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CONLEY INDEX AND PERMANENCE IN DYNAMICAL SYSTEMS

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1. Introduction

The motivation for our problem comes from permanence theory, which plays an important role in mathematical ecology. Roughly speaking, a flow f on $\mathbb{R}^n \times [0,\infty)$ is said to be permanent (or uniformly persistent) whenever $\mathbb{R}^n \times \{0\}$ is a repeller (see [7]). Other closely related terminology includes cooperativity, persistence and ecological stability. For a discussion of how these terms are related, see [1], [9]. The criterion of permanence for biological systems is a condition ensuring the long-term survival of all species. Sufficient conditions for permanence have been given for a wide variety of models. For more details and extensive bibliographies concerning the problem, we refer the reader to [2], [8].

In this paper we show that if $S \subset \mathbb{R}^n \times \{0\}$ is an isolated invariant set with nonzero homological Conley index, then there exists an x in $\mathbb{R}^n \times (0,\infty)$ such that $\omega(x)$ is contained in S. This may be understood as a strong violation of permanence.

We first give a brief account of the Conley index theory.

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2. Isolating blocks and Conley index

The Conley index theory is a very elegant and useful tool in the study of qualitative properties of nonlinear dynamical systems. Generalizing the Morse index of a non-degenerate critical point of a differentiable function it associates with an isolated invariant set of a flow a homotopy type of a space with base point.

Let X be a locally compact, metric space. By a *flow* on X we mean a continuous function

$$X \times \mathbb{R} \ni (x,t) \to xt \in X$$

such that $x0 = x$ and $x(s+t) = (xs)t$. The *backward flow* is defined as the map

$$X \times \mathbb{R} \ni (x,t) \to x(-t) \in X.$$

A set $S \subset X$ is called *invariant* if $S\mathbb{R} = S$. If $N \subset X$, then the set $\operatorname{inv}(N) = \{x \in N : x\mathbb{R} \subset N\}$ is the maximal invariant set contained in N. N is called an *isolating neighborhood* if $\operatorname{inv}(N) \subset \operatorname{int} N$. An invariant set S is said to be *isolated* if there exists an isolating neighborhood N such that $S = \operatorname{inv}(N)$. The basis of the Conley index theory is the notion of an isolating block. We recall that a set $\Sigma \subset X$ is called a *δ-section* provided $\Sigma(-\delta,\delta)$ is an open set in X and the map

$$\Sigma \times (-\delta,\delta) \ni (x,t) \to xt \in \Sigma(-\delta,\delta)$$

is a homeomorphism. Let B be a compact subset X. B is called an *isolating block* if there exists a $\delta > 0$ and two δ-sections Σ^+ and Σ^- such that

(i) $$\operatorname{cl}(\Sigma^+ \times (-\delta,\delta)) \cap \operatorname{cl}(\Sigma^- \times (-\delta,\delta) = \emptyset,$$

(ii) $$B \cap (\Sigma^+(-\delta,\delta)) = (B \cap \Sigma^+)[0,\delta),$$
$$B \cap (\Sigma^-(-\delta,\delta)) = (B \cap \Sigma^-)(-\delta,0],$$

(iii) $$\forall x \in \partial B \setminus (\Sigma^+ \cup \Sigma^-)\ \exists \mu < 0 < \nu : x\mu \in \Sigma^+, x\nu \in \Sigma^- \text{ and } x[\mu,\nu] \subset \partial B.$$

We put $B^+ = B \cap \Sigma^+$, $B^- = B \cap \Sigma^-$, $a^+ = \{x \in B^+ : x[0,\infty) \subset B\}$, $a^- = \{x \in B^- : x(-\infty,0] \subset B\}$, $A^+ = \{x \in B : x[0,\infty) \subset B\}$ and $A^- = \{x \in B : x(-\infty,0] \subset B\}$.

THEOREM 1. *If S is an isolated invariant set, then each isolating neighborhood of S contains a block which is a neighborhood of S. If B_1 and B_2 are two blocks which isolate S then the homotopy types of the pointed spaces $(B_1/B_1^-, [B_1^-])$ and $(B_2/B_2^-, [B_2^-])$ coincide.*

For the proof see [4], [5].

