letting h tend to zero, we therefore find

$$\langle u'(t_0), u(t_0) - x_0\rangle \le 0$$

for $t_0 \in [0,T]$ outside a fixed set of measure zero and for all x_0 in a countable dense subset of $C(t_0, u(t_0))$, hence for all $x_0 \in C(t_0, u(t_0))$. This concludes the proof that u is a solution of (1.1). □

It should be remarked that according to Lemma 2.1 and the above proof, (2.9) only needs to hold for R up to some sufficiently large R_0 determined through T, L_1, and $(1-L_2)^{-1}$, but in fact it is not necessary to assume (2.9) for all $R > 0$.

Theorem 3.3 also has a local version, in case that $C(t,u)$ is defined only in a neighbourhood of $(0,u_0)$.

Theorem 3.4. *Let $\delta > 0$ and $[0,\delta] \times \overline{B}_\delta(u_0) \ni (t,u) \mapsto C(t,u)$ be a multifunction with nonempty closed convex values such that* (1.3) *holds for*

$$(t,u),(s,v) \in [0,\delta] \times \overline{B}_\delta(u_0) \quad \text{with some } L_2 < 1,$$

and such that (2.9) *is satisfied for $t \in [0,\delta]$, $A \subset \overline{B}_\delta(u_0)$, and $R > 0$. If $u_0 \in C(0,u_0)$, then* (1.1) *has a local solution on some time interval $[0,T]$, with T depending on δ.*

Proof. Analogous to Theorem 3.3. □

According to the remarks in the introduction, Theorem 3.3 in particular yields an existence result for (1.5). Note that $\gamma(A+x) = \gamma(A)$ for $A \subset H$ bounded and $x \in H$, and thus condition (3.24) below implies (2.9) for $C(t,u) = \Gamma(u - \int_0^t f(s)\,ds) + \int_0^t f(s)\,ds$.

Theorem 3.5. *Let* (1.7) *be satisfied, $f \in L^\infty([0,T];H)$, and suppose that* (1.8) *holds with $0 \le L < 1$. If in addition*

$$\gamma(\Gamma(A)) < \gamma(A) \quad \text{for } A \subset H \text{ bounded with } \gamma(A) > 0, \tag{3.24}$$

where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff (*ball-*) *measure of noncompactness, and if $v_0 \in \Gamma(v_0)$, then* (1.5) *has a solution on* $[0,T]$.

According to the above results, the picture is clear now both for (1.1) and (1.5) in the cases $0 \le L_2 < 1$ and $L_2 > 1$ in (1.3) resp. $0 \le L < 1$ and $L > 1$ in (1.8). We next give a one-dimensional example showing that (1.1) may have no solution for $L_2 = 1$.

Example 3.6. Let $H = \mathbb{R}$ and $C(t,u) = [t+u,\infty[$ for $t \in [0,T]$ and $u \in \mathbb{R}$. Then $u_0 = 0 \in [0,\infty[= C(0,u_0)$, and (1.3) holds with $L_1 = L_2 = 1$. In this case (1.1) can have no solution, since this requires $u(t) \in C(t,u(t))$ for $t > 0$, meaning here that $t + u(t) \le u(t)$, a contradiction. □

In case of $\dim H \ge 2$, it is also possible to give a counterexample to the existence of solutions to (1.5) with $L = 1$ in (1.8). Obviously it is enough to find such an example for $\dim H = 2$.

Example 3.7. Let $H = \mathbb{R}^2$ and $\Gamma(u) = \{|u|\} \times [-1,1]$ for $u \in \mathbb{R}^2$, and $f(t) = (0,-1)$. Take $u_0 = (0,0) \in C(0)$, and notice that Γ is 1-Lipschitz. Suppose that (1.5) has a solution u on some interval $[0,T]$. Then $u(t) = (u_1(t),u_2(t)) \in \Gamma(u(t))$ implies $u_1(t) = |u(t)|$, and therefore $u_2(t) = 0$ in $[0,T]$. For every $x = (x_1,x_2) \in \Gamma(u(t))$ we have $u_1'(t)(x_1 - u_1(t)) + (u_2'(t)+1)(x_2 - u_2(t)) \ge 0$, and since necessarily $x_1 = u_1(t)$, this reduces to $x_2 \ge 0$ for all $x_2 \in [-1,1]$, a contradiction. □

Nevertheless, contrary to the more general (1.1), (1.5) has a solution for $L = 1$ in (1.8) if $H = \mathbb{R}$.