The homotopy type determined by Theorem 1 is denoted by $h(S)$ and is called the *Conley index* of S. Unfortunately, working with homotopy classes of spaces is difficult. To get around this, it is useful consider the homological Conley index. If H denotes an arbitrary homology or cohomology functor, then $H(h(S)) \cong H(B, B^-)$. This is proved in [11, p. 57]. By $h^*(S)$ we denote the Conley index of S with respect to the backward flow. Obviously $H(h^*(S)) = H(B, B^+)$. In this paper we denote by H the singular homology functor with coefficients in $\mathbb{Z}$ (or any field), but this is not an essential assumption.

3. Main result

For brevity, we write $E^+ = \mathbb{R}^n \times [0, \infty)$. The main result of this paper is the following

Theorem 2. *Assume that f is a continuous flow on E^+ (observe that ∂E^+ is invariant for f). Let $S \subset \partial E^+$ be an isolated invariant set* (in E^+) *with nonzero homological Conley index (in the whole phase space E^+). Then there exists an x in $\operatorname{int} E^+ = \mathbb{R}^n \times (0, \infty)$ such that $\emptyset \neq \omega(x) \subset S$.*

Remark 3. The same problem was first investigated by A. Capietto and B. M. Garay in [3]. Their approach works for flows induced by vector fields and with the assumption that S is a saturated invariant set with nontrivial Conley index with respect to the flow f restricted to ∂E^+. Geometrically, the saturatedness of S means that there is a neighbourhood N of S such that the trajectories run downward inside $N \setminus \partial E^+$. By application of the time-duality of the Conley index (see [10]), the results of [12] extend Theorem 1 of [3] to the case in which the set S is of attracting type (see definition below) with nonzero Conley index on the boundary and any continuous flow. Actually, Proposition 11 of [12] is a special case of our Theorem 2. Indeed, for attracting type sets the Conley indices with respect to f and f restricted to the boundary are the same, by Remark 9 of [12].

We use the notion of the repelling type set introduced in [12]. An isolated invariant set $S \subset \partial E^+$ is called of *repelling type* if and only if the stable set $W^+(S) = \{x \in E^+ : \emptyset \neq \omega(x) \subset S\}$ is contained in ∂E^+. Set of attracting type are defined by reversal of time.

We shall need the following fact:

Lemma 4. *Assume $S \subset \partial E^+$ is an isolated invariant set for f and B is an isolating block such that $S = \operatorname{inv} B$. Then*

(1) *$H(B, B \setminus S) = 0$,*
(2) *if S is of repelling type then $H(B^+, B^+ \setminus a^+) = 0$.*

Proof. (1) Using excision property we have $H(B, B \setminus S) \cong H(E^+, E^+ \setminus S)$ and any point $p \in \operatorname{int} E^+$ is a strong deformation retract of both E^+ and $E^+ \setminus S$ (by radial deformation).

(2) Let Σ^+ be a δ-section from the definition of the isolating block (hence $B^+ = \Sigma^+ \cap B$). Since $a^+ \subset \operatorname{int} B^+ (\operatorname{rel} \Sigma^+)$ (see [4]), we have

$$H(B^+, B^+ \setminus a^+) \cong H(\Sigma^+, \Sigma^+ \setminus a^+),$$

by the excision property. We show that $H(\Sigma^+, \Sigma^+ \setminus a^+) = 0$. As S is of repelling typ, $a^+ \subset \operatorname{int} B^+ \cap \partial E^+$ and there is a compact neighborhood K of a^+ such that $K \subset \operatorname{int} B^+ \cap \partial E^+$. For $0 < \delta_1 < \delta$ we put

$$U = \Sigma^+(-\delta, \delta) \cap \operatorname{int} E^+,$$
$$V = K(-\delta_1, \delta_1) \subset \partial E^+,$$
$$W = (K \setminus a^+)(-\delta_1, \delta_1) \subset \partial E^+.$$