Theorem 3.8. *Let $H = \mathbb{R}$, and let* (1.7) *be satisfied, $f \in L^\infty([0,T])$, and assume that* (1.8) *holds with $0 \le L \le 1$. If $v_0 \in \Gamma(v_0)$, then* (1.5) *has a solution on* $[0,T]$.

Proof. Let

$$\mathcal{V} = \{v : [0,T] \to \mathbb{R} : v(0) = v_0,\ |v(t) - v(s)| \le K|t-s| \quad \text{for } t,s \in [0,T]\}$$

with $K = |f|_{L^\infty([0,T])}$. Then $\emptyset \ne \mathcal{V} \subset C([0,T])$ is compact and convex. Define $S : \mathcal{V} \to C([0,T])$ by letting $u = Sv$ be the unique solution of the inhomogeneous classical sweeping process with $C_v(t) = \Gamma(v(t))$, i.e.,

$$-u'(t) \in N_{C_v(t)}(u(t)) + f(t) \quad \text{a.e. in } [0,T],\ u(0) = v_0 \in C_v(0). \tag{3.25}$$

Note that in fact $v_0 \in C_v(0)$, since $v_0 \in \Gamma(v_0)$ and $v(0) = v_0$ by definition of $\mathcal{V}$. Moreover, by (1.8),

$$\mathrm{d}_H(C_v(t), C_v(s)) \le L|v(t) - v(s)| \le LK|t-s| \le K|t-s|, \quad t,s \in [0,T],$$

and thus Lemma 2.4 implies that $u = Sv$ is Lipschitz with constant

$$\max\{|f|_{L^\infty([0,T])}, K\} = |f|_{L^\infty([0,T])} = K,$$

whence $u(0) = v_0$ yields $S(\mathcal{V}) \subset \mathcal{V}$. Hence, to find a fixed point of S which will be a solution of (1.5), it is enough to show that S is continuous. Fix $v, \overline{v} \in \mathcal{V}$ and let $u = Sv \in \mathcal{V}$ and $\overline{u} = S\overline{v} \in \mathcal{V}$. In particular, $|u'(t)| \le K$ a.e. and $|\overline{u}'(t)| \le K$ a.e. Since $\mathrm{d}_H(C_v(t), C_{\overline{v}}(t)) \le L|v - \overline{v}|_{C([0,T])}$, we find continuous functions $r_1, r_2 : [0,T] \to H = \mathbb{R}$ such that $u(t) \in C_v(t) \subset C_{\overline{v}}(t) + r_1(t)$ and $\overline{u}(t) \in C_{\overline{v}}(t) \subset C_v(t) + r_2(t)$, as well as $|r_i(t)| \le L|v - \overline{v}|_{C([0,T])}$, $i = 1,2$. By (3.25), for u and the corresponding equation for $\overline{u}$, we thus obtain

$$\langle u'(t), u(t) - \overline{u}(t) + r_2(t)\rangle \le 0 \quad \text{and} \quad \langle \overline{u}'(t), \overline{u}(t) - u(t) + r_1(t)\rangle \le 0.$$

Hence a.e. in $[0,T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t) - \overline{u}(t)|^2 &= \langle u'(t), u(t) - \overline{u}(t) + r_2(t)\rangle + \langle \overline{u}'(t), \overline{u}(t) - u(t) + r_1(t)\rangle \\ &\quad - \langle u'(t), r_2(t)\rangle - \langle \overline{u}'(t), r_1(t)\rangle \le 2KL|v - \overline{v}|_{C([0,T])}.\end{aligned}$$

This in turn yields by integration

$$|Sv - S\overline{v}|_{C([0,T])} \le \left(4KLT|v - \overline{v}|_{C([0,T])} + |v - \overline{v}|^2_{C([0,T])}\right)^{1/2},$$

and therefore the continuity of S. □

The fixed-point approach described in Theorem 3.8 would give an alternative proof of Theorem 3.3 for $0 \le L_2 < 1$. We want to close with some further remarks.