We define

$$N = U \cup V, \quad N_1 = U \cup W \quad \text{and} \quad K^+ = (\Sigma^+ \cap \operatorname{int} E^+) \cup K.$$

It follows by the definition of δ-section that K^+ is a strong deformation retract of N and $K^+ \setminus a^+$ is a strong deformation retract of N_1, so

$$H(N, N_1) \cong H(K^+, K^+ \setminus a^+).$$

By the excision property we have

$$H(K^+, K^+ \setminus a^+) \cong H(\Sigma^+, \Sigma^+ \setminus a^+),$$
$$H(N, N_1) \cong H(\operatorname{int} E^+ \cup V, \operatorname{int} E^+ \cup W).$$

Hence

$$H(\Sigma^+, \Sigma^+ \setminus a^+) \cong H(\operatorname{int} E^+ \cup V, \operatorname{int} E^+ \cup W) = 0,$$

because any point $p \in \operatorname{int} E^+$ is a strong deformation retract of both $\operatorname{int} E^+ \cup V$ and $\operatorname{int} E^+ \cup W$ (by radial deformation).

Theorem 2 is a simple consequence of the following

Proposition 5. *If $S \subset \partial E^+$ is of repelling type then $H(h(S)) = 0$.*

Proof. Let B be any isolating block for S. By Proposition 3.7 of [4], B^- (or B^+) is a strong deformation retract of $B \setminus A^+$ ($B \setminus A^-$, respectively) so we must prove that $H(h(S)) \cong H(B, B^-) \cong H(B, B \setminus A^+)$ is trivial. Consider the Mayer–Vietoris exact sequence for the triple $(B, B \setminus A^+, B \setminus A^-)$ (see [6]):

$$\cdots \to H(B, B \setminus A) \to H(B, B \setminus A^+) \oplus H(B, B \setminus A^-) \to H(B, B \setminus S) \to \cdots,$$

where $A = A^+ \cup A^-$.

By Lemma 4, $H(B, B \setminus S) = 0$, so that

$$H(B, B \setminus A) \cong H(B, B \setminus A^+) \oplus H(B, B \setminus A^-).$$

We show that $H(B, B \setminus A) \cong H(B, B \setminus A^-)$ and this gives $H(B, B \setminus A^+) = 0$ because we are dealing with finitely generated abelian groups. By Proposition

3.5 of [4] $B^+ \setminus a^+$ is a strong deformation retract of $B \setminus A$, so it is sufficient to prove that $H(B, B^+ \setminus a^+)$ is isomorphic to $H(B, B^+)$. For this we take the long exact sequence of the triple $(B, B^+, B^+ \setminus a^+)$:

$$\cdots \to H(B^+, B^+ \setminus a^+) \to H(B, B^+ \setminus a^+) \to H(B, B^+) \to \cdots,$$

in which the first term is trivial by Lemma 4.

Remark 6. In [12] it was proved that the Euler characteristic of the Conley index of a repelling type set is zero.

Remark 7. Let (X, d) be a locally compact, metric space and $\emptyset \neq E$ be a closed subset of X. Assume that f is a flow on E with invariant boundary ∂E. Suppose that an isolated invariant set $S \subset \partial E$ admits an isolating block for which there are a set $K \subset B \cap \operatorname{int} E$ and a deformation $F : B \times [0, 1] \to B$ such that:

(1) $F_0 = \mathrm{id}_B$,
(2) $F_1(B) = K$,
(3) $F_t(x) = x$ for all $x \in K$, $t \in [0, 1]$,
(4) $F_t(x) \in B \setminus \partial E$ for all $x \in B$, $t \in (0, 1]$.

Then, if the homological Conley index $H(h_E(S))$ of S in E is nontrivial then there is an x in $E \setminus \partial E$ such that $\emptyset \neq \omega(x) \subset S$. For the proof suppose that $W^+(S) \subset \partial E$. Then $H(h_E(S)) \cong H(B, B^-) \cong H(B, B \setminus A^+)$. But A^+ is contained in ∂E, so K is a strong deformation retract of both B and $B \setminus A^+$, hence $H(h_E(S)) = 0$.

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