Remark 3.9. In some applications there appear quasi-variational inequalities of type (1.5) with a badly behaved moving set $v \mapsto \Gamma(v)$, as there is no way to show that the dependence is Lipschitz continuous w.r.to d_H; cf. [7] for a stationary example, where $v_n \to v$ only implies $\Gamma(v_n) \to \Gamma(v)$ in the sense of Mosco. A corresponding time-dependent problem was introduced in [9] to model the evolution of sandpiles. In those cases, the very specific properties of the underlying PDE model have to be taken into account to prove the existence of solutions.

Remark 3.10. We have seen in Example 3.7 above that in general there will be no solution of (1.5) when $L = 1$ in (1.8). Nevertheless, one can obtain a weak-$*$ limit in $L^\infty([0,T];H)$ of a sequence $(v_n)_{n\in\mathbb{N}}$ of approximate solutions obtained by taking $(1-\varepsilon_n)\Gamma$ instead of Γ (which is $(1-\varepsilon_n)$-Lipschitz) and by deriving an L^∞-bound for these approximations. Perhaps the corresponding weak-$*$ limit can be interpreted as some kind of weak solution of (1.5). In our last lemma we show how the L^∞-bound is proved (under an additional assumption).

Lemma 3.11. *Let* (1.7) *hold*, $f \in L^\infty([0,T];H)$, *and assume that* (1.8) *is satisfied with* $L = 1$. *In addition, suppose that* (3.24) *is true, and*

$$\bigcap_{v\in H} \Gamma(v) \neq \emptyset.$$

If $v_0 \in \Gamma(v_0)$, *then there are "approximate solutions"* v_n (*in the above sense*) *of* (1.5) *such that* $\sup_{n\in\mathbb{N}} |v_n|_{L^\infty([0,T];H)} < \infty$.

Proof. Fix $a \in \bigcap_{v\in H} \Gamma(v)$. Changing if necessary to $\widetilde{v}(t) = v(t) - a$ and $\widetilde{\Gamma}(\widetilde{v}) = \Gamma(\widetilde{v} + a) - a$, we may assume that $a = 0$. Consider $\Gamma_n(v) = (1-\varepsilon_n)\Gamma(v)$ with $\varepsilon_n \to 0^+$. Then (1.7) and (3.24) hold for Γ_n, and Γ_n is d_H-Lipschitz with constant $(1-\varepsilon_n) < 1$. Hence Theorem 3.4 yields solutions $v_n : [0,T] \to H$ of

$$(3.26) \qquad -v_n'(t) \in N_{\Gamma_n(v_n(t))}(v_n(t)) + f(t) \quad \text{a.e. in } [0,T],\ v_n(0) = (1-\varepsilon_n)v_0 \in \Gamma_n(v_0).$$

Since $a = 0$, in particular $0 \in \Gamma_n(v)$ for all $n \in \mathbb{N}$ and $v \in H$. Hence, by (3.26), for a.e. $t \in [0,T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_n(t)|^2 &= \langle v_n'(t) + f(t), v_n(t)\rangle - \langle f(t), v_n(t)\rangle \\ &\le -\langle f(t), v_n(t)\rangle \le |f|_{L^\infty([0,T];H)}|v_n(t)|.\end{aligned}$$

Therefore

$$\frac{1}{2}\,|v_n(t)|^2 \le \frac{1}{2}|v_n(0)|^2 + |f|_{L^\infty([0,T];H)} \int_0^t |v_n(s)|\,ds$$

yields, cf. [2, Lemme A.5, p. 157],

$$|v_n(t)| \le |v_n(0)| + t|f|_{L^\infty([0,T];H)} \le |v_0| + T|f|_{L^\infty([0,T];H)} \quad \text{for } n \in \mathbb{N},\ t \in [0,T],$$

and this concludes the proof. □

